THE BOGOMOLOV MULTIPLIER OF FINITE SIMPLE GROUPS

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1. Results

Let G be a finite group, and let $\mathrm{M}(G) := H^2(G, \mathbb{Q}/\mathbb{Z})$ be its Schur multiplier. Denote by $B_0(G)$ the subgroup of $\mathrm{M}(G)$ consisting of the cohomology classes whose restriction to any abelian subgroup of G is zero. We call $B_0(G)$ the *Bogomolov multiplier* of G. This subgroup was introduced in [Bo87] in order to provide an explicit expression for the unramified Brauer group of the quotient V/G where V stands for any faithful linear representation of G over $\mathbb{C}$. This birational invariant had earlier on been used by Saltman to give a negative answer to Noether's problem [Sa]. The reader interested in historical perspective and geometric context is referred to [Sh], [CTS], [Bo07].

We say that G is *quasisimple* if G is perfect and its quotient by the centre $L = G/Z$ is a nonabelian simple group. We say that G is *almost simple* if for some nonabelian simple group L we have $L \subseteq G \subseteq \operatorname{Aut} L$. Our first observation is

Theorem 1.1. *If G is a finite quasisimple group, then $B_0(G) = 0$.*

As a particular case, Theorem 1.1 contains the assertion on vanishing of $B_0(G)$ for all finite simple groups stated as a conjecture in [Bo92] and proved for the groups of Lie type A_n in [BMP].

Theorem 1.1 implies

Theorem 1.2. *If G is a finite almost simple group, then $B_0(G) = 0$.*

Remark 1.3. In Theorems 1.1 and 1.2 we consider top and bottom decorations of simple groups, respectively. Apparently one can complete the picture, allowing both perfect central extensions and outer automorphisms, by deducing from Theorems 1.1 and 1.2 that $B_0(G) = 0$ for all *nearly simple* groups G (see the definition in Section 2.3 below). In particular, this statement holds true for all finite "reductive" groups such as the general linear group $GL(n, q)$, the general unitary group $GU(n, q)$, and the like.

Our notation is standard and mostly follows [GLS]. Throughout below "simple group" means "finite nonabelian simple group". Our proofs heavily rely on the classification of such groups.

2. Preliminaries

In order to make the exposition as self-contained as possible, in this section we collect the group-theoretic information needed in the proofs. All groups are assumed finite (although some of the notions discussed below can be defined for infinite groups as well).

2.1. **Schur multiplier.** The material below (and much more details) can be found in [Ka].

The group $\mathrm{M}(G) := H^2(G, \mathbb{Q}/\mathbb{Z})$, where G acts on $\mathbb{Q}/\mathbb{Z}$ trivially, is called the Schur multiplier of G. It can be identified with the kernel of some central extension

$$1 \to \mathrm{M}(G) \to \widetilde{G} \to G \to 1.$$

The covering group $\widetilde{G}$ is defined uniquely up to isomorphism provided G is perfect (i.e. coincides with its derived subgroup $[G,G]$).

We will need to compute $\mathrm{M}(G)$ in the case where G is a semidirect product of a normal subgroup N and a subgroup H. If A is an ableian group on which G acts trivially, the restriction map $\mathrm{Res}_H \colon H^2(G,A) \to H^2(H,A)$ gives rise to a split exact sequence [Ka, Prop. 1.6.1]

$$1 \to K \to H^2(G,A) \to H^2(H,A) \to 1.$$

The kernel K can be computed from the exact sequence [Ka, Th. 1.6.5(ii)]

$$1 \to H^1(H, \mathrm{Hom}(N,A)) \to K \stackrel{\mathrm{Res}_N}{\to} H^2(N,A)^H \to H^2(H, \mathrm{Hom}(N,A)).$$

If H is perfect and $A = \mathbb{Q}/\mathbb{Z}$, we have $\mathrm{Hom}(H,A) = 1$ and thus [Ka, Lemma 16.3.3]

$$\mathrm{M}(G) \cong \mathrm{M}(N)^H \times \mathrm{M}(H). \tag{2.1}$$

2.2. **Bogomolov multiplier.** The following properties of $B_0(G) := \ker[H^2(G,\mathbb{Q}/\mathbb{Z}) \to \prod_A H^2(A,\mathbb{Q}/\mathbb{Z})]$ are taken from [Bo87], [BMP].

(1) $B_0(G) = \ker[H^2(G,\mathbb{Q}/\mathbb{Z}) \to \prod_B H^2(B,\mathbb{Q}/\mathbb{Z})]$, where the product is taken over all bicyclic subgroups B of G [Bo87], [BMP, Cor. 2.3].

(2) For an abelian group A denote by A_p its p-primary component. We have

$$B_0(G) = \bigoplus_p B_{0,p}(G),$$

where $B_{0,p}(G) := B_0(G) \cap \mathrm{M}(G)_p$. For any Sylow p-subgroup S of G we have $B_{0,p}(G) \subseteq B_0(S)$. In particular, if all Sylow subgroups of G are abelian, $B_0(G) = 0$ [Bo87], [BMP, Lemma 2.6].

(3) If G is an extension of a cyclic group by an abelian group, then $B_0(G) = 0$ [Bo87, Lemma 4.9].

(4) For $\gamma \in \mathrm{M}(G)$ consider the corresponding central extension:

$$1 \to \mathbb{Q}/\mathbb{Z} \xrightarrow{i} \widetilde{G}_\gamma \to G \to 1,$$

and denote

$$K_\gamma := \{h \in \mathbb{Q}/\mathbb{Z} \,|\, i(h) \in \bigcap_{\chi \in \mathrm{Hom}(\widetilde{G}_\gamma, \mathbb{Q}/\mathbb{Z})} \ker(\chi)\}.$$

Then γ does not belong to $B_0(G)$ if and only if some element of K_γ can be represented as a commutator of a pair of elements of $\widetilde{G}_\gamma$ [BMP, Cor. 2.4].

(5) If $0 \neq \gamma \in \mathrm{M}(G)$, we say that G is γ-minimal if the restriction of γ to all proper subgroups $H \subset G$ is zero. A γ-minimal group must be a p-group. We say that a γ-minimal nonabelian p-group G is a γ-minimal factor if for any quotient map $\rho\colon G \to G/H$ there is no $\gamma' \in B_0(G/H)$ such that $\gamma = \rho^*(\gamma')$ and γ' is G/H-minimal. A γ-minimal factor G must be a metabelian group (i.e. $[[G,G],[G,G]] = 0$), with a central series of length at most p, and the order of γ in $\mathrm{M}(G)$ equals p [Bo87, Theorem 4.6]. Moreover, if G is a γ-minimal p-group which is a central extension of $G^{\mathrm{ab}} := G/[G,G]$ and $G^{\mathrm{ab}} = (\mathbb{Z}_p)^n$, then $n = 2m$ and $n \geq 4$ [Bo87, Lemma 5.4].

2.3. **Finite simple groups.** We need the following facts concerning finite simple groups (see, e.g., [GLS]) believing that the classification of finite simple groups is complete.

(1) *Classification.* Any finite simple group L is either a group of Lie type, or an alternating group, or one of 26 sporadic groups.

(2) *Schur multipliers.* As L is perfect, it has a unique covering group $\widetilde{L}$, and $L \cong \widetilde{L}/\,\mathrm{M}(L)$. The Schur multipliers $\mathrm{M}(L)$ of all finite simple groups L are given in [GLS, 6.1].

(3) *Automorphisms.* The group of outer automorphisms $\mathrm{Out}(L) := \mathrm{Aut}(L)/L$ is solvable. It is abelian provided L is an alternating or a sporadic group. For groups of Lie type defined over a finite field F the structure of $\mathrm{Out}(L)$ can be described as follows.

Every automorphism of L is a product $idfg$ where i is an inner automorphism (identified with an element of L), d is a diagonal automorphism (induced by conjugation by an element h of a maximal torus which normalizes L), f is a field automorphism (arising from an automorphism of the field $\overline{F}$), and g is a graph automorphism (induced by an automorphism of the Dynkin diagram corresponding to L); see [GLS, 2.5] for more details.

The group $\mathrm{Out}(L)$ is a split extension of $\mathrm{Outdiag}(L) := \mathrm{Inndiag}(L)/L$ by the group $\Phi\Gamma$, where Φ is the group of field automorphisms and Γ is the group of graph automorphisms of L. The group $\mathcal{O} = \mathrm{Outdiag}(L)$ is isomorphic to the center of $\widetilde{L}$ by the isomorphism preserving the action of $\mathrm{Aut}(L)$ and is nontrivial only in the following cases:

L is of type $A_n(q)$; $\mathcal{O} = \mathbb{Z}_{(n+1,q)}$;
L is of type ${}^2A_n(q)$; $\mathcal{O} = \mathbb{Z}_{(n+1,q-1)}$;
L is of type $B_n(q)$, $C_n(q)$, or ${}^2D_{2n}(q)$; $\mathcal{O} = \mathbb{Z}_{(2,q-1)}$;
L is of type $D_{2n}(q)$; $\mathcal{O} = \mathbb{Z}_{(2,q-1)} \times \mathbb{Z}_{(2,q-1)}$;
L is of type ${}^2D_{2n+1}(q)$; $\mathcal{O} = \mathbb{Z}_{(4,q-1)}$;
L is of type ${}^2E_6(q)$; $\mathcal{O} = \mathbb{Z}_{(3,q-1)}$;
L is of type $E_7(q)$; $\mathcal{O} = \mathbb{Z}_{(2,q-1)}$.

If L is of type ${}^d\Sigma(q)$ for some root system Σ ($d = 1, 2, 3$), the group Φ is isomorphic to $\mathrm{Aut}(\mathbb{F}_{q^d})$. If $d = 1$, then Γ is isomorphic to the group of symmetries of the Dynkin diagram of Σ and $\Phi\Gamma = \Phi \times \Gamma$ provided Σ is simply-laced; otherwise, $\Gamma = 1$ except if $\Sigma = B_2, F_4$, or G_2 and q is a power of 2, 2, or 3, respectively, in which cases $\Phi\Gamma$ is cyclic and $[\Phi\Gamma : \Phi] = 2$. If $d \neq 1$, then $\Gamma = 1$.

The action of $\Phi\Gamma$ on $\mathcal{O}$ is described as follows. If $L \ncong D_{2n}(q)$, then Φ acts on the cyclic group $\mathcal{O}$ as $\mathrm{Aut}(\mathbb{F}_{q^d})$ does on the multiplicative subgroup of $\mathbb{F}_{q^d}$ of the same order as $\mathcal{O}$; if $L \cong D_{2n}(q)$, then Φ centralizes $\mathcal{O}$. If $L \cong A_n(q)$, $D_{2n+1}(q)$, or $E_6(q)$, then $\Gamma = \mathbb{Z}_2$ acts on $\mathcal{O}$ by inversion; if $L \cong D_{2n}(q)$ and q is odd, then Γ, which is isomorphic to the symmetric group S_3 (for $m = 2$) or to $\mathbb{Z}_2$ (for $m > 2$) acts faithfully on $\mathcal{O} = \mathbb{Z}_2 \times \mathbb{Z}_2$.

(4) *Decorations.* It is often useful to consider groups close to finite simple groups, namely, *quaisimple* and *almost simple* groups, as

in the statements of Theorems 1.1 and 1.2 above. As an example, if the simple group under consideration is $L = PSL(2, q)$, the group $SL(2, q)$ is quasisimple and the group $PGL(2, q)$ is almost simple. More generally, one can consider *semisimple* groups (central products of quasisimple groups) and *nearly simple* groups G, i.e. such that the generalized Fitting subgroup $F^*(G)$ is quasisimple. $F^*(G)$ is defined as the product $E(G)F(G)$ where $E(G)$ is the layer of G (the maximal semisimple normal subgroup of G) and $F(G)$ is the Fitting subgroup of G (or the nilpotent radical, i.e. the maximal nilpotent normal subgroup of G). The general linear group $GL(n, q)$ is an example of a nearly simple group.

3. Proofs

Proof of Theorem 1.1. As G is perfect, there exists a unique universal central covering $\widetilde{G}$ of G whose centre $Z(\widetilde{G})$ is isomorphic to $\mathrm{M}(G)$ and any other perfect central extension of G is a quotient of $\widetilde{G}$. So we can argue exactly as in [Bo87, Remark after Lemma 5.7] and [BMP]. Namely, $B_0(G)$ coincides with the collection of classes whose restriction to any bicyclic subgroup of G is zero, see 2.2(1). Therefore, to establish the assertion of the theorem, it is enough to prove that any $z \in Z(\widetilde{G})$ can be represented as a commutator $z = [a, b]$ of some $a, b \in \widetilde{G}$. Moreover, it is enough to prove that such a representation exists for all elements z of prime power order, see 2.2(2).

It remains to apply the results of Blau [Bl] who classified all elements z having a fixed point in the natural action on the set of conjugacy classes of $\widetilde{G}$ (such elements evidently admit a needed representation as a commutator):

Theorem 3.1. ([Bl, Theorem 1]) *Assume that G is a quasisimple group and let $z \in Z(G)$. Then one of the following holds:*

(i) $\mathrm{order}(z) = 6$ *and* $G/Z(G) \cong A_6, A_7, Fi_{22}, PSU(6, 2^2)$, *or* ${}^2E_6(2^2)$*;*

(ii) $\mathrm{order}(z) = 6$ *or* 12 *and* $G/Z(G) \cong PSL(3, 4), PSU(4, 3^2)$ *or* M_{22}*;*

(iii) $\mathrm{order}(z) = 2$ *or* 4, $G/Z(G) \cong PSL(3, 4)$, *and* $Z(G)$ *is noncyclic;*

(iv) there exists a conjugacy class C of G such that $Cz = C$.

This theorem implies that the only possibility for an element of G of prime power order to act on the set of conjugacy classes without fixed points is case (iii) where $\widetilde{G}/Z(\widetilde{G}) \cong PSL(3,4)$ and z is an element of order 2 or 4. So the classes $\gamma \in H^2(G, \mathbb{Q}/\mathbb{Z})$ corresponding to such z's are the only candidates for nonzero elements of $B_0(G)$.

A more detailed analysis of the case $PSL(3,4)$ is sketched in [Bl, Remark (2) after Theorem 1]. Namely, in that case $Z(\widetilde{G}) \cong \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4$. Of the twelve elements of order 4 in $Z(\widetilde{G})$ exactly six fix a conjugacy class of $\widetilde{G}$. If z is one of the remaining six elements, we consider $y = z^2$. According to the same remark from [Bl], the only case when a central element of order 2 in a quasisimple group does not fix a conjugacy class is y acting on the conjugacy classes of the subgroup G_0 of index 2 in $\widetilde{G}$. However, the action of y on the conjugacy classes of $\widetilde{G}$ has a fixed point, so y can be represented as a commutator in $\widetilde{G}$. By 2.2(4), the element $\gamma \in H^2(G, \mathbb{Q}/\mathbb{Z})$ corresponding to z does not belong to $B_0(G)$.

□

Remark 3.2. It is interesting to compare [BMP, Lemma 3.1] with a theorem from the PhD thesis of Robert Thompson [Th, Theorem 1].

Proof of Theorem 1.2. Let $L \subseteq G \subseteq \mathrm{Aut}(L)$ where L is a simple group. Clearly, it is enough to prove the theorem for $G = \mathrm{Aut}(L)$. The group $\mathrm{Out}(L) = \mathrm{Aut}(L)/L$ of outer automorphisms of L acts on $\mathrm{M}(L)$, and since L is perfect, we have an isomorphism

$$\mathrm{M}(G) \cong \mathrm{M}(L)^{\mathrm{Out}(L)} \times \mathrm{M}(\mathrm{Out}(L)) \tag{3.1}$$

(see (2.1)).

Lemma 3.3. $B_0(\mathrm{Out}(L)) = 0$.

Proof of Lemma 3.3. We maintain the notation of Section 2.3. If $\mathrm{Out}(L)$ is abelian, the statement is obvious. This includes the cases where L is an alternating or sporadic group. So we may assume L is of Lie type. If $\mathcal{O} = 1$, i.e. L is of type E_8, F_4, or G_2, the result follows immediately. If the group $\Phi\Gamma$ is cyclic, the result follows because $\mathcal{O}$ is abelian (see 2.3(3)). This is the case for all groups having no graph automorphisms, in particular, for all groups of type B_n or C_n ($n \geq 3$), E_7, and for all twisted forms. For the groups of type B_2, the group $\Phi\Gamma$ is always cyclic. It remains to consider the cases A_n, D_n, and E_6. In the case $L = E_6$ all Sylow p-subgroups of $\mathrm{Out}(L)$ are abelian, and the result holds. Let $L = D_{2m}(q)$. If q is even, we have $\mathcal{O} = 1$, $\Gamma = \mathbb{Z}_2$ (if $m > 2$) or S_3 (if $m = 2$); in both cases the Sylow p-subgroups of $\mathrm{Out}(L)$ are abelian, and we are done. If q is odd, we have $\mathcal{O} = \mathbb{Z}_2 \times \mathbb{Z}_2$, and Φ

centralizes $\mathcal{O}$ (see Section 2.3), so every Sylow p-subgroup of $\mathrm{Out}(L)$ can be represented as an extension of a cyclic group by an abelian group, and we conclude as above. Finally, let L be of type $A_n(q)$ or $D_{2m+1}(q)$. Then we have $\mathcal{O} = \mathbb{Z}_h$, $h = (n+1, q-1)$ or $h = (4, q-1)$, respectively, $\Gamma = \mathbb{Z}_2$, $\Phi = \mathrm{Aut}(\mathbb{F}_q)$. The action of both Γ and Φ on $\mathcal{O}$ may be nontrivial: Γ acts by inversion, Φ acts on $\mathcal{O}$ as $\mathrm{Aut}(\mathbb{F}_q)$ does on the multiplicative subgroup of $\mathbb{F}_q$ of the same order as $\mathcal{O}$. Hence we can represent the metabelian group $\mathrm{Out}(L)$ in the form

$$1 \to V \to \mathrm{Out}(L) \to A \to 1, \tag{3.2}$$

where V, the derived subgroup of $\mathrm{Out}(L)$, is isomorphic to a cyclic subgroup $\mathbb{Z}_c$ of $\mathcal{O}$, and the abelian quotient A is of the form $\mathbb{Z}_a \times \mathbb{Z}_b \times \mathbb{Z}_2$ for some integers a, b, c. Since it is enough to establish the result for the Sylow 2-subgroup, we may assume that a, b and c are powers of 2. Then the statement of the lemma follows from the properties of γ-minimal elements described in Section 2.2. Indeed, if γ is a nonzero element of $B_0(G)$ and G is γ-minimal, then G is metabelian, both V and A are of exponent p, and in any representation of G in the form (3.2) the group A must have even number $s = 2t$ of direct summands $\mathbb{Z}_p$ with $t \geq 2$. However, if G is the Sylow 2-subgroup of $\mathrm{Out}(L)$, this is impossible because A contains only three direct summands. Thus $B_0(Syl_2(\mathrm{Out}(L))) = 0$, and so $B_0(\mathrm{Out}(L)) = 0$. The lemma is proved. □

We can now finish the proof of the theorem. Let γ be a nonzero element of $B_0(G)$. Using the isomorphism (3.1), we can represent γ as a pair (γ_1, γ_2) where $\gamma_1 \in \mathrm{M}(L)$, $\gamma_2 \in \mathrm{M}(\mathrm{Out}(L))$. Restricting to the bicyclic subgroups of G, we see that $\gamma_1 \in B_0(L)$, $\gamma_2 \in B_0(\mathrm{Out}(L))$, and the result follows from Theorem 1.1 and Lemma 3.3. □

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References

[Bl] H. I. Blau, *A fixed-point theorem for central elements in quasisimple groups*, Proc. Amer. Math. Soc. **122** (1994) 79–84.

[Bo87] F. A. Bogomolov, *The Brauer group of quotient spaces by linear group actions*, Izv. Akad. Nauk. SSSR Ser. Mat. **51** (1987) 485–516; English transl. in Math. USSR Izv. **30** (1988) 455–485.

[Bo92] F. A. Bogomolov, *Stable cohomology of groups and algebraic varieties*, Mat. Sb. **183** (1992) 1–28; English transl. in Sb. Math. **76** (1993) 1–21.

[Bo07] F. Bogomolov, *Stable cohomology of finite and profinite groups*, "Algebraic Groups" (Y. Tschinkel, ed.), Universitätsverlag Göttingen, 2007, pp. 19–49.

[BMP] F. Bogomolov, J. Maciel, T. Petrov, *Unramified Brauer groups of finite simple groups of Lie type* A_ℓ, Amer. J. Math. **126** (2004) 935–949.

[CTS] J.-L. Colliot-Thélène, J.-J. Sansuc, *The rationality problem for fields of invariants under linear algebraic groups (with special regards to the Brauer group)*, Proc. Intern. Colloquium on Algebraic Groups and Homogeneous Spaces (Mumbai 2004) (V. Mehta, ed.), TIFR Mumbai, Narosa Publishing House, 2007, pp. 113–186.

[GLS] D. Gorenstein, R. Lyons, R. Solomon, *The Classification of the Finite Simple Groups*, Number 3, Math. Surveys and Monographs, vol. 40, no. 3, Amer. Math. Soc., Providence, RI, 1998.

[Ka] G. Karpilovsky, *Group Representations*, vol. 2, North-Holland Math. Studies **177**, North-Holland, Amsterdam et al., 1993.

[Sa] D. J. Saltman, *Noether's problem over an algebraically closed field*, Invent. Math. **77** (1984) 71–84.

[Sh] I. R. Shafarevich, *The Lüroth problem*, Trudy Mat. Inst. Steklov **183** (1990) 199–204; English transl. Proc. Steklov Inst. Math. **183** (1991) 241–246.

[Th] R. C. Thompson, *Commutators in the special and general linear groups*, Trans. Amer. Math. Soc. **101** (1961) 16–33.

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