On weakly stable Yang-Mills fields over positively pinched manifolds and certain symmetric spaces

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ON WEAKLY STABLE YANG-MILLS FIELDS OVER POSITIVELY PINCHED MANIFOLDS AND CERTAIN SYMMETRIC SPACES

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Abstract. In this paper it is proved that for $n \geq 5$ there exists a constant $\delta(n)$ with $\delta \leq \delta(n) < 1$ such that any weakly stable Yang-Mills connection over a simply connected compact Riemannian manifold M with $\delta(n)$-pinched sectional curvatures is always flat. The pinching constants are possible to compute by elementary functions. Moreover we give some remarks on stability of Yang-Mills connections over certain symmetric spaces.

Introduction.

Let M be an n-dimensional compact Riemannian manifold with a metric g and G be a compact Lie group with the Lie algebra $\mathbf{g}$. Let E be a Riemannian vector bundle over M with structure group G, and let $\mathcal{C}_E$ denote the space of G-connections in E, which is an affine space modeled on the vector space $\Omega^1(\mathbf{g}_E)$ of smooth 1-forms with values in the adjoint bundle $\mathbf{g}_E$ of E. The Yang-Mills functional $\mathcal{YM} : \mathcal{C}_E \longrightarrow \mathbf{R}$ is

$$\mathcal{YM}(\nabla) = \frac{1}{2}\int_M \|F^\nabla\|^2 \mathrm{dvol},$$

for each $\nabla \in \mathcal{C}_E$, where F^∇ is the curvature form of the connection ∇. Note that F^∇ is a smooth section of $\Omega^2(\mathbf{g}_E)$. The Yang-Mills connection $\nabla \in \mathcal{C}_E$ is a critical point of $\mathcal{YM}$. A Yang-Mills connection ∇ is called *weakly stable* if , for each variation $\nabla^t \in \mathcal{C}_E$ with $\nabla = \nabla^0$,

$$(d^2/dt^2)\mathcal{YM}(\nabla^t)|_{t=0} \geq 0.$$

M is called *Yang-Mills unstable* (cf. [K-O-T]) if for every vector bundle (E, G) over M, any weakly stable Yang-Mills connection on E is always flat. First Simons proved that the Euclidean n-sphere S^n for $n \geq 5$ is Yang-Mills unstable ([B-L]). Ever since several persons have investigated the instability of Yang-Mills fields over various Riemannian manifolds ; convex hypersurfaces, submanifolds, compact symmetric spaces (cf. [Ka],[K-O-T],[Pa1],[Sh],[Ta],[We]). In [K-O-T] it was shown that the Cayley projective plane $P_2(\mathbf{Cay})$ and the compact symmetric space of exceptional type E_6/F_4 are Yang-Mills unstable.

In this paper we first establish the instability theorem for Yang-Mills fields over a simply connected compact Riemannian manifold with sufficiently pinched sectional curvatures. Okayasu [Ok] used the construction and results of Ruh, Grove and Karcher ([Ru],[G-K-R1],[G-K-R2]) to show the instability of harmonic maps into a Riemannian manifold with sufficiently pinched sectional curvatures. By using the same idea, the second named author [Pa2] showed an instability theorem for harmonic maps from a Riemannian manifold with sufficiently pinched sectional curvatures to an arbitrary Riemannian manifold. We will also use it. Next we shall prove some results on weakly stable Yang-Mills fields over certain symmetric spaces. Some of them were stated in [K-O-T] without proof. They supplement results of Laquer [La] which determined the stablity of canonical connections over simply connected compact irreducible symmetric spaces. Moreover we prove that a weakly stable Yang-Mills field satisfying a certain condition over a quaternionic projective space $P_m(\mathbf{H})$ is a B_2-connection in a sense of [Ni], or equivalently a self-dual connection in a sense of [C-S], and hence it minimizes the Yang-Mills functional.

1. Preliminaries on Yang-Mills fields.

Let $\nabla \in \mathcal{C}_E$. For any $B \in \Omega^1(\mathbf{g}_E)$, set $\nabla^t = \nabla + tB \in \mathcal{C}_E$. The second variational formula for the Yang-Mills functional is given as follows ([B-L]);

$$(1.1)\qquad \begin{aligned}(d^2/dt^2)\mathcal{YM}(\nabla^t)|_{t=0} &= \mathcal{I}^\nabla(B,B)\\ &= \int_M (\mathcal{S}_0^\nabla(B),B)\mathrm{dvol}\\ &= \int_M \{(\mathcal{S}^\nabla(B),B)-(\delta^\nabla B,\delta^\nabla B)\}\mathrm{dvol},\end{aligned}$$

where $\mathcal{S}_0^\nabla(B)=\delta^\nabla d^\nabla B+\mathcal{F}^\nabla(B)$ and $\mathcal{S}^\nabla(B)=\Delta^\nabla(B)+\mathcal{F}^\nabla(B)$. Here d^∇ and δ^∇ denote the exterior covariant differentiation induced by the connection $\nabla\in\mathcal{C}_E$ and its adjoint differential operator, and $\mathcal{F}^\nabla$ is a symmetric bundle endomorphism of $T^*M\otimes\mathbf{g}_E$ defined by $(\mathcal{F}^\nabla(b))(X)=\sum_{i=1}^n[F^\nabla(e_i,X),b(e_i)]$ for $b\in T_x^*M\otimes(\mathbf{g}_E)_x$ and $X\in T_xM$, where $\{e_i\}$ is an orthonormal basis of T_xM.

Let $\{\omega^i\}$ be the dual frame of a local orthonormal frame field $\{e_i\}$ in M. Throughout this paper we use the summation convention. Set $B=B_i\omega^i$ and $F^\nabla=(1/2)F_{ij}\omega^i\wedge\omega^j$. Then we have

$$\begin{aligned}d^\nabla B &= (\nabla_i B_j-\nabla_j B_i)\omega^i\wedge\omega^j,\\ \delta^\nabla d^\nabla B &= (\nabla_j\nabla_i B_j-\nabla_j\nabla_j B_i)\omega^i,\\ \mathcal{F}^\nabla(B) &= [F_{ij},B_i]\omega^j,\\ \|F^\nabla\|^2 &= (F_{ij},F_{ij})/2\end{aligned}$$

And (1.1) becomes

$$\begin{aligned}&(d^2/dt^2)\mathcal{YM}(\nabla^t)|_{t=0}\\&=\int_M\{(\nabla_j\nabla_iB_j,B_i)-(\nabla_j\nabla_jB_i,B_i)+([F_{ij},B_i],B_j)\}\mathrm{dvol}.\end{aligned}$$

Let D be a Riemannian connection of M and let R denote the curvature tensor field of D ; $R(e_i,e_j)e_k=R_{ijkl}e_l$. The Ricci tensor field Ric of M is defined by $R_{ij}=R_{ikkj}$. The scalar curvature R of M is defined by $R=R_{ii}$. The Ricci identities are as follows:

$$\begin{aligned}&D_kD_jX^i-D_jD_kX^i=R_{kjli}X^l\quad\text{for}\quad X=X^ie_i,\\&\nabla_l\nabla_kF_{ij}-\nabla_k\nabla_lF_{ij}=-F_{mj}R_{lkij}-F_{im}R_{lkjm}+[F_{lk},F_{ij}].\end{aligned}$$

The curvature form F^∇ always satisfies the Bianchi identity $d^\nabla F^\nabla=0$, or equivalently

$$\nabla_kF_{ij}+\nabla_iF_{jk}+\nabla_jF_{ki}=0. \tag{1.2}$$

The Yang-Mills equation is $\delta^\nabla F^\nabla=0$, namely

$$\nabla_jF_{ij}=0. \tag{1.3}$$

Let $\nabla\in\mathcal{C}_E$. Assume that $\varphi=(1/2)\varphi_{ij}\omega^i\wedge\omega^j\in\Omega^2(\mathbf{g}_E)$ is harmonic with respect to ∇, that is, $d^\nabla\varphi=0$ and $\delta^\nabla\varphi=0$. Note that if ∇ is a Yang-Mills connection, we can take $\varphi=F^\nabla$. Let $V\in C^\infty(TM)$ with $V=V^ie_i$. Set $B=i_V\varphi=B_i\omega^i\in\Omega^1(\mathbf{g}_E)$. Here $B_i=V^j\varphi_{ji}$. Then by the harmonicity of φ and the Bochner-Weitzenböck formula (cf. [B-L]) we compute

$$\begin{aligned}(\mathcal{S}^\nabla(B))(X)&=\varphi(D^*DV,X)-2\sum_{i=1}^n(\nabla_{e_i}\varphi)(D_{e_i}V,X)\\&+\varphi(V,\mathrm{Ric}(X))-\{\varphi\circ(\mathrm{Ric}\wedge I-2\mathcal{R})\}(V,X)\\&-\sum_{i=1}^n\{[F^\nabla(e_i,V),\varphi(e_i,X)]+[F^\nabla(e_i,X),\varphi(e_i,V)]\},\end{aligned} \tag{1.4}$$

where $D^*DV=-\sum_{i=1}^nD^2V(e_i,e_i)$, and $\mathcal{R}$ denotes the curvature operator of (M,g) acting on $\bigwedge^2TM$. We define a quadratic form Q_φ on $C^\infty(TM)$ as

$$Q_\varphi(V)=(d^2/dt^2)\mathcal{YM}(\nabla^t)|_{t=0}=\int_Mq_\varphi(V)\mathrm{dvol},$$

where $\nabla^t = \nabla + t(i_V\varphi) \in \mathcal{C}_E$. By straightforward computations we have

$$\begin{aligned}(1.5)\quad q_\varphi(V) = & D_jD_iV^kV^l(\varphi_{kj},\varphi_{li}) - D_jD_jV^kV^l(\varphi_{ki},\varphi_{li}) \\ & + D_jV^kV^l(\nabla_i\varphi_{kj},\varphi_{li}) - 2D_jV^kV^l(\nabla_j\varphi_{ki},\varphi_{li}) \\ & + V^kV^l([F^\nabla_{jk},\varphi_{ij}] + [F^\nabla_{ji},\varphi_{kj}],\varphi_{li}) \\ & + V^kV^l\{R_{ikmj}(\varphi_{mj},\varphi_{li}) - R_{jikm}(\varphi_{mj},\varphi_{li}) + R_{km}(\varphi_{im},\varphi_{li})\}.\end{aligned}$$

2. The construction of Ruh for a δ-pinched manifold.

We recall the idea and construction of Ruh ([Ru],[G-K-R1],[G-K-R2]). Let (M,g) be an n-dimensional simply connected compact Riemannian manifold with δ-pinched sectional curvature, namely $\delta < K \leq 1$. We fix a normalized Riemannian metric $g_0 = \{(1+\delta)/2\}g$ on M. Then we have $2\delta/(1+\delta) < K_{g_0} \leq 2/(1+\delta)$. Consider a vector bundle $\Xi = TM \oplus \varepsilon(M)$ with a fibre metric $\langle\ ,\ \rangle$ over M. Here $\varepsilon(M)$ is a trivial line bundle with a fiber metric and it is orthogonal to TM. Let e denote a smooth section of lengh 1 in $\varepsilon(M)$. Now we define a metric connection D'' in Ξ as follows;

$$D''_XY = D_XY - g_0(X,Y)e.$$
$$D''_Xe = X$$

for $X, Y \in C^\infty(TM)$. It was proved that if δ is sufficiently close to 1, there exists a flat connection D' in Ξ close to D'' ([G-K-R1]). Define

$$\begin{aligned}&\|D' - D''\| \\ &:= Max\{\|D'_XY - D''_XY\|; X \in T_xM, g_0(X,X) = 1, Y \in \Xi_x, \|Y\| = 1\}.\end{aligned}$$

Note that it is a half of that one in [G-K-R2]. Set

$$\begin{aligned}&k_1(\delta) = (4/3)(1-\delta)\delta^{-1}\{1 + (\delta^{1/2}\sin(1/2)\pi\delta^{-1/2})^{-1}\},\\ &k_2(\delta) = \{(1+\delta)/2\}^{-1}k_1(\delta),\\ &k_3(\delta) = k_2(\delta)[1 + \{1 - (1/24)\pi^2k_2(\delta)^2\}^{-2}]^{1/2}.\end{aligned}$$

[G-K-M 2] proved that $\|D' - D''\| \leq k_3(\delta)/2$. The curvature form R'' of the connection D'' is

$$(2.1)\qquad R''(X,Y)Z = R(X,Y)Z - \langle Y,Z\rangle X + \langle X,Z\rangle Y,$$

$$(2.2)\qquad R''(X,Y)e = 0$$

for $X, Y, Z \in T_xM$.

3. Trace formula for second variations of Yang-Mills fields over a δ-pinched manifold.

Assume that M is a simply connected compact Riemannian manifold with δ-pinched sectional curvatures. Let $P = \{v \in C^\infty(\Xi); D'v = 0\}$, which is linerly isometric to $\mathbf{R}^{n+1}$. For each $v \in P$, we denote by $V = v^T$ the TM-component of v in Ξ. Set $\mathcal{V} = \{V \in C^\infty(TM); V = v^T$ for some $v \in P\}$, which has a natural inner product so that it is linearly isometric to P. Choose an orthonormal basis $\{V_\alpha\}_{\alpha=0,\dots,n}$ of $\mathcal{V}$. Set $V_\alpha = (v_\alpha)^T$. Then $\sum_{\alpha=0}^n V_\alpha^k V_\alpha^l = \delta^{kl}$. In this section we compute the trace $\mathrm{Tr}_\mathcal{V} Q_\varphi = \sum_{\alpha=0}^n Q_\varphi(V_\alpha)$ of Q_φ on $\mathcal{V}$ relative to the inner product.

A straightforward computation shows

Lemma 3.1.

$$D_jV^k = \langle D''_{e_j}v, e_k\rangle - \langle v, e\rangle\delta_{jk}. \tag{3.1}$$

$$\begin{aligned} &D_jD_iV^k \\ &= \langle (D''^2v)(e_i, e_j), e_k\rangle - \delta_{jk}\langle D''_{e_i}v, e\rangle - \delta_{ik}\langle D''_{e_j}v, e\rangle - \delta_{ik}\langle v, e_j\rangle. \end{aligned} \tag{3.2}$$

Lemma 3.2.

$$\begin{aligned} &\int_M \{D_jD_iV^kV^l(\varphi_{kj}, \varphi_{li}) + D_jV^kV^l(\nabla_i\varphi_{kj}, \varphi_{li})\}\mathrm{dvol} \\ &= \int_M \{R_{jimk}V^mV^l(\varphi_{kj}, \varphi_{li}) - D_jV^kD_iV^l(\varphi_{kj}, \varphi_{li})\}\mathrm{dvol}. \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\int_M -2D_jV_\alpha^kV_\alpha^l(\nabla_j\varphi_{ki}, \varphi_{li})\mathrm{dvol} \\ &= \int_M \{-2D_kD_jV_\alpha^kV_\alpha^l(\varphi_{ij}, \varphi_{li}) - 2D_jV_\alpha^kD_kV_\alpha^l(\varphi_{ij}, \varphi_{li}) \\ &\quad - D_iD_jV_\alpha^kV_\alpha^l(\varphi_{ij}, \varphi_{kl}) - D_jV_\alpha^kD_iV_\alpha^l(\varphi_{ij}, \varphi_{kl}) \\ &\quad - 2D_iD_jV_\alpha^kV_\alpha^l(\varphi_{jk}, \varphi_{li}) - 2D_jV_\alpha^kD_iV_\alpha^l(\varphi_{jk}, \varphi_{li})\}\mathrm{dvol}. \end{aligned} \tag{3.4}$$

Proof. (3.3) is due to the Ricci identity and the divergence theorem. We show (3.4). By $d^\nabla\varphi = 0$, we have

$$\begin{aligned} &-2D_jV_\alpha^kV_\alpha^l(\nabla_j\varphi_{ki}, \varphi_{li}) \\ &= 2D_jV_\alpha^kV_\alpha^l(\nabla_k\varphi_{ij}, \varphi_{li}) + 2D_jV_\alpha^kV_\alpha^l(\nabla_i\varphi_{jk}, \varphi_{li}). \end{aligned}$$

By using the divergence theorem, we get

$$\begin{aligned}&\int_M 2D_jV_\alpha^kV_\alpha^l(\nabla_i\varphi_{jk},\varphi_{li})\mathrm{dvol}\\&=\int_M\{-2D_iD_jV_\alpha^kV_\alpha^l(\varphi_{jk},\varphi_{li})-2D_jV_\alpha^kD_iV_\alpha^l(\varphi_{jk},\varphi_{li})\}\mathrm{dvol}.\end{aligned}$$

We compute

$$\begin{aligned}&2D_jV_\alpha^kV_\alpha^l(\nabla_k\varphi_{ij},\varphi_{li})\\&=2D_k\{D_jV_\alpha^kV_\alpha^l(\varphi_{ij},\varphi_{li})\}-2D_kD_jV_\alpha^kV_\alpha^l(\varphi_{ij},\varphi_{li})\\&\quad-2D_jV_\alpha^kD_kV_\alpha^l(\varphi_{ij},\varphi_{li})-2D_jV_\alpha^kV_\alpha^l(\varphi_{ij},\nabla_k\varphi_{li}).\end{aligned}$$

Since

$$D_jV_\alpha^kV_\alpha^l=-V_\alpha^kD_jV_\alpha^l, \tag{3.6}$$

we have

$$D_jV_\alpha^kV_\alpha^l(\varphi_{ij},\nabla_k\varphi_{li})=D_jV_\alpha^kV_\alpha^l(\varphi_{ij},\nabla_l\varphi_{ik}).$$

Hence by Bianchi identity we get

$$-2D_jV_\alpha^kV_\alpha^l(\varphi_{ij},\nabla_k\varphi_{li})=D_jV_\alpha^kV_\alpha^l(\varphi_{ij},\nabla_i\varphi_{kl}).$$

Thus by using the divergence theorem we obtain

$$\begin{aligned}&\int_M 2D_jV_\alpha^kV_\alpha^l(\nabla_k\varphi_{ij},\varphi_{li})\mathrm{dvol}\\&=\int_M\{-2D_kD_jV_\alpha^kV_\alpha^l(\varphi_{ij},\varphi_{li})-2D_jV_\alpha^kD_kV_\alpha^l(\varphi_{ij},\varphi_{li})\\&\quad-D_iD_jV_\alpha^kV_\alpha^l(\varphi_{ij},\varphi_{kl})-D_jV_\alpha^kD_iV_\alpha^l(\varphi_{ij},\varphi_{kl})\}\mathrm{dvol}.\end{aligned}$$

q.e.d.

By (1.5),(3.3) and (3.4), we get

$$\begin{aligned}&\mathrm{Tr}_{\mathcal V}\,Q_\varphi\\&=\int_M\{-D_jV_\alpha^kD_iV_\alpha^l(\varphi_{kj},\varphi_{li})-D_jD_jV_\alpha^kV_\alpha^l(\varphi_{ki},\varphi_{li})\\&\quad-2D_kD_jV_\alpha^kV_\alpha^l(\varphi_{ij},\varphi_{li})-2D_jV_\alpha^kD_kV_\alpha^l(\varphi_{ij},\varphi_{li})\\&\quad-D_iD_jV_\alpha^kV_\alpha^l(\varphi_{ij},\varphi_{kl})-D_jV_\alpha^kD_iV_\alpha^l(\varphi_{ij},\varphi_{kl})\\&\quad-2D_iD_jV_\alpha^kV_\alpha^l(\varphi_{jk},\varphi_{li})-2D_jV_\alpha^kD_iV_\alpha^l(\varphi_{jk},\varphi_{li})\\&\quad+R_{jilk}(\varphi_{kj},\varphi_{li})+R_{ikmj}(\varphi_{mj},\varphi_{ki})\\&\quad-R_{jikm}(\varphi_{mj},\varphi_{ki})+R_{km}(\varphi_{im},\varphi_{ki})\}\mathrm{dvol}.\end{aligned} \tag{3.7}$$

Lemma 3.3.

$$(3.8)\qquad \begin{aligned} &-2D_iD_jV_\alpha^kV_\alpha^l(\varphi_{jk},\varphi_{li})\\ &= D_jV_\alpha^kD_iV_\alpha^l(\varphi_{jk},\varphi_{li}) + D_iV_\alpha^kD_jV_\alpha^l(\varphi_{jk},\varphi_{li})\\ &\quad + R_{jimk}V_\alpha^mV_\alpha^l(\varphi_{jk},\varphi_{li}),\end{aligned}$$

$$(3.9)\qquad -D_iD_jV_\alpha^kV_\alpha^l(\varphi_{ij},\varphi_{kl}) = -(1/2)R_{ijmk}V_\alpha^mV_\alpha^l(\varphi_{ij},\varphi_{kl}).$$

Proof. (3.9) is due to the Ricci identity. We show (3.8). Differentiating covariantly (3.6), we have

$$(3.10)\qquad \begin{aligned} &D_iD_jV_\alpha^kV_\alpha^l + V_\alpha^kD_iD_jV_\alpha^l\\ &\quad + D_jV_\alpha^kD_iV_\alpha^l + D_iV_\alpha^kD_jV_\alpha^l = 0.\end{aligned}$$

(3.8) follows from (3.10) and the Ricci identity. q.e.d.

Lemma 3.4.

$$(3.11)\qquad \begin{aligned} -D_jD_jV_\alpha^kV_\alpha^l(\varphi_{ki},\varphi_{li}) &= \langle D''_{e_i}v_\alpha, D''_{e_i}v_\beta\rangle V_\beta^kV_\alpha^l(\varphi_{ki},\varphi_{li})\\ &\quad + \{2\langle D''_{e_k}v_\alpha, e\rangle + \langle v_\alpha, e_k\rangle\}V_\alpha^l(\varphi_{ki},\varphi_{li}).\end{aligned}$$

Proof. From $\langle v_\alpha, v_\beta\rangle = \delta_{\alpha\beta}$, we have

$$(3.12)\qquad \begin{aligned} &\langle (D''^2v_\alpha)(e_i,e_j), v_\beta\rangle + \langle (D''^2v_\beta)(e_i,e_j), v_\alpha\rangle\\ &= -\langle D''_{e_i}v_\alpha, D''_{e_j}v_\beta\rangle - \langle D''_{e_j}v_\alpha, D''_{e_i}v_\beta\rangle.\end{aligned}$$

Using (3.2) and (3.12), we obtain (3.11). q.e.d.

Lemma 3.5.

$$(3.13)\qquad \begin{aligned} &\int_M -2D_kD_jV_\alpha^kV_\alpha^l(\varphi_{ij},\varphi_{li})\mathrm{dvol}\\ &= \int_M [2\langle D''_{e_j}v_\alpha, e\rangle V_\alpha^l(\varphi_{ij},\varphi_{li})\\ &\quad + 2\langle D''_{e_k}v_\alpha, e_k\rangle\langle D''_{e_j}v_\alpha, e_l\rangle(\varphi_{ij},\varphi_{li})\\ &\quad + 2\{(2-(n/2))\langle D''_{e_k}v_\alpha, e_k\rangle\langle v_\alpha, e\rangle - (1/4)\langle R''(e_l,e_k)e_k, e_l\rangle\\ &\qquad - (1/4)\langle D''_{e_k}v_\alpha, D''_{e_l}v_\beta\rangle\langle v_\beta, e_k\rangle\langle v_\alpha, e_l\rangle\\ &\qquad - (1/4)\langle D''_{e_k}v_\alpha, D''_{e_l}v_\beta\rangle\langle v_\beta, e_l\rangle\langle v_\alpha, e_k\rangle\\ &\qquad - (1/2)\langle D''_{e_k}v_\alpha, e\rangle V_\alpha^k + (1/2)\langle D''_{e_k}v_\alpha, e_k\rangle\langle D''_{e_l}v_\alpha, e_l\rangle\}\|\varphi\|^2\\ &\quad - 2\langle R''(e_k,e_j)e_l, e_k\rangle(\varphi_{ij},\varphi_{li}) + 2(n+1)\langle D''_{e_j}v_\alpha, e\rangle V_\alpha^l(\varphi_{ij},\varphi_{li})\\ &\quad + 2\langle v_\alpha, e_j\rangle V_\alpha^l(\varphi_{ij},\varphi_{li})]\mathrm{dvol}.\end{aligned}$$

Proof. By (3.2), we have

$$(3.14)\qquad \begin{aligned} &- 2D_kD_jV_\alpha{}^kV_\alpha{}^l(\varphi_{ij},\varphi_{li}) \\ &= -2\{\langle (D''^2 v_\alpha)(e_j,e_k),e_k\rangle - (n+1)\langle D''_{e_j}v_\alpha, e\rangle \\ &\quad - \langle v_\alpha, e_j\rangle\}V_\alpha^l(\varphi_{ij},\varphi_{li}). \end{aligned}$$

By using the Ricci identity we get

$$(3.15)\qquad \begin{aligned} &\langle (D''^2 v_\alpha)(e_j,e_k),e_k\rangle V_\alpha^l(\varphi_{ij},\varphi_{li}) \\ &= \{\langle (D''^2 v_\alpha)(e_k,e_j),e_k\rangle + \langle R''(e_k,e_j)v_\alpha, e_k\rangle\}V_\alpha^l(\varphi_{ij},\varphi_{li}). \end{aligned}$$

We compute

$$(3.16)\qquad \begin{aligned} &\langle (D''^2 v_\alpha)(e_k,e_j),e_k\rangle V_\alpha^l(\varphi_{ij},\varphi_{li}) \\ &= D_j\{\langle D''_{e_k}v_\alpha, e_k\rangle V_\alpha^l(\varphi_{ij},\varphi_{li})\} - \langle D''_{e_j}v_\alpha, e\rangle V_\alpha^l(\varphi_{ij},\varphi_{li}) \\ &\quad - \langle D''_{e_k}v_\alpha, e_k\rangle\langle D''_{e_j}v_\alpha, e_l\rangle(\varphi_{ij},\varphi_{li}) - \langle D''_{e_k}v_\alpha, e_k\rangle\langle v_\alpha, e\rangle(\varphi_{ij},\varphi_{ij}) \\ &\quad - \langle D''_{e_k}v_\alpha, e_k\rangle V_\alpha{}^l(\varphi_{ij},\nabla_j\varphi_{li}). \end{aligned}$$

By the Bianchi identity we get

$$(3.17)\qquad -\langle D''_{e_k}v_\alpha, e_k\rangle V_\alpha{}^l(\varphi_{ij},\nabla_j\varphi_{li}) = (1/4)\langle D''_{e_k}v_\alpha, e_k\rangle V_\alpha{}^l D_l\|\varphi\|^2.$$

We compute

$$(3.18)\qquad \begin{aligned} &\langle D''_{e_k}v_\alpha, e_k\rangle V_\alpha{}^l D_l\|\varphi\|^2 \\ &= D_l\{\langle D''_{e_k}v_\alpha, e_k\rangle V_\alpha{}^l\|\varphi\|^2\} - \langle (D''^2 v_\alpha)(e_k,e_l),e_k\rangle V_\alpha{}^l\|\varphi\|^2 \\ &\quad + \langle D''_{e_k}v_\alpha, e\rangle V_\alpha{}^k\|\varphi\|^2 - \langle D''_{e_k}v_\alpha, e_k\rangle\langle D''_{e_l}v_\alpha, e_l\rangle\|\varphi\|^2 \\ &\quad + n\langle D''_{e_k}v_\alpha, e_k\rangle\langle v_\alpha, e\rangle\|\varphi\|^2. \end{aligned}$$

By using (3.12) and the Ricci identity we get

$$(3.19)\qquad \begin{aligned} &\langle (D''^2 v_\alpha)(e_k,e_l),e_k\rangle V_\alpha{}^l \\ &= -(1/2)\{\langle R''(e_l,e_k)e_k,e_l\rangle + \langle D''_{e_k}v_\alpha, D''_{e_l}v_\beta\rangle V_\beta^k V_\alpha^l \\ &\quad + \langle D''_{e_k}v_\alpha, D''_{e_l}v_\beta\rangle V_\beta^l V_\alpha^k\}. \end{aligned}$$

Hence, by the divergence theorem, (3.13) follows from (3.14), (3.15), (3.16), (3.17), (3.18) and (3.19). q.e.d.

Therefore, by (2.1), (3.8), (3.9), (3.11) and (3.13), (3.7) reduces to the follwing trace formula.

(3.20) $\mathrm{Tr}_{\mathcal{V}}\ Q_\varphi$

$$\begin{aligned}
&= \int_M [2\{5-2n+(n(n-1)-R)/4\}\|\varphi\|^2 + R_{jl}(\varphi_{ij},\varphi_{il}) \\
&+ \langle D''_{e_i}v_\alpha, D''_{e_i}v_\beta\rangle V^k_\beta V^l_\alpha(\varphi_{ki},\varphi_{li}) - 2\langle D''_{e_k}v_\alpha, e_k\rangle\langle D''_{e_j}v_\alpha, e_l\rangle(\varphi_{ij},\varphi_{il}) \\
&+ 2\{(2-(n/2))\langle D''_{e_k}v_\alpha, e_k\rangle\langle v_\alpha, e\rangle \\
&\quad - (1/4)\langle D''_{e_k}v_\alpha, D''_{e_l}v_\beta\rangle\langle v_\beta, e_k\rangle\langle v_\alpha, e_l\rangle \\
&\quad - (1/4)\langle D''_{e_k}v_\alpha, D''_{e_l}v_\beta\rangle\langle v_\beta, e_l\rangle\langle v_\alpha, e_k\rangle \\
&\quad - (1/2)\langle D''_{e_k}v_\alpha, e\rangle V^k_\alpha + (1/2)\langle D''_{e_k}v_\alpha, e_k\rangle\langle D''_{e_l}v_\alpha, e_l\rangle\}\|\varphi\|^2 \\
&- 2(n+1)\langle D''_{e_j}v_\alpha, e\rangle V^l_\alpha(\varphi_{ij},\varphi_{il}) - 8\langle D''_{e_j}v_\alpha, e_k\rangle\langle v_\alpha, e\rangle(\varphi_{ij},\varphi_{ik}) \\
&+ 2\langle D''_{e_j}v_\alpha, e_k\rangle\langle D''_{e_k}v_\alpha, e_l\rangle(\varphi_{ij},\varphi_{il}) - \langle D''_{e_j}v_\alpha, e_k\rangle\langle D''_{e_i}v_\alpha, e_l\rangle(\varphi_{ij},\varphi_{kl}) \\
&+ \langle D''_{e_l}v_\alpha, e_j\rangle\langle D''_{e_i}v_\alpha, e_k\rangle(\varphi_{ij},\varphi_{kl})]\mathrm{dvol}.
\end{aligned}$$

4. Instability theorem for Yang-Mills fields over a δ-pinched Riemannian manifold.

Note that if δ=1, then $D' = D''$, hence (3.20) becomes

$$\mathrm{Tr}_{\mathcal{V}}\ Q_\varphi = 2(4-n)\int_M \|\varphi\|^2.$$

Since the sectional curvatures of M are δ-pinched, we have

$$\begin{aligned}
&2\{5-2n+(1/4)(n(n-1)-R)\}\|\varphi\|^2 + R_{jl}(\varphi_{ij},\varphi_{il}) \\
&\leq 2[5-2n+(1/4)n(n-1)\{1-2\delta/(1+\delta)\} + 2(n-1)/(1+\delta)]\|\varphi\|^2.
\end{aligned}$$

We can make estimates for each other term of (3.20) as follows:

$$
\begin{aligned}
&\langle D''_{e_i} v_\alpha, D''_{e_i} v_\beta\rangle V^k_\beta V^l_\alpha(\varphi_{ki}, \varphi_{li}) \le (n/2)k_3(\delta)^2\|\varphi\|^2, \\
&- 2\langle D''_{e_k} v_\alpha, e_k\rangle\langle D''_{e_j} v_\alpha, e_l\rangle(\varphi_{ij}, \varphi_{il}) \le n(n+1)k_3(\delta)^2\|\varphi\|^2, \\
&(2-(n/2))\langle D''_{e_k} v_\alpha, e_k\rangle\langle v_\alpha, e\rangle \le n(n/4-1)k_3(\delta), \\
&- (1/4)\langle D''_{e_k} v_\alpha, D''_{e_l} v_\beta\rangle\langle v_\beta, e_k\rangle\langle v_\alpha, e_l\rangle \le (n^2/16)k_3(\delta)^2, \\
&- (1/4)\langle D''_{e_k} v_\alpha, D''_{e_l} v_\beta\rangle\langle v_\beta, e_l\rangle\langle v_\alpha, e_k\rangle \le (n^2/16)k_3(\delta)^2, \\
&- (1/2)\langle D''_{e_k} v_\alpha, e\rangle V^k_\alpha \le (n/4)k_3(\delta), \\
&(1/2)\langle D''_{e_k} v_\alpha, e_k\rangle\langle D''_{e_l} v_\alpha, e_l\rangle \le (n^2/8)k_3(\delta)^2, \\
&- 2(n+1)\langle D''_{e_j} v_\alpha, e\rangle V^l_\alpha(\varphi_{ij}, \varphi_{il}) \le 2(n+1)k_3(\delta)\|\varphi\|^2, \\
&- 8\langle D''_{e_j} v_\alpha, e_k\rangle\langle v_\alpha, e\rangle(\varphi_{ij}, \varphi_{ik}) \le 8k_3(\delta)\|\varphi\|^2, \\
&2\langle D''_{e_j} v_\alpha, e_k\rangle\langle D''_{e_k} v_\alpha, e_l\rangle(\varphi_{ij}, \varphi_{il}) \le nk_3(\delta)\|\varphi\|^2, \\
&\langle D''_{e_l} v_\alpha, e_j\rangle\langle D''_{e_i} v_\alpha, e_k\rangle(\varphi_{ij}, \varphi_{kl}) \\
&\quad - \langle D''_{e_j} v_\alpha, e_k\rangle\langle D''_{e_i} v_\alpha, e_l\rangle(\varphi_{ij}, \varphi_{kl}) \le k_3(\delta)\|\varphi\|^2.
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
(4.1)\qquad &\mathrm{Tr}_{\mathcal{V}}\, Q_\varphi \\
&\le 2[5-2n+(1/4)n(n-1)\{1-2\delta/(1+\delta)\}+2(n-1)/(1+\delta) \\
&\quad + (1/4)(n^2+n+20)k_3(\delta)+(1/4)(3n^2+5n+2)k_3(\delta)^2]\int_M \|\varphi\|^2.
\end{aligned}
$$

Therefore we obtain

Theorem 4.1. *If $n \ge 5$ and*

$$
\begin{aligned}
(4.2)\qquad &5-2n+(1/4)n(n-1)\{1-2\delta/(1+\delta)\}+2(n-1)/(1+\delta) \\
&\quad + (1/4)(n^2+n+20)k_3(\delta)+(1/4)(3n^2+5n+2)k_3(\delta)^2 < 0,
\end{aligned}
$$

then M is Yang-Mills unstable.

Corollary 4.2. *For $n \ge 5$, there exists a constant $\delta(n)$, which depends only on n, with $1/4 \le \delta(n) < 1$ such that any n-dimensional simply connected compact Riemannian manifold M with $\delta(n)$-pinched sectional curvatures is Yang-Mills unstable.*

Remark. As n tends to the infinity, the right hand side of (4.2) divided by $(1/4)(3n^2+5n+2)$ tends to $(1/3)\{1-2\delta/(1+\delta)\}+(1/3)k_3(\delta)+k_3(\delta)^2 > 0$. In our argument it is not possible to find a pinching constant δ independent of the dimension of the base manifold M such that M is Yang-Mills unstable.

5. Trace formula for second variations of Yang-Mills fields over submanifolds in Euclidean space.

Assume that M is isometrically immersed in a Euclidean space $\mathbf{R}^N$. Let Φ denote the immersion. We may assume that $\Phi(M)$ is not contained in any hyperplane of $\mathbf{R}^N$. Set $\mathcal{U} = \{U \in C^\infty(TM); U = \text{grad } f_u \quad \text{for some } u \in \mathbf{R}^N\}$. Here f_u denotes the hight function on M defined by $f_u(x) = \langle \Phi(x), u \rangle$. Suppose that ∇ is a connection on a Riemannian vector bundle (E, G) over M and $\varphi \in \Omega^2(\mathbf{g}_E)$ is harmonic with respect to ∇. Then we recall

Proposition 5.1 ([K-O-T]). *For* $U = \text{grad } f_u \in \mathcal{U}$,

$$\begin{aligned} &\mathcal{S}^\nabla(i_U\varphi)(X) \\ &= -\{\varphi \circ (\text{Ric} \wedge I - 2\mathcal{R})\}(U, X) \\ &\quad + n\varphi(A_\eta(U), X) + \varphi(U, \text{Ric}(X)) - \varphi(\text{Ric}(U), X) \\ &\quad - \sum_{i=1}^{n}\{[F^\nabla(e_i, U), \varphi(e_i, X)] + [F^\nabla(e_i, X), \varphi(e_i, U)]\} \\ &\quad - 2\sum_{i,j=1}^{n} \langle B(e_i, e_j), u\rangle (\nabla_{e_j}\varphi)(e_i, X) - n\sum_{i=1}^{n}\langle D^\perp{}_{e_i}\eta, u\rangle \varphi(e_i, X). \end{aligned} \tag{5.1}$$

$$\text{tr}_{\mathcal{U}} Q_\varphi = 2\int_M (\varphi \circ \{(n/2)(A_\eta \wedge I) - \text{Ric} \wedge I + 2\mathcal{R}\}, \varphi)\text{dvol}, \tag{5.2}$$

where $\mathcal{R}$, B, A, η *and* $D^\perp$ *denote the curvature operator of* M *acting on* $\bigwedge^2 TM$, *the second fundamental form, the shape operator, the mean curvature and the normal connection of* Φ, *respectively.*

Consider a compact Riemannian homogeneous space with irreducible isotropy representation M.

Lemma 5.2. *If* ∇ *is a weakly stable Yang-Mills connection, then we have*

$$\sum_{i=1}^{n}\{[F^\nabla(e_i, Y), \varphi(e_i, X)] + [F^\nabla(e_i, X), \varphi(e_i, Y)]\} = 0 \tag{5.3}$$

for every $X, Y \in T_xM$.

Proof. Let K be the group of isometries of M and let $\mathbf{k}$ be its Lie algebra of Killing vector fields on M. Since M has irreducible isotropy representation, we can fix a K-invariant inner product on $\mathbf{k}$ which induces the K-invariant Riemannian metric of M. By [B-L, (10.4) Lemma], for each $V \in \mathbf{k}$

$$\mathcal{S}_0^{\nabla}(i_V\varphi)(X) = -\sum_{i=1}^{n}\{[F^{\nabla}(e_i,V),\varphi(e_i,X)] + [F^{\nabla}(e_i,X),\varphi(e_i,V)]\}.$$

Hence $\mathrm{tr}_{\mathbf{k}} Q_{\varphi} = 0$. Since ∇ is weakly stable, we have $\mathcal{I}^{\nabla}(i_V\varphi, i_V\varphi) = 0$ for all $V \in \mathbf{k}$. For any $B \in \Omega^1(\mathbf{g}_E)$,

$$0 \le \mathcal{I}^{\nabla}(i_V\varphi + tB, i_V\varphi + tB) = 2t\mathcal{I}^{\nabla}(i_V\varphi, B) + t^2\mathcal{I}^{\nabla}(B,B),$$

hence $\mathcal{I}^{\nabla}(i_V\varphi, B) = 0$. Thus $\mathcal{S}_0^{\nabla}(i_V\varphi) = 0$ for all $V \in \mathbf{k}$. q.e.d.

Consider $\Phi : M \longrightarrow S^{N-1}(\sqrt{n/\lambda_1}) \subset \mathbf{R}^N$ be the first standard minimal immersion of M (cf. [K-O-T]). Since M is an Einstein manifold and Φ is a minimal immersion onto a sphere of radius $\sqrt{n/\lambda_1}$, if $\varphi = F^{\nabla}$, then (5.1) becomes

$$\begin{aligned}\mathcal{S}^{\nabla}(i_U\varphi)(X) &= [\varphi \circ \{(\lambda_1 - 2c)I + 2\mathcal{R}\}](U,X) \\ &\quad - 2\sum_{i,j=1}^{n}\langle B(e_i,e_j),u\rangle(\nabla_{e_j}\varphi)(e_i,X),\end{aligned} \tag{5.4}$$

where c and λ_1 denote the Einstein constant of M and the first eigenvalue of the Laplace-Beltrami operator of M acting on functions, respectively.

Assume that M is a compact irreducible symmetric space. Let

$$\bigwedge^2 T_xM = \mathbf{h}_0 + \mathbf{h}_1 + \ldots + \mathbf{h}_p \tag{5.5}$$

be the orthogonal decomposition into eigenspaces of $\mathcal{R}$, where $\mathbf{h}_0$ is the eigenspace with eigenvalue 0 and $\mathbf{h}_s$ is the eigenspace with eigenvalue $\mu_s > 0$. We decompose $\varphi = \varphi_0 + \varphi_1 + \ldots + \varphi_p$ along (5.5). Note that $\nabla\varphi = 0$ if and only if $\nabla\varphi_s = 0$ for each $s = 0,\ldots,p$. Assume that $\nabla\varphi = 0$. If ∇ is weakly stable Yang-Mills field, then by (5.3) we have

$$\mathcal{S}^{\nabla}(i_V\varphi_s) = (\lambda_1 - 2c + 2\mu_s)(i_V\varphi_s) \quad \text{for each } s = 0,\ldots,p. \tag{5.6}$$

6. Remarks on Yang-Mills fields over compact symmetric spaces.

First we remark on the stability of the canonical connections over compact globally Riemannian symmetric spaces. Laquer [La] determined the indices and nullities of the canonical connection on the standard principal bundle of each simply connected compact irreducible symmetric spaces. We denote by $\mathrm{i}(\nabla)$ and $\mathrm{n}(\nabla)$ the index and nullity of a Yang-Mills connection ∇ (cf. [B-L] for their definitions).

Theorem 6.1 ([La]). *Let $M = K/H$ be a simply connected compact irreducible symmetric space associated with a symmetric pair (K, H) and let ∇ the canonical connection of the principal bundle $K \longrightarrow K/H$.*
(1) If M is a group manifold, then $\mathrm{i}(\nabla) = 1$ *and* $\mathrm{n}(\nabla) = 0$.
(2) If $M = S^n$ $(n \geq 5), P_2(\mathbf{Cay}), E_6/F_4$, then $\mathrm{i}(\nabla) = n+1, 26, 54$ *and* $\mathrm{n}(\nabla) = 0$, *respectively.*
(3) If $M = P_m(\mathbf{H})$ $(m \geq 1)$, then $\mathrm{i}(\nabla) = 0$, $\mathrm{n}(\nabla) = 10$ $(m = 1)$ *or* $m(2m+3)$ $(m \geq 2)$.
(4) If M is otherwise, then $\mathrm{i}(\nabla) = \mathrm{n}(\nabla) = 0$.

We should note that the values $\mathrm{i}(\nabla)$ for $M = S^n$ $(n \geq 5), P_2(\mathbf{Cay}), E_6/F_4$ and $\mathrm{n}(\nabla)$ for $M = P_m(\mathbf{H})$ $(m \geq 2)$ are equal to the dimension of the first eigenspace of the Laplace-Beltrami operator of M acting on functions, and $n(\nabla)$ for $M = P_1(\mathbf{H}) = S^4$ is equal to its twice. It is known that, in the cases of $M = S^n, P_m(\mathbf{H})$, $P_2(\mathbf{Cay})$, the space of all gradient vector fields for the first eigenfunctions on M coincides with the space of all proper infinitesimal conformal transformations or projective transformations on M.

We observe the case when M is a non-simply connected, compact irreducible symmetric space. From [La] we see that if M is a group manifold, then $\mathrm{i}(\nabla) = 1, \mathrm{n}(\nabla) = 0$. Suppose that M is not a group manifold. We easily check that if the canonical connection of the universal covering $\tilde{M}$ of M has $\mathrm{i}(\nabla) = \mathrm{n}(\nabla) = 0$, then the canonical connection of M also has $\mathrm{i}(\nabla) = \mathrm{n}(\nabla) = 0$. When $\tilde{M} = S^n$, by virtue of [B-L, (9.1) Theorem], we have $\mathrm{i}(\nabla) = \mathrm{n}(\nabla) = 0$. From the theory of symmetric spaces (cf. [He]) we know that if $\tilde{M} = P_n(\mathbf{H})$ or $P_2(\mathbf{Cay})$, then $\tilde{M} = M$, and if $\tilde{M} = E_6/F_4$, then $M = E_6/F_4 \cdot \mathbf{Z}_3$. We show that the canonical connection of $M = E_6/F_4 \cdot \mathbf{Z}_3$ has $\mathrm{i}(\nabla) = \mathrm{n}(\nabla) = 0$. From Theorem 6.1 we see $\mathrm{n}(\nabla) = 0$. First we recall the realization of E_6/F_4 and $E_6/F_4 \cdot \mathbf{Z}_3$ (cf. [Yo]). Consider the Jordan algebra $\mathcal{T} = \{u \in \mathrm{M}(3, \mathbf{Cay}); u^* = u\}$ of (real) dimension 27. Let $\mathbf{R}^{54} = \mathbf{C}^{27} = \mathcal{T}^{\mathbf{C}}$ be the complexification of $\mathcal{T}$ with a natural real inner product $\langle\ ,\ \rangle$. Let $S^{53} = \{u \in \mathbf{R}^{54}; \langle u, u\rangle = 3\}$, a hypersphere of $\mathcal{T}^{\mathbf{C}}$. Set $\tilde{M} = \{u \in S^{53}; \det(u) = 1\}$ and let Φ denote the embedding $\tilde{M} \longrightarrow S^{53} \subset \mathbf{R}^{54}$.

Proposition 6.2. *(1) $\tilde{M}$ is isometric to a simply connected compact irreducible symmetric space E_6/F_4 (cf. [Yo]).*
(2) The embedding Φ is the first standard minimal immersion of $\tilde{M} = E_6/F_4$ (cf. [Oh]).

Now we define a finite group Γ acting freely and isometrically on $\mathbf{R}^{54} - \{0\}$ and $\tilde{M}$ by

$$\Gamma = \{1, \sigma, \sigma^2\} \cong \mathbf{Z}_3,$$
$$\sigma(u) = e^{(2/3)\pi\sqrt{-1}}u \quad \text{for each } u \in \mathbf{R}^{54}.$$

Then the quotient $M = \tilde{M}/\Gamma$ is isometric to the symmetric space $E_6/F_4 \cdot \mathbf{Z}_3$.

Set $K = E_6, H = F_4$ and $N = 54$. Let R^∇ be the curvature form of the canonical connection ∇ for (K, H). Then we have

$$\bigwedge^2 T_x\tilde{M} = \mathbf{so}(T_x\tilde{M}) = \mathbf{h}_0 + \mathbf{h}_1,$$

where $\mathbf{h}_1$ is isometric to the Lie algebra of F_4, which is the holonomy algebra of $\tilde{M}$. Since $\lambda_1 - 2c + 2\mu_1 < 0$ by virtue of the result of [K-O-T], from (5.4) we see that

$$\Theta = \{i_U R^\nabla; U = \text{grad } f_u \quad \text{for some } u \in \mathbf{R}^N\}$$

is an eigenspace of $\mathcal{S}^\nabla$ of dimension 54 with a negative eigenvalue. From Theorem 6.1 we see $\mathrm{i}(\nabla) = \dim \Theta$. In order to show that the canonical connection of M has $\mathrm{i}(\nabla) = 0$, it suffices to show that if $i_U R^\nabla \in \Theta$ is invariant by Γ, then $U = 0$. It follows from the following two lemmas.

Lemma 6.3. *Let $V \in C^\infty(TM)$. If*

$$\gamma(i_V R^\nabla) = i_V R^\nabla \quad \textit{for each } \gamma \in \Gamma,$$

then $\gamma_* V = V$ for each $\gamma \in \Gamma$.

Proof. For any $X \in T_x M$,

$$\begin{aligned} R^\nabla(V_x, X) &= \gamma(i_V R^\nabla)(X) = \gamma(R^\nabla(V_{\gamma^{-1}(x)}, \gamma_*^{-1} X)) \\ &= R^\nabla(\gamma_* V_{\gamma^{-1}(x)}, X), \end{aligned}$$

hence $R^\nabla(\gamma_* V_{\gamma^{-1}(x)} - V_x, X) = 0$. If we let the canonical decomposition $\mathbf{k} = \mathbf{h} + \mathbf{m}$ at $x \in \tilde{M}$ and we use the identification $\mathbf{m} = T_x M$, then $R^\nabla(X, Y) = -\mathrm{ad}_{\mathbf{m}}[X, Y]$ (cf. [K-N]). Thus $\mathrm{ad}_{\mathbf{m}}[\gamma_* V_{\gamma^{-1}(x)} - V_x, X] =$ for each $X \in \mathbf{m}$. Since $\mathbf{h} = [\mathbf{m}, \mathbf{m}]$ and $\mathbf{k}$ is semisimple, $\gamma_* V_{\gamma^{-1}(x)} - V_x = 0$. q.e.d.

Lemma 6.4. *Let $U = \text{grad } f_u \in C^\infty(TM)$ for some $u \in \mathbf{R}^N$. If $\gamma \in \Gamma - \{1\}$ and $\gamma_* U = U$, then $u = 0$.*

Proof. For each $x \in \tilde{M}$ and $X \in T_x M$,

$$\langle \gamma_* U, X \rangle = \langle U, \gamma_*^{-1} X \rangle = \langle \gamma^{-1}(X), u \rangle = \langle X, \gamma(u) \rangle = \langle U, X \rangle = \langle X, u \rangle,$$

hence $\langle X, \gamma(u) - u \rangle = 0$ Thus $\langle x, \gamma(u) - u \rangle$ is constant in $x \in \tilde{M}$. Since $\Phi(\tilde{M})$ is not contained in any hyperplane of $\mathbf{R}^N$, we have $\gamma(u) = u$. Since Γ acts freely on $\mathbf{R}^N - \{0\}$, we get $u = 0$. q.e.d.

Next we remark on weakly stable Yang-Mills fields over a quaternionic projective space $M = P_m(\mathbf{H})$. Generally let M be a quaternionic Kähler manifold. The $\mathbf{Sp}(m) \cdot \mathbf{Sp}(1)$-structure induces the orthogonal decomposition

$$\bigwedge^2 T^*M = W_0 + W_1 + W_2,$$

where $(W_0)_x, (W_1)_x \cong \mathbf{sp}(1), (W_2)_x \cong \mathbf{sp}(m)$ are irreducible $\mathbf{Sp}(m)\cdot\mathbf{Sp}(1)$-modules. The curvature form $F^\nabla = F_0^\nabla + F_1^\nabla + F_2^\nabla$ of a connection ∇ on the vector bundle E over M splits into components F_i^∇ to $End(E) \otimes W_i$ at each point. A connection ∇ with $F^\nabla = F_2^\nabla$ (resp. $F^\nabla = F_1^\nabla$) is called a B_2-connection (resp. A_1'-connection) as in [Ni], or a *self-dual* conncetion (resp. an *anti-self-dual* connection) as in [C-S]. They are Yang-Mills connections which minimizes the Yang-Mills functional ([C-S],[Ni]).

Proposition 6.5. *Let E be a Riemannian vector bundle over $P_m(\mathbf{H})$. If ∇ is a weakly stable Yang-Mills connection on E satisfying $F_1^\nabla = 0$, then ∇ is a B_2-connection (self-dual).*

Proof. We may suppose that g is an $\mathbf{Sp}(m+1)$-invariant Riemannian metric on $P_m(\mathbf{H}) = \mathbf{Sp}(m+1)/\mathbf{Sp}(m) \times \mathbf{Sp}(1)$ induced by the Killing form of the Lie algebra of $\mathbf{Sp}(m+1)$. From [K-O-T] we know

$$\begin{aligned} &\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1 + \mathcal{R}_2, \\ &\mathcal{R}_0 = 0, \\ &\mathcal{R}_1 = (m/2(m+2))I, \\ &\mathcal{R}_2 = (1/2(m+2))I. \end{aligned} \tag{6.1}$$

Hence by virtue of (5.2), we get

$$\begin{aligned} &\mathrm{Tr}_{\mathcal{U}}\ Q_{F^\nabla} \\ &= 2\int_M (F^\nabla \circ \{2\mathcal{R} - (1/(m+2))I\}, F^\nabla)\mathrm{dvol} \\ &= 2\{-1/(m+2)\int_M (F_0^\nabla, F_0^\nabla)\mathrm{dvol} + (m-1)/(m+2)\int_M (F_1^\nabla, F_1^\nabla)\mathrm{dvol}\}. \end{aligned}$$

Proposition 6.5 follows from this equation. q.e.d.

From the proof of Proposition 6.5, we see that if ∇ satisfies the assumption, then

$$\sum_{i,j=1}^{n} \langle B(e_i, e_j), u\rangle (\nabla_{e_j} F^\nabla)(e_i, X) = 0, \tag{6.2}$$

for all $u \in \mathbf{R}^N$ and all $X \in T_xM$. Using the properties of the second fundamental form of Φ and the curvature tensor field of $P_m(\mathbf{H})$, we can check that (6.2) implies that the restriction of F^∇ to every quaternionic projective line $P_1(\mathbf{H}) \subset P_m(\mathbf{H})$ is always a Yang-Mills field. Hence by (5.6) and (6.1) we obtain that, for any B_2-connection ∇ over $P_m(\mathbf{H})$ and any infinitesimal projective transformation U on $P_m(\mathbf{H})$, we have $\mathcal{S}^\nabla(i_U F^\nabla) = 0$. This means the existence of an infinitesimal action of the projective transformation group of $P_m(\mathbf{H})$ on the space of all B_2-connections over $P_m(\mathbf{H})$. In fact, it is known that the projective transformation group of $P_m(\mathbf{H})$ acts on the moduli space of all B_2-connections on E.

By (5.4), (5.6) and (6.1) we obtain that the indices $\mathrm{i}(\nabla)$ and the nullity $\mathrm{n}(\nabla)$ of the canonical connection of $M = S^n$ $(n \geq 5), P_2(\mathbf{Cay})$ and E_6/F_4 come from $\mathrm{span}_{\mathbf{R}}\{i_U R^\nabla; U \in \mathcal{U}\}$, and the nullities for $M = P_1(\mathbf{H}) = S^4$ and $P_m(\mathbf{H})$ $(m \geq 2)$ comes from $\mathrm{span}_{\mathbf{R}}\{i_U R_1^\nabla, i_U R_2^\nabla; U \in \mathcal{U}\}$ and $\mathrm{span}_{\mathbf{R}}\{i_U R_2^\nabla; U \in \mathcal{U}\}$, respectively. We do not know whether each weakly stable canonical connection over a compact symmetric space minimizes the Yang-Mills functional. And it is interesting to investigate relationships of Yang-Mills fields with holonomy groups and the classification of vector bundles with Yang-Mills connections satisfying $\nabla F^\nabla = 0$ over compact symmetric spaces. From results of [B-L, p. 211] and [K-O-T] we can find gap phenomena for Yang-Mills fields over every compact irreducible symmetric space which is not locally Hermitian symmetric. The classification of such Yang-Mills connections may also be useful to establish accurately isolation theorems for Yang-Mills fields over compact symmetric spaces.

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