On k-spannedness for projective surfaces

by

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INTRODUCTION. Let L be a line bundle on a smooth connected projective surface S. In this paper we make a general study of pairs (S,L) where L is k-spanned. K-spannedness of a line bundle L is natural notion of higher order embedding for the map associated to L that was introduced in [3], e.g. L is 0-spanned (1-spanned) iff L is spanned by global sections (very ample).

In § 0 we recall the definition of k-spannedness and the main result of [3], a Reider type material criterion for a line bundle to be k-spanned. We also collect a number of results, we continually use.

In § 1 we study k-spannedness on curves proving a number of inequalities linking invariants of the line bundle, the curve and k. We characterize k-spannedness of L on S in terms of the restriction of L to curves on S.

In § 2 we study lower bounds for $h^0(L)$ in terms of k. For $k = 2$ we get the very strong result that $h^0(L) \geq 6$, while for $k \geq 3$ we only get $h^0(L) \geq k+3$. Our prove is based on jet bundle arguments.

In § 3 we give sufficient conditions for a line bundle L on a $\mathbb{P}^1$ bundle over a curve to be k-spanned. The conditions are necessary for $h^1(\mathcal{O}_S) \leq 1$ and almost necessary in general.

In § 4 we use the results of [13] to study when $kK_S + L$ is spanned by global sections. This gives very strong numerical relat a k-spanned line bundle must satisfy.

In § 5 we use the results obtained to classify the pairs (S,L) with $g(L) \leq 5$ and $k \geq 2$ where L is k-spanned and $g(L)$ is the genus of a smooth $C \in |L|$ (see also [7], [9], [10]).

In § 6 we study the dependence of the inequalities between $g(L)$, $c_1(L)^2$, k for k-spanned L with $k \geq 2$ and the birational geography of S.

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§ 0. Notation and background material.

We work over the complex numbers $\mathbb{C}$. Throughout the paper, S always denotes a smooth connected projective surface. We denote its structure sheaf by $\mathcal{O}_S$ and the canonical sheaf of the holomorphic 2-forms by K_S. For any coherent sheaf F on S we shall denote by $h^i(F)$ the complex dimension of $H^i(S,F)$.

Let L be a line bundle on S. L is said to be numerically effective, nef for short, if $L\cdot C \geq 0$ for each irreducible curve C on S, and in this case L is said to be big if $c_1(L)^2 > 0$, where $c_1(L)$ is the first Chern class of L. We say that L is spanned if it is spanned by the space of its global sections $\Gamma(L)$.

(0.1) We fix some more notation.

$\sim$ (resp. $\approx$) the numerical (resp. linear) equivalence of divisors;

$\chi(L) = \sum(-1)^i h^i(L)$, the Euler characteristic of a line bundle L;

$|L|$, the complete linear system associated to L;

$q(S) = h^1(\mathcal{O}_S)$, the irregularity of S;

$p_g(S) = h^2(\mathcal{O}_S)$, the geometric genus of S;

$\kappa(S)$, the Kodaira dimension of S;

$e(S)$, the topological Euler characteristic of S.

We denote by $J_t(S,L)$, $J_t(C,L)$ the t-th holomorphic jet bundles of a line bundle L on S (resp. on a smooth curve C). Recall that $J_t(S,L)$, $J_t(C,L)$ are vector bundles of rank $(t+1)(t+2)/2$, $t+1$ respectively (for general properties of jet bundles we refer to [8] and [11]).

As usual we don't distinguish between locally free sheaves and vector bundles, nor between line bundles and Cartier divisors. Hence we shall freely switch from the multiplicative to the additive notation and viceversa.

(0.2) The genus formula. Let L be a nef and big line bundle on S. Then the sectional genus, $g(L)$, of L is defined by the equality $2g(L)-2 = (K_S+L)\cdot L$.

It can be easily seen that $g(L)$ is an integer. Furthermore if there exists an irreducible reduced curve C in $|L|$, $g(L)$ is simpl the arithmetic genus $p_a(C)=1-\chi(\mathcal{O}_C)$ of C.

(0.3) Let L be a nef and big line bundle on S. We say that the (generically) polarized pair (S,L) is geometrically ruled if S is a $\mathbb{P}^1$ bundle, $p : S \longrightarrow R$, over a nonsingular curve R and the restriction L_f of L to a fibre f of p is $\mathcal{O}_f(1)$. We shall denote by E a fundamental section of p. We say that (S,L) is a scroll (resp. a conic bundle) over a nonsingular curve R if there is a surjective morphism with connected fibres $p : S \longrightarrow R$, with the property that L is relatively ample with respect to p and there exists some very ample line bundle M on R such that $K_S \otimes 2L \approx p^*M$ (resp. $K_S \otimes L \approx p^*M$). We also say that (S,L) is a geometrically ruled conic bundle if S is a $\mathbb{P}^1$ bundle $p : S \longrightarrow$ and the restriction of L to a fibre f of p is $\mathcal{O}_f(2)$.

We denote the rational $\mathbb{P}^1$ bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, $n \geq 0$, by $\mathbb{F}_n$, the Hirzebruch surface. Note that the only $\mathbb{P}^1$ bundle which is not a scroll in the above sense is $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ with $L = \mathcal{O}_{\mathbb{F}_0}(1,1)$. We say that S is a Del Pezzo surface if $-K_S$ is ample.

(0.4) Castelnuovo's bound. Let L be a very ample line bundle on a smooth surface S and let C be a general element in $|L|$.

Assume that $|L|$ embeds S in a projective space $\mathbb{P}^N$ and let $d = L\cdot L$. Then $g(L) = g(C)$ and Castelnuovo's Lemma says that (see e.g. [1])

$$(0.4.1) \qquad g(C) \leq \left[\frac{d-2}{N-2}\right](d-N+1-(\left[\frac{d-2}{N-2}\right]-1)\ \frac{N-2}{2})$$

where $[x]$ means the greatest integer $\leq x$. From (0.4.1), writing $[\ d-2\ /N-2] = d-2-\varepsilon/N-2,\ 0 \leq \varepsilon \leq N-3$, we find that

$$d \geq N/2 + \sqrt{2(N-2)g(L)+((N-4/2)-\varepsilon)^2}$$

this leading to

$$(0.4.2) \qquad d \geq \begin{cases} N/2+\sqrt{2(N-2)g(L)+1/4} & \text{if } N-4 \text{ is odd;} \\ N/2+\sqrt{2(N-2)g(L)} & \text{if } N-4 \text{ is even.} \end{cases}$$

(0.5) <u>k-spannedness</u>. Let L be a line bundle on S (resp. on a nonsingular curve C). We say that L is k-<u>spanned</u> for $k \geq 0$ if for any distinct points $z_1,\ldots,z_t$ on S (resp. on C) and any positive integers $k_1,\ldots,k_t$ with $\sum_{i=1}^{t} k_i = k+1$, the map $\Gamma(L) \longrightarrow \Gamma(L \otimes \mathcal{O}_Z)$ is onto, where $(Z,\mathcal{O}_Z)$ is a 0-dimensional subscheme defined by the ideal sheaf I_Z where $I_Z\mathcal{O}_{S,z}$ is $\mathcal{O}_{S,z}$ (resp. $\mathcal{O}_{C,z}$) for $z \notin \{z_1,\ldots,z_t\}$ and $I_Z\mathcal{O}_{S,z_i}$ is generated by $(x_i,y_i^{k_i})$ at z_i, with (x_i,y_i) local coordinates at z_i on S, $i = 1,\ldots,t$ (resp. I_Z is generated by $y_i^{k_i}$, y_i local coordinate at z_i on C). We call a 0-cycle Z as above <u>admissible</u>.

Note that 0-spanned is equivalent to L being spanned by $\Gamma(L)$ and 1-spanned is equivalent to very ample.

(0.5.1) If L is k-spanned on S , then $L\cdot C \geq k$ for every effective curve C on S , with equality only if $C \cong \mathbb{P}^1$. Further either $C \cong \mathbb{P}^1$ or $p_a(C) = 1$ if $\deg L_C = k+1$.

The fact that $L\cdot C \geq k$ is clear from the definition, as well as $h^0(L_C) \geq k+1$. Now, looking at the embedding of C in $\mathbb{P}^N$ given by $\Gamma(L_C)$ one has $\deg L_C = \deg C \geq N \geq k$, so that $C \cong \mathbb{P}^1$ whenever $\deg L_C = k$ or $\deg L_C = N = k+1$ and $p_a(C) = 1$ if $\deg L_C = k+1$.

(0.5.2) Let L be a k-spanned line bundle on either S or a smoo- curve C . We say that $V \subset \Gamma(L)$ k-spans L (or V is a k-spanning set of L) if for all admissible 0-cycles $(Z,\mathcal{O}_Z)$ with length $(\mathcal{O}_Z)$ $= k+1$, the map $V \longrightarrow \Gamma(L \otimes \mathcal{O}_Z)$ is onto.

For a given admissible 0-cycle $(Z,\mathcal{O}_Z)$ on S we say that a smooth curve C is compatible with $(Z,\mathcal{O}_Z)$ if

- $C \supset Z_{red}$;
- for any point $z \in Z_{red}$, where $I_Z\mathcal{O}_{S,z} = (x,y^n)$, x,y local parameters at z , then $f-x \in m_z^n$ where f is the local equation of C at z and m_z is the maximal ideal of $\mathcal{O}_{S,z}$.

Thus we get the following characterization of k-spannedness, we need to prove the key-Lemma below.

> $V \subset \Gamma(L)$ k-spans L on S if and only if for all smooth connected compatible curves C on S, $\mathrm{Im}(V \longrightarrow \Gamma(L_C))$ k-spans the restriction L_C .

(0.5.3) LEMMA. Let L_i be k_i-spanned line bundles either on S or on a smooth curve C and let $V_i \subset \Gamma(L_i)$ k_i-spans L_i for

$i = 1,\dots,m$. Then the image V of $V_1 \otimes \dots \otimes V_m$ in $\Gamma(L_1 \otimes \dots \otimes L_m$ $(k_1 + \dots + k_m)$ - spans $L_1 \otimes \dots \otimes L_m$.

Proof. In view of the characterization of k-spannedness given in (0.5.2) we easily see that one can reduce to the curves case. Further the result is clearly reduced to the case $m = 2$. Thus we have to show that, given a 0-cycle $Z = \sum_{i=1}^{t} n_i p_i$ on C where $n_i > 0$, $\sum_{i=1}^{t} n_i = k+1$, the map $V \longrightarrow \Gamma(L_1 \otimes L_2 \otimes \mathcal{O}_Z)$ is onto.

To see this fix an index i . Write $n_i = a_i+b_i$ where $a_i > 0$, $b_i > 0$ and $n_j = a_j+b_j$, $j \neq i$, where $a_j \geq 0$, $b_j \geq 0$. Then $\sum_{r=1}^{t} a_r = k_1+1$, $\sum_{r=1}^{t} b_r = k_2+1$ and let $Z_1 = \sum_{r=1}^{t} a_r p_r$, $Z_2 = \sum_{r=1}^{t} b_r p_r$. By the fact that V_1 k_1-spans L_1 we can choose elements $s_1,\dots,s_{i_{a_i}}$ of V_1 whose images in $\Gamma(L_1 \otimes \mathcal{O}_{Z_1})$ vanish at p_j to the a_j-th order for $j \neq i$ and which have prescribed a_i-1 jet at p_i . Similarly we can choose elements $u_1,\dots,u_{i_{b_i}}$ of V_2 whose images in $\Gamma(L_2 \otimes \mathcal{O}_{Z_2})$ vanish at p_j to the b_j-th order for $j \neq i$ and which have prescribed b_i-1 jet at p_i . Note that the tensor powers of these sections give a space of sections W_i of $L_1 \otimes L_2$ which vanish at p_j to the n_j-th order for $j \neq i$ and which have prescribed n_i-1 jet at p_i . Now $W_1,\dots,W_t$ clearly generate $\Gamma(L_1 \otimes L_2 \otimes \mathcal{O}_Z)$, so we are done.

□

Let us recall the following numerical characterization of k-spannedness.

(0.6) THEOREM ([3], (3.1)). Let L be a nef and big line bundle on a surface S and let $L \cdot L \geq 4k+5$. Then either K_S+L is k-spanned or there exists an effective divisor D such that L-2D is $\mathbb{Q}$-effective, D contains some admissible 0-cycle of degree $t+1 \leq k+1$ where the k-spannedness fails and

$$L \cdot D - t - 1 \leqq D \cdot D < L \cdot D/2 < t+1 .$$

□

We need the following consequence of the result above (compare with (5.3.4)).

(0.7) PROPOSITION. Let S be a Del Pezzo surface which is a blowi up of $\mathbb{F}_0$ or $\mathbb{F}_1$. Let L be the spanned line bundle on S, obtai by pulling back to S the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ to $\mathbb{F}_i$ under the bundle projection $\mathbb{F}_i \longrightarrow \mathbb{P}^1$, $i = 0,1$. Further assume $K_S \cdot K_S = 1$. Then $K_S^{-t} \otimes L^q$ is k-spanned if $q > 1$ and $t \geq k$.

Proof. Let $M = K_S^{-t-1} \otimes L^q$. Note that M is ample and $M \cdot M = (t+1)^2 + 2(t+1)q \geq (k+1)^2 + 4(k+1)q$ since $t \geq k$, $K_S^{-1} \cdot L = 2$. Thus $M \cdot M \geq 4k+5$. If $K_S \otimes M = K_S^{-t} \otimes L^q$ is not k-spanned, by the Theorem above one has

$$M \cdot D - k - 1 \leqq D \cdot D < M \cdot D/2 < k+1$$

for some effective divisor D on S. Now, since $t \geq k$, $M \cdot D/2 < k+$ gives $K_S^{-1} \cdot D = 1$ and hence

$$D \cdot D \geq M \cdot D - k - 1 \geq qL \cdot C \geq 0 .$$

Since $K_S \cdot D = 1$, $D \cdot D \geq 1$ by the genus formula. Then by the Hodge index theorem we get $K_S \cdot K_S = D \cdot D = 1$ and also $K_S^{-1} \sim D$. This lead to the contradiction $qL \cdot D = 2q \leq D \cdot D = 1$.

□

Note that

(0.7.1) $g(K_S^{-t} \otimes L^q) = t(t-1)/2+(2t-1)/q + 1$.

□

To the reader's convenience we recall here the following result from [3], we use several times.

(0.8) PROPOSITION ([3], (3.6)). *Let* S *be a Del Pezzo surface. Then* K_S^{-t} *is* k-*spanned for* $k \geq 0$ *if and only if*:

(0.8.1) $t \geq k/3$ *if* $S = \mathbb{P}^2$;

(0.8.2) $t \geq k/2$ *if* $S = \mathbb{P}^1 \times \mathbb{P}^1$;

(0.8.3) $t \geq k+2$ *if* $K_S \cdot K_S = 1$;

(0.8.4) $t \geq k$ *if* $K_S \cdot K_S \geq 3$ *or* $K_S \cdot K_S = 2$ *and* $k \neq 1$.

Further, if $K_S \cdot K_S = 2$, K_S^{-t} *is very ample iff* $t \geq 2$.

(0.9) k-*reduction*. Let L be a line bundle on S . A pair (S',L') is said to be a k-*reduction* of (S,L) if there is a morphism $\pi : S \longrightarrow S'$ expressing S as S' with a finite set F blown up and $L \approx \pi^*L' - k\pi^{-1}(F)$. Note that $K_S^k \otimes L \approx \pi^*(K_{S'}^k \otimes L')$.

□

Apart from some cases where $k \geq 2$ is explicitely needed, we carry out for completeness most results for k-spanned line bundles with $k \geq 1$, even though in the "classical" case $k = 1$ they don't give something new.

In § 4 we use extensively the results of [13]. We refer directly the reader to [13] instead of reporting here the results we need. Through the paper we also use well known results describing polarized pairs (S,L) with L of sectional genus $g(L) = 0,1$; for this we refer e.g. to [5] and [7].

§ 1. k-spannedness on curves.

Throughout this section we denote by C a nonsingular irreduci curve of genus $g(C)$ and by K_C the canonical divisor of C. Our aim is to express the k-spannedness on C in terms of some useful numerical conditions.

(1.1) LEMMA. Let $\mathcal{L}$ be a line bundle on C. Then:

(1.1.1) $\mathcal{L}$ is k-spanned if $\deg \mathcal{L} \geq 2g(C) + k$;

(1.1.2) if $\deg \mathcal{L} = 2g(C)+k-1$, $\mathcal{L}$ is k-spanned if and only if $h^0(\mathcal{L}-K_C) = 0$.

Proof. (1.1.1) follows from the definition. Indeed, let $z_1,\dots,z_r$ be r distinct points on C and let $k_1,\dots,k_r$ r non negative integers such that $\sum_{i=1}^{r} k_i = k+1$. Then $h^0(K_C-\mathcal{L}+\sum_{i=1}^{r} k_i z_i) = 0$, so that we have a surjective map $\Gamma(\mathcal{L}) \longrightarrow \Gamma(\mathcal{L} \otimes \mathcal{O}_Z)$, where Z is the 0-cycle defined as $Z = \sum_{i=1}^{r} k_i z_i$; this means that $\mathcal{L}$ is k-spanned.

To prove (1.1.2), note that, since $\deg \mathcal{L} = 2g(C)+k-1$, we can write $\mathcal{L} = K_C \otimes L$ for some line bundle L of degree $k+1$. Then $h^1(\mathcal{L}) = 0$ and hence $\mathcal{L}$ is k-spanned if and only if

$$*) \qquad h^1(\mathcal{L}-D) = h^0(D-L) = 0 ,$$

for every effective divisor D on C with $\deg D = k+1$. We claim that condition *) is equivalent to $h^0(L) = h^0(\mathcal{L}-K_C) = 0$. In fact, for any divisor D as above, $\deg(\mathcal{L}-D) = 2g(C)-2$; hence $h^1(\mathcal{L}-D) \neq$ implies that $\mathcal{L}-D \sim K_C$ that is $L \sim D$, so $h^0(L) \neq 0$. Viceversa, $h^1(\mathcal{L}-L) = h^1(K_C) = 0$ if $h^0(L) \neq 0$, a contradiction.

□

The following plays a relevant role in the sequel.

(1.2) THEOREM. Let L be a k-spanned line bundle on C and let $h^1(L) \neq 0$. Then :

(1.2.1) K_C is k-spanned;

(1.2.2) $h^0(L) \leq 1$ for any line bundle L with $\deg L \leq k+1$;

(1.2.3) $g(C) \geq 2k+1$.

Proof. First, we can assume $h^1(L) = 1$. Indeed, if $h^1(L) = h^0(K_C-L) \geq 2$, we can write $K_C-L \approx \Delta+M$ where $h^0(\Delta) = 1$ and the moving part M is base points free. Then $L' = L+M$ is k-spanned by (0.5.3) and $h^1(L') = h^0(\Delta)$.

To prove (1.2.1), note that K_C is k-spanned if and only if $h^1(K_C-Z) = h^1(K_C) = 1$, for every length k+1 0-cycle Z on C . This easily follows by looking at the exact sequence

$$0 \longrightarrow K_C \otimes \mathcal{O}_C(-Z) \longrightarrow K_C \longrightarrow K_C \otimes \mathcal{O}_Z \longrightarrow 0 .$$

Now, if $h^1(K_C-Z) \geq 2$, clearly $h^0(K_C-L+Z) \geq 2$ since K_C-L is effective and hence by duality $h^1(L-Z) \geq 2$. Again, the k-spannedness of L can be expressed as $h^1(L-Z) = h^1(L)$, this leading to a contradiction. Thus K_C is k-spanned and $h^1(K_C-Z) = h^0(Z) = 1$ for every length $t \leq k+1$ 0-cycle Z on C . This gives (1.2.1) and (1.2.2). From the Existence Theorem (see [1], p. 206) we know that for any integer $t \geq (g(C)+2)/2$ there exists a line bundle L on C of degree t and with $h^0(L) \geq 2$. Therefore it has to be $k+1 < (g(C)+2)/2$, which gives (1.2.3).

(1.3) KEY-LEMMA. Let L be a k-spanned line bundle on C . Then $h^0(L) \geq k+2$ if $g(C) > 0$.

Proof. Since L is very ample one sees that the k-th holomorphic jet bundle $J_k(C,L)$ is spanned by the image of $\Gamma(L)$ under the natural map $j_k : L \longrightarrow J_k(C,L)$. Since $J_0(C,L) = L$ and $T^*_C \otimes L$ are ample vector bundles we see from the exact sequence

$$0 \longrightarrow T^{*(k)}_C \otimes L \longrightarrow J_k(C,L) \longrightarrow J_{k-1}(C,L) \longrightarrow 0$$

that $J_k(C,L)$ is an ample vector bundle of rank $k+1$. Then $h^0(L) \geq \mathrm{rk}\, J_k(C,L) + \dim C \geq k+2$.

(1.4) COROLLARY. Let L be a k-spanned line bundle on C with $g(C) > 0$ and let $d = \deg L$. Then:

(1.4.1) $d \geq k+2$;

(1.4.2) $d \geq 2k+2$ if $d \leq 2g(C)$ with equality only if either $d = 2g(C)$ or $L \approx K_C$, $k = 1$, $g(C) = 3$.

Proof. If $h^1(L) = 0$, then $d-g(C)+1 = h^0(L) \geq k+2$ gives $d \geq k+g(C)+1 \geq k+2$. If $h^1(L) \neq 0$ Clifford's theorem and (1.3) yie[ld] $d/2+1 \geq h^0(L) \geq k+2$, whence $d \geq 2k+2$ and (1.4.1) is proved.

Note that Clifford's inequality holds true also if $h^1(L) = 0$ whenever $d \leq 2g(C)$. Therefore $d \geq 2k+2$ by (1.4.1). Now, $d = 2k+2$ gives the equality in the Clifford's theorem, so we find that $d = 2g(C)$ if $h^1(L) = 0$, and either $L \approx K_C$ or C is a hyperelliptic curve with L a multiple of the unique g^1_2 on C if $h^1(L) \neq 0$. If $L \approx K_C$, $d = 2k+2 = 2g(C)-2$ and $g(C) \geq 2k+1$ by (1.2.3), this leading to $k = 1$, $g(C) = 3$. In the remaining case $d \leq 2g(C)-2$, $L \approx ng^1_2$ and $K_C \approx mg^1_2$ for some positive integers m,n $m \geq n$. Then $K_C \approx L+(m-n)g^1_2$ would be very ample, a contradiction to hyperellipticity. This proves (1.4.2).

(1.5) REMARK (compare with § 6). Note that if L is a k-spanned line bundle on S with $p_g(S) > 0$, then for a general element $C \in |L|$ the restriction $\mathcal{L} = L_C$ verifies the condition $h^1(\mathcal{L}) (= h^0(K_{S|C})) \neq 0$, so $\deg \mathcal{L} = L \cdot L \leq 2g(C)-2$. Hence $g(L) \geq 2k+1$ by (1.2.3) and $L \cdot L \geq 2k+3$ if $k \geq 2$ by (1.4.2).

§ 2. A lower bound for $h^0(L)$.

Let L be a k-spanned line bundle on S . In this section we show that the k-spannedness condition forces S to be embedded by $|L|$ in a projective space of dimension at least 5. First of al note that from Lemma (1.3), we have

$$(2.1) \qquad h^0(L) \geq k+3 ,$$

so the claim is clear if $k \geq 3$.

(2.2) Let L be a k-spanned line bundle on S with $k \geq 2$. Take a point $x \in S$ and let $V_2 \subset \Gamma(L)$ denote the space of the section of L that vanish to the 2-nd order at x . We claim that after choosing a trivialization of L at x , a basis of V_2 can be written in the form $s_1 = q_1+O(3),\ldots,s_t = q_t+O(3)$ where the q_α's are quadratic functions in the local parameters at x and at least 2 of the q_α's are not (identically) zero and have no common facto $\alpha = 1,\ldots,t$.

Indeed, set $I = \{i,q_i \neq 0\}$, $J = \{j,q_j = 0\}$ and assume that the q_i's have a linear common factor, say u . The maximal ideal m_x of $\mathcal{O}_{S,x}$ can be assumed to be of the form $m_x = (u,v)$ for some linear factor v .

We can also assume that on the open set $U_0 = \{x \in S,\ s_0(x) \neq s_0 \in \Gamma(L)\}$ a basis for $\Gamma(S,L)$ on U_0 consists of the $h^0(L)-1$ elements $\{u,v,\ldots,s_i,\ldots s_j\ldots\}$. Now L is 2-spanned by the assumption so that the map

$$\rho : \Gamma(L) \longrightarrow \Gamma(L \otimes \mathcal{O}_x/(u,v^3))$$

is onto. Since clearly u and the s_i's, s_j's belong to $\mathrm{Ker}\rho$ we find $\dim \mathrm{Ker}\rho \geq h^0(L)-2$, which contradicts $\dim \Gamma(L \otimes \mathcal{O}_x/(u,v^3)) = 3$

Note that the claim we proved here shows that *the kernel of the evaluation map* $j_{2,x} : (S \times \Gamma(L))_x \longrightarrow J_2(S,L)_x$ *at* x *induced by* $j_2 : L \longrightarrow J_2(S,L)$ *has dimension at most* $h^0(L)-5$.

□

We can now prove the following general result.

(2.3) THEOREM. *If the* 2-th *jets bundles of a* $k \geq 2$ *spanned line bundle* L *on* S *don't span* $J_2(S,L)$ *at at least one point, then*:

(2.3.1) $c_1(S)^2 = 2c_2(S)$ *and the tangent bundle of either* S *or an unramified double cover of* S *splits as a direct sum of line bundles*;

(2.3.2) Cokernel $(j_2 : S \times \Gamma(L) \longrightarrow J_2(S,L)) \cong K_S \otimes L$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} S \times \Gamma(L) & \xrightarrow{\ j_2\ } & J_2(S,L) \\ & \searrow^{j_1} & \downarrow r \\ & & J_1(S,L) \end{array}$$

where r denotes the surjective restriction map, whose kernel is $T_S^{*(2)} \otimes L$. Since L is very ample, j_1 is onto, the restriction j of j_2 to the kernel K of j_1 has image contained in $T_S^{*(2)} \otimes L$, so one has a morphism

$$j : K \longrightarrow T_S^{*(2)} \otimes L .$$

Fix a point $x \in S$, local coordinates at x and a trivialization of L at x. By the earlier argument (2.2), the image of j in $T_S^{*(2)} \otimes L$ at x is of the form

$$(\mathrm{Im} j)_x = \{\lambda\varphi(dz,dw)+\mu\psi(dz,dw);\ \lambda,\mu \in \mathbb{C}, \varphi,\psi \text{ homogeneous quadratic functions without common factors}\}\ .$$

It is easy to see that any such a special pencil has precisely 2 distinct elements which are squares, e.g. the map $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$ given by (φ,ψ) has degree 2 and has precisely two branch points by Hurwitz's theorem. Thus the pencil is given at x by

$$\{\lambda\omega_1^2 + \mu\omega_2^2;\ \lambda,\mu \in \mathbb{C},\ \omega_1,\omega_2 \in T^*_{S,x}\}\ .$$

Hence two directions on T_S are determined at x. It is easy to check that they vary holomorphically and give a submanifold $A \subseteq \mathbb{P}(T_S^*) = [T_S - S]/\mathbb{C}^*$, S the 0-section of $T_S \longrightarrow S$, which is a two to one unramified cover of S under π_A, the restriction to A of $\pi : \mathbb{P}(T_S^*) \longrightarrow S$. Now, either A is a union of 2-sections of π or $q_A^{-1}(A) \subseteq \mathbb{P}(T_A^*)$ is a union of 2 sections of $\mathbb{P}(T_A^*) \longrightarrow A$ where $q_A : \mathbb{P}(T_A^*) \longrightarrow \mathbb{P}(T_S^*)$ is the induced map. In the former case $T_S^* \cong L_1 \oplus L_2$ for 2 line bundles L_1, L_2 on S; in the latter case $T_A^* \cong L_1' \oplus L_2'$ for 2 line bundles L_1', L_2' on A. Note that by a well known result of Bott [4], $c_1(L_i)^2 = c_1(L_i')^2 = 0$. Thus $c_1(S)^2 = 2c_2(S)$ in the former case and $c_1(A)^2 = 2c_2(A)$ in the latter case. Since π_A is an unramified 2-sheeted cover in the latter case, it follows that $c_1(A)^2 = 2c_1(S)^2$, $c_2(A) = 2c_2(S)$. Thus in any case $c_1^2(S) = 2c_2(S)$, which proves (2.3.1).

Note that the image $j(K)$ in $T_S^{*(2)} \otimes L$ in the former case is $(L_1^2 \otimes L) \oplus (L_2^2 \otimes L)$ and hence the cokernel is $L_1 \otimes L_2 \otimes L = = K_S \otimes L$. In the latter case L_1, L_2 are locally still well defined as in the former case. Nonetheless choosing any open set U such that L_1, L_2 are well defined, $T_U^* = L_1 \oplus L_2$ and $L_1 \otimes L_2$ is canonically identified with K_S. Thus the cokernel of j is always $K_S \otimes L$. So we are done by noting that $\operatorname{coker} j = \operatorname{coker} j_2$.

□

As a consequence, we get the result claimed at the beginning of this section.

(2.4) THEOREM. <u>Let</u> L <u>be a</u> k-<u>spanned line bundle on</u> S, $k \geq 2$. <u>Then</u> $h^0(L) \geq 6$.

<u>Proof</u>. From (2.1) we know that $h^0(L) \geq 5$ and $h^0(L) \geq 6$ if $k \geq 3$. So let $k = 2$ and assume $h^0(L) = 5$. Then previous argument (2.2) shows that the evaluating map $j_2 : S \times \Gamma(L) \longrightarrow J_2(S,L)$ induced by $j_2 : L \longrightarrow J_2(S,L)$ is injective. Hence we have an exact sequence of vector bundles

$$0 \longrightarrow S \times \Gamma(L) \longrightarrow J_2(S,L) \longrightarrow K_S \otimes L \longrightarrow 0$$

by the above Theorem. Thus $\det J_2(S,L) \cong K_S \otimes L$. Now a direct computation, by looking at the exact sequences, $t = 1,2$,

$$(2.4.1) \quad 0 \longrightarrow T_S^{*(t)} \otimes L \longrightarrow J_t(S,L) \longrightarrow J_{t-1}(S,L) \longrightarrow 0$$

shows that $\det J_2(S,L) = K_S^4 \otimes L^6$. Therefore $K_S^3 \otimes L^5 \cong \mathcal{O}_S$ and hence there exists a line bundle M on S such that $M^{-5} \cong K_S$, $M^3 \cong L$. Since $M^3 \cong L$, M is ample so $p(t) = \chi(M^t)$ is a non degenerate degree 2 polynomial. But $M^{-5} \cong K_S$ implies, by Kodaira' vanishing theorem, $p(t) = 0$ for $t = -1,-2,-3,-4$, a contradiction This proves that $h^0(L) \geq 6$.

□

Next we show that if $h^0(L) = 6$, then $J_2(S,L)$ is genericall spanned.

(2.5) PROPOSITION. <u>With the notation as in</u> (2.4), <u>there exists at least one point</u> $x \in S$ <u>such that</u> $j_{2,x} : (S \times \Gamma(L))_x \longrightarrow J_2(S,L)_x$ <u>is onto</u>.

<u>Proof</u>. Note that if $(S,L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ it is well known that $j_2 : S \times \Gamma(L) \longrightarrow J_2(S,L)$ is onto (see e.g. [8] or [11]). Thus we can assume $(S,L) \not\cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ and let us suppose $j_{2,x}$ to be not onto for any $x \in S$. Then by (2.3.1) there is an exact sequenc of vector bundles

$$0 \longrightarrow \operatorname{Ker} j_2 \longrightarrow S \times \Gamma(L) \longrightarrow J_2(S,L) \longrightarrow K_S \otimes L \longrightarrow 0$$

and hence $\operatorname{Ker} j_2$ has rank 1, since $\operatorname{rk}(S \times \Gamma(L)) = \operatorname{rk} J_2(S,L) = 6$ The total Chern classes verify the relation, where $K = \operatorname{Ker} j_2$,

$$(2.5.1) \quad (1+K)\cdot c(J_2(S,L)) = 1+K_S+L .$$

We know that $c_1(J_2(S,L)) = \det J_2(S,L) = 4K_S+6L$ while a long but standard computation, by using sequences (2.4.1), gives us

$$(2.5.2) \quad c_2(J_2(S,L)) = 5c_2(S)+5K_S\cdot K_S+20K_S\cdot L+15L\cdot L .$$

Furthermore from (2.5.1) we obtain

$$K\cdot c_1(J_2(S,L)) + c_2(J_2(S,L)) = 0$$

and hence

(2.5.3) $c_2(J_2(S,L)) = (3K_S+5L)\cdot(4K_S+6L) = 12K_S\cdot K_S+38K_S\cdot L+30L\cdot L$.

By combining (2.5.2) with (2.5.3), and noting that $c_1(S)^2 = 2c_2(S)$ by (2.3.1), we find

(2.5.4) $L\cdot L = 3c_2(S)+12(g(L)-1)$.

Note also that K_S+L is nef. Otherwise (S,L) would be either $(\mathbb{P}^2,\mathcal{O}(2))$, $(\mathbb{P}^2,\mathcal{O}(1))$, $(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}(1,1))$ or a scroll, contradicting $(S,L) \neq (\mathbb{P}^2,\mathcal{O}(2))$ or the fact that L is at least 2-spanned. Therefore $K_S\cdot K_S+4(g(L)-1) \geq L\cdot L$; hence (2.5.2) and $K_S\cdot K_S = 2c_2(S)$ lead to

(2.5.5) $c_2(S)+8(g(L)-1) \leq 0$.

Clearly $g(L) \neq 0$ since $k \geq 2$ and $(S,L) \not\cong (\mathbb{P}^2,\mathcal{O}(2))$. Similarly $g(L) \neq 1$: otherwise (S,L) would be either a scroll, contradicting again $k \geq 2$, or a Del Pezzo surface, contradicting $c_2(S) \leq 0$. Thus $g(L) \geq 2$, so $2c_2(S) = K_S\cdot K_S < 0$ and therefore $\chi(\mathcal{O}_S) < 0$. This implies that S is birationally ruled, so $K_S\cdot K_S \leq 8(1-q(S))$, and the Riemann-Roch theorem yields

(2.5.6) $c_2(S) \geq 4-4q(S)$.

Hence from (2.5.5), (2.5.6) we infer that $g(L) \leq (q(S)+1)/2$. Now, since (S,L) is neither $(\mathbb{P}^2,\mathcal{O}(1))$, $(\mathbb{P}^2,\mathcal{O}(2))$, $(\mathbb{P}^1 \times \mathbb{P}^1,\mathcal{O}(1,1))$ nor a scroll, it has to be $g(L) > q(S)$ (see e.g. [12]). So we get $q(S) = 0$, contradicting $\chi(\mathcal{O}_S) < 0$. This proves the Proposition.

□

Now, certain arguments that we have not been able to make rigorous, together with the fact that $(S,L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ whenever j_2 is an isomorphism by a result due to Sommese [11], suggest the following

(2.6) Conjecture. Let L be a k-spanned line bundle on S, $k \geq 2$. Then $h^0(L) = 6$ if and only if $(S,L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

§ 3. k-spannedness on geometrically ruled surfaces.

Throughout this section, S is assumed to be a geometrically ruled surface over a nonsingular curve R of genus $g(R)$. As usual, E,f denote a section of minimal self-intersection $E^2 = -e$ and a fibre of the ruling. Here we find some sufficient numerical conditions for a line bundle L on S to be k-spanned. In some case, such conditions come out to be also necessary.

First we consider the case $g(R) = 0$.

(3.1) PROPOSITION. Let $S = \mathbb{F}_r$ be a Hirzebruch surface of invariant $r \geqq 1$ and let $L \sim aE+bf$ be a line bundle on S. Then L is k-spanned if and only if $a \geqq k$ and $b \geqq ar+k$.

Proof. If L is k-spanned, then $L \cdot f = a \geqq k$ and $L \cdot E = -ar+b \geqq k$. To show the converse, write

$$L \sim k(E+(r+1)f)+(a-k)(E+rf)+(b-(ar+k))f$$

and note that $E+(r+1)f$ is very ample and $E+rf$, f are spanned (see e.g. [6], p. 379, 382). Then we are done by (0.5.3).

(3.2) REMARK. On a quadric $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ is clear that a line bundle L of type (a,b) is k-spanned if and only if $a \geqq k$, $b \geqq k$. Indeed, $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)$ is $k = \min(a,b)$-spanned.

□

Thus we can assume $g(R) > 0$. Recall that $K_S \sim -2E+(2g(R)-2-e)f$ where $e = -E\cdot E$ is the invariant of S.

(3.3) PROPOSITION. Let S be a geometrically ruled surface with invariant $e \geqq 0$ and $q(S) > 0$. Let $L \sim aE+bf$ be a line bundle

<u>on</u> S . <u>Then</u> L <u>is k-spanned if</u>

$$a \geq k;\ b \geq ae+2q(S)-2+\max(k+2,e)\ .$$

<u>Proof</u>. First note that $E+ef$ is nef; indeed we see that $(E+ef)\cdot B$ for every irreducible curve B on S, recalling that for such a curve B, $B \neq E,f$, $B \sim \alpha E+\beta f$ with $\alpha > 0$, $\beta \geq \alpha e$ ([6], p. 382). No let

$$M = L-K_S \sim (a+2)E+(b+e-2(q(S)-1))f\ .$$

Then $M\cdot M = (2b-4(q(S)-1)-ae)(a+2) \geq 2(k+2)^2$ and hence $M\cdot M \geq 4k+5$ for $k \geq 1$. Further M is nef; indeed

$$M \sim (a+2)(E+ef)+(b-ae-e-2(q(S)-1))f$$

and both $E+ef$, f are nef. Thus if L is not k-spanned, Theorem (0.6) applies to say that there exists an effective divisor D su that

$$M\cdot D-k-1 \leq D\cdot D < M\cdot D/2 < k+1\ .$$

We can write $D \sim xE+yf$ where $x = D\cdot f \geq 0$, $y = D\cdot(E+ef) \geq 0$. Now $M\cdot D = x(b-ae-e-2(q(S)-1))+y(a+2)$, then from $M\cdot D/2 < k+1$ and the assumptions made on a and b we get $y(k+2) < 2(k+1)$ which lead to $y = 0,1$. If $y = 0$, $D\cdot D = -ex^2+2xy \geq M\cdot D-k-1$ yields

$$-ex+x(k+2)-k-1 \leq -ex^2$$

and $x \geq 1$ since $y = 0$. Hence $ex(x-1)+2x \leq 0$, a contradiction. If $y = 1$, $D\cdot D \geq M\cdot D-k-1$ gives

$$-ex+1 \leq -ex^2$$

that is $xe(x-1)+1 \leqq 0$, again a contradiction.

(3.4) PROPOSITION. Let S be a geometrically ruled surface of invariant $e < 0$ and $q(S) > 0$. Let $L \sim aE+bf$ be a line bundle on S . Then L is k-spanned if

$$a \geqq k;\ b \geqq ae/2+2q(S)+k \ .$$

Proof. First, note that $E+(e/2)f$ is nef. Indeed one sees that $(E+(e/2)f)\cdot B \geqq 0$ for every irreducible curve B on S , recalling that for such a curve B, $B \neq E,f$, $B \sim \alpha E+\beta f$ with either $\alpha = 1$, $\beta \geqq 0$ or $\alpha \geqq 2$, $\beta \geqq \alpha e/2$ ([6], p. 382). Let

$$M = L-K_S \sim (a+2)E+(b-(2q(S)-2)+e)f \ .$$

Then $M\cdot M = (2b-4(q(S)-1)+2e-(a+2)e)(a+2) \geqq 2(k+2)^2$ and hence $M\cdot M \geqq 4k+5$ for $k \geqq 1$. Further, by writing

$$M \sim (a+2)\left(E+\frac{e}{2}f\right) + \left(\frac{2b-4(q(S)-1)-ae}{2}\right)f$$

we see that M is nef. Thus if L is not k-spanned, there exists by (0.6) an effective divisor D such that

$$M\cdot D-k-1 \leqq D\cdot D < M\cdot D/2 < k+1 \ .$$

We can write $D \sim x(E+(e/2)f)+yf$ where $x \in \mathbb{Z}$, $2y \in \mathbb{Z}$ and further $x = D\cdot f \geqq 0$, $D\cdot(E+(e/2)f) = y \geqq 0$. Here

$$M\cdot D = x\left(\frac{-ae}{2}+b-2(q(S)-1)\right)+y(a+2) \ ,$$

then from $M\cdot D/2 < k+1$ and the assumptions made on a and b we find $(k+2)(x+y)/2 < k+1$, which gives $x+y < 1$. Therefore, since $y \in \mathbb{Z}$ if $x = 0$, the only possible cases are $(x,y) = (1,1/2)$, (1,0), (0,1) and an easy check shows that they contradict $D \cdot D = 2xy \geq M \cdot D - k - 1$.

□

In the special case when $q(S) = 1$ something more can be said.

(3.5) REMARK ($q(S)=1$). Assume S is a geometrically ruled surface over a curve of genus $g(R) = 1$. Then a standard but rather long computation, following the lines of the previous proofs, gives us necessary and sufficient conditions for a line bundle $L \sim aE+bf$ on S to be k-spanned. We state here the results, omitting the proof for shortness.

(3.5.1) if $e = -1$, L is k-spanned if and only if $a \geq k$, $2b+a \geq k+2$;

(3.5.2) if $e \geq 0$, L is k-spanned if and only if $a \geq k$, $b \geq k+2+ae$.

(3.6) REMARK (conic bundle case). It is worth to point out that if (S,L) is a conic bundle on a curve R of genus $q(S)$ and L is a k-spanned line bundle on S, then $k \leq 2$. Further, if $q(S) \geq 1$, $k = 2$ and S is geometrically ruled then

$$g(L) \geq q(S)+3 .$$

Indeed $L \sim 2E+bf$, so $L \cdot f = 2$ and hence $k \leq 2$. If $k = 2$, $\deg L_E = L \cdot E = b-2e \geq 4$ by (1.4.1) while the genus formula gives us $g(L) = b+2q(S)-1-e$. Therefore $g(L) \geq q(S)+e+3$, so we are done since $e \geq -q(S)$.

§ 4. The k-th adjunction mapping.

Let L be a k-spanned line bundle on S. It is useful to use [13] to find for which positive integers t, $t \leq k$, the line bundle tK_S+L is very ample or spanned. The results of this section are essentially corollaries of the analogous results for very ample line bundles contained in [13], we used over and over (especially Theorems (0.1), (0.2) and (2.1)).

For $t = 1$, we have the following

(4.1) THEOREM. *Let* L *be a k-spanned line bundle on* S *with* $k \geq 2$. *Then* (S,L) *contains no lines and* K_S+L *is very ample unless either*:

(4.1.1) $k = 2$ *and* (S,L) *is a geometrically ruled conic bundle;*

(4.1.2) $k = 2$ *and either* $(S,L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, $g(L)=0$, *or* $(S,L) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,2))$, $g(L)=1$;

(4.1.3) $k = 2$ *and* S *is a Del Pezzo surface with* $K_S \cdot K_S = 2$, $L \approx K_S^{-2}$, $g(L)=3$;

(4.1.4) $k = 3$ *and* $(S,L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$, $g(L)=1$.

Proof. First note that L is in fact 2-spanned or 3-spanned in all cases listed above. This is clear if (S,L) is either as in (4.1.1), (4.1.2) or (4.1.4), while (0.8) shows that L is 2-spanned in case (4.1.3).

Now if L is k-spanned, clearly (S,L) contains no lines since $L \cdot C \geq k$ for any curve C on S. Then (S,L) cannot be either $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, a scroll, nor not relatively minimal. Thus by looking over the lists in [13] we see that K_S+L is very ample unless either (S,L) is in one of the cases listed above or

i) S is a $\mathbb{P}^1$ bundle over an elliptic curve with invariant $e = -1$ and $L \cong \xi^3$, ξ the tautological line bundle;

ii) S is a Del Pezzo surface with $K_S^{-3} \approx L$, $K_S \cdot K_S = 1$;

iii) S is a Del Pezzo surface with $K_S^{-1} \approx L$.

In case i), ξ is a effective elliptic curve and $L \cdot \xi = 3$, wh contradicts (1.4.1).

In case ii), the general element $E \in |K_S^{-1}|$ is an elliptic curve and $L \cdot E = \deg L_E = 3$ contradicts again (1.4.1).

In case iii), since (S,L) is relatively minimal it has to be either $(S,L) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2,2))$ or $(S,L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ as in class (4.1.2) or (4.1.4) respectively. This completes the proof.

(4.2) REMARK. Let L be a k-spanned line bundle on S with $k \geq$ and assume K_S+L very ample. Then $(K_S+L) \cdot L \geq k$ so by the genus formula

$$g(L) \geq 1 + k/2 \ .$$

Note also that if (S, K_S+L) is not relatively minimal and ℓ is a line with $\ell^2 = -1$, then $(K_S+L) \cdot \ell = 1$ gives $L \cdot \ell = 2$, hence $k = 2$.

(4.3) Special classes. To go on it is convenient to analyze first the very ampleness and the spannedness of $t\,K_S+L$, L k-spanned line bundle and t positive integer, in three particular cases.

(4.3.1) Let $S = \mathbb{P}^2$. Then $L \approx \mathcal{O}_S(k)$; hence $t\,K_S+L \approx \mathcal{O}_S(-3t+k)$ is very ample iff $t < k/3$ and spanned if $t \leq k/3$.

(4.3.2) Let $S = \mathbb{P}^1 \times \mathbb{P}^1$. Then $L \cong \mathcal{O}_S(a,b)$ with $k = \min(a,b)$. Therefore $t\,K_S+L \approx \mathcal{O}_S(a-2t, b-2t)$ is very ample if and only if $t < k/2$ and spanned if and only if $t \leq k/2$.

(4.3.3) Let S be a $\mathbb{P}^1$ bundle and let f be a fibre. Here we can assume by induction on t that $(t-1)K_S+L$ is very ample. Then by [13] (see in particular (1.4)) we know that:

i) tK_S+L is very ample unless either $((t-1)K_S+L)\cdot f = 1,2$ or S is a $\mathbb{P}^1$ bundle over an elliptic curve with invariant $e = -1$, $(t-1)K_S+L \approx 3\xi$, ξ the tautological line bundle and $((t-1)K_S+L)\cdot f = 3$

ii) tK_S+L is ample and spanned unless $((t-1)K_S+L)\cdot f = 1$ or 2 and is spanned unless $((t-1)K_S+L)\cdot f = 1$.

Thus, recalling that $L\cdot f \geq k$, an easy check gives us the following

(4.3.3.1) for a positive integer t, tK_S+L is:

- very ample if $t < k/2$, unless S is a $\mathbb{P}^1$ bundle over an elliptic curve of invariant $e = -1$ and L is described as in i) above; further in this case tK_S+L is very ample if $t \leq (k-2)/2$;
- ample and spanned if $t < k/2$;
- spanned if $t \leq k/2$.

□

Thus we can assume now that S is neither $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$ nor a $\mathbb{P}^1$ bundle.

(4.4) THEOREM. <u>Let</u> L <u>be a</u> k-<u>spanned line bundle on</u> S <u>with</u> $k \geq 2$ <u>and let</u> S <u>be neither</u> $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$ <u>nor a</u> $\mathbb{P}^1$ <u>bundle. Then,</u> <u>for a positive integer</u> t, $t \leq k-1$, <u>we have</u>:

(4.4.1) tK_S+L <u>is very ample unless</u> $t = k-1$, S <u>is a Del Pezzo surface with</u> $K_S\cdot K_S = 2$ <u>and</u> $L \approx K_S^{-k}$;

(4.4.2) $(t+1)K_S+L$ <u>is spanned unless</u> $t = k-1$ <u>and</u> (S,L) <u>is as in</u> (4.3.3.1). <u>Further if</u> $(t+1)K_S+L$ <u>is spanned, the morphism associated</u>

<u>to</u> $\Gamma((t+1)K_S+L)$ <u>has a</u> 2-<u>dimensional image unless</u> $t = k-1$ <u>and</u> <u>either</u> S <u>is a Del Pezzo surface with</u> $L \approx K_S^{-k}$, $K_S \cdot K_S \neq 1$ <u>or</u> $(S,(k-1)K_S+L)$ <u>is a conic bundle</u>, $L \cdot f = k$.

<u>Proof</u>. By induction on t , we can assume $(t-1)K_S+L$ to be very ample. Note that by [13], (0.1) we can also assume $t K_S+L$ to be spanned: otherwise S is as in one of the above examples. Note also that S does not contain lines ℓ such that $\ell^2 = -1$ and $((t-1)K_S+L)\cdot\ell = 1$ since $t \leq k-1$ and $L\cdot f \geq k$. If $t K_S+L$ is not very ample we see by [13], (0.2) and (2.1) that the only possibiliti are:

i) $K_S^{-1} \approx (t-1)K_S+L$;

ii) $(S,(t-1)K_S+L)$ is a conic bundle;

iii) $K_S^{-2} \approx (t-1)K_S+L$ and $K_S\cdot K_S = 2$;

iv) $K_S^{-3} \approx (t-1)K_S+L$ and $K_S\cdot K_S = 1$.

In case i), either S contains a line $\ell, \ell^2 = K_S\cdot\ell = -1$, and hence $L\cdot\ell = t K_S^{-1}\cdot\ell = t \geq k$, $S = \mathbb{P}^2$ or $S = \mathbb{P}^1 \times \mathbb{P}^1$, a contradict

In case ii) we can assume $L\cdot f \geq 2k$, f a fibre of the ruling; otherwise each fibre would be irreducible and hence S would be a $\mathbb{P}^1$ bundle. Then $((t-1)K_S+L)\cdot f = 2$ contradicts $t \leq k-1$.

In case iv), $L \approx K_S^{-(t+2)}$ is k-spanned if and only if $t+2 \geq k+2$ by (0.8), this contradicting once again $t \leq k-1$.

In case iii), $L \approx K_S^{-(t+1)}$ is k-spanned if and only if $t+1 \geq k$, again by (0.8). Hence $t = k-1$ and $L \approx K_S^{-k}$. This proves (4.4.1).

Now, by [13], (0.1) we see that $(t+1)K_S+L$ is spanned whenever $t K_S+L$ is very ample under the assumptions made on S .

Finally by [13], (0.2) we see that, if $(t+1)K_S+L$ is spanned, the morphism associated to $\Gamma((t+1)K_S+L)$ has a 2-dimensional image unless either $K_S^{-1} \approx t\,K_S+L$ or $(S, t\,K_S+L)$ is a conic bundle. In both cases we find $t = k-1$ and we are done. Note that if $K_S^{-1} \approx t\,K_S+L$, there exists some line ℓ with $\ell^2 = K_S \cdot \ell = -1$ since S is neither $\mathbb{P}^2$ nor $\mathbb{P}^1 \times \mathbb{P}^1$.

□

The above result gives us rather strong numerical conditions for k-spannedness.

(4.5) COROLLARY. *Let L be a k-spanned line bundle on S, write $d = L \cdot L$ and let $g(L)$ be the sectional genus of L. Further assume S be neither $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, a $\mathbb{P}^1$ bundle, nor a Del Pezzo surface with $K_S \cdot K_S = 2$, $L \approx K_S^{-k}$ as in (4.3.3.1). Then we have:*

(4.5.1) $d \leq k^2 K_S \cdot K_S + 4k(g(L)-1)/(2k-1)$;

(4.5.2) $d \leq 2k(g(L)-1)/(k-1)$;

(4.5.3) $d \leq k\,K_S \cdot K_S + 2(g(L)-1)$ *if* $\kappa(S) \geq 0$.

Proof. By (4.4.2), $k\,K_S+L$ is nef. Then $(k\,K_S+L)^2 \geq 0$, $(k\,K_S+L) \cdot L \geq 0$ and $K_S \cdot (k\,K_S+L) \geq 0$, together genus formula (0.2), give (4.5.1), (4.5.2) and (4.5.3) respectively.

§ 5. A classification of (S, L) for small values of $g(L)$.

As an application of the previous result we classify in this section the polarized pairs (S, L) where L is a k-spanned line bundle on S with $k \geq 2$ and sectional genus $g(L) \leq 5$. Note that in [9], [10] a complete classification is carried out for $g(L) \leq 6$ if $k = 1$ (see also [7]).

The cases $g(L) \leq 3$ are easy consequences of Theorem (4.1).

(5.1) PROPOSITION. *Let* L *be a* k-*spanned line bundle on* S *with* $k \geq 2$ *and sectional genus* $g(L) \leq 3$. *Then we have*:

(5.1.1) $g(L) = 0$, $k = 2$, $(S, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$;

(5.1.2) $g(L) = 1$; *either* $k = 2$ *and* $(S, L) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2$ *or* $k = 3$ *and* $(S, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$;

(5.1.3) $g(L) = 2$, $k = 2$ *and either* $(S, L) \cong (\mathbb{F}_0, \mathcal{O}_{\mathbb{F}_0}(a, b))$ *with* $(a, b) = (2,3), (3,2)$, *or* $(S, L) \cong (\mathbb{F}_1, 2E + 4f)$

(5.1.4) $g(L) = 3$; *either* $k = 4$ *and* $(S, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))$ *or* $k = 2$ *and either* (S, L) *is isomorphic to* $(\mathbb{F}_0, \mathcal{O}_{\mathbb{F}_0}(a, b))$, *with* $(a, b) = (2,4), (4,2)$, $(\mathbb{F}_1, 2E +$ $+ 5f)$, $(\mathbb{F}_2, 2E + 6f)$, *or* S *is a Del Pezzo surface with* $K_S^{-2} \approx L$, $K_S \cdot K_S = 2$.

Proof. Let $g(L) \leq 1$. Then $K_S + L$ is not very ample in view of (4.2), so that (S, L) is as in (5.1.1) or (5.1.2) by Theorem (4.1).

Let $g(L) = 2$. Note that $L \cdot L \geq 4$ by (1.4.1) and hence $p_g(S) = 0$ by the genus formula. Therefore

$$h^0(K_S + L) = \chi(K_S + L) = 2 - q(S) ,$$

so that $K_S + L$ is not very ample. Thus by combining (4.1) and (3.6) we see that (S, L) is a geometrically ruled conic bundle over $\mathbb{P}^1$ and $k = 2$. We have $K_S \sim 2E - (2 + r)f$; if $L \sim 2E + bf$, $b \geq 2r + 2$ by (3.1) and the genus formula $2 = (K_S + L) \cdot L = 2(b - 2 - r)$ gives $b = r + 3$. Then either $r = 0$, $b = 3$ or $r = 1$, $b = 4$.

Let $g(L) = 3$. We know that $h^0(L) \geq 6$ by (2.4), so $d = L \cdot L \geq 7$ by Castelnuovo's bound (0.4.2). The same argument as above shows that $p_g(S) = 0$ and $h^0(K_S + L) = 3$, hence if $K_S + L$ is very ample we have $(S, K_S + L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ that is $(S, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))$ and $k = 4$. If $K_S + L$ is not very ample, again by (4.1) we see that $k = 2$ and either S is a Del Pezzo surface with $K_S \cdot K_S = 2$, $L \approx K_S^{-2}$ or (S, L) is a geometrically ruled conic bundle. In this case $S = \mathbb{F}_r$ by (3.6). If $L \sim 2E + bf$, $b \geq 2r + 2$ by (3.1) and the genus formula $4 = (K_S + L) \cdot L = 2(b - 2 - r)$ gives $b = 4 + r$. Then $r \leq 2$ and we are done.

(5.2) PROPOSITION. <u>Let</u> L <u>be a</u> k-<u>spanned</u> <u>line bundle on</u> S <u>with</u> $k \geq 2$ <u>and sectional genus</u> $g(L) = 4$. <u>Then either</u>:

(5.2.1) $k = 3$, $S = \mathbb{P}^1 \times \mathbb{P}^1$ <u>and</u> $L \cong \mathcal{O}_S(3,3)$;

(5.2.2) $k = 2$, S <u>is a cubic surface in</u> $\mathbb{P}^3$ <u>and</u> $L \cong \mathcal{O}_S(2)$; <u>or</u>,

(5.2.3) $k = 2$, <u>either</u> $S = \mathbb{F}_r$ <u>with</u> $r \leq 3$, $L \sim 2E + (5 + r)f$ <u>or</u> S <u>is a</u> $\mathbb{P}^1$ <u>bundle over an elliptic curve of</u>

<u>invariant</u> $e = -1$ <u>and</u> $L \sim 2E + 2f$.

<u>Proof</u>. One has $h^0(L) \geq 6$ by (2.4) then $d = L \cdot L \geq 8$ by Castelnuovo's bound (0.4.2). Therefore the genus formula and the Riemann-Roch theorem give us

$$h^0(K_S + L) = \chi(K_S + L) = 4 - q(S) .$$

Then if $K_S + L$ is very ample, it has to be $q(S) = 0$ and $|K_S +$ embeds S as a surface of degree $d' = (K_S + L)^2$ in $\mathbb{P}^3$. Hence $K_S \approx \mathcal{O}_S(d' - 4)$ and $L \approx \mathcal{O}_S(5 - d')$. Now since $p_g(S) = 0$ and L is at least 2-spanned the only possible cases are $d' = 2,3$ If $d' = 2$ we get class (5.2.1). If $d' = 3$, S is a cubic in $\mathbf{P}^3$ and $L \cong \mathcal{O}_S(2)$. Note that L is 2-spanned since $L \cdot \ell = 2$ for a line ℓ on S , so we find class (5.2.2).

If $K_S + L$ is not very ample (S , L) is a geometrically ruled conic bundle by (4.1) and $q(S) = 0,1$ by (3.6). Let $L \sim 2E + bf$.

If $q(S) = 0$, $S = \mathbb{F}_r$ and $b \geq 2r + 2$ by (3.1). The genus formula $6 = (K_S + L) \cdot L = 2(b - 2 - r)$ gives $b = 5 + r$. Then $r \leq 3$.

If $q(S) = 1$, $K_S \sim -2E - ef$, $e = -E^2$, $b - 2e \geq 4$ by (1.4.1) and the equality $6 = (K_S + L) \cdot L = 2(b - e)$ yields $b =$ $= 3 + e$, hence $e = -1$. An easy check by using (0.6) shows that $L \sim 2E + 2f$ is 2-spanned (see also (3.5.1)).

□

In the remaining case $g(L) = 5$, Theorem (2.4) plays a relevant role.

(5.3) PROPOSITION. *Let* L *be a* k-*spanned line bundle on* S *with* $k \geq 2$ *and sectional genus* $g(L) = 5$. *Then either*:

(5.3.1) $k = 2$ *and* $|L|$ *embeds* S *in* $\mathbb{P}^5$ *as a* K3 *surface of degree* 8, *a complete intersection of three quadrics*;

(5.3.2) $k = 2$, $(S, L) \cong (\mathbb{F}_1, 3E + 5f)$;

(5.3.3) $k = 2$, S *is a Del Pezzo surface*, $L \approx -2K_S$, $K_S \cdot K_S = 4$;

(5.3.4) $k = 2$, S *is the blowing up* $\sigma : S \longrightarrow \mathbb{F}_r$ *of* $\mathbb{F}_r$, $r = 0,1$, *along* 7 *distinct points* p_i, $L \approx \sigma^*(4E + (2r+5)f) - 2 \sum_{i=1}^{7} P_i$, $P_i = \sigma^{-1}(p_i)$;

(5.3.5) $k = 2$, $(S, L) \cong (\mathbb{F}_r, 2E + (6+r)f)$, $r \leq 4$; *or*,

(5.3.6) $k = 2$, S *is a* $\mathbb{P}^1$ *bundle over an elliptic curve*, $L \sim -2E + (e+4)f$, $e = 0, -1$.

Proof. Since $h^0(L) \geq 6$ by (2.4), Castelnuovo's inequality (0.4.2) gives now $d = L \cdot L \geq 8$.

First, let us assume $K_S + L$ very ample. We distinguish two cases, according to the value of $p_g(S)$.

If $p_g(S) > 0$ it has to be $d = 8$ by the genus formula and hence $K_S \cdot L = 0$ so that $K_S \sim 0$. From [12], § 3 we know that $5 = g(L) \geq h^0(L) + q(S) - 1$ and hence $h^0(L) = 6$, $q(S) = 0$. Thus $|L|$ embeds S as a degree 8 K3 surface in $\mathbb{P}^5$. Further $k = 2$ in view of (1.2.3). Note that S is a complete intersection of three quadrics. Indeed, if not, it is known that a general element $C \in |L|$ contains a g^1_3 (see e.g. [2], p. 142). Now $K_C \approx L_C$ is 2-spanned and $h^0(D) \leq 1$ for any divisor D on C with $\deg D \leq 3$ by (1.2), this contradicting C to be trigonal.

If $p_g(S) = 0$, the Riemann-Roch theorem yields $h^0(K_S + L) =$
$= \chi(K_S + L) = 5 - q(S)$, which gives $q(S) = 0$, $h^0(K_S + L) = 5$.
Then $|K_S + L|$ embeds S in $\mathbb{P}^4$ as a surface of degree $d' =$
$= (K_S + L)^2$ and one has (see [6], p. 434)

(5.3.7) $d'^2 - 5d' - 10(g(L) - 1) + 12\chi(\mathcal{O}_S) = 2K_S \cdot K_S$.

Now the usual Hodge index theorem yields $dd' \leq [L \cdot (K_S + L)]^2 = 64$
that $d' \leq 6$. From (5.3.7) and the equalities

$$g(K_S+L) = d' - g(L) + 2 ;$$
$$d' = K_S \cdot K_S + 2(2g(L)-2) - d = K_S \cdot K_S + 16 - d$$

a purely numerical check gives us for d , d' , $K_S \cdot K_S$,
$g(K_S+L)$ the values as in the table below

cases	$K_S \cdot K_S$	d'	$g(K_S + L)$	d
i)	8	3	0	21
ii)	4	4	1	16
iii)	1	5	2	12
iv)	− 1	6	3	9

In case i), $(S, K_S + L)$ is a $\mathbb{P}^1$ bundle $\mathbb{F}_r$ over $\mathbb{P}^1$ wit
$L \sim 3E + bf$ and $d = 21$ leads to $2b - 3r = 7$, hence $r \neq 0$.
Since $b \geq 3r + k$ by (3.1) we find $3r \leq 7 - 2k$ which gives $k =$
$r = 1$; so we obtain class (5.3.2).

In case ii), $(S, K_S + L)$ is either a Del Pezzo surface or a scroll over an elliptic curve, this contradicting $q(S) = 0$. Then $L \approx K_S^{-2}$ and we know from (0.8) that K_S^{-2} is 2-spanned. So we get class (5.3.3).

In case iii), let C be a general element of $|K_S + L|$. Then from the exact sequence

$$0 \longrightarrow K_S \longrightarrow K_S^2 \otimes L \longrightarrow K_C \longrightarrow 0$$

we find $h^0(2K_S + L) = h^0(K_C) = g(K_S + L) = 2$. By [13], (0.1), (0.2) we know that $2K_S + L$ is spanned and, since $(2K_S + L)^2 = 0$, $(S, K_S + L)$ is a conic bundle over $\mathbb{P}^1$. Further, since $K_S \cdot K_S = 1 = K_{\mathbb{F}_r} \cdot K_{\mathbb{F}_r} - 7$, we see that S is obtained as the blowing up $\sigma : S \longrightarrow \mathbb{F}_r$ of $\mathbb{F}_r$ at 7 distinct points p_i's. Then $K_S + L \approx \sigma^* M - \sum_{i=1}^{7} P_i$, $P_i = \sigma^{-1}(P_i)$, for some ample line bundle $M \sim 2E + bf$ on $\mathbb{F}_r$. Therefore $b \geq 2r + 1$ (see [6], p. 380) and $g(M) = g(K_S + L) = 2$, so the genus formula for M gives $b = r + 3$. Thus $r \leq 2$ and $L \approx \sigma^*(4E + (2r+5)f) - 2\sum_{i=1}^{7} P_i$. Now, case $r = 2$ is clearly excluded since $4E + 9f$ is not 2-spanned by (0.7) and, if E' denotes the proper transform under σ, $L \cdot E' \leq (4E + 9f) \cdot E = 1$. Thus Proposition (0.7) applies to say that L is 2-spanned. Indeed L is of the form $K_S^{-t} \otimes \mathcal{L}^q$ with $t = 2$, $q = 1$ and $\mathcal{L}$ the inverse image under σ of the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ to $\mathbb{F}_r$ under a bundle projection $\mathbb{F}_r \longrightarrow \mathbb{P}^1$. This gives class (5.3.4).

In case iv), again from [13], (0.1), (0.2) we know that $2K_S + L$ is spanned and, since $(2K_S + L)^2 = 1$, $|2K_S + L|$ gives a birational morphism $\sigma : S \longrightarrow \mathbb{P}^2$. Further since $K_S \cdot K_S = -1$ we see that σ is the blowing up of $\mathbb{P}^2$ along 10 distinct point p_i's and $L \approx \sigma^*\mathcal{O}_{\mathbb{P}^2}(7) - 2 \sum_{i=1}^{10} P_i$, P_i's the exceptional divisor Let γ be a cubic plane curve passing through 9 of the points p_i's . Note that γ does not contain the remaining point. Otherwise the proper transform γ' of γ under σ belongs to $|-K_S$ and hence $h^0(-K_S) \geq 1$. Since $k \geq 2$ this contradicts the fact that $-L \cdot K_S = 1$. Then $\gamma' \cdot L = 3$, so γ' is irreducible. Fix now four points of the p_i's and take six plane cubic curves C_j as above, passing through the four fixed points and with $C_j' \cdot L =$ $j = 1, \ldots ,6$. Since $h^0(L) = 6$, we can choose an element $A \in$ whose image $\sigma(A)$ passes through the four fixed points, so that $A \cdot C_j \geq 4$, $j = 1, \ldots ,6$. It thus follows that A contains th cubics C_j's and this clearly contradicts $L \cdot A = L \cdot L = 9$.

Thus we can assume $K_S + L$ not very ample. Then (S , L) is geometrically ruled conic bundle by (4.1) with irregularity $q(S) = 0,1$ or 2 in view of (3.6).

Note that the case $q(S) = 2$ does not occur. Indeed the equalities $(K_S + L)^2 = 0$, $(K_S + L) \cdot L = 8$, $K_S \cdot K_S = 8(1 - q(S))$ give $d = 8$ if $q(S) = 2$, a contradiction.

If $q(S) = 0$, the genus formula $8 = (K_S + L) \cdot L = 2(b - 2 - r$ yields $b = 6 + r$ where $L \sim 2E + bf$, $r = -E^2$. Then, since $b \geq$ $\geq 2r + 2$ by (3.1), we find $r \leq 4$ and we are in class (5.3.5).

If $q(S) = 1$, by using again the genus formula one has $b =$ $= 4 + e$, where $L \sim 2E + bf$, $e = -E^2$ and $\deg L_E = b - 2e \geq 4$ by (1.4.1). Thus we find either $e = 0$, $b = 4$ or $e = -1$, $b = 3$. Note that in both cases L is 2-spanned in view of

(3.3), (3.4) and we are in class (5.3.6.).

(5.5) REMARK. If the conjecture (2.6) is true, then (5.3.1) does not occur. We attempted without success to show that the restriction L of $\mathcal{O}_{\mathbb{P}^5}(1)$ to S, S equal to the complete intersection of three quadrics in $\mathbb{P}^5$, is only 1-spanned. It should be noted that there exist such S which contain a line, ℓ, of $\mathbb{P}^5$ and for these, since $L \cdot \ell = 1 < 2$, it follows that L is not 2-spanned. In general though there are no lines on such an intersection of quadrics.

(5.5) REMARK (compare with § 6). Let L be a k-spanned line bundle on S with $k \geqq 2$ and assume $p_g(S) \geqq 2$. Then $g(L) \geqq$ $\geqq 2k+1$ by (1.2.3). In the extremal case $g(L) = 2k+1$ the inequality

$$p_g(S) \leqq k-3$$

holds true, hence in particular $\chi(\mathcal{O}_S) \leqq k-2$. To see this, recall that $L \cdot L \geqq 2k+3$ by (1.5), so the genus formula reads $K_S \cdot L \leqq$ $\leqq 2k-3$. Thus we are done after showing that $K_S \cdot L \geqq p_g(S)+k$. Indeed, $h^0(K_S - L) = 0$ since $(K_S - L) \cdot L < 0$ so that $h^0(K_{S|C}) \geqq$ $\geqq p_g(S)$. Now if the p_i's are $p_g(S)-2$ different points, on S, we have $h^0(K_{S|C} - \sum_i p_i) \geqq 2$. Therefore $\deg K_{S|C} - p_g(S) + 2 =$ $K_S \cdot L - p_g(S) + 2 \geqq k+2$ by (1.2.2).

§ 6. Geography of surfaces and k-spannedness.

In this section we study the relation between k-spannedness of a line bundle L on S and the birational geometry of S. We aim for a broad picture. The arguments we use clearly give much sharper bounds in particular cases. Since the case of very ample line bundles is well studied we make the blanket assumption that $k \geq 2$. Through this section we shall use repeatedly almost all the results we stated in § 1 as well as the genus formula (0.2) and property (0.5.1). We also use a number of well known results on the birational classification of surfaces for which we refer to [2]. We shall write d instead of $L \cdot L$.

(6.1) Let $\pi : S \longrightarrow S'$ be a morphism of S to a minimal model S'. Let $L' = (\pi_* L)^{**} = [\pi(C)]$ where C is a smooth element of $|L|$. Note $K_S \approx \pi^* K_{S'} + \sum_{i=1}^{r} n_i P_i$ where the P_i's are the irreducible components of the positive dimensional fibres of π, $n_i \geq 1$, $r = e(S) - e(S')$. Further $n_i = 1$ for all i if and only if $\pi : S \longrightarrow S'$ is a simple blowing up of a finite set of r points. From this we easily obtain the following simple lemma.

(6.1.1) LEMMA. <u>One has</u> $L \cdot K_S \geq k(e(S) - e(S')) + L \cdot \pi^* K_{S'}$, <u>with equality if and only if</u> (S', L') <u>is a</u> k-<u>reduction</u> <u>of</u> (S, L).

(6.1.2) COROLLARY. <u>If</u> $\kappa(S) \geq 0$, <u>then</u> $L \cdot K_S \geq k(e(S) - e(S'))$. <u>If further</u> $\kappa(S) \geq 1$ <u>and</u> $h^0(K_S^t) > 0$ <u>for some</u> $t > 0$ <u>then</u> $L \cdot K_S \geq k(e(S) - e(S')) + (k+1)/t$.

Proof. It follows from (6.1.1) by noting that $K_{S'}$ is nef and the general element A of $|\pi^* tK_{S'}|$ has positive arithmetic genus, so that $L \cdot A \geq k+1$.

(6.2) THEOREM. Assume $\kappa(S) \geq 0$. Then $d \geq 2k+3$. Further $g(L) \geq 2k+1$ unless possibly if S is minimal, $p_g(S) = 0$ and $q(S) = 0$ or 1. If $g(L) \leq 2k$ and $\kappa(S) = 2$ then $q(S) = 0$, $1 \leq K_S \cdot K_S \leq 9$, $d \geq (5k+10)/2$ and $g(L) \geq (3k+8)/2$.

Proof. Let C be a general element of $|L|$. Since $\kappa(S) \geq 0$, $d \leq 2g(L) - 2$ so it follows that $d \geq 2k+3$. If $h^1(L_C) \neq 0$ then $g(L) \geq 2k+1$. Thus we can assume that $h^1(L_C) = 0$ and therefore $p_g(S) = 0$. Since $\chi(\mathcal{O}_S) \geq 0$ we conclude that $q(S) = 0$ or 1. Further, by the Riemann-Roch theorem

$$(6.2.1) \quad d = h^0(L_C) + g(L) - 1 = 2h^0(L_C) + K_S \cdot L$$

whence

$$(6.2.1)' \quad g(L) - 1 = h^0(L_C) + K_S \cdot L .$$

If S were non minimal, $K_S \cdot L \geq k$ by (6.1.1). Hence $g(L) \geq k+2+k$ by (6.2.1)'. Therefore we can assume further that S is minimal. Now, let $g(L) \leq 2k$. If $\kappa(S) = 2$, then $K_S \cdot K_S \geq 1$ and $\chi(\mathcal{O}_S) > 0$, while $p_g(S) = 0$ implies $q(S) = 0$ and hence $\chi(\mathcal{O}_S) = 1$. Thus $K_S \cdot K_S \leq 9$ by the Miyaoka-Yau inequality. The Riemann-Roch theorem gives $h^0(K_S^2) \geq 2$. It thus follows that $K_S \cdot L \geq (k+1)/2$ by (6.1.1). Actually $K_S \cdot L \geq (k+2)/2$ since otherwise we would have a pencil of rational or elliptic curves on S. Then by (6.2.1), (6.2.1)' we find $d \geq (5k+10)/2$ and

$g(L) \geq (3k+8)/2$.

(6.3) THEOREM. Let S be a $\mathbb{P}^1$ bundle $p : S \longrightarrow R$ over a cur R of genus $q(S)$. Then

(6.3.1) $d \geq 2k^2$ and $g(L) \geq (k-1)^2$ if $q(S) = 0$;

(6.3.2) $d \geq k(k+2)$ and $g(L) \geq k(k+1)/2$ if $q(S) = 1$;

(6.3.3) $d \geq 2k+4$ and $g(L) \geq 2k+1$ if $q(S) \geq 2$.

Proof. Let E be a section of p of minimal self-intersection and f a fibre of p , so $L \sim aE + bf$.

If $q(S) = 0$, $E^2 = -r$ and $b \geq ar + k$, $a \geq k$ by (3.1). Hence $d = L \cdot L = a(2b - ar) \geq k(b+k) \geq 2k^2$. Similarly $g(L) \geq$ $\geq 2(k-1)^2$.

Let $q(S) = 1$, $E^2 = -e$. Here $a \geq k$ and either $b \geq ae +$ if $e \geq 0$ or $2b - ae \geq k+2$ if $e = -1$ (see (3.5)). So $d = L \cdot$ $= a(2b - ae) \geq 2(k+2)k$ if $e \geq 0$ and $d \geq k(k+2)$ if $e = -1$. Further $2g(L) - 2 = (a-1)(2b-ae) \geq 2(k-1)(k+2)$ if $e \geq 0$ an $2g(L) - 2 \geq (k-1)(k+2)$ if $e = -1$. In either case $d \geq k(k+2)$ and $g(L) \geq k(k+1)/2$.

Let $q(S) \geq 2$. We know from (4.3.3) that $kK_S + 2L$ is nef. Hence $k^2K_S \cdot K_S + 4kK_S \cdot L + 4L \cdot L > 0$. Now $K_S \cdot K_S = 8 - 8q(S)$, then

$$4k(2g(L) - 2) \geq (4k-4)d + (8q(S) - 8)k^2$$

and also

$$g(L) - 1 \geq (k-1)d/2k + (q(S) - 1)k$$

If $h^1(L_C) \neq 0$ we are done. Hence we can assume $h^1(L_C) = 0$, so that $d = h^0(L_C) + g(L) - 1 \geq g(L) + k + 1$. Thus

$$g(L) - 1 \geq (k-1)(g(L) + k + 1)/2k + (q(S) - 1)k$$

which gives

$$(k+1)(g(L) - 1)/2k \geq (k-1)(k+2)/2k + (q(S) - 1)k$$

or

$$g(L) \geq (k-1)(k+2)/(k+1) + 2k^2/(k+1) + 1 = 3k - 1 \geq 2k + 1 .$$

Finally $d \geq g(L) + k + 1$ yields $d \geq 4k > 2k + 3$.

(6.4) THEOREM. _If_ $K_S \cdot K_S \leq -x < 0$ _and_ S _is not a_ $\mathbb{P}^1$ _bundle then_

$$d \geq 2k + 3 \ ; \quad g(L) \geq k(1 + x/4) + 3/4k .$$

Proof. Now $kK_S + L$ is nef by (4.4), so we find

$$-k^2x + 2k(2g(L) - 2) - (2k-1)d \geq 0$$

and also

(6.4.1) $$g(L) - 1 \geq kx/4 + (2k-1)d/4k .$$

If $d > 2g(L) - 2$ we get

$$g(L) - 1 \geq kx/4 + (2k-1)(2g(L) - 2)/4k + (2k-1)/4k$$

or

$$(g(L) - 1)/2k \geq kx/4 + (2k-1)/4k$$

and also

$$g(L) \geqq k^2x/2 + k + 1/2$$

which gives $d \geqq k^2x + 4k - 1 \geqq 2k + 3$. If $d < 2g(L) - 2$, then $d \geqq 2k + 3$ and, by (6.4.1), $g(L) \geqq k(1 + x/4) + 3/4k$. Note that $k^2x/2 + k + 1/2 \geqq k(1 + x/4) + 3/4k$.

(6.5) THEOREM. *If* $\chi(\mathcal{O}_S) < 0$ *and* S *is not a* $\mathbb{P}^1$ *bundle. Then*

$$d \geqq 2k + 3 \ ; \quad g(L) > k(2q(S) - 1) + k/4 \ .$$

Proof. Since $\chi(\mathcal{O}_S) < 0$, S is a ruled surface with $q(S) > 1$ and $K_S \cdot K_S < 8(1 - q(S)) < 0$. Use Theorem (6.4) with $x = 8q(S)-1$

It mainly remains to consider rational and elliptic surfaces.

(6.6) LEMMA. *If* S *is rational and* $K_S \cdot K_S \geqq 0$ then $h^0(K_S^{-1}) >$

Proof. S is either $\mathbb{P}^2$ or a blowing up of $\mathbb{F}_r$. An easy calculation shows that $h^0(K_{\mathbb{F}_r}^{-1}) \geqq 9$. Each time a point is blown up on a surface, the number of sections of the anticanonical line bundle decreases by at most 1, so that $h^0(K_S^{-1}) \geqq h^0(K_{\mathbb{F}_r}^{-1}) - \#$ where # denotes the number of blowing ups. Thus since $\# = K_{\mathbb{F}_r} \cdot K_{\mathbb{F}_r} - K_S \cdot K_S \leqq 8$ the Lemma is proven.

(6.7) PROPOSITION. *Assume* S *is not a* $\mathbb{P}^1$ *bundle. Then*:

(6.7.1) $d \geqq k^2$ *and* $g(L) \geqq k(k - 1)/2 + 1$ *if* $K_S \cdot K_S \geqq 0$ *and* S *is rational*;

(6.7.2) $d \geqq 2k + 3$ *and* $g(L) \geqq 2k + 1$ *if* $K_S \cdot K_S \leqq -4$;

(6.7.3) $d \geqq 4k$ *and* $g(L) \geqq 2k - 1$ *if* $K_S \cdot L \leqq -4$.

Proof. Since $kK_S + L$ is nef by (4.4) one has

(6.7.4) $$d \geq -kK_S \cdot L .$$

If S is rational and $K_S \cdot K_S \geq 0$, Lemma (6.6) gives $h^0(K_S^{-1}) > 0$, hence $-K_S \cdot L \geq k$ so $d \geq k^2$. Further

(6.7.5) $$2g(L) - 2 \geq -(k-1)K_S \cdot L \geq k(k-1)$$

whence $g(L) \geq k(k-1)/2 + 1$. This proves (6.7.1).

Now (6.7.2) follows from (6.4) while (6.7.4) and (6.7.5) yield (6.7.3).

(6.8) REMARK. Note that if $kK_S + L$ is nef, by writing $(kK_S + L)^2 = k^2K_S \cdot K_S + (2k-1)K_S \cdot L + 2g(L) - 2 \geq 0$ we find

$$2g(L) - 2 \geq -k^2K_S \cdot K_S - (2k-1)K_S \cdot L .$$

Therefore if $K_S \cdot L \leq 0$ and $K_S \cdot K_S < 0$ one has $g(L) \geq k^2/2 + 1$ and $d\ (\geq -k^2K_S \cdot K_S - 2kK_S \cdot L) \geq k^2$.

(6.9) THEOREM. *If S is rational, $d \geq k^2$. Further $g(L) \geq k(k-1)/2 + \min(1, k-2)$ if $K_S \cdot L \leq 0$ and $g(L) > 5k/4$ if $K_S \cdot L > 0$.*

Proof. If S is a $\mathbb{P}^1$ bundle use (6.3). If S is not a $\mathbb{P}^1$ bundle, use (6.7.1) if $K_S \cdot K_S \geq 0$; (6.8) if $K_S \cdot K_S < 0$ and $K_S \cdot L \leq 0$; (6.4) with $x = 1$ if $K_S \cdot K_S < 0$ and $K_S \cdot L > 0$.

(6.10) THEOREM. *Let S be an elliptic ruled surface but not a $\mathbb{P}^1$ bundle. Then $d \geq k^2$, $g(L) \geq (k^2+2)/2$ unless $K_S \cdot L > 0$. If $K_S \cdot L > 0$ then $d \geq 2k+3$ and $g(L) > 5k/4$.*

Proof. We know that $kK_S + L$ is nef by (4.4). Then

$$2g(L) - 2 \geqq - k^2 K_S \cdot K_S - (2k - 1) K_S \cdot L$$

as in (6.8) with $K_S \cdot K_S < 0$. So if $K_S \cdot L \leqq 0$ we find $g(L) \geqq (k^2 + 2)/2$ and also $d \geqq 2g(L) - 2 \geqq k^2$. If $K_S \cdot L > 0$ we use (6.4).

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