Periods of Enriques Surfaces

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§0. Introduction.

The aim of this article is twofold. The first is to give simplified and more arithmetic proof of the Torelli theorem and the surjectivity theorem of the period map for Enriques surfaces due to Horikawa [5]. The main theorem is precisely stated in §1 (1.14). Our statement is even more refined than Horikawa's in two minor points (Remark 1.15). The second is to apply the Torelli theorem in studying Enriques surfaces more precisely.

Such a theory was developed extensively in the case of K3 surfaces by Pjateckiĭ-Šapiro and Šafarevič and others. Here we take a way to follow this model as far as possible and reduce the problem to the corresponding one in the case of K3 surface. It turns out that this method is remarkably far-reaching to obtain better results than those by our predecessors. We should say, however, that this was made possible thanks to a deep study in lattice theory mainly due to Nikulin. The lattice theory plays an essential role here, and instead the only essential geometric fact we use other than the results on K3 surface is that the universal covering of an Enriques surface is isomorphic to a K3 surface. To the contrary we deduce other important geometric properties of Enriques surface from our results (see §§5,6).

In §1 we recall fundamental properties of Enriques surfaces and state the main result precisely. Roughly saying, we can define the socalled period space $\mathcal{D}/\Gamma$ of Enriques surface (1.7) and with each Enriques surface S we can associate a point ω_S in $\mathcal{D}/\Gamma$ called the period of

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S (1.13). The Torelli theorem (in a weaker form) asserts that the period determines S uniquely and the surjectivity theorem asserts that except for an explicitly defined irreducible divisor all points in $\mathcal{D}/\Gamma$ correspond to periods of Enriques surface (1.14). The irreducibility of this exceptional set was noticed here for the first time.

The next §2 is devoted to a lattice-theoretic study of the cohomology group of an Enriques surface which is the essential part of our method. One of the key fact is Theorem 1.4. We collect in §3 the corresponding results for K3 surfaces which we need for the proof of our main result. Here we also give a lattice-theoretic characterization of a finite automorphism group of a K3 surface in the group of isometries of K3 lattice, which is inspired by Nikulin [11] and is a key to reduce the surjectivity theorem for Enriques surfaces to that for K3 surfaces.

The Torelli theorems for Enriques surface are formulated and proved in §4. The proof in this method was announced by Nikulin in [10] . Our formulation is as similar as possible to that for K3 surface.

The next two sections are devoted to applications of the Torelli theorem. Thanks to our formulation we can obtain synthetically several recent results on the automorphism group of an Enriques surface due to Dolgachev [4] and Barth-Peters [1], which we exhibit in §5. We here make no use of the structure of a double covering over a rational surface as [5] or [2], but we can prove the existence of such a structure purely with lattice theory ((5.11) and (5.15)). In §6 we study smooth rational curves and smooth elliptic curves. A generic Enriques surface contains no smooth rational curves and the set of periods of Enriques surfaces containing rational curves forms an irreducible divisor in the period space defined explicitly (Theorem 6.4). In contrast with this any

Enriques surface contains smooth elliptic curves forming a pencil (Theorem 6.7). However in any case there are only a finitely many rational and elliptic curves up to Aut(S) and linear equivalence (Theorem 6.5, 6.7). We count this number for one interesting example of Enriques surface which was treated in [5], [4], [1].

The surjectivity theorem of the period map is proved in the last section §7. A proof essentially in the same line is given also in [2].

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§1. Definitions and statement of the main theorem

Definition (1.1). A complex analytic surface S is called an *Enriques surface* if i) the geometric genus p_g of S and the irregularity q of S vanish, and ii) with K denoting the canonical divisor of S, 2K is linearly equivalent to zero.

(1.2) An Enriques surface S has an unramified covering $p : \tilde{S} \to S$ with the covering transformation i where $\tilde{S}$ is isomorphic to a K3 surface. Let $\iota = i^*$ be the involution of $H^2(\tilde{S}, \mathbb{Z})$ induced from i. If we denote by ω the cohomology class of a holomorphic 2-form on S (unique up to constant) in $H^2(\tilde{S}, \mathbb{Z})$, called the *period* of $\tilde{S}$, then $\iota\omega = -\omega$ since there is no global holomorphic 2-form on S.

(1.3) A lattice H is a free abelian group of finite rank endowed with an integral quadratic form. For a lattice H and an integer n we denote by H(n) the lattice whose quadratic form is the one on H multiplied by n.

The second cohomology group $H^2(S, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{10} \oplus (\mathbb{Z}/2)$ where $\mathbb{Z}/2$ is generated by $c_1(K)$. The free part $H^2(S,\mathbb{Z})_0$ admits a canonical structure of a lattice induced from the cup product $\langle , \rangle$ which is a symmetric bilinear form. It is even, unimodular and of index (1, 9), hence isomorphic to $E = E_8(-1) \perp U$ where E_8 is the positive definite lattice associated with the Dynkhin diagram of type E_8 and U is the hyperbolic lattice $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

In the same way $H^2(\tilde{S}, \mathbb{Z})$ is also an even unimodular lattice isomorphic to $L = (\overset{2}{\perp} E_8(-1)) \perp (\overset{3}{\perp} U)$. The induced map $p^* : H^2(S, \mathbb{Z})_0 \to H^2(\tilde{S}, \mathbb{Z})$ has a homothetic property that $\langle p^*(x), p^*(y)\rangle = 2\langle x,y\rangle$, hence $p^*H = E_8(-2) \perp U(2)$ as a lattice. The covering involution ι determines two primitive sublattices (i.e. quotients have no torsion)

$$M = \{x \in L;\ \iota x = x\} \quad \text{and} \quad N = \{x \in L;\ \iota x = -x\}$$

such that $M \perp N$ and $\operatorname{rk} M + \operatorname{rk} N = 22$. Clearly $p^*E \subset M$, but it turns out that the equality holds (Prop. (2.3)) in fact. Hence $M \cong E(2) = E_8(-2) \perp U(2)$ and $N \cong E_8(-2) \perp U(2) \perp U$.*) In particular M and N are 2-primitive (M^*/M and N^*/N are 2-torsion groups), from which we can deduce the following important uniqueness property of the embedding of M into L.

Theorem (1.4). Let $j_1, j_2 : E(2) \to L$ be two primitive embeddings. Then any isometry $\varphi : E(2) \to E(2)$ extends to an isometry $\tilde{\varphi} : L \to L$ with $\tilde{\varphi}\cdot j_1 = j_2\cdot\varphi$.

The same property holds for $E(2) \perp U$ also.

The proof is given in §2. In [5] this arithmetic theorem was proved with geometric method. This way is, however, much longer, though it has a nice biproduct [4].

(1.5) Now we consider the abstract lattices L and E defined in (1.3). By virtue of Theorem (1.4), without loss of generality we may choose and fix the following embedding of $E(2)$ into L.

For $x \in L$ we write its decomposition as

$$\begin{array}{ccccccccccc} L & = & E_8(-1) & \perp & E_8(-1) & \perp & U & \perp & U & \perp & U \\ \cup & & & & \cup & & & & & & \\ x & = & x_1 & + & x_2 & + & u_1 & + & u_2 & + & u_3. \end{array}$$

We define an involution ι on L by sending $(x_1, x_2, u_1, u_2, u_3)$ to $(x_2, x_1, u_2, u_1, -u_3)$. Let M (resp. N) be the $(+1)$- (resp. (-1)-) eigenspace of ι in L. We have two emdeddings of $E(2) = E_8(-2) \perp U(2)$:

*) Actually this follows first from the proof of Theorem 1.4 in (2.9).

$$\begin{array}{rcl} E_8(-2) \perp U(2) & \to & L \\ \cup & & \cup \\ i_1 : (x, \quad u) & \to & (x, x, u, u, 0), \\ i_2 : (x, \quad u) & \to & (x, -x, u, -u, 0), \end{array}$$

which give isomorphisms $M \simeq E(2)$ and $N \simeq E(2) \perp U$ respectively.

(1.6) A K3 surface $\tilde{S}$ has a holomorphic 2-form Ω unique up to multiplicative constants. Denote by ω the cohomology class of Ω in $H^2(\tilde{S},\mathbb{C})$ defined by the De Rham isomorphism. This is equivalent to say that ω is the cohomology class determined by the periods:

$$\begin{array}{rcl} \int_\Omega : H_2(\tilde{S},\mathbb{Z}) & \to & \mathbb{C} \\ \cup & & \cup \\ \gamma & \to & \int_\gamma \Omega. \end{array}$$

This ω satisfies Riemann's equality and inequality

$$(*) \qquad \langle\omega,\omega\rangle = 0, \quad \langle\omega,\bar{\omega}\rangle > 0$$

where $\langle,\rangle$ denotes the bilinear form on $H^2(\tilde{S},\mathbb{C})$ coming from the cup product and "$\bar{\ }$" the complex conjugate in $H^2(\tilde{S},\mathbb{C})$ with respect to $H^2(\tilde{S},\mathbb{R})$. Moreover we have

$$(**) \qquad \iota\omega = -\omega$$

i.e. $\omega \in N_{\mathbb{C}}$ since the quotient surface $S = \tilde{S}/\langle\iota\rangle$ has no global holomorphic 2-forms.

<u>Definition</u> (1.7). We construct the so-called period domain as follows. We use the notation above in (1.6).

i) $\quad \mathcal{D} = \{(v) \in \mathbb{P}(N_{\mathbb{C}}) \ ; \langle v,v\rangle = 0, \langle v,\bar{v}\rangle > 0\},$

$$\tilde{\mathcal{D}} = \{ (v) \in \mathbb{P}(L_{\mathbb{C}}) \ ; \ \langle v,v\rangle = 0, \ \langle v,\bar{v}\rangle > 0 \}.$$

The former (resp. the latter) is called the period domain of Enriques surfaces (resp. of K3 surfaces). They are of dimension 10 and 20 respectively. Moreover $\mathcal{D}$ is a union of 2 copies of bounded symmetric domain of type IV. There is a natural embedding $\mathcal{D} \to \tilde{\mathcal{D}}$ induced from $N \subset L$, by which we consider $\mathcal{D}$ as a closed subset of $\tilde{\mathcal{D}}$.

On $\mathcal{D}$ (resp. $\tilde{\mathcal{D}}$) the orthogonal group $O(N_{\mathbb{R}})$ (resp. $O(L_{\mathbb{R}})$) acts transitively.

ii) Let $\Gamma = O(N)$ (resp. $\Gamma' = O(M)$) be the group of isometries of N (resp. M) and $\tilde{\Gamma}$ the group of isometries of L which commute with ι, or equivalently to say, preserve subspaces M and N. We have a canonical homomorphism $\rho : \tilde{\Gamma} \to \Gamma$ (resp. $\rho' : \tilde{\Gamma} \to \Gamma'$) defined by the restriction.

iii) $\mathcal{D}/\Gamma$ is called the *period space* of Enriques surfaces. Denote by π the canonical surjection of $\mathcal{D}$ onto $\mathcal{D}/\Gamma$.

(1.8) Since Γ is arithmetic, Γ acts properly discontinuously on $\mathcal{D}$, hence by H. Cartan's theorem $\mathcal{D}/\Gamma$ admits a canonical structure of a normal analytic space such that the canonical surjection $\pi : \mathcal{D} \to \mathcal{D}/\Gamma$ is holomorphic. It is even a quasi-projective algebraic variety. It is also connected ($\sigma = (+1)_{|E(2)} \oplus (-1)_{|U} \in O(N)$ interchanges the two connected components of $\mathcal{D} \subset \mathbb{P}(N_{\mathbb{C}})$), hence irreducible as an algebraic variety.

<u>Definition</u> (1.9). We set

$$\mathcal{D}_o = \{(v) \in \mathcal{D} \ ; \ \text{for each} \ \ \ell \subset N \ \ \text{with} \ \ \langle \ell,\ell \rangle = -2, \ \langle \ell,v \rangle \neq 0\}.$$

In other words $\mathcal{D}_o$ is the complement of the union of all reflexion hyperplanes $H_\ell = \{(v) \ ; \ \langle v,\ell \rangle = 0\}$ in $\mathcal{D}$.

(1.10) For a root ℓ (i.e. $\langle \ell,\ell \rangle = -2$) in N we set $\Gamma_\ell = \{g \in O(N); g(\ell) = \pm\ell\}$. $\mathcal{D}_\ell = \mathcal{D} \cap H_\ell$ is again a union of 2 copies of bounded symmetric domains of type IV but of dimension 9 on which Γ_ℓ acts properly discontinuously. $\mathcal{D}_\ell/\Gamma_\ell$ is an irreducible closed subvariety of $\mathcal{D}/\Gamma$. Theorem 2.13 in §2 implies that

$$\mathcal{D}_o/\Gamma \amalg \mathcal{D}_\ell/\Gamma_\ell = \mathcal{D}/\Gamma.$$

Definition (1.11). A *marked Enriques surface* is a pair (S,ψ) of an Enriques surfaces S and an isometry (called *marking*) $\psi : H^2(S,\mathbb{Z})_o \to E$. A *morphism* of marked Enriques surfaces is an isomorphism $a : S_1 \to S_2$ such that $\psi_1 \cdot a^* = \psi_2$ with the induced isometry $a^* : H^2(S_2,\mathbb{Z})_o \to H^2(S_1,\mathbb{Z})_o$.

A marking of K3 surface is defined similarly by using L.

(1.12) By virtue of Theorem 1.4, for a marked Enriques surface (S,ψ), we can extend the marking to that of the covering K3 surface $(\tilde{S},\tilde{\psi})$ satisfying the following commutative diagram

$$\begin{array}{ccc} H^2(S,\mathbb{Z})_o & \xrightarrow{\psi} & E \\ & & \downarrow id \\ p^* \downarrow & & E(2) \\ & & \tilde{\psi} \ \ \downarrow i_1 \\ H^2(\tilde{S},\mathbb{Z}) & \to & L \end{array}$$

under the notation in (1.5).

Then the cohomology class ω of $\tilde{S}$ (1.6) defines a point $(\tilde{\psi}_{\mathbb{C}}(\omega))$ in $\mathbb{P}(L_{\mathbb{C}})$, but in fact in $\mathcal{D}$ by the properties (*) and (**) in (1.6). This point depends on the marking $\tilde{\psi}$, but it is uniquely determined by (S,ψ) modulo Γ. Moreover by Corollary 2.9 (for ρ') $\omega_S = (\tilde{\psi}_{\mathbb{C}}(\omega))$ mod Γ depends only on S and not on the marking ψ.

<u>Definition</u> (1.13). The point $\omega_S = (\tilde{\psi}_{\mathbb{C}}(\omega))$ mod Γ in $\mathcal{D}/\Gamma$ is called the *period* of S.

<u>Main Theorem</u> (1.14). The correspondence: $\{S\} \to \omega_S$ gives a bijection between the set of isomorphy classes of Enriques surfaces and $\mathcal{D}_o/\Gamma$ (not the whole $\mathcal{D}/\Gamma$).

<u>Remark</u> (1.15). There are two improvements in our statement than in [5]. The first is that the arithmetic group of quotient Γ is $O(N)$ itself (by virtue of Theorem 1.4). The second is the remark that the complement of $\mathcal{D}_o/\Gamma$ is irreducible (1.10).

More important remark would be that *as a corollary* of the above theorem we obtain the following.

<u>Corollary</u> (1.16). Any two Enriques surfaces can be deformed to each other.

§2. Enriques lattice.

(2.1) In this section we prove some properties of the Enriques lattice which play an essential role in the proof of the first part of the main Theorem.

(2.2) Let S be an Enriques surface and $p : \tilde{S} \to S$ be the universal covering with the covering involution i (1.2). We consider the induced homomorphism $p^* : H^2(S,\mathbb{Z}) \to H^2(\tilde{S},\mathbb{Z})$ and the (+1)-eigenspace $M = \{x \in H^2(\tilde{S},\mathbb{Z});\ \iota x = x\}$ of $\iota = i^*$ (1.3).

Proposition (2.3). $p^*(H^2(S,\mathbb{Z})) = M$.

Proof (due to Mukai). With the Poincaré dual $p_* : H^2(\tilde{S},\mathbb{Z}) \to H^2(S,\mathbb{Z})_0$ we have $p^*p_*(x) = x + \iota(x)$ for $x \in H^2(\tilde{S},\mathbb{Z})$, hence $p^*(H^2(S,\mathbb{Z})) \subset M$ and $M/p^*(H^2(S,\mathbb{Z}))$ is at most 2-torsion group.

Since S is algebraic, there is an ample cycle $a \in H^2(S,\mathbb{Z})$. Note that $p^*(a)$ is also ample on $\tilde{S}$.

Let x be an element in M. By adding $mp^*(a)$ for $m >> 0$ we may assume that x is a class of a curve C whose linear system $|C|$ has no fixed points. On $|C|$ the involution ι acts linearly, hence has a fixed point t. Then $p_*(C_t)$ (C_t is the member of $|C|$ corresponding to t) is in the form $2C'$ for a (maybe reducible) curve C'. Therefore $2p^*([C']) = 2x$, which implies $p^*([C']) = x$ since $H^2(\tilde{S},\mathbb{Z})$ has no torsion.

Q.E.D.

Remark (2.4). One can prove the above proposition also purely topologically by using the fact that $H^1(\tilde{S},\mathbb{Z}) = 0$.

Corollary (2.5). M is isomorphic to $E(2) = E_8(-2) \perp U(2)$, and in particular M^*/M is a 2-torsion group (such lattice is called *2-primitive* in [12]).

<u>Corollary (2.6)</u>. We use the notation in (1.7) ii). The homomorphisms ρ and ρ' are surjective and their kernels are isomorphic respectively to the groups $\Gamma'(2)$, $\Gamma(2)$ of isometries of M and N which acts trivially on M^*/M ($\simeq M/2M$) and N^*/N.

Proof. The first part is a special case of Theorem 1.4. The second is a mere restatement of Proposition 1.5.1 in [12].

(2.7) To prove the next results (Proof of Theorem 1.4, and Theorems 2.13, 2.15) we make use of Nikulin's deep study on integral quadratic forms [12]. The most important fact is the following:

<u>Theorem</u> (2.8) ([12] Theorem 1.14.2). Let T be an even indefinite (non-degenerate) lattices satisfying the following conditions: i) rank $T \geq \ell((T^*/T)_p)+2$ for each prime $p \neq 2$ where $\ell(A_p)$ denotes the number of minimal generators of the p-part of a finite abelian group A, and ii) if rank $T = \ell((T^*/T)_2)$ then $q_{T,2} \simeq u(2) \perp q_2'$ or $v(2) \perp q_2'$ where $q_{T,p}$ denotes the p-component of the discriminant form q of T^*/T and $u(2^k)$ (resp. $v(2^k)$) denotes the discriminant form of $U(2^k)$ (resp. $V(2^k)$ with $V = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$). Then the genus of T contains only one isomorphy class, and the homomorphism $O(T) \to O(q_T)$ is surjective.

(2.9) *Proof of Theorem* 1.4.

By (2.5) M is 2-primitive, hence so is $N = M^{\perp}$, and their discriminant forms are isomorphic to $\overset{5}{\perp} u(2)$. Therefore we can apply for N the above Theorem 2.8 to conclude that the genus of N contains a unique lattice and $O(N) \to O(q_N)$ is surjective. By (1.5) we see that

$N = E_8(-2) \perp U(2) \perp U$. A similar result holds for M also by (2.8).

Then combining Proposition 1.14.1 in [12] with the above fact we obtain Theorem 1.4.

Remark (2.10). One can also apply James' theorem ([6], Theorem 6) to M but not to N.

(2.11) As the last topic we study vectors ℓ in N with $\langle \ell,\ell \rangle = -2$ and -4. The former are called *roots* in N. Our main result is that there are only one ℓ with $\langle \ell,\ell \rangle = -2$ (Theorem 2.13) and two ℓ's with $\langle \ell,\ell \rangle = -4$ (Theorem 2.15) up to $O(N)$.

The next fundamental fact we use is again due to Nikulin.

Proposition (2.12)([12], Proposition 1.15.1). A primitive embedding of Λ into an even lattice N is determined by the set $(H_\Lambda, H_N, \gamma; K, \gamma_K)$ where $H_\Lambda \subset \Lambda^*/\Lambda$ and $H_N \subset N^*/N$ are subgroups, $\gamma: q_\Lambda|H_\Lambda \to q_N|H_N$ is an isometry preserving the restrictions of the quadratic forms to these subgroups, K is an even lattice with the complementary index and discriminant form $-\delta$ where $\delta = (q_\Lambda \oplus (-q_N)|\Gamma_\gamma^\perp)/\Gamma_\gamma$ where Γ_γ is the "graph" of γ in $\Lambda^*/\Lambda \oplus N^*/N$, and $\gamma_K: q_K \to (-\delta)$ is an isomorphism of quadratic forms.

Two such sets $(H_\Lambda, H_N, \gamma: K, \gamma_K)$, $(H'_\Lambda, H'_N, \gamma'; K', \gamma'_{K'})$ determine isomorphic primitive embeddings if and only if i) $H_\Lambda = H'_\Lambda$ and ii) there exist $\xi \in O(q_N)$ and $\psi \in \mathrm{Isom}(K,K')$ for which $\gamma' = \xi \cdot \gamma$ and $\bar{\xi} \cdot \gamma_K = \gamma'_{K'} \cdot \bar{\psi}$ where $\bar{\xi}$ is the isomorphism of discriminant forms δ and δ' induced by ξ.

Theorem (2.13). For any two roots ℓ and ℓ' in N there is an isometry $\phi \in \Gamma = O(N)$ with $\phi(\ell) = \ell'$.

Proof. We consider the sublattice Λ generated by ℓ and show that the embedding $j:\Lambda \to N$ is unique up to $O(N)$.

We calculate the invariants of the embedding in (2.12). First $\Lambda^*/\Lambda \cong \mathbb{Z}/2$ with $q_\Lambda = \langle -1/2 \rangle$ and $N^*/N \cong (\mathbb{Z}/2)^{10}$ with $q_N \cong \overset{5}{\perp} u(2)$ where $u(2)$ is the discriminant form of $U(2)$. This lattice is $\mathbb{Z}$-valued, hence q_Λ cannot be embedded in q_M. Therefore $H_\Lambda = H_N = (0)$ and $\delta \cong \overset{5}{\perp} u(2) \perp \langle -1/2 \rangle$. By Theorem 2.8 $K = \Lambda^\perp$ is unique, hence isomorphic to $E(2) \perp \langle 2 \rangle$ and the canonical homomorphism $O(K) \to O(q_K)$ is surjective. Again by Proposition 2.12 this implies that the embedding $j:\Lambda \to N$ is unique up to $O(N)$. Q. E. D.

__Remark__ (2.14). We do not know an elementary direct proof. It is easy to see that the similar result holds for $U \perp U(2)$.

__Theorem__ (2.15). Let ℓ be a vector in N with $\langle \ell,\ell \rangle = -4$. Then $\ell^\perp$ is isomorphic to either $E_8(-2) \perp U \perp \langle 4 \rangle$ or $E_8(-2) \perp U(2) \perp \langle 4 \rangle$. In the former case we call ℓ *of even type* and in the latter case *of odd type.*

Two vectors in N with length -4 are equivalent modulo $O(N)$ if and only if they have the same type.

Proof. Again let Λ be the sublattice generated by ℓ and we calculate the invariants of the embedding $j:\Lambda \to N$ in (2.12). In this case $\Lambda^*/\Lambda \cong \mathbb{Z}/4$ with $q_\Lambda = \langle -1/4 \rangle$. Since q_N is $\mathbb{Z}$-valued, $H_\Lambda \cong \mathbb{Z}/2$ or 0, and then $\delta \cong \overset{4}{\perp} u(2) \perp \langle -1/4 \rangle$ or $\overset{5}{\perp} u(2) \perp \langle -1/4 \rangle$ respectively. (Note that, when $H_N \cong H_\Lambda \cong \mathbb{Z}/2$, such H_N is generated by a vector n in N^*/N with $q_N(n) = 1$ and it is easy to see such n is unique up to $O(q_N)$.)

Therefore we can apply Nikulin's Theorem 2.8 for $\ell^{\perp}$ to conclude that $\ell^{\perp}$ is unique in each case and $O(\ell^{\perp}) \to O(q_{\ell^{\perp}})$ is surjective. By choosing ℓ in $U(2)$ or U we see that $\ell^{\perp}$ is isomorphic to one of the lattices given above.

Again by (2.12) the uniqueness of the embedding in each type follows from the uniqueness of H_N modulo $O(q_N)$ and the surjectivity of $O(\ell^{\perp}) \to O(q_{\ell^{\perp}})$ above. Q. E. D.

We close this section by giving a characterization of vectors of length -4 of even type.

Proposition (2.16). Let ℓ be a vector in $N \subset L$ with $\langle \ell,\ell \rangle = -4$. Then ℓ is of even type if and only if there is a vector m in $M = N^{\perp}$ with $\langle m,m \rangle = -4$ and $(\ell+m)/2 \in L$.

Proof. Since any isometry in N lifts to that in L (Corollary 2.6), the above property is invariant under the action of $O(N)$, hence it suffices to see this for concrete examples.

We use the notation in (1.5) and write with a basis as $u_i = \alpha_i e_i + \beta_i f_i$ with $\alpha_i, \beta_i \in \mathbb{Z}$ and with $\langle e_i,e_i \rangle = 0 = \langle f_i,f_i \rangle$ and $\langle e_i,f_i \rangle = 1$ $(i = 1,2,3)$. Then $\ell_0 = e_1-f_1-e_2+f_2 \in N$ is of even type and $\ell_1 = e_3-2f_3 \in N$ is of odd type. Now with $m = e_1-f_1+e_2-f_2$ the former ℓ_0 satisfies the condition and it is easily seen that ℓ_1 does not satisfy the condition because all vectors in M have zero coefficient at e_3. Q. E. D.

§3. Period of K3 surfaces

(3.1) In this section we recall the corresponding facts on periods for K3 surfaces to which we reduce our main theorem. For the complete proof we refer the reader to [13], [3], [7] on Torelli theorem and [15], [9] for the surjectivity theorem.*)

For the reduction we make use of the theory of hyperbolic reflexion groups as in [7]. This enables us to see the key of the proof more transparently. We also prove a general theorem on a characterization of isometries in the K3 lattice which come from finite automorphisms of a K3 surface modulo reflexions. This is a slight generalization of Nikulin's and from it we derive the surjectivity theorem for Enriques surfaces (§7). Another application is given in [8] (cf. Remark 5.4).

(3.2) Let X be a K3 surface and ω the period in $H^2(X,\mathbb{C})$ (1.6). We define

$$H^{1,1}(X) = \{x \in H^2(X, \mathbb{R});\ \langle\omega,x\rangle = 0\},$$

$$H^{1,1}_{\mathbb{Z}}(X) = H^{1,1}(X) \cap H^2(X,\mathbb{Z}).$$

On $H^{1,1}(X)$ the bilinear form $\langle,\rangle$ has index (1.19).

We set

$$V = \{x \in H^{1,1}(X);\ \langle x,x\rangle > 0\}$$

which is a union of 2 opposite homogeneous convex cones V^+ and V^- $(= -V^+)$. We set also

$$P = \{\delta \in H^{1,1}_{\mathbb{Z}}(X);\ \langle\delta,\delta\rangle = -2\},$$

and

$$P^+ = \{\delta \in P;\ \delta \text{ is an effective algebraic cycle}\},$$

*) See also a very beautiful survey on this subject by Beauville in Séminaire Bourbaki.

$$P^- = \{\delta \in P;\ -\delta \in P^+\}.$$

A member of P is called a *root*. By the Riemann-Roch theorem $P = P^+ \amalg P^-$.

Lastly we define the Kähler cone K as

$$K = \{x \in V^+;\ \langle x,\delta\rangle > 0 \text{ for all } \delta \in P^+\}.$$

Clearly any Kähler class falls into K, but conversely an element in K corresponds to a Kähler form (see Theorem 3.13 below), since every K3 surface admits a Kähler metric as Siù has proved recently [14].

(3.3) One of the important aspects in the proof of Torelli theorem for K3 surfaces is that we interprete the Kähler cone in a different way.

Each $\delta \in P$ defines a reflexion $s_\delta : x \to x + \langle x,\delta\rangle\delta$ which is an isometry in $H^2(X,\mathbb{Z})$ and fixes the period ω, hence preserves V and P. Let W be the group generated by s_δ, $\delta \in P$. Since $s_\delta = s_{-\delta}$, we can repalce P by P^+. Now the theory of reflexion group asserts that W acts properly discontinuously on V^+ and $\overline{K}$ is a fundamental domain of W in V^+.

(3.4) A point which we want to emphasize here, though known and used by specialists, is to consider a subset P_o^+ of P^+ defined as

$$P_o^+ = \{\delta \in P^+;\ \delta \text{ is an effective cycle of a smooth rational curve in } X\}.$$

We set $P_o^- = \{\delta \in P^-;\ -\delta \in P_o^+\}$.

The fact that $H^{1,1}(X)$ is of index (1.19) implies that

(*) $\langle x,y\rangle > 0$ for $x \in V^+$, $y \in \overline{V}^+ - \{0\}$.

Also the genus formula says that for any irreducible curve C we have $C^2 \geq -2$ and $C^2 = -2$ if and only if C is a smooth rational curve. By these two properties we see that the following conditions are equivalent for $x \in H^{1,1}(X)$:

i) $\langle x,e\rangle > 0$ for all effective algebraic cycles e on X;

ii) $\langle x,\delta\rangle > 0$ for all $\delta \in P^+$;

iii) $\langle x,\delta\rangle > 0$ for all $\delta \in P_o^+$;

iv) $\langle x,C\rangle > 0$ for all irreducible algebraic curves C on X.

Hence we can replace P^+ by P_o^+ in the definition of K.

The last property implies that W is in fact generated by P_o^+.

Remark (3.5). i) We can show moreover that P_o^+ is a minimal generator of W, or equivalently $\{\langle x,\delta\rangle > 0$ for $\delta \in P_o^+\}$ is the minimal set of inequalities defining K. This can be seen as follows. Choose an element $a \in K$. For each $\delta \in P_o^+$ consider $x(\lambda) = \lambda\delta + a$ with $\lambda > 0$. Then $\langle x(\lambda),\delta'\rangle > 0$ for any $\delta' \in P_o^+$ different from δ since $\langle\delta',\delta\rangle \geq 0$. Therefore if we substitute $\lambda = (\langle\delta,a\rangle+\epsilon)/2$ for sufficiently small ϵ, then $\langle x(\lambda),x(\lambda)\rangle = \langle\delta,a\rangle^2/2 + \langle a,a\rangle - \epsilon^2 > 0$ and $\langle x(\lambda),\delta\rangle = -\epsilon < 0$.

ii) The set P_o^+ is a "fundamental root system" and satisfies the condition $\langle\delta,\delta'\rangle \geq 0$ for different $\delta, \delta' \in P_o^+$, which enables us to have an algorithm to find an element $w \in W$ which transforms a given element $x \in V$ to $w(x) \in \overline{K}$ analogous to the Kac-Moody theory.

(3.6) The Torelli theorem for K3 surfaces which we use here is as follows:

<u>Theorem</u> (Global Torelli theorem for Kähler K3 surfaces). Let X and X' be two Kähler K3 surfaces, and $\phi : H^2(X',\mathbb{Z}) \to H^2(X,\mathbb{Z})$ an isometry of lattices such that i) $\phi_{\mathbb{C}}\omega_{X'} = \lambda\omega_X$ for $\lambda \in \mathbb{C} - \{0\}$ where ω_X (resp. $\omega_{X'}$) is the period of X(resp. X') (1.6); ii) $\phi_{\mathbb{R}}(V^+(X')) = V^+(X)$ where $V^+(X)$ and $V^+(X')$ are the connected components of $V(X)$ and $V(X')$ which contain Kähler classes; iii) ϕ maps effective cycles to effective cycles. Then ϕ is induced by a unique isomorphism $a : X \to X'$.

<u>Proposition</u> (3.7). Let X, X' as above. For an isometry ϕ : $H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ satisfying i) above in (3.6) the following conditions are equivalent:

a) ii) and iii) in (3.6);

b) ii) and $\phi(P^+(X')) \subset P^+(X)$;

c) ii) and $\phi(P_0^+(X')) \subset P^+(X)$;

d) $\phi(K(X')) = K(X)$;

e) $\phi(K(X')) \cap K(X) \neq \phi$.

Proof. Clear from (3.3), (3.4) (cf. [7]).

(3.8) Together with the above (3.3) we can derive a modified form.

<u>Theorem</u> (Modified global Torelli theorem). Let X and X' be two Kähler K3 surfaces and $\phi : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ an isometry of lattices which satisfies the condition i) (resp. i) and ii)) above in (3.6). Then there exists unique isomorphism $a : X \to X'$ and a unique element $w \in W(X)$ such that $w \cdot \phi = a^*$ or $-a^*$ (resp. a^*).

(3.9) For the proof of the surjectivity theorem we also need the global Torelli theorem. We apply this to prove the following lattice-theoretical characterization of finite automorphism group of a K3 surface, which is a slight generalization of Nikulin's [11] (4.2) and (4.15).

<u>Theorem</u> (3.10). Let X be a K3 surface and G a finite subgroup of the group of isometries in $L = H^2(X, \mathbb{Z})$. Denote by ω the period of

X in $L_{\mathbb{C}} = H^2(X,\mathbb{C})$ and set $S_{G,X} = (L^G)^{\perp} \cap \{\mathbb{C}\omega\}^{\perp}$ ($= (L^G)^{\perp}$ in $H^{1,1}_{\mathbb{Z}}(X)$). Then there exists an element $w \in W(X)$ such that $wGw^{-1} \subset \mathrm{Aut}(X)$ if and only if i) $\mathbb{C}\omega$ is G-invariant, ii) $S_{G,X}$ contains no element ℓ of length -2, and iii) (in case $L^G_{\mathbb{C}} \ni \omega$, i.e. ω is G-invariant) $S_{G,X}$ is non-degenerate and negative definite if $S_{G,X} \neq 0$, or iii)' (in case $L^G_{\mathbb{C}} \not\ni \omega$) L^G contains an element x with $\langle x,x\rangle > 0$. (If $wGw^{-1} \subset \mathrm{Aut}(X)$, then iii) and iii)' hold always.)

Proof. *Necessity*. Since each reflexion is an isometry which preserves ω, it suffices to prove for $G \subset \mathrm{Aut}(X)$. The property i) is clear. Suppose that $\ell^2 \geq -2$ for some $\ell \in S_{G,X}$. Then ℓ or $-\ell$ is effective. Say $\ell = [C]$. Then $\tilde{C} = \sum_{g \in G} g(C) \in L^G \cap S_{G,X} = 0$, which is impossible. Hence follow ii) and iii). Lastly take an element x_0 in V^+. Then $x = \sum_{g \in G} g^*(x_0) \in (L_{\mathbb{R}})^G \cap V^+$. Since $L^G = \sum_{g \in G} \mathrm{Ker}(g^* - 1)$, it is rational, hence $(L_{\mathbb{R}})^G = (L^G)_{\mathbb{R}}$, in which $(L^G)_{\mathbb{Q}}$ is dense. Therefore we have iii)'.

Sufficiency. First we show that iii) or iii)' imply that $V^+ \cap L^G_{\mathbb{R}} \neq \phi$. In case $\omega \in L^G_{\mathbb{C}}$, the condition iii) says that L^G has index (3, 19 - rank $S_{G,X}$), hence $H^{1,1}_{\mathbb{R}} \cap L^G_{\mathbb{R}}$ has 1 positive eigenvalue. In case $\omega \notin L^G_{\mathbb{C}}$, it is enough to observe that $L^G \subset H^{1,1}_{\mathbb{Z}}$. The latter holds since there is $g \in G$ with $g(\omega) = \alpha\omega$ with $\alpha \neq 1$, hence $\langle x,\omega\rangle = \langle g(x), g(\omega)\rangle = \alpha\langle x,\omega\rangle$ for $x \in L^G$.

Then by the condition ii) there exists a G-invariant $x \in H^{1,1}_{\mathbb{R}}$ with $\langle x,x\rangle > 0$ and $\langle x,\ell\rangle \neq 0$ for all $\ell \in P$. We set $P^+_x = \{\ell \in P;\ \langle x,\ell\rangle > 0\}$, which is clearly G-invariant. There is a reflexion $w \in W(X)$ which transforms P^+_x to P^+. Then take $G' = wGw^{-1}$.

We now check the conditions in (3.6) for $g' \in G'$. The period ω is relative invariant by i). The cone V^+ is preserved because $V^+ \cap L_{\mathbb{R}}^{G'} \neq \phi$. The set P^+ is also preserved by the above modification. Therefore our theorem follows from the global Torelli theorem (3.6).

Remark (3.11). i) The above gives another proof for the fact that for a non-algebraic K3 surface ω is always G-invariant ([11], Theorem 0.1 a)).

ii) By the above theorem the condition of representability of a group G by automorphisms is closed if ω is G-invariant but not closed if ω is not G-invariant. Because of this the real surjectivity of the period map of Enriques surface is false (cf. Theorem 6.2).

(3.12) Now we formulate the surjectivity theorem for the period map of K3 surface. We use the notation in (1.7). Moreover for $p = (v) \in \mathfrak{D}$ we set:

$$H_p^{1,1} = \{x \in L_{\mathbb{R}};\ \langle x,v\rangle = 0\},$$

$$V_p = \{x \in H_p^{1,1};\ \langle x,x\rangle > 0\},$$

$$P_p = \{\delta \in H_p^{1,1} \cap L;\ \langle \delta,\delta\rangle = -2\},$$

$$V_p^o = \{x \in V_p;\ \text{for all } \delta \in P_p,\ \langle x,\delta\rangle \neq 0\},$$

and let W_p be the subgroup of $O(L)$ generated by the reflexions s_δ for $\delta \in P_p$ (cf. (3.2), (3.3)).

Then by the same reason as (3.3) W_p acts on V_p properly discontinuously and it gives a bijection between W_p and the set of connected components of V_p^o.

The surjectivity theorem essentially due to Todorov (for algebraic case essentially due to Kulikov) is stated as follows.

<u>Theorem</u> (3.13) (Surjectivity theorem for the period map of K3 surfaces). For any given $p \in \tilde{\mathcal{D}}$ and $\rho \in V_p^o$ there exists a (unique) marked K3 surface (X, ψ) and a Kähler metric μ on X such that the period $(\psi_{\mathbb{C}}(\omega_X)) = p$ and $\psi_{\mathbb{R}}(\omega_\mu) = \rho$ where ω_X is the period of X in $H^2(X, \mathbb{C})$ and ω_μ is the Kähler class of μ in $H^2(X, \mathbb{R})$.

§4. Global Torelli theorem for Enriques surfaces.

(4.1) In this section we prove the global Torelli theorem for Enriques surfaces, by reducing it to that for K3 surfaces (3.6). For the reduction we use again the theory of hyperbolic reflexions.

(4.2) Let S be an Enriques surface. We define similar notations as (3.2). Recall that $H^2(S, \mathbb{R})$ has index (1,9) (since $p_g = 0$, we may regard H^2 as $H^{1,1}$ itself).

$$V(S) = \{x \in H^2(S, \mathbb{R});\ \langle x, x\rangle > 0\}.$$

$V(S)$ has two connected components V^+ and V^- opposite to each other. We choose V^+ as the one containing ample classes.

$$P^+(S) = \{\delta \in H^2(S, \mathbb{Z})_o;\ \langle\delta,\delta\rangle = -2,\ \delta \text{ is effective}\},$$
$$P^-(S) = \{\delta \in H^2(S, \mathbb{Z})_o;\ -\delta \in P^+(S)\},$$
$$P(S) = P^+(S) \amalg P^-(S).$$

This modification of P is necessary because it can happen that neither δ nor $-\delta$ is effective for $\delta \in H^2(S, \mathbb{Z})_o$ with $\langle\delta,\delta\rangle = -2$.

$$P_o^+(S) = \{\delta \in H^2(S, \mathbb{Z})_o;\ \delta \text{ is an effective cycle of a smooth rational curve on } S\},$$
$$P_o^-(S) = \{\delta \in H^2(S, \mathbb{Z})_o;\ -\delta \in P_o^+(S)\},$$
$$K(S) = \{x \in V^+;\ \langle x, \delta\rangle > 0 \text{ for all } \delta \in P^+\}.$$

Let W be the subgroup of $O(H^2(S, \mathbb{Z})_o)$ generated by the reflexions s_δ for $\delta \in P(S)$. With the same reason as (3.4) W is generated by s_δ for $\delta \in P_o^+(S)$ and $\overline{K}(S)$ is a fundamental domain of W with respect to its action on V^+.

(4.3) Let $p : \tilde{S} \to S$ be the universal covering. For $X = \tilde{S}$ we have also the notions defined in (3.2), (3.3). To avoid the confusion we put " $\sim$ " for all the notations concerning $\tilde{S}$ ($\tilde{V}$, $\tilde{P}$, $\tilde{W}$ etc.).

We want to study their relations through $p^* : H^2(S, \mathbb{Z})_o \to H^2(\tilde{S}, \mathbb{Z})$. We identify $H^2(S, \mathbb{Z})_o$ with the invariant subspace M of $H^2(\tilde{S}, \mathbb{Z})$ with respect to the covering involution $\iota = i^*$ by (2.3). We should, however, take note on the fact that the quadratic form is doubled by p^*.

Lemma (4.4). We have

$$V = \tilde{V} \cap M_{\mathbb{R}},$$
$$V^{\pm} = \tilde{V}^{\pm} \cap M_{\mathbb{R}}.$$

Proof. Trivial.

Proposition (4.5). There is a canonical bijection between P_o^+ and the set of pairs $\{(\tilde{\delta}_1, \tilde{\delta}_2) ; \tilde{\delta}_i \in \tilde{P}_o^+, \langle\tilde{\delta}_1, \tilde{\delta}_2\rangle = 0, \iota(\tilde{\delta}_1) = \tilde{\delta}_2\}$ such that $p_*\tilde{\delta}_i = \delta \in P_o^+$ and $p^*\delta = \tilde{\delta}_1 + \tilde{\delta}_2$.

Proof. If C is a smooth rational curve on S, then $p^{-1}(C)$ is a disjoint union of two smooth rational curves $\tilde{C}_1$ and $\tilde{C}_2$ on S since p is unramified. The converse is also clear.

Remark (4.6). i) More generally, if an effective cycle $\delta \in P^+$ is a tree, then $p^*\delta$ is a disjoint union of cycles isomorphic to δ, which i interchanges one to another. The proof is easy by induction.

ii) Such decomposition is *not* possible for a general $\delta \in P^+$, because Im p_* is not primitive in $H^2(S, \mathbb{Z})_o$. For example consider elliptic pencil and take the elliptic curve e in a multiple fibre and a smooth rational curve δ in a fibre. Then we have $p^*(e + \delta) = \tilde{e} + \tilde{\delta}_1 + \tilde{\delta}_2$.

An element $\delta \in H^2(S, \mathbb{Z})$ with $\langle\delta, \delta\rangle = -2$ is in $P^*(S)$ and has such decomposition if and only if there is a vector ℓ in $H^{1,1}_{\mathbb{Z}}(\tilde{S}) \cap N$ (i.e. $\langle\ell, \omega_{\tilde{S}}\rangle = 0$ and $\iota(\ell) = -\ell$) with $\langle\ell, \ell\rangle = -4$ such that $(p^*\delta + \ell)/2 \in H^2(\tilde{S}, \mathbb{Z})$. Recall that such ℓ is called of even type in Theorem 2.15.

Note that, since all vectors in $P_0(S)$ are as such, this gives a characterization of the Weyl group W only in terms of lattice and the period.

iii) The above remark i) shows that no Enriques surface can contain smooth rational curves with the configuration $T_{2,3,7}$ (Figure 1), for they would induce rational curves on S with 2 copies of $T_{2,3,7}$ as the configuration, which is impossible by the Hodge index theorem ($T_{2,3,7} \cong E = E_8(-1) \perp H$). This gives rise to an interesting problem on automorphism groups of Enriques surfaces (cf. Remark 5.4, ii)).

Figure 1

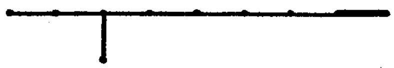


Figure 2

Therefore Horikawa's example ([5], 1, §5, [4]) is an Enriques surface which contains the most smooth rational curves, whose configuration is as Figure 2. The lattice generated by these cycles has index 2 in E. See (6.6) for further discussion on this example.

Proposition (4.7). We have

$$K = \tilde{K} \cap M_{\mathbb{R}} .$$

Proof. It is clear that $K \supset \tilde{K} \cap M_{\mathbb{R}}$ by the above Proposition 4.5. Take $x \in K$. We must show that $\langle x, \gamma\rangle > 0$ for all $\gamma \in \mathcal{P}_0^+$. Note that $\langle \gamma, \iota(\gamma)\rangle \geq 0$ since γ and $\iota(\gamma)$ are different irreducible cycles. If $\langle \gamma, \iota(\gamma)\rangle = 0$, then $\langle x, \gamma\rangle > 0$ again by Proposition 4.5. If $\langle \gamma, \iota(\gamma)\rangle > 0$, then $\langle \gamma + \iota(\gamma), \gamma + \iota(\gamma)\rangle = -4+2\langle \gamma, \iota(\gamma)\rangle$. Since $\gamma + \iota(\gamma) \in M$, we have $\langle \gamma, \iota(\gamma)\rangle \equiv 0 \pmod 2$, a fortiori $\langle \gamma, \iota(\gamma)\rangle \neq -1$, hence $\langle \gamma + \iota(\gamma), \gamma + \iota(\gamma)\rangle \geq 0$, i.e. $\gamma + \iota(\gamma) \in \overline{V}^+ - \{0\}$. Therefore $0 < \langle \gamma + \iota(\gamma), x\rangle = 2\langle \gamma, x\rangle$ (cf. (3.4)*)). Q.E.D.

(4.8) Let $\delta \in P_o^+$ and decompose $P^*\delta$ into γ_1 and γ_2 as in Proposition 4.5. Consider $\tilde{s}_\delta = \tilde{s}_{\gamma_1} \cdot \tilde{s}_{\gamma_2} = \tilde{s}_{\gamma_2} \cdot \tilde{s}_{\gamma_1}$, and the subgroup $\mathring{W}$ of $W \subset O(L)$ generated by such $\tilde{s}_\delta$'s for all $\delta \in P_o^+$.

Recall ((1.7), ii)) the group Υ of isometries of $H^2(\tilde{S}, \mathbb{Z})$ which preserve M and the canonical homomorphism $\rho' : \Upsilon \to O(M)$ defined by the restriction, which we know to be surjective (2.9). Since $p^* : H^2(S, \mathbb{Z})_o \to M$ is a homothety, their groups of isometries are canonically isomorphic, hence we have an epimorphism $\mathring{\rho} : \Upsilon \to O(H^2(S, \mathbb{Z})_o)$. The key lemma is the following:

<u>Lemma</u> (4.9) i) $\mathring{W} \subset \Upsilon$.

ii) $\mathring{\rho}(\tilde{s}_\delta) = s_\delta$, hence $\mathring{\rho}(\mathring{W}) = W$.

Proof. i) By definition we have

$$\tilde{s}_\delta(x) = x + \langle x, \gamma_1 \rangle \gamma_1 + \langle x, \gamma_2 \rangle \gamma_2,$$

hence clearly $\tilde{s}_\delta$ preserves M (if $x \in M$, then $\langle x, \gamma_1 \rangle = \langle \iota(x), \iota(\gamma_1) \rangle = \langle x, \gamma_2 \rangle$).

ii) Note that, for $x \in H^2(S, \mathbb{R})$ and $\delta \in P_o^+$,

$$\langle p^*x, \gamma_1 + \gamma_2 \rangle = \langle p^*x, p^*\delta \rangle = 2\langle x, \delta \rangle,$$

hence $\langle p^*x, \gamma_1 \rangle = \langle p^*x, \gamma_2 \rangle = \langle x, \delta \rangle$. Therefore

$$\begin{aligned} p^*\mathring{\rho}(\tilde{s}_\delta)(x) &= \tilde{s}_\delta(p^*(x)) \\ &= p^*(x) + \langle p^*(x), \gamma_1 \rangle \gamma_1 + \langle p^*(x), \gamma_2 \rangle \gamma_2 \\ &= p^*(x) + \langle x, \delta \rangle p^*(\delta). \end{aligned}$$

Q.E.D.

(4.10) Now we can state and prove :

<u>Theorem</u> (Global Torelli theorem for Enriques surfaces). Let S and

S' be two Enriques surfaces and $\phi : H^2(S', \mathbb{Z})_o \to H^2(S, \mathbb{Z})_o$ an isometry of lattices such that i) ϕ extends to an isometry $\tilde{\phi} : H^2(\tilde{S}', \mathbb{Z}) \to H^2(\tilde{S}, \mathbb{Z})$ of universal coverings which preserves the period (i.e., $\tilde{\phi}_{\mathbb{C}}(\omega_{\tilde{S}'}) = \lambda\omega_{\tilde{S}}$), ii) ϕ maps each effective cycle on S' to an effective cycle on S. Then ϕ is induced from an isomorphism $a : S \to S'$.

<u>Remark</u> (4.11). i) As an analogy of (3.7) we have the following equivalent statements under the condition i) in (4.10),

a') ii) in (4.10),

b') $\phi_{\mathbb{R}}(V^+(S')) \subset V^+(S)$ and $\phi(P^+(S')) \subset P^+(S)$,

c') $\phi_{\mathbb{R}}(V^+(S')) \subset V^+(S)$ and $\phi(P_o^+(S')) \subset P^+(S)$,

d') $\phi_{\mathbb{R}}(K(S')) = K(S)$,

e') $\phi_{\mathbb{R}}(K(S')) \cap K(S) \neq \phi$.

ii) The uniqueness of an isomorphism a holds "generically", i.e. except for three cases (cf. Remark 5.4, i)).

(4.12) *Proof. of* (4.10). We consider the isometry $\tilde{\phi}$ and apply Theorem 3.6, which is possible because $\tilde{S}$ and $\tilde{S}'$ are algebraic hence Kähler. Since $\phi_{\mathbb{R}}(K(S')) = K(S)$ and $K(S') = \tilde{K}(\tilde{S}') \cap M_{\mathbb{R}}$, $K(S) = \tilde{K}(\tilde{S}) \cap M_{\mathbb{R}}$, (4.7), the above condition (4.11), d') implies (3.7) e). Hence $\tilde{\phi}$ is induced from an isomorphism $\tilde{a} : \tilde{S} \to \tilde{S}'$ which commutes with the covering involutions. Therefore ϕ is induced from an isomorphism $a : S \to S'$ induced from $\tilde{a}$. Q.E.D.

(4.13) From the global Torelli theorem we derive a modified form whose Corollary 4.14 is the most definite result in our case.

<u>Theorem</u> (Modified global Torelli theorem for Enriques surfaces). Let S and S' be two Enriques surfaces and $\phi : H^2(S', \mathbb{Z})_o \to H^2(S, \mathbb{Z})_o$ an isometry of lattices which extends to an isometry $\tilde{\phi} : H^2(\tilde{S}', \mathbb{Z}) \to$

$H^2(\tilde{S}, \mathbb{Z})$ which preserves the period (resp. and $\phi_{\mathbb{R}}(V^+(S')) \subset V^+(S)$). Then there is an isomorphism $a : S \to S'$ and a unique element $w \in W(S)$ such that a^* or $-a^*$ (resp. a^*) $= w\cdot\phi$.

Proof. By replacing ϕ by $-\phi$ if necessary, we may assume that $\phi(V^+(S')) \subset V^+(S)$. Since $\overline{K}(S)$ is a fundamental domain with respect to the operation of $W(S)$, there is a unique element w in $W(S)$ with $w(\phi(K(S))) = K(S)$. Note that this w has an extension $\tilde{w}$ in $\tilde{W}(\tilde{S})$ which preserves the period by virtue of Lemma 4.9, ii). Now apply (4.10) for $w\cdot\phi$.

<u>Corollary</u> (4.14) (Weak Torelli theorem). If two Enriques surfaces S and S' have the same period in $\mathcal{D}/\Gamma$ (cf. (1.7), (1.13)), then they are isomorphic.

Proof. Let ψ and ψ' be (arbitrary) markings of S and S' respectively and $\tilde{\psi}$ and $\tilde{\psi}'$ their respective extensions to those of the universal coverings $\tilde{S}$ and $\tilde{S}'$. Corollary 2.9 asserts that by changing the marking suitably we may assume that $\tilde{\psi}_{\mathbb{C}}(\omega_{\tilde{S}}) = \tilde{\psi}'_{\mathbb{C}}(\omega_{\tilde{S}'})$. Then apply Theorem 4.13 for $\phi = \psi^{-1}\cdot\psi' : H^2(S', \mathbb{Z})_0 \to H^2(S, \mathbb{Z})_0$ with extension $\tilde{\phi} = \tilde{\psi}^{-1}\cdot\tilde{\psi}'$.

§5. Automorphisms of Enriques surfaces

(5.1) In this section we exhibit some properties on the automorphism group of an Enriques surface. Most of the results have been already known essentially (due to Barth-Peters [1] and Dolgachev [4]), but we hope our treatment would give clearer insight to them.

(5.2) Let S be an Enriques surface. We make free use of the notation in §4. We define moreover a subgroup $G = G(S)$ of $O(M)$ ($= O(H^2(S, \mathbb{Z})_0)$) consisting of elements which have *period preserving* extensions to $O(H^2(\tilde{S}, \mathbb{Z}))$. Denote also by $A^*(S)$ the image of the homomorphism $\mathrm{Aut}(S) \to O(H^2(S, \mathbb{Z})_0)$ $= O(M)$, and by $D = \{\pm 1\}$ the centre of $O(M)$.

The first result is a small refinement of Dolgachev's main theorem in [4].

Theorem (5.3) (cf. [13], §6, Theorem 1). i) The group $W(S)$ is a normal subgroup of $G(S)$.

ii) $G(S) = A^*(S)\cdot W(S)\cdot D$ (semi-direct product).

iii) $G(S)$ contains $\Gamma'(2)$, hence in particular $G(S)$ is of finite index in $\Gamma' = O(M)$.

Proof. i) We know that $W(S)$ is normal in $O(M)$. Each generator s_δ, $\delta \in P_0^+(S)$, has an extension $\tilde{s}_\delta = \tilde{s}_{\tilde{\delta}_1}\cdot\tilde{s}_{\tilde{\delta}_2}$ (4.8) in $O(H^2(\tilde{S}, \mathbb{Z}))$ which preserves the period ($\tilde{s}_{\tilde{\delta}_i} \in \tilde{W}(\tilde{S})$), hence $W(S) \subset G(S)$.

ii) This is nothing but a restatement of (4.13).

iii) Let ϕ be an element in $\Gamma'(2)$. By definition it acts trivially on M^*/M. Then by [11] Proposition 1.1 ϕ on M and id on N extend to an isometry $\tilde{\phi}$ of $H^2(\tilde{S}, \mathbb{Z})$, which clearly preserves the period of $\tilde{S}$.

Remark (5.4). i) The homomorphism $\mathrm{Aut}(S) \to H^2(X, \mathbb{Z})_0$ is *not* injective in three cases which can be described explicitly ([8]). We here say only that two cases are involutions and have 2-dimensional moduli, and the last case is an automorphism of order 4 having 1-dimensional moduli. The involutions act trivially even on $H^2(S, \mathbb{Z})$ (including torsion). A geometric construction of such automorphisms is due to Barth-Peters [1] and Lieberman

ii) The claims ii) and iii) in (5.3) imply that the auromorphism group Aut(S) of S is finite if and only if W(S) is of finite index in O(M). Such an example is given by Dolgachev [4], in which case the automorphism group turns out to be a dihedral group D_4 [1] (see 6.10 below).

Recently I was informed from Barth that Nikulin had classified all Enriques surfaces with finite auromorphism groups.

(5.5) Now we consider "generic" Enriques surfaces, by which we mean those whose periods are contained in the complement of countably many closed analytic sets in $\mathcal{D}_o/\Gamma$.

<u>Proposition</u> (5.6). For a generic Enriques surface S its covering K3 surface has N as the space of transcendental cycles and M as the space of algebraic cycles.

Proof. Since the period is contained in $N_{\mathbb{C}}$, the space T of transcendental cycles is contained in N (by a marking $\tilde{\phi} : H^2(\tilde{S}, \mathbb{Z}) \to L$). Take $n \in N$. For a marked K3 surface S the cycle corresponding to n is algebraic in $\tilde{S}$ if and only if it is orthogonal to the period, hence the set of periods corresponding to such surfaces is a divisor in $\mathcal{D}/\Gamma$ (a hyperplane section in $\mathcal{D}$). Then T = N if and only if the period is not contained in this countable union of such divisors in $\mathcal{D}/\Gamma$.

<u>Corollary</u> (5.7). A generic Enriques surface S does not contain a

smooth rational curve (i.e. $P(S) = \phi$).

Proof. Take an Enriques surface S whose universal covering $\tilde{S}$ has M as the space of algebraic cycles. Then $\tilde{S}$ does not contain any smooth rational curves (since M contains no element of length -2), hence neither S.

Remark (5.8) In the next section (Proposition 6.2) we shall make the statement (5.7) more precise.

(5.9) Next we determine the automorphism group of a generic Enriques surface. This result is due to Barth and Peters [1]. We state it in a slightly more precise form than theirs.

Take $\phi(\neq \pm 1) \in \Gamma = O(N)$, and let E_ϕ be the union of eigenspaces of ϕ in $N_{\mathbb{C}}$. Denote by E the union of E_ϕ for all $\phi(\neq \pm 1) \in \Gamma$ which consists of countably many proper linear subspaces of $N_{\mathbb{C}}$ and is invariant by Γ. Again E/Γ is a union of countably many closed subvarieties in $\mathcal{D}/\Gamma$.

Note that, since ϕ is an element of an orthogonal group defined over $\mathbb{Q}$, all eigenvalues are roots of unity, and that the degree does not exceed 12 = the rank of N, hence the possible orders are $1,\cdots,16,18,20,21,22,24,$ $26,28,30,36,42$.

The set E is hence written also as

$$E = \bigcup_{\substack{\lambda_i : \ell\text{-th roots of unity} \\ \text{with } \ell \leq 42}} \ \bigcup_{\phi \in \Gamma - \{\pm 1\}} \mathrm{Ker}_{N_{\mathbb{C}}}(\phi_{\mathbb{C}} - \lambda_i 1_N).$$

<u>Theorem</u> (5.10). The automorphism group of an Enriques surface S is isomorphic to $\Gamma'(2)/\pm 1$ if and only if S contains no smooth rational curves (this condition can be stated in terms of the period — cf. Theorem 6.4) and the period of S is not contained in E/Γ.

Proof. The surface S contains no smooth rational curve if and only if $W(S) = \{1\}$.

On the other hand take $\phi \in \Gamma' = O(M)$ with $\phi \notin \Gamma'(2)$. For an arbitrary extension $\tilde{\phi}$ of ϕ to $\tilde{\Gamma}$ we have $\tilde{\phi}|N \neq \pm 1$ since $\phi \notin \Gamma'(2)$. Then by definition $\tilde{\phi}$ preserves the period of $(\tilde{S}, \tilde{\psi})$ if and only if $\tilde{\psi}_{\mathbb{C}}(\omega_{\tilde{S}}) \in E_{\tilde{\phi}|N}$.

Combining these two observations together and applying (5.3) ii), we obtain the desired result. Q.E.D.

As the last topic we show a theorem on the existence of automorphisms of Enriques surfaces in general.

<u>Theorem</u> (5.11). Every Enrique surface S admits an involutive automorphism.

Proof. Let $\tilde{S}$ be the covering K3 surface of S with the involution i, which induces an isometry ι in $L = H^2(\tilde{S}, \mathbb{Z})$, (2.2).

We shall construct another involution on $\tilde{S}$ which commutes with i. For that purpose we construct an isometry $\sigma \in O(L)$ commuting with ι so that Proposition (3.10) can be applied.

We use the notations in the previous sections such as M, N (2.2), $H_{\mathbb{Z}}^{1,1}$, P for $\tilde{S}$ (3.2).

Recall that $M = U(2) \perp E_8^-(-2)$. We choose a sublattice isomorphic to $U(2)$ and fix it. We denote by V the orthogonal complements of $U(2)$ in $H_{\mathbb{Z}}^{1,1}$. It has the following properties:

(*) i) even and negative definite; ii) ι-invariant; iii) if x is ι-invariant, then $\langle x,x\rangle \equiv 0 \pmod 4$; iv) $P(V) = \{\delta \in P;\ \delta \in V\}$ are the roots of union of Dynkhin diagrams. In particular it has a decomposition $\{\delta \in P^+;\ \delta \in V\} \amalg \{\delta \in P^-;\ \delta \in V\}$ which is ι-invariant.

Lemma (5.12). Let V be a primitive sublattice of $\mathbb{L}$ satisfying (*). Then, if $P(V) \neq \phi$, there is $\delta \in P(V)$ with $\langle\delta,\iota\delta\rangle = 0$.

Proof. Take a δ which corresponds to a fundamental root. Since ι preserves root system, $\langle\delta,\iota\delta\rangle \geq 0$. By i) and iii) we have also

$$0 > \langle\delta+\iota\delta,\delta+\iota\delta\rangle = -4 + 2\langle\delta,\iota\delta\rangle \equiv 0 \pmod 4.$$

Hence $\langle\delta,\iota\delta\rangle = 0$. Q. E. D.

The crucial step in our proof is to show the following claim.

Proposition (5.13). Let V be as above. Then V admits an orthogonal decomposition $V_+ \perp V_-$ (by this we mean that V_+ and V_- are primitive sublattices in V orthogonal to each other and $[V: V_+ \perp V_-] < \infty$) satisfying the following conditions: i) ι preserves the decomposition; ii) V_+ is 2-primitive (i.e. $V_+ \subset V_+^* \subset \frac{1}{2}V_+$); iii) the primitive hull E of $(N \cap V_+) \oplus (M \cap V_-)$ is isomorphic to $E_8(-2)$; iv) the primitive hull H of $N \cap V_+ \oplus V_-$ contains no element δ with $\langle\delta,\delta\rangle = -2$.

Proof. At first we replace iv) by a weaker condition: iv') V_- contains no element δ with $\langle\delta,\delta\rangle = -2$.

We construct inductively a series of decreasing sublattices

$$V_-^{(1)} \supsetneq V_-^{(2)} \supsetneq \cdots \supsetneq V_-^{(n)},$$

satisfying i), ii), iii) $(V_+^{(k)} = (V_-^{(k)})^\perp)$. After a finite number of steps $V_-^{(n)}$ satisfies iv'), then we put $V_- = V_-^{(n)}$ (and $V_+ = V_+^{(n)}$).

First let $V_-^{(1)} = V$.

Suppose that we have constructed $V_-^{(k)}$ and it still contains roots δ $(\langle\delta,\delta\rangle = -2)$. Since $V_-^{(k)}$ also has the property (*), we can choose $\delta \in V_-^{(k)}$ with $\langle\delta,\iota\delta\rangle = 0$ by Lemma (5.12). Then we set

$V_+^{(k+1)}$ = the primitive hull of $V_+^{(k)} + \mathbb{Z}\delta + \mathbb{Z}\iota\delta$.

The conditions i) and ii) hold clearly for $V_-^{(k+1)}$.

To prove iii) it is enough to note that

$$N \cap V_+^{(k+1)} = H(N \cap V_+^{(k)} \perp \mathbb{Z}(\delta-\iota\delta))$$

$$M \cap V_-^{(k)} = H(M \cap V_-^{(k+1)} \perp \mathbb{Z}(\delta+\iota\delta))$$

(where $H(Q)$ denotes the primitive hull of Q) since $\delta - \iota\delta \in V_-^{(k)}$ and $\delta + \iota\delta \in N$, hence we have

$$s_\delta(H((N \cap V_+^{(k)}) + (M \cap V_-^{(k)})))$$

$$= H((N \cap V_+^{(k+1)}) + (M \cap V_-^{(k+1)})).$$

(note that $s_\delta(\iota\delta - \delta) = \delta + \iota\delta$ and s_δ is the identity on $\delta^\perp$) where s_δ is the reflexion defined by δ (3.3).

Now we shall prove that under i) - iii) the condition iv') implies iv) in fact (hence actually true for each $V_-^{(k)}$).

Take δ in H and decompose it into orthogonal components as

$$\begin{array}{ccccccc} H\otimes\mathbb{Q} = & (N\cap V_+)\otimes\mathbb{Q} & \perp & (M\cap V_-)\otimes\mathbb{Q} & \perp & (N\cap V_-)\otimes\mathbb{Q} \\ \cup & & & \cup & & \\ \delta = & \delta_0 & + & \delta_+ & + & \delta_- . \end{array}$$

We have $2\delta_+ = \delta + \iota\delta \in M \cap V_-$ and $\langle\delta_+,\delta_+\rangle = -1$. Next $2\delta_0 \in V_+ \subset L$ since, for all $x \in V_+$, $\langle x,\delta_0\rangle = \langle x,\delta\rangle \in \mathbb{Z}$, hence $\delta_0 \in V_+^* \subset \frac{1}{2}V_+$ by ii). Hence $2\delta_0 \in V_+ \cap N$ and $\langle 2\delta_0,2\delta_0\rangle \equiv 0 \pmod 4$, or $\langle\delta_0,\delta_0\rangle \in \mathbb{Z}$.

On the other hand $\delta_- \neq 0$ since $\delta \notin E_8(2)$, hence $\langle\delta_-,\delta_-\rangle < 0$.

Thus we have

$$-2 = \langle\delta,\delta\rangle = \langle\delta_0,\delta_0\rangle + \langle\delta_+,\delta_+\rangle + \langle\delta_-,\delta_-\rangle < \langle\delta_0,\delta_0\rangle - 1,$$

which is possible only when $\langle\delta_0,\delta_0\rangle = 0$. Q. E. D.

(5.14) Now we are ready to prove the theorem.

Define M_σ as the primitive hull of $U(2) \perp V_+$ and N_σ its orthogonal complement in L. By the construction M_σ and N_σ are invariant by ι and 2-primitive, hence there is an involution σ of L with $\sigma|_{M_\sigma} = 1$ and $\sigma|_{N_\sigma} = -1$ which commutes with ι.

Note that $\sigma\iota = \iota\sigma$ is also an involution and $L^{\sigma\iota} = E^\perp$.

Consider a finite group $G = \langle\iota,\sigma\rangle$ ($\subset O(L)$) which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. We check the conditions to apply Theorem (3.10) for G. The condition i) is clear since $\iota\omega = \sigma\omega = -\omega$. The condition ii) holds since $S_{G,\tilde{S}} = H \subset V$ in Proposition (5.13) by definition. The last condition iii) is also clear since $L^G \supset U(2)$.

Hence by changing the marking with a suitable reflexion w, G can be realized as a group of automorphisms of $\tilde{S}$. By Torelli theorem for Enriques surfaces the involution $w\iota w^{-1}$ induces the same Enriques surfaces S and $w\sigma w^{-1}$ induces a non-trivial involution on S.

Therefore the theorem is proved.

Remark (5.15). The above theorem is a lattice-theoretic version of the socalled U(2)-marking [2] or presentation of an Enriques surface as a double covering of rational surfaces [5]. The involution σ corresponds to the covering involution and two primitive elements e, f of length 0 in U(2) correspond to fibres in two rulled fibre structures on the rational surface. The (quasi) polarization defined by $e+f$ gives an quartic embedding into $\mathbb{P}^3$ of (eventually singular) quotient $\tilde{S}/\sigma$ where (-2)-curves in L^G are contracted. For detail see the literature cited above.

§6. Rational curves and elliptic curves on an Enriques surface.

(6.1) In this section we study smooth rational curves and smooth elliptic curves on an Enriques surface.

First we give the characterization of the periods of Enriques surfaces containing smooth rational curves*) and show that they form an *irreducible* divisor in the period space. Next we show that the number of smooth rational curves and smooth elliptic curves on an Enriques surface S is finite modulo Aut(S) and linear equivalence. This fact was first pointed out by Looijenga in the case of K3 surfaces. By virtue of the similitude of (5.3) to the corresponding theorem for K3 surfaces, the way of proof is also the same. Lastly we calculate the concrete number of such curves in one example.

Proposition (6.2). Let S be an Enriques surface and $\omega \in H^2(\tilde{S}, \mathbb{C})$ the period of its covering K3 surface $\tilde{S}$ (1.6). Then S contains a smooth rational curve if and only if there is a vector ℓ in $N \cap H^{1,1}_{\mathbb{Z}}(\tilde{S})$ (i.e. $\langle \ell, \omega \rangle = 0$ and $\iota(\ell) = -\ell$) with $\langle \ell, \ell \rangle = -4$ and of even type (2.15).

Proof. The surface S contains a smooth rational curve if and only if the Weyl group W(S) is non-trivial (4.2). Then the above follows from the characterization of W(S) with the period given in Remark 4.6 ii).

Q.E.D.

(6.3) Using the notations in (1.5) and (1.7), we denote by N the union of all hyperplane sections $H_\ell \cap D$ with ℓ being of length -4 and of even type. This set N is invariant by Γ.

Then with the same reason as (1.10), by virtue of Theorem 2.15 (instead

*) Other interesting characterizations were found by F. Cossec.

of 2.13) together with the above Proposition 6.2 we obtain the following theorem.

Theorem (6.4). The set N/Γ is a closed irreducible divisor in the period space D/Γ. An Enriques surface S contains a smooth rational curve if and only if the corresponding period ω_S is in N/Γ.

Theorem (6.5). On an Enriques surface S there are only a finite number of nonsingular rational curves modulo automorphisms of S.

Proof. We consider the group G(S) in (5.2), which is a subgroup of O(M) of finite index, hence is arithmetic. Therefore there is a fundamental domain F in $V(S) = \{x \in \bar{H}^2(S, \mathbb{Z}); \langle x, x\rangle > 0\}$ (4.2).

Consider a set of roots $G\cdot P_o^+ = \{g\delta;\ g \in G,\ \delta \in P_o^+\}$. Then for only a finite number of γ_i's in $G\cdot P_o^+$; $H_{\gamma_i} \cap F \neq \phi$, for F has only a finite number of faces and if $H_\gamma \cap F^\circ \neq \phi$ then $s_\gamma \cdot F^\circ \cap F \neq \phi$.

Therefore $F - \bigcup_i H_{\gamma_i}$ decomposes into a finite number of connected components $\{F_r\}_r$ each of which has also only a finite number of faces. For each F_r there is $w_r \in W(S)$ such that $F_r' = w_r F_r \subset K$. Let $\Delta = \{\delta_a\}_a$ be the set of smooth rational curves in P_o^+ with $H_{\delta_a} \cap F_r' \neq \phi$ for some r. This is a *finite* set.

We claim that

$$A^*\cdot\Delta = P_o^+.$$

To see this we note first that $A^*(\bigcup F_r')$ fills up K. Then for each $\delta \in P_o^+$, H_δ appears as a face of K of codimension 1 (3.5), hence so does as a face of $g\cdot F_r'$ for some $g \in A^*$ and r, which implies $g^{-1}\delta \in \Delta$.

Q.E.D.

Next we treat elliptic curves. We use the notation in §4. The following result was more or less known and partly proved in [5] with a different method.

<u>Proposition</u> (6.6). Let S be an Enriques surface. There is a bijective correspondence between the set of elliptic pencils on S and the set of isotropic lines in $H^2(S, \mathbb{Z})_o$ (through 0) and contained in $K(S)^-$ (i.e. there is a generating vector $e \in H^2(S, \mathbb{Z})$ (with $\langle e, e\rangle = 0$) and $\langle e, a\rangle \geq 0$ for all $a \in P_o^+(S)$ (hence for every effective cycles a on S). (Note that this condition is nothing but the numerical effectivity of a divisor.)

For each isotropic line ℓ as above there is a unique generator e in $H^2(S, \mathbb{Z})_o$ such that $2e$ is the cycle of general fibres of the corresponding elliptic pencil and e is the cycle of a multiple fibre in it (in this case, if considered in $H^2(S, \mathbb{Z})$, e and $e+K$ are both effective and each corresponds to one of the two multiple fibres).

Proof. Let e be a primitive isotropic vector in $H^2(S, \mathbb{Z})_o$. Replacing e by $-e$ if necessarily, we may assume that $p^*e \in H^2(\tilde{S}, \mathbb{Z})$ is effective.

i) By the same argument as in the proof of Proposition 4.7, p^*e is also numerically effective. Then by [13], §3, Theorem 1, p^*e is a cycle of an irreducible elliptic curves which form a pencil. Since p^*e is also invariant by ι, ι acts on the pencil and have two fixed points. This induces an elliptic pencil on S with two multiple fibres (of multiplicity 2).

Conversely if there is an elliptic pencil on an Enriques surface S, then the inverse image $p^{-1}(E)$ of a general fibre E is a disjoint union of two irreducible elliptic curves $E_1 + E_2$ where the involution i interchanges the components, for otherwise (i.e. if $p^{-1}(E)$ is irreducible) on the induced elliptic pencil in $\tilde{S}$ the involution i acts trivially on parameter and as translations in general fibres (since i has no fixed points), and this implies that the non-zero holomorphic 2-form on $\tilde{S}$ is i-invariant, which is absurd. Therefore $p^*([E]) = 2[E_1]$ and $[E_2]$ is primitive in $H^2(\tilde{S}, \mathbb{Z})$. Note that E is numerically effective.

It is clear that the above two correspondence are inverse to each other. Q.E.D.

Theorem (6.7). On an Enriques surface S there are only a finite number of non-singular elliptic curves up to $\mathrm{Aut}(S)$ and linear equivalence.

Moreover this number is always positive.

Proof. By (6.6) before the claim is equivalent to say that there are only a finite number of rational isotropic lines in $K(S)^-$ modulo $A^*(S)$ (the notation in (5.2)). On the other hand by Theorem 5.3 $G(S)$ (cf. (5.2)) is an arithmetic subgroup, hence there are only a finitely many rational isotropic vectors modulo $G(S)$.

Moreover we know that a fundamental domain of $V^+(S)$ with respect to $O(H^2(S, \mathbb{Z})_o)$ contains unique rational isotropic vector modulo constant multiple in its closure ([16], p. 340). Since $G(S)$ is a subgroup of $O(H^2(S, \mathbb{Z})_o)$ any fundamental domain with respect to $G(S)$ also contains rational isotropic vectors. Q.E.D.

Remark (6.8). A generic Enriques surface S has exactly 527 elliptic fibrations modulo Aut(S) ([1]).

Before calculating an example we make one observation as preliminary.

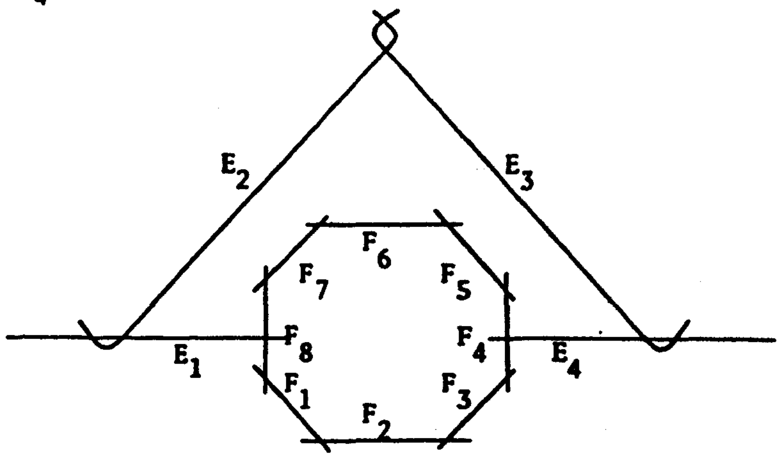
Proposition (6.9). Let S be an Enriques surface and $D_1, \cdots, D_s$ mutually different smooth rational curves on S. Denote by δ_i the class corresponding to D_i in $H^2(S, \mathbb{Z})_o$. Suppose that the polyhedral cone $\overline{K} = \{x \in H^2(S, \mathbb{R}); \langle x, \delta_i\rangle \geq 0$ for all $i\}$ is contained in $\overline{V}^+(S)$ the closure of $V^+(S)$. Then D_i's are the only smooth rational curves on S and Aut(S) is finite.

Proof. If there exists another smooth rational curve D, then $\langle D, D_i\rangle \geq 0$ for all i, hence by assumption ($\overline{K} \subset \overline{V}^+$) $\langle D_i, D_i\rangle \geq 0$, which is impossible.

Now $\overline{K}$ is a fundamental domain with respect to $W(S)$ with finite volume, hence $W(S)$ is of finite index in $O(H^2(S, \mathbb{Z})_o)$ ([16]). Therefore by Theorem 5.3 Aut(S) is finite (cf. Remark 5.4 ii)). Q.E.D.

Example (6.10). We shall consider the Enriques surface S in [1] §4, Case 3. Barth and Peters have shown that Aut(S) = D_4 (the dihedral group). Hence the numbers of smooth rational curves and elliptic curves are also finite. We want to determine them. We mention here that this example was first considered by Horikawa [5] and he is Dolgachev who observed that its automorphism is finite. His proof in [4] can be slightly simplified by the argument given here. (cf. Remark 4.6, iii), 5.4, ii).)

In [1] it is shown that this surface S contains smooth rational curves $F_1, F_2, \cdots, F_8, E_1, E_2, N$ (cf. ibid. §4 for the precise definition). There is, however, one more smooth rational curve E_3 which is the irreducible component other than E_2 in the fibration with parameter u. Summing up, we have obtained the rational curves on S with the following configuration (we write E_4 instead of N) :



By direct calculation one can check that these rational curves satisfy the assumption in Proposition 6.5. Hence these are the only rational curves on S. The action of Aut(S) on the set of rational curves $P_o^+(S)$ is nothing but the symmetry of the above configuration, which is clearly isomorphic to $(\mathbb{Z}/2)^2$ ($\cong D_4$/centre). Therefore $\# P_o^+(S) = 12$ and $\# (P_o^+(S)/\mathrm{Aut}(S)) = 5$.

The same computation shows also that there are the following 9 elliptic fibrations on S (we list the cohomology class of singular fibres of each fibration) :

$G_1 : 3E_4+2F_2+4F_3+6F_4+5F_5+4F_6+3F_7+2F_8+E_1$,

$G_2 : 3E_4+2F_6+4F_5+6F_4+5F_5+4F_6+3F_7+2F_8+E_1$,

$G_3 : 3E_1+2F_6+4F_7+6F_8+5F_1+4F_2+3F_3+2F_4+E_4$,

$G_4 : 3E_1+2F_2+4F_1+6F_8+5F_7+4F_6+3F_5+2F_4+E_4$,

$G_5 : 2E_4+F_1+2F_2+3F_3+4F_4+3F_5+2F_6+F_7 \sim 2(E_1+E_2)$ ($\sim E_Y'$ in [1]),

$G_6 : 2E_1+F_5+2F_6+3F_7+4F_8+3F_1+2F_2+F_3 \sim 2(E_3+E_4)$

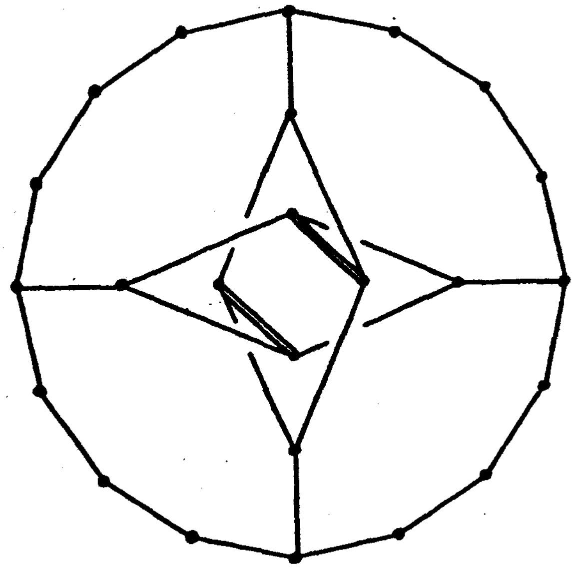
$G_7 : E_1+F_7+2F_8+2F_1+2F_2+2F_3+2F_4+F_5+E_4$

G_8 : $E_4+F_3+2F_4+2F_5+2F_6+2F_7+2F_8+F_1+E_1$

G_9 : $E_2+E_3 \sim 2(F_1+F_2+F_3+F_4+F_5+F_6+F_7+F_8)$ ($\sim F_Y$ in [1]).

They are equivalent modulo Aut(S) if and only if the above configurations of the singular fibres are the same; hence there are 4 classes of elliptic pencils up to Aut(S).

Lastly we remark that these rational curves on S induces those in the covering K3 $\tilde{S}$ with the following configuration (this is a dual graph):



This has symmetries isomorphic to D_4, from which we can reprove that $\mathrm{Aut}(S) \cong D_4$. The lattice generated by these curves is isomorphic to $\overset{2}{\perp} E_8(-1) \perp U \perp \langle -4 \rangle$ and primitive in $H^2(\tilde{S}, \mathbb{Z})$.

§7. Surjectivity of the period map for Enriques surfaces.

(7.1) Recall the notation in §1:

$$\mathcal{D} = \{(v) \in \mathbf{P}(N_{\mathbb{C}})\ ;\ \langle v,v\rangle = 0,\ \langle v,\bar{v}\rangle > 0\} \quad (1.7),$$

$$\mathcal{D}_0 = \{(v) \in \mathcal{D}\ ;\ \langle v,\ell\rangle \neq 0 \text{ for all } \ell \in N \text{ with } \langle \ell,\ell\rangle = -2\} \quad (1.9).$$

What we prove in this section is the following :

Theorem (7.2) (surjectivity theorem of the period map for Enriques surfaces). For any point $p \in \mathcal{D}_0/\Gamma$ there exists an Enriques surface S whose period ω_S (1.13) is p.

Proof. Take a representative $(\omega) \in \mathcal{D}$ of p. By the inclusion $N \subset L$, $\mathcal{D}$ is embedded into the period domain $\tilde{\mathcal{D}}$ of K3 surfaces (1.7). The surjectivity of the period for K3 surfaces (3.13) asserts that there exists a marked K3 surface $(\tilde{S}, \tilde{\psi})$ such that $\tilde{\psi}_{\mathbb{C}}(\omega_{\tilde{S}}) = \omega$.

Consider the involution induced from ι in L via $\tilde{\psi}$. We denote it by the same letter ι. We see that we can apply Theorem 3.10 for $G = \langle\iota\rangle$ and $X = \tilde{S}$. Note that $S_{G,X} \simeq \omega^{\perp}$ in N. Since $\omega \in N_{\mathbb{C}}$, we have $\iota(\omega_{\tilde{S}}) = -\omega_{\tilde{S}}$ hence $\mathbb{C}\omega_{\tilde{S}}$ is ι-invariant. The property that $S_{G,X}$ contains no roots is exactly the condition that $(\omega) \in \mathcal{D}_0$ by definition. Lastly $S_{G,X}$ is negative definite because N has index (2.10), $N_{\mathbb{R}} \cap (\mathbb{C}\omega \oplus \mathbb{C}\bar{\omega})$ is a positive definite subspace of dimension 2 and $S_{G,X}$ is its orthogonal complement in N.

Therefore by Theorem 3.10, changing the marking by a suitable $w \in W(\tilde{S})$ the involution $w\iota w^{-1}$ comes from an automorphism i of $\tilde{S}$. Let S be the quotient $\tilde{S}/\langle i\rangle$ with the canonical holomorphic map $p : \tilde{S} \to S$.

It remains to prove that S is an Enriques surface, i.e., i has no fixed locus. The involution can be expressed near a fixed point in the form

$$(z_1, z_2) \to (\pm z_1, \pm z_2).$$

Since $i^*(dz_1 \wedge dz_2) = -dz_1 \wedge dz_2$, it must be the form

$$(z_1, z_2) \to (z_1, -z_2) \quad \text{or} \quad (-z_1, z_2),$$

in other words i may have only a fixed curve. Suppose that there exists a fixed curve. Then S is non-singular and $q(S) = p_g(S) = 0$, since there are no holomorphic i-invariant 1-forms nor 2 forms on $\tilde{S}$. Observe that S is minimal, for an exceptional curve C on S induces an i-invariant algebraic cycle $p^*[C]$ of length $\langle p^*[C], p^*[C] \rangle = 2C^2 = -2$, which is impossible in M.

Denote the irreducible (= connected) components of the fixed curve by $C_1, \cdots, C_s$. Via p each C_i is isomorphic to $\overline{C}_i$ on S. Since $\tilde{S}$ is a ramified double covering, we have $2K_S = -\sum_{i=1}^{s} \overline{C}_i$. In particular S is rational, since S is algebraic and $-2K_S$ is effective. An elementary calculation shows that $c_2(S) = 12 + \frac{1}{2} \sum_{i=1}^{s} \chi(C_i)$ where $\chi(C_i)$ is the Euler number of C_i. On the other hand the (generalized) Lefschetz fixed point formula says that $\Sigma\chi(C_i) = \sum_{k=0}^{4} (-1)^k \mathrm{Tr}(i^* | H^k(\tilde{S}, \mathbb{Z})) = 2 + \mathrm{Tr}(i^* | H^2) = 0$. Hence $c_2(S) = 12$, which is impossible for this number should be 3 or 4 for minimal rational surfaces (according to $\mathbb{P}^2$ or ruled surface).

<u>Remark</u> (7.3). As an analogy to (3.13) we can say that each element in $K \subset V^+ \subset H^2(S, \mathbb{R})$ corresponds to a Kähler class, for $K \cap H^2(S, \mathbb{Z})$ is nothing but the set of ample classes by Nakai criterion and $K \cap H^2(S, \mathbb{Q})$ is dense in K.

A complete analogue to (3.13) is possible, but difficult to formulate (especially defining the set V_p^o) because the lattice-theoretic characterization of the set P^+ is rather complicated (cf. Remark 4.6, ii)).

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