Linearisation of isotropic automorphisms of non-quadratic elliptic CR-manifolds in $\mathbb{C}^4$ [*†]

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January 30, 2001

Abstract

For non-exceptional elliptic CR-manifolds in $\mathbb{C}^4$ (introduced in [ES4]) we prove that all isotropic automorphisms can be simultaneously linearised in normal coordinates. This result is a higher codimensional analogue of theorems by Webster, Kružilin-Loboda for non-umbilic and strictly pseudoconvex hypersurfaces ([W1, KL]). Together with the result by Schmalz [S1] in the hyperbolic case it establishes the linearisation property for real-analytic CR-manifolds of codimension 2 in $\mathbb{C}^4$ with stable Levi form.

[*] **Mathematics Subject Classification:** primary 32C16; secondary 32C05, 32H02

[†] **Keywords and Phrases:** CR-manifolds, invariants, classification

[‡] Research of the first author was supported by ARC, MPI Bonn, and SFB 256.

[‡‡] Research of the second author was supported by SFB 256 of Deutsche Forschungsgemeinschaft.

1 Introduction

Let M be an embedded real-analytic, elliptic, non-quadratic CR-manifold in $\mathbb{C}^4$, i.e., in normal coordinates $(z^1, z^2, \omega^1, \omega^2)$ it can be written as a complex equation

$$V - z^1\bar{z}^2 = N(z^1, \bar{z}^1, z^2, \bar{z}^2, \eta, \bar{\eta}),$$

where $V = \frac{\omega^1 - \bar{\omega}^2}{2\,\mathrm{i}}$, $\eta = \frac{\omega^1 + \bar{\omega}^2}{2}$ and $N \not\equiv 0$ uniquely splits into the sum

$$N = N^{CM}(z^1, \bar{z}^2, \eta) + N^{tor}(z^1, \bar{z}^1, z^2, \bar{z}^2, \eta, \bar{\eta}).$$

Here, the restriction of the Chern-Moser component N^{CM} to $z^1 = z^2$ and $\eta \in \mathbb{R}$ is a power series in $(z^1, \bar{z}^1, \eta)$ whose real and imaginary part are in classical Chern-Moser normal form (in $\mathbb{C}^2$) and in the torsion component $N^{tor} = \sum_{\substack{k,\ell \geq 1 \\ k+\ell \geq 3}} N^{tor}_{k\ell}$
the homogeneous polynomials $N^{tor}_{k\ell}$ of degree k in z and ℓ in $\bar{z}$ meet the conditions

$$\begin{aligned} N^{tor}_{k,0} &= 0, \quad k \geq 0, & N^{tor}_{1,1} &= 0 \\ \frac{\partial N^{tor}_{k1}}{\partial \bar{z}^2} &= 0, \quad k \geq 0, & \frac{\partial^2 N^{tor}_{21}}{\partial z^1 \partial z^2} &= 0 \\ \frac{\partial^4 N^{tor}_{22}}{\partial z^1 \partial z^2 \partial \bar{z}^1 \partial \bar{z}^2} &= 0 \end{aligned}$$

(see [ES1], and further, [EIS]).
We note that it was proved in [CS] that partially integrable abstract elliptic CR-manifolds of CR dimension and codimension 2 without torsion are automatically real-analytic, embeddable and N^{tor} vanishes identically.
The object of our main interest is the group $\mathrm{Aut}_0\, M$ of germs of biholomorphic isotropic (at 0) automorphisms of M. In [ES4] we were considering the subgroup $\mathrm{Aut}_{0,\mathrm{id}}\, M$ of nil-isotropic automorphisms $\psi = (f, g): \quad z^* = f(z, \omega), \quad \omega^* = g(z, \omega)$ with

$$\frac{\partial f}{\partial z}\Big|_{\substack{z=0 \\ \omega=0}} = \mathrm{id}$$

and proved that $\mathrm{Aut}_{0,\mathrm{id}}\, M = \{\mathrm{id}\}$ or $\mathbb{C}$. The case $\mathrm{Aut}_{0,\mathrm{id}}\, M = \mathbb{C}$ is extremely restrictive and occurs only for exceptional manifolds parametrised by at most one complex and two integer parameters. The existence and explicit description of such exceptional manifolds are yet to be established. Note that for all hypersurfaces or 2-codimensional hyperbolic manifolds in $\mathbb{C}^4$ the group $\mathrm{Aut}_{0,\mathrm{id}}$ is trivial which is a necessary condition for the linearisability of the entire group Aut_0. For strictly pseudoconvex or non-umbilic hypersurfaces and hyperbolic manifolds it was proved that this condition is also sufficient ([KL], [W1], [S1]). However, the construction in [E2] shows that there exist hypersurfaces with indefinite Levi-form such that Aut_0 is not linearisable (even in projective sense).

In this paper we prove the linearisability of Aut_0 for non-exceptional elliptic manifolds.

Let us introduce some useful notations. Recall that for any $\psi \in \mathrm{Aut}_0 M$ the linear map

$$L_\psi : \quad (z, \omega) \mapsto (Cz, \rho\omega)$$

with

$$C = \left.\frac{\partial f}{\partial z}\right|_0, \qquad \rho = \left.\frac{\partial g}{\partial \omega}\right|_0$$

is an automorphism of the quadric $Q: \quad V - z^1\bar{z}^2 = 0$. By $\mathrm{Aut}_0^0 M$ we denote the subgroup of $\mathrm{Aut}_0 M$ that consists of automorphisms ψ such that L_ψ belongs to the connected component of the identity in $\mathrm{Aut}_0 Q$. We note that for $\psi \in \mathrm{Aut}_0^0 M$ the matrices C and ρ are diagonal.

Let $\psi = (f, g) \in \mathrm{Aut}_0^0 M$. Define $a = (a^1, a^2) \in \mathbb{C}^2$ and $r \in \mathbb{C}$ via

$$\begin{aligned} \left.\frac{\partial f}{\partial \omega}\right|_0 &= C \begin{pmatrix} a^1 & 0 \\ 0 & a^2 \end{pmatrix} \\ \frac{1}{2}\left(\left.\frac{\partial^2 g^1}{(\partial\omega^1)^2}\right|_0 + \overline{\left.\frac{\partial^2 g^2}{(\partial\omega^2)^2}\right|_0} \right) &= \rho_{11} r. \end{aligned} \tag{1}$$

Since M is given in normal form we can consider ψ as a normalisation with parameters C, ρ, a, r (see [ES1]). Therefore, ψ equals up to terms of order 3 in z, ω to a fractional linear automorphisms of the elliptic CR-quadric $V - z^1\bar{z}^2 = 0$:

$$
\begin{aligned}
\tilde{z}^1 &= \frac{z^1 + a^1\omega^1}{1 - 2\,\mathrm{i}\,\bar{a}^2 z^1 - (r + \mathrm{i}\,a^1\bar{a}^2)\omega^1} \\
\tilde{z}^2 &= \frac{z^2 + a^2\omega^2}{1 - 2\,\mathrm{i}\,\bar{a}^1 z^2 - (\bar{r} + \mathrm{i}\,a^2\bar{a}^1)\omega^2} \\
\tilde{\omega}^1 &= \frac{\omega^1}{1 - 2\,\mathrm{i}\,\bar{a}^2 z^1 - (r + \mathrm{i}\,a^1\bar{a}^2)\omega^1} \\
\tilde{z}^2 &= \frac{\omega^2}{1 - 2\,\mathrm{i}\,\bar{a}^1 z^2 - (\bar{r} + \mathrm{i}\,a^2\bar{a}^1)\omega^2}.
\end{aligned} \tag{2}
$$

If we plug ψ into the equation of M we obtain an identity which is satisfied for all points of M. To analyse this identity we introduce the notion of weight γ and defects:

$$
\begin{aligned}
\gamma(z_j) &= 1, & \gamma(\omega_j) &= \gamma(\eta) = \gamma(\bar{\eta}) = 2 \\
\gamma\left(\frac{\partial}{\partial z_j}\right) &= -1, & \gamma\left(\frac{\partial}{\partial \omega_j}\right) &= \gamma\left(\frac{\partial}{\partial \eta}\right) = \gamma\left(\frac{\partial}{\partial \bar{\eta}}\right) = -2,
\end{aligned}
$$

and the weight of a product is the sum of the weights of its factors.
For any monomial Π in $z_1, z_2, \bar{z}_1, \bar{z}_2, \eta, \bar{\eta}$ the 1-defect $d_1(\Pi)$ is defined as the difference of the degrees with respect to z_1 and with respect to $\bar{z}_2$.
By $\bar{1}$-defect $d_{\bar{1}}(\Pi)$ we will denote the difference of the degrees with respect to $\bar{z}_1$ and with respect to z_2.
The total defect $\delta(\Pi)$ is the sum of d_1 and $d_{\bar{1}}$.
According to these definitions we expand

$$
N(z, \bar{z}, \eta, \bar{\eta}) = \sum_{\gamma=\gamma_0}^{\infty} N_\gamma(z, \bar{z}, \eta, \bar{\eta}),
$$

where N_γ is the component weight γ in N and $N_{\gamma_0} \neq 0$. We can further split N_γ into components $N_{\gamma, d_1, d_{\bar{1}}}$ of 1-defect d_1 and $\bar{1}$-defect $d_{\bar{1}}$. We are interested in certain properties of 3 consecutive components N_{γ_0}, N_{γ_0+1}, N_{γ_0+2} that arise from the components of weights $\gamma_0, \gamma_0+1, \gamma_0+2$ in the mentioned above identity.
For N_{γ_0} the identity implies

$$\rho_{11} N_{\gamma_0}(z,\bar z,\eta,\bar\eta) = N_{\gamma_0}(Cz,\bar C\bar z,\rho_{11}\eta,\rho_{22}\bar\eta). \tag{3}$$

Suppose that $a_1 = 0$ and let $N_{\gamma_0,\delta}$ be the component of maximal total defect in N_{γ_0}. Making use of the notation of the vector field

$$\chi_1^1 = i\bar a_2 \left(2z_1^2 \frac{\partial}{\partial z_1} + 2(\eta + iV) z_1 \frac{\partial}{\partial \omega_1} - i(\eta - iV) \frac{\partial}{\partial \bar z_2} \right)$$

we derive from the component of weight-defect $(\gamma_0+1,\delta+1)$ in the identity

$$\begin{aligned}\rho_{11} N_{\gamma_0+1,\delta+1}(z,\bar z,\eta,\bar\eta) = {}&\\ N_{\gamma_0+1,\delta+1}(Cz,\bar C\bar z,\rho_{11}\eta,\rho_{22}\bar\eta) &+ \rho_{11} C_{11} \chi_1^1 N_{\gamma_0,\delta}(z,\bar z,\eta,\bar\eta).\end{aligned} \tag{4}$$

And, finally, if $a = 0$ then the (γ_0+2,δ)-component of the identity provides

$$\rho_{11} N_{\gamma_0+2,\delta}(z,\bar z,\eta,\bar\eta) = N_{\gamma_0+2,\delta}(Cz,\bar C\bar z,\rho_{11}\eta,\rho_{22}\bar\eta) + 2\rho_{11}^2 (\operatorname{Re}\chi_2) N_{\gamma_0,\delta} \tag{5}$$

where χ_2 is the vector field

$$\chi_2 = \sum_{j=1,2} r_j z_j \omega_j \frac{\partial}{\partial z_j} + r_j \omega_j^2 \frac{\partial}{\partial \omega_j},$$

with $r_1 = r = \bar r_2$.
Now we are ready to prove the individual linearisation of isotropic automorphisms.

2 Linearisation of a single automorphism

Let M be a non-exceptional manifold. We may assume that M is given in normal form. Then any automorphism

$$\psi : z^* = f(z,\omega) \quad \omega^* = g(z,\omega)$$

from $\operatorname{Aut}_0^0 M$ can be considered as a normalisation, and, therefore, it is uniquely determined by the parameters $\lambda \in \mathbb{R}_{>0}$, $\mu \in \mathbb{R}_{>0}$, $-\pi < \theta \le \pi$, $-\frac{\pi}{2} < \phi \le \frac{\pi}{2}$ defined by

$$C = \left.\frac{\partial f}{\partial z}\right|_0 = \lambda\,\mathrm{e}^{\mathrm{i}\,\phi} \begin{pmatrix} \mu\,\mathrm{e}^{\mathrm{i}\,\theta} & 0 \\ 0 & \frac{1}{\mu}\,\mathrm{e}^{-\mathrm{i}\,\theta} \end{pmatrix}$$

and by the parameters a, r defined in (1). Though, all these parameters are free for the quadric, this is no longer the case for automorphisms of non-quadratic manifolds.

Lemma 1 *Parameters (a, r, λ) are uniquely determined by (μ, ϕ, θ).*

Proof. Suppose that two automorphisms in $\mathrm{Aut}_0^0\, M$ have the same parameters $\lambda, \mu, \phi, \theta$. Then according to the Main Theorem of [ES4] automorphism $\psi_0 = \psi_1 \circ \psi_2^{-1} \in \mathrm{Aut}_{0,\mathrm{id}}$ is the identical map. Thus, $\psi_1 = \psi_2$ and the corresponding parameters a and r coincide. To show that λ is also determined by (μ, ϕ, θ) we consider the equation of M in normal form

$$V = z^1 \bar{z}^2 + N(z, \bar{z}, \eta, \bar{\eta}) \tag{6}$$

By T_C we denote the action of C on N (or homogeneous polynomials from the expansion of N) induced by the linear mapping

$$z^* = Cz, \quad \omega^* = \rho\omega,$$

where $\rho = \lambda^2\,\mathrm{e}^{2\,\mathrm{i}\,\theta}\,\mathrm{id}$, i.e. the operator

$$T_C N(z, \bar{z}, \eta, \bar{\eta}) = \frac{\mathrm{e}^{-2\,\mathrm{i}\,\theta}}{\lambda^2} N(Cz, \overline{Cz}, \lambda^2\,\mathrm{e}^{2\,\mathrm{i}\,\theta}\,\eta, \lambda^2\,\mathrm{e}^{-2\,\mathrm{i}\,\theta}\,\bar{\eta}).$$

Let N_{γ_0} be the first non-vanishing weighted component of N. Since ψ preserves the equation (6) then

$$T_C N_{\gamma_0} = N_{\gamma_0}.$$

Any monomial

$$\Pi = c(z^1)^{\alpha}(\bar{z}^1)^{\bar{\alpha}}(z^2)^{\beta}(\bar{z}^2)^{\bar{\beta}}\eta^{\delta}\bar{\eta}^{\bar{\delta}}$$

in N_{γ_0} is an eigenvector of T_C with the eigenvalue

$$\mathrm{e}^{\mathrm{i}(\alpha+\beta-\bar{\alpha}-\bar{\beta})\phi}\,\mathrm{e}^{\mathrm{i}(\alpha+\bar{\beta}-\bar{\alpha}-\beta+2\delta-2\bar{\delta}-2)\theta}\,\lambda^{\gamma_0-2}\mu^{\alpha+\bar{\alpha}-\beta-\bar{\beta}}.$$

Hence, λ is determined by $\lambda = \mu^{\frac{\alpha+\bar{\alpha}-\beta-\bar{\beta}}{2-\gamma_0}}$. $\square$

Remark 1 *If $\alpha+\bar\beta+2\delta=\bar\alpha+\beta+2\bar\delta+2$ then, in general, it is impossible to determine θ via ϕ. If $\alpha+\beta=\bar\alpha+\bar\beta$ then ϕ cannot be determined by θ.*

Corollary 1 *The dimension of $\operatorname{Aut}_0 M$ does not exceed 3.*

Since parameters a, r depend on the other parameters in C it will be our first aim to choose normal coordinates so that in these coordinates $a=0$ and $r=0$ (this is the simplest possible dependence).

Lemma 2 *For each $\psi\in\operatorname{Aut}_0^0 M$ there exist normal coordinates such that $a=0$.*

Proof. According to Theorem 3 in [ES1] it suffices to find coordinates such that the chain $\Gamma_0=\{z=0, v=0\}$ is invariant with respect to ψ. Therefore, we will look for an invariant chain Γ and pass to such normal coordinates where Γ will be mapped to Γ_0. Since Γ is uniquely defined by $T_0\Gamma$, ψ-invariance of Γ means that

$$d\psi|_0 T_0\Gamma = T_0\Gamma. \tag{7}$$

Applying Jacobian

$$d\psi|_0=\left(\begin{array}{cc|cc}\lambda\mu\,\mathrm{e}^{\mathrm{i}(\phi+\theta)} & 0 & \lambda\mu\,\mathrm{e}^{\mathrm{i}(\phi+\theta)}\,a^1 & 0\\ 0 & \frac{\lambda}{\mu}\,\mathrm{e}^{\mathrm{i}(\phi-\theta)} & 0 & \frac{\lambda}{\mu}\,\mathrm{e}^{\mathrm{i}(\phi-\theta)}\,a^2\\ \hline 0 & 0 & \lambda^2\,\mathrm{e}^{2\,\mathrm{i}\,\theta} & 0\\ 0 & 0 & 0 & \lambda^2\,\mathrm{e}^{-2\,\mathrm{i}\,\theta}\end{array}\right)$$

to the basis $\tau_1=(b^1,0,1,0)^t$, $\tau_2(0,b^2,0,1)^t$ of $T_0\Gamma$ we obtain

$$d\psi|_0\begin{pmatrix} b^1 & 0\\ 0 & b^2\\ 1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\lambda\mu\,\mathrm{e}^{\mathrm{i}(\phi+\theta)}(b^1+a^1) & 0\\ 0 & \frac{\lambda}{\mu}\,\mathrm{e}^{\mathrm{i}(\phi-\theta)}(b^2+a^2)\\ \lambda^2\,\mathrm{e}^{2\,\mathrm{i}\,\theta} & 0\\ 0 & \lambda^2\,\mathrm{e}^{-2\,\mathrm{i}\,\theta}\end{pmatrix}$$

Now, equation (7) implies that the columns in the matrix at the right hand side are linear combinations of the two basis vectors. This can only happen if the first one is τ_1 scaled by $\lambda^2\,\mathrm{e}^{2\,\mathrm{i}\,\theta}$ and the second one is τ_2 scaled by $\lambda^2\,\mathrm{e}^{-2\,\mathrm{i}\,\theta}$ and therefore

$$
\begin{aligned}
a^1 &= \left(\frac{\lambda}{\mu} \mathrm{e}^{\mathrm{i}(\theta-\phi)} - 1\right) b^1 \\
a^2 &= \left(\lambda\mu \, \mathrm{e}^{-\mathrm{i}(\theta+\phi)} - 1\right) b^2.
\end{aligned}
$$

Using the splitting $C = \Lambda U$ with $\Lambda = \lambda \begin{pmatrix} \mathrm{e}^{\mathrm{i}\,\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\,\theta} \end{pmatrix}$, $U = \mathrm{e}^{\mathrm{i}\,\phi} \begin{pmatrix} \mu & 0 \\ 0 & \frac{1}{\mu} \end{pmatrix}$ this can be written as

$$
(\Lambda U^{-1} - \mathrm{id}) b = a. \tag{8}
$$

This system is always soluble for b if ΛU^{-1} does not have eigenvalue 1. Then, solving (8) for b we find the unique ψ-invariant chain tangent to the span of τ_1, τ_2.
Now, suppose that 1 is an eigenvalue of ΛU^{-1}. Then either $\lambda = \frac{1}{\mu}$ and $\theta = -\phi$, or $\lambda = \mu$ and $\theta = \phi$. Without loss of generality we consider the first possibility. System (8) will have a (non-unique) solution if and only if $a^2 = 0$. In this case we find a complex 1-parameter family of ψ-invariant chains.
We are going to show by contradiction that, in fact, $a^2 = 0$ if ψ is an automorphism of a non-exceptional manifold. Then (3) and (4) takes the form

$$
\begin{aligned}
T_C N_{\gamma 0} &= N_{\gamma 0} & (9)\\
(T_C - \mathrm{id}) N_{\gamma 0+1,\delta+1} &= \# \chi_1^1 N_{\gamma 0,\delta}. & (10)
\end{aligned}
$$

Here and everywhere below # substitutes a non-vanishing constant.

¿From the form of χ_1^1 we read that $\chi_1^1 N_{\gamma_0} \neq 0$ belongs to the kernel of $T_C - \mathrm{id}$ since N_{γ_0} belongs to that kernel by (9). It follows that the right hand side of (10) equals $\#\chi_1^1 N_{\gamma 0,\delta}$ and lies in the kernel of $T_C - \mathrm{id}$ whereas the left hand side of (10) lies in the range of $T_C - \mathrm{id}$. As $T_C - \mathrm{id}$ is diagonalisable (the monomials form an eigenbasis) its range is spanned by the eigenvectors with eigenvalues different from 0 and therefore intersects with the kernel by $\{0\}$. Thus, $\chi_1^1 N_{\gamma 0,\delta} = 0$.
On the other hand, in [ES4] it was proved that for a non-exceptional manifold $\chi_1^1 N_{\gamma_0} \neq 0$ if $a^2 \neq 0$. Contradiction. □

Having chosen coordinates where Γ_0 is an ψ-invariant chain, we can now change the coordinates corresponding to a reparametrisation of this chain in order to make $r = 0$:

Lemma 3 *Let $\psi \in \operatorname{Aut}_0^0 M$ and $a = 0$ then in some normal coordinates $r = 0$.*

Proof. Suppose that $\lambda^2 \mathrm{e}^{2\mathrm{i}\theta} \neq 1$. Consider the transformation

$$\begin{aligned} z^* &= (\mathrm{id} - R)^{-1} z \\ \omega^* &= (\mathrm{id} - R)^{-1} \omega \end{aligned} \tag{11}$$

where

$$R = \begin{pmatrix} r'\omega^1 & 0 \\ 0 & \bar{r}'\omega^2 \end{pmatrix}, \quad z = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}, \quad \omega = \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}.$$

One computes that in $*$-coordinates the new parameter

$$r^* = r - (\lambda^2 \mathrm{e}^{2\mathrm{i}\theta} - 1) r'.$$

Taking

$$r' = \frac{r}{1 - \lambda^2 \mathrm{e}^{2\mathrm{i}\theta}}$$

we obtain $r^* = 0$.
Suppose now that

$$\lambda^2 \mathrm{e}^{2\mathrm{i}\theta} = 1, \tag{12}$$

i.e., $\lambda = 1$, and $\mathrm{e}^{2\mathrm{i}\theta} = 1$. We show that then $r = 0$ automatically. Identity (5) can be written as

$$\begin{aligned} T_C N_{\gamma_0} &= N_{\gamma_0} \\ (T_C - \mathrm{id}) N_{\gamma_0 + 2} &= \# \operatorname{Re} \chi_2 N_{\gamma_0}. \end{aligned}$$

It is a consequence of (12) that the right hand side of the last equation belongs to the kernel of $T_C - \mathrm{id}$ though the left hand side belongs to the range of

$T_C - \mathrm{id}$. By the same argument as in the proof of the previous lemma this implies $\operatorname{Re}\chi_2 N_{\gamma_0} = 0$ which is impossible if $r \neq 0$. $\square$

Combining Lemmata 2 and 3 we easily prove that any $\psi \in \operatorname{Aut}_0^0 M$ can be linearised.

Theorem 1 *For each individual automorphism $\psi \in \operatorname{Aut}_0^0 M$ there exist normal coordinates such that ψ takes the form of a linear transformation*

$$\begin{aligned} z^* &= \Lambda U z \\ \omega^* &= \Lambda^2 \omega \end{aligned}$$

Proof. According to Lemmata 2 and 3 we can choose normal coordinates such that the parameters a and r of ψ considered as a normalisation of M vanish. By Theorem 3 in [ES1] the vanishing of a means that ψ is a mapping of the form

$$\begin{aligned} z^* &= \Lambda U (\mathrm{id} - R)^{-1} z \\ \omega^* &= \Lambda^2 (\mathrm{id} - R)^{-1} \omega. \end{aligned}$$

Condition $r = 0$ implies that $R = 0$ and therefore ψ has the required form. $\square$

3 Simultaneous linearisation

The goal of this section is to prove

Theorem 2 *For every non-exceptional manifold M there exist normal coordinates (z, ω) such that all the elements of $\operatorname{Aut}_0 M$ take the form*

$$\begin{aligned} z^* &= \Lambda U z \\ \omega^* &= \Lambda^2 \omega \end{aligned} \tag{13}$$

The proof of this Theorem requires several preliminary results.

Lemma 4 *The group $\operatorname{Aut}_0^0 M$ is Abelian.*

Proof. This is an immediate consequence of the fact that an automorphism from $\mathrm{Aut}_0^0 M$ is uniquely determined by the parameter C which is the restriction of the differential in 0 to the complex tangent space and that the matrices C are diagonal. □

Lemma 5 *Every $\psi \in \mathrm{Aut}_0^0 M$ splits into the composition of two automorphisms ψ_1 with parameter*

$$C_1 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}(\phi+\theta)} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}(\phi-\theta)} \end{pmatrix}, \tag{14}$$

referred to as a rotation, and ψ_2 with parameter

$$C_2 = \begin{pmatrix} \lambda\mu & 0 \\ 0 & \frac{\lambda}{\mu} \end{pmatrix}$$

referred to as a scaling.

Proof. Let $\psi \in \mathrm{Aut}_0^0 M$ and $(\lambda, \mu, \phi, \theta, a, r)$ be the corresponding set of parameters. According to the Theorem on individual linearisation from the previous section we can start with a normal coordinate system such that $a = 0$, $r = 0$ and ψ is a linear map (13).
Analysing the action of the linear map ψ on the equation of M we observe that ψ splits into the required composition of two automorphisms

$$\psi_1 : \quad z^* = \begin{pmatrix} \mathrm{e}^{\mathrm{i}(\phi+\theta)} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}(\phi-\theta)} \end{pmatrix} z, \quad \omega^* = \omega \tag{15}$$

and

$$\psi_2 : \quad z^* = \begin{pmatrix} \lambda\mu & 0 \\ 0 & \frac{\lambda}{\mu} \end{pmatrix} z, \quad \omega^* = \lambda^2 \omega. \tag{16}$$

□

It follows now that the group $\mathrm{Aut}_0^0 M$ decomposes into the direct product of the subgroup S of all scalings and the subgroup K of all rotations.

Lemma 6 *The subgroup S is either trivial or one-parametric.*

Proof. Let $\mathrm{id} \neq \psi \in S$. We choose normal coordinates such that ψ is linear of the form (16). Set $\ell := \ln \lambda$ and $m := \ln \mu$. Consider the one-parametric group

$$\psi^t : \quad z^* = \begin{pmatrix} \mathrm{e}^{(\ell+m)t} & 0 & \\ 0 & \mathrm{e}^{(\ell-m)t} & 0 \end{pmatrix} z, \quad \omega^* = \mathrm{e}^{2\ell t}\,\omega.$$

ψ^t acts on a monomial Π of the equation of M by multiplication with e^{ht} where h is an integers that depends only on Π. Since $\psi^1 = \psi$ is an automorphism this integer must vanish. Thus, all ψ^t are automorphisms. Since the dimension of S does not exceed 1 it follows that S is either trivial or one-parametric. $\square$

We prove that all automorphisms from K can be simultaneously linearised.

Lemma 7 *There exist normal coordinates such that all automorphisms $\psi \in K$ take the form (15).*

Proof. At first we prove that all automorphisms in K have a common invariant chain. If K contains ψ with parameter C such that neither $\mathrm{e}^{\mathrm{i}(\phi+\theta)} = 1$ nor $\mathrm{e}^{\mathrm{i}(\phi-\theta)} = 1$ then there is a unique ψ-invariant chain Γ. Since K is Abelian Γ is then also invariant with respect to the other automorphisms from K. If such ψ does not exists then either for all elements in K the parameter ϕ equals θ or for all elements ϕ equals $-\theta$, i.e. the group K is either one-parametric or finite cyclic. In both cases it suffices to find an invariant chain for a generator.
Without loss of generality we may now assume that the standard chain Γ_0 is invariant for all $\psi \in K$. For all automorphisms with $\theta = 0$ the parameter r vanishes automatically, according to Lemma 3, regardless to parametrisation of Γ_0. These automorphisms form, obviously, a subgroup of K. The factor group of K by this subgroup is either one-parametric or finite cyclic. In both cases we can find coordinates such that the parameter r vanishes for an an automorphism from a generating coset. Then $r = 0$ for all elements of K. $\square$

Corollary 2 *K is a compact Lie group.*

We consider now the situation when $\mathrm{Aut}_0^0\, M$ contains a scaling σ since, otherwise, by the lemma above we know that $\mathrm{Aut}_0^0\, M = K$ can be linearised.
Let Γ_K be the family of all K-invariant chains.

Lemma 8 *The family Γ_K contains a σ-invariant chain Γ_σ.*

Proof. Assume K is not trivial (otherwise there is nothing to prove). We choose normal coordinates such that K is linearised. Family Γ_K may consist of single chain Γ_0 (this is the case if for some $\psi \in K$ $\phi \neq \theta$ and $\phi \neq -\theta$) or else

$$T_0\Gamma = \mathrm{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \\ 0 \\ 1 \end{pmatrix} \right\}, \quad b \in \mathbb{C}, \tag{17}$$

(this is the case if $\phi = \theta$ for every $\psi \in K$), or

$$T_0\Gamma = \mathrm{span}\left\{ \begin{pmatrix} b \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad b \in \mathbb{C},$$

(this is the case if $\phi = -\theta$ for every $\psi \in K$).
If Γ_0 is the only K-invariant chain we are done because σ commutes with K and, therefore, preserves Γ_0.
Since the two remaining cases are similar we consider only the case $\phi = \theta$ for all $\psi \in K$. The Jacobian $d\sigma|_0$ has the form

$$d\psi|_0 = \left(\begin{array}{cc|cc} \lambda\mu & 0 & \lambda\mu a^1 & 0 \\ 0 & \frac{\lambda}{\mu} & 0 & \frac{\lambda}{\mu}a^2 \\ \hline 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & \lambda^2 \end{array}\right).$$

Since σ commutes with K it maps any chain from Γ_K into a chain from Γ_K. From the first component of

$$d\psi|_0 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \lambda^2 \\ 0 \end{pmatrix},$$

one reads that $a^1 = 0$. If $a^2 = 0$ then $\Gamma_\sigma = \Gamma_0$ and we are done. If $a^2 \neq 0$ then, as it had been shown in the proof of Lemma 2, $\lambda \neq \mu^{-1}$, and Γ_σ is the chain tangent to (17) for $b = \frac{a^2}{\lambda\mu - 1}$. $\square$

Let now Γ_0 be a common invariant chain for $\operatorname{Aut}_0^0 M$.

Lemma 9 *There exist normal coordinates such that* $\operatorname{Aut}_0^0 M$ *is linearised.*

Proof. If the parameter λ of the scaling σ is equal to 1 then $r = 0$ automatically. So, it remains to choose coordinates such that Γ_0 is a common invariant chain and K is linearised.
Let $\lambda \neq 1$. Choose normal coordinates such that Γ_0 remains a common invariant chain and σ is linearised. It turns out that K is then also linearised automatically. Indeed, if for $\psi \in K$ the parameter $r \neq 0$ then N_{γ_0} and N_{γ_0+2} satisfy the conditions:

$$\begin{aligned}
T_{C_\sigma} N_{\gamma_0} &= N_{\gamma_0} \\
T_{C_\psi} N_{\gamma_0} &= N_{\gamma_0} \\
T_{C_\sigma} N_{\gamma_0+2} &= N_{\gamma_0+2} \\
T_{C_\psi} N_{\gamma_0+2} - N_{\gamma_0+2} &= \# T_{C_\psi} \operatorname{Re} \chi_2 N_{\gamma_0}
\end{aligned}$$

Since $2 \operatorname{Re} \chi_2 N_{\gamma_0} \neq 0$ the right hand side of the last equation contains monomials Π such that $T_{C_\sigma} \Pi = \lambda^2 \Pi$, though, the left hand side contains only monomials form the kernel of $T_{C_\sigma} - \mathrm{id}$, by the third equation. Contradiction. $\square$

Lemma 10 $\operatorname{Aut}_0 M$ *has a common invariant chain.*

Proof. Aut_0 consists at most of two cosets with respect to $\operatorname{Aut}_0^0 M$. Therefore, a chain that is invariant for $\operatorname{Aut}_0^0 M$ and one element of $\psi \in \operatorname{Aut}_0 M \setminus \operatorname{Aut}_0^0 M$ will be invariant for the whole group.
We start with normal coordinates where $\operatorname{Aut}_0^0 M$ is linearised. The Jacobian of ψ has the form

$$d\psi|_0 = \left(\begin{array}{cc|cc}
0 & \lambda\mu\, e^{i(\phi+\theta)} & 0 & \lambda\mu\, e^{i(\phi+\theta)}\, a^1 \\
\frac{\lambda}{\mu} e^{i(\phi-\theta)} & 0 & \frac{\lambda}{\mu} e^{i(\phi-\theta)}\, a^2 & 0 \\
\hline
0 & 0 & 0 & \lambda^2 e^{2i\theta} \\
0 & 0 & \lambda^2 e^{-2i\theta} & 0
\end{array}\right) \tag{18}$$

Notice that ψ^2 already belongs to Aut_0^0 and therefore it is linear. So, the right upper block of the square of (18) must vanish. This gives the equations

$$\begin{aligned} a_1 + \frac{\lambda}{\mu}\,\mathrm{e}^{\mathrm{i}(\theta-\phi)} a_2 &= 0 \\ \lambda\mu\,\mathrm{e}^{-\,\mathrm{i}(\theta+F)} a_1 + \qquad\qquad a_2 &= 0. \end{aligned} \tag{19}$$

If $\lambda \neq 1$ or $\phi \neq 0$ this immediately implies that $a_1 = a_2 = 0$, i.e., the standard chain is invariant for ψ. Assume now $\lambda = 1$ and $\phi = 0$. Then

$$d\psi|_0 = \left(\begin{array}{cc|cc} 0 & \mu\,\mathrm{e}^{\mathrm{i}\,\theta} & 0 & \mu\,\mathrm{e}^{\mathrm{i}\,\theta}\,a^1 \\ \frac{1}{\mu}\,\mathrm{e}^{-\,\mathrm{i}\,\theta} & 0 & e^{-2\,\mathrm{i}\,\theta} a^1 & 0 \\ \hline 0 & 0 & 0 & \mathrm{e}^{2\,\mathrm{i}\,\theta} \\ 0 & 0 & \mathrm{e}^{-2\,\mathrm{i}\,\theta} & 0 \end{array}\right)$$

.

Since $\operatorname{Aut}_0^0 M$ is a normal subgroup of $\operatorname{Aut}_0 M$ then $d\psi|_0 T_0\Gamma_0$ is invariant for $\operatorname{Aut}_0 M$. It follows that either $a^1 = 0$ (and therefore Γ_0 is invariant for ψ, as well), or $\operatorname{Aut}_0^0 M = \{\mathrm{id}\}$ (because we would obtain more than a one-parametric family of invariant chains). In this case we find an invariant chain for ψ by solving the equation (8). $\square$

It follows that there exist normal coordinates such that $\operatorname{Aut}_0^0 M$ is linearised and any $\psi \in \operatorname{Aut}_0 M \setminus \operatorname{Aut}_0^0 M$ has the form

$$\begin{aligned} z^* &= J\Lambda U(\mathrm{id} - R)^{-1} z \\ \omega^* &= J\Lambda^2(\mathrm{id} - R)^{-1}\omega \end{aligned} \tag{20}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} r\omega^1 & 0 \\ 0 & \bar{r}\omega^2 \end{pmatrix}.$$

It suffices to find a transformation of the form (11) such that $\operatorname{Aut}_0^0 M$ remains linearised and one (and therefore every) element from $\operatorname{Aut}_0 M \setminus \operatorname{Aut}_0^0 M$ becomes linear.

Lemma 11 *If for some* $\psi \in \operatorname{Aut}_0 M$ *the parameter* λ *is different from* 1 *then* $\operatorname{Aut}_0 M$ *is automatically linearised if* $\operatorname{Aut}_0^0 M$ *is linearised.*

Proof. Without loss of generality we assume that $\lambda \neq 1$ for some $\psi \in \operatorname{Aut}_0 M \setminus \operatorname{Aut}_0^0 M$. Taking the square of ψ in the form (20) we obtain a fractional linear mapping with parameter

$$r' = r + \lambda^2 \, \mathrm{e}^{-2\,\mathrm{i}\,\theta} \, \bar{r} \tag{21}$$

On the other hand we know that ψ^2 is already linear, thus $r' = 0$. But this is only possible if r was already 0. □

Lemma 12 *If* $\mathrm{e}^{2\,\mathrm{i}\,\theta}$ *is different from* 1 *for some* $\psi \in \operatorname{Aut}_0^0 M$ *then* $\operatorname{Aut}_0 M$ *is automatically linearised if* $\operatorname{Aut}_0^0 M$ *is linearised.*

Proof. Let $\psi_1 \in \operatorname{Aut}_0 M \setminus \operatorname{Aut}_0^0 M$ have parameters θ_1 and r. Take $\psi_2 \in \operatorname{Aut}_0^0 M$ with $\mathrm{e}^{2\,\mathrm{i}\,\theta_2} \neq 1$. The condition $r' = 0$ in (21) implies that $2(\theta_1 - \arg r) = \pi$, or $r = 0$. Considering (21) for the composition $\psi_1 \circ \psi_2$ leads to $2(\theta_1 - \theta_2 - \arg r) = \pi$, or $r = 0$. By the assumption of the lemma it follows that $r = 0$. □

Theorem 2 is now a consequence of the lemma below.

Lemma 13 *If* $\lambda = 1$ *and* $\mathrm{e}^{2\,\mathrm{i}\,\theta} = 1$ *for all* $\psi \in \operatorname{Aut}_0^0 M$ *then* $\operatorname{Aut}_0 M$ *is can be linearised.*

Proof. According to the proof of Lemma 3, any coordinate change of the form (11) preserves the linear form of Aut_0^0. Therefore it suffices to find a transformation (11) that linearises one $\psi \in \operatorname{Aut}_0 M \setminus \operatorname{Aut}_0^0 M$. This changes the parameter r of ψ into $r + r' + \mathrm{e}^{-2\,\mathrm{i}\,\theta} \, \bar{r}'$. If $r \neq 0$ then (21) ensures that there exists r' such that $r + r' + \mathrm{e}^{-2\,\mathrm{i}\,\theta} \, \bar{r}' = 0$ □

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