SUPERCONFORMAL ALGEBRAS AND LIE SUPERALGEBRAS OF THE HODGE THEORY

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Introduction

In this work we will follow the definition of a superconformal algebra given by V. G. Kac and J. W. van de Leur in [KL].

A *superconformal algebra* is a simple complex Lie superalgebra, such that it contains the centerless Virasoro algebra (i.e. Witt algebra) $Witt = \oplus_{n\in\mathbb{Z}}\mathbb{C}L_n$ as a subalgebra, and has growth 1. The $\mathbb{Z}$-graded superconformal algebras are ones for which adL_0 is diagonalizable with finite-dimensional eigenspaces.

The main series of superconformal algebras are $W(N)$ $(N \geq 0)$, $S'(N,\alpha)$ $(N \geq 2)$, and $K'(N)$, $(N \geq 1)$. The corresponding central extensions were classified in [KL]. $W(N)$ is the Lie superalgebra of all derivations of $\mathbb{C}[t,t^{-1}]\otimes\Lambda(N)$, where $\Lambda(N)$ is the Grassmann algebra in N odd variables. $W(N)$ has no non-trivial central extensions if $N > 2$. If $N = 1$ or 2, then there exists one non-trivial central extension. In particular, a central extension of $W(1)$ is isomorphic to the $N = 2$ *superconformal algebra*. Recall that this superalgebra is spanned by the Virasoro generators, a $U(1)$ current algebra, two fermionic fields, and a central element, see [FST].

$S(N,\alpha)$, where $\alpha \in \mathbb{C}$, is a one-parameter family of subalgebras of $W(N)$. In particular, $S(N,0)$ is the subalgebra of divergence-free derivations. If $\alpha \notin \mathbb{Z}$, then $S(N,\alpha)$ is simple (assuming that $N \geq 2$), and if $\alpha \in \mathbb{Z}$, then the derived superalgebra $S'(N,\alpha)$ is a simple ideal of $S(N,\alpha)$ of codimension 1. For $\alpha \in \mathbb{C}$, $S'(N,\alpha)$ has a non-trivial central extension if and only if $N = 2$. Notice that for every $\alpha \in \mathbb{C}$ the $N = 2$ superconformal algebra is embedded into a central extension $\hat{S}'(2,\alpha)$. In this way one can realize the $N = 2$ superconformal algebra as a one-parameter family of isomorphic superalgebras related by the spectral flow transformation, see [FST]. Thus $\hat{S}'(2,\alpha)$ is spanned by the generators of the $N = 2$ superconformal algebra together with two additional series of even elements, and two additional series of odd elements. Notice that $\hat{S}'(2,0)$ is isomorphic to the $N = 4$ *superconformal algebra* ([A]).

The superalgebra $K(N)$ is a subalgebra of $W(N)$ characterized by its action on a contact differential 1-form ([KL], [CK]). Notice that $K(N)$ is simple except when $N = 4$. In this case $K'(4) = [K(4), K(4)]$ is simple. If $N > 4$, then there is no non-trivial central extensions. If $N \leq 3$, then there exists one non-trivial central extension. In particular, $\hat{K}(1)$ is isomorphic to the *Neveu-Schwarz algebra* ([NS]), and $\hat{K}(2)$ is isomorphic to the $N = 2$ superconformal algebra. The superalgebra $K'(4)$ has 3 independent central extensions. V. G. Kac and S.-J. Cheng have shown that the $N = 4$ superconformal algebra can be realized inside $K(4)$, using the construction of CK_6 inside $K(6)$ ([CK]).

In this work we construct an inclusion of a central extension of $K'(4)$ into the super-analogue of the Lie algebra of pseudodifferential symbols on the circle with 2 odd variables.

Recall that the Lie algebra R of pseudodifferential symbols on the circle consists of the formal series

$$A(t, \xi) = \sum_{-\infty}^{n} a_i(t)\xi^i, \tag{0.1}$$

where $a_i(t) \in \mathbb{C}[t, t^{-1}]$, and the variable ξ corresponds to $\partial/\partial t$ ([KK]). One can naturally define *the superalgebra of pseudodifferential symbols on the circle* $R(N)$ by introducing N odd variables.

We consider the following inclusion of Lie superalgebras:

$$\text{the centerless } N = 2 \text{ SCA } \subset S'(2, 0) \subset \hat{K}'(4), \tag{0.2}$$

spanned by 2 even and 2 odd, 4 even and 4 odd, 8 even and 8 odd series of generators together with a central element, respectively. We realize the superalgebras (0.2) as subalgebras of $R(2)$. First we prove that the exterior derivations of $S'(2, \alpha)$, where $\alpha \in \mathbb{Z}$, form an $\mathfrak{sl}(2)$. Notice that an obvious exterior derivation is defined by an element (a differential operator) from $S(2, \alpha)$, which is not in $S'(2, \alpha)$. We fail to represent all the exterior derivations as some analogues of regular differential operators on the circle. However, we obtain an inclusion of the exterior derivations into $R(2)$. We also observe that the exterior derivations of $S'(2, \alpha)$ for all $\alpha \in \mathbb{Z}$ generate a subalgebra of $R(2)$ isomorphic to the loop algebra $\tilde{\mathbb{S}\mathfrak{L}}(2)$ ($\mathbb{S}\mathfrak{L}(2)$ corresponds to $\alpha = 1$). The elements of $\tilde{\mathbb{S}\mathfrak{L}}(2)$ are certain series of pseudodifferential symbols, which can be identified with formulas of integration by parts in the space $V_\beta = t^\beta\mathbb{C}[t, t^{-1}] \otimes \Lambda(2)$, where $\beta \in \mathbb{R} \setminus \mathbb{Z}$. Finally, we prove that the family $S'(2, \alpha)$, where $\alpha \in \mathbb{Z}$, together with $\tilde{\mathbb{S}\mathfrak{L}}(2)$ generate a central extension

$$\hat{K}'(4) \subset R(2). \tag{0.3}$$

In this way we obtain one of 3 non-trivial central extensions of $K'(4)$. The corresponding central element is the identity element $\mathbb{1} \in R(2)$.

It is known that the Lie algebra of pseudodifferential symbols on the circle has 2 independent central extensions, which correspond to $\log \xi$ and $\log t$, see [KK], [KZ]. There exist the analogous central extensions of $R(N)$. If $N = 2$ we can consider the restrictions of these extensions to $K'(4)$. Then we obtain two other independent central extension of $K'(4)$. One can note that the inclusion (0.3) should correspond to the fact that the Lie (super)algebra of pseudodifferential symbols on the circle contracts to the Poisson (super)algebra ([KLR], [OR]).

In this work we observe a correspondence between the superconformal algebras (0.2) and Lie superalgebras of the Hodge theory. Recall that given a compact Kähler manifold M, there exist a number of classical operators on the complex of differential forms on M, such as the differentials $\partial, \bar{\partial}, d, d_c$, their adjoint operators and associated Laplacians. There also exists an action of $\mathfrak{sl}(2)$ on $H^*(M)$ according to the Lefschetz theorem. All these operators satisfy a series of identities known as Hodge identities (or Kodaira's identities), see [GH]. Naturally, the classical operators form a Lie superalgebra. There exist the analogous operators in the case of a compact hyperKähler manifold. They were constructed by M. Verbitsky, see [V1], [V2]. Earlier N. Hitchin and his collaborators have pointed out that a hyperKähler manifold has $N = 4$ supersymmetry ([HKLR]). J. M. Figueroa-O'Farrill, C. Köhl, and B. Spence have recently shown that the Hodge-Lefschetz theory for (hyper)Kähler manifolds is a natural consequence of supersymmetry ([FKS]).

Recall that by definition a superconformal algebra $\mathfrak{g}$ includes the centerless Virasoro algebra as a subalgebra, and the element L_0 defines a $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{n\in\mathbb{Z}}\mathfrak{g}_n$. We show that the degree zero parts of the $\mathbb{Z}$-gradings of the centerless $N = 2$ superconformal algebra, $S'(2,0)$ and $K'(4)$ are related to the (finite-dimensional) Lie superalgebras of operators on the de Rham complex in the case of a compact Riemannian, Kähler, and hyperKähler manifold, respectively.

It is tempting to consider superconformal algebras as affinizations of finite-dimensional Lie superalgebras of operators acting on the differential forms on a manifold. Recall that the Weil algebra is used for definition of the equivariant differential forms. The infinite-dimensional generalization of the classical Weil complex is the semi-infinite Weil complex introduced by B. Feigin and E. Frenkel ([FF]). Recall that this complex is based on the space

$$W^{\frac{\infty}{2}+*}(\mathfrak{g}) = S^{\frac{\infty}{2}+*}(\mathfrak{g}) \otimes \Lambda^{\frac{\infty}{2}+*}(\mathfrak{g}), \tag{0.4}$$

where $\mathfrak{g}$ is a $\mathbb{Z}$-graded Lie algebra, and $S^{\frac{\infty}{2}+*}(\mathfrak{g})$ and $\Lambda^{\frac{\infty}{2}+*}(\mathfrak{g})$ are some semi-infinite analogues of the symmetric and exterior power modules. As in the classical case, the analogues of the differential d for the Lie algebra cohomology and the Koszul differential are defined in the semi-infinite case. The cohomology of the complex $\{W^{\frac{\infty}{2}+*}(\mathfrak{g}), d\}$ is called *the semi-infinite cohomology* $H^{\frac{\infty}{2}+*}(\mathfrak{g}, S^{\frac{\infty}{2}+*}(\mathfrak{g}))$.

In [P1]-[P3] we showed that the semi-infinite cohomology of the loop algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes [t, t^{-1}]$ of a complex finite-dimensional Lie algebra $\mathfrak{g}$ admits an action of the $N = 2$ superconformal algebra. In the case, when $\mathfrak{g}$ is endowed with a non-degenerate invariant symmetric bilinear form, there exists a representaion of a central extension of the Lie superalgebra of derivations of $S'(2, 0)$ in the *relative* semi-infinite cohomology of $\tilde{\mathfrak{g}}$. It was observed by I. Frenkel, H. Garland and G. Zuckerman in [FGZ], that under certain conditions the relative semi-infinite cohomology has some structures, which are similar to that of the de Rham cohomology in Kähler geometry. We describe the analogues of the classical operators on the relative semi-infinite Weil complex of the loop algebra $\tilde{\mathfrak{g}}$. In this case the $\mathfrak{sl}(2)$ action (stated in the Lefschetz theorem) is defined by the exterior derivations of $S'(2, 0)$. On the other hand, the degree zero part of the $\mathbb{Z}$-grading of $S'(2, 0)$ gives an additional set of operators on the relative semi-infinite Weil complex of a loop algebra, which are analogous to the classical ones, and include the semi-infinite Koszul differential and the semi-infinite analogue of the homotopy operator.

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1. Superconformal algebras

Recall that a *superconformal algebra* is a complex $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g}$ such that

1) $\mathfrak{g}$ is simple,

2) $\mathfrak{g}$ contains the Witt algebra $Witt = der\mathbb{C}[t, t^{-1}] = \oplus_{n\in\mathbb{Z}}\mathbb{C}L_n$ with the well-known commutation relations

$$[L_n, L_m] = (n - m)L_{n+m} \tag{1.1}$$

as a subalgebra,

3) $\mathfrak{g}$ has growth 1, i.e. adL_0 is diagonalizable with finite-dimensional eigenspaces:

$$\mathfrak{g} = \oplus_j \mathfrak{g}_j, \mathfrak{g}_j = \{x \in \mathfrak{g} \mid [L_0, x] = jx\}, \tag{1.2}$$

so that $dim\mathfrak{g}_j < C$, where C is a constant independent of j, see [KL]. In general, a superconformal algebra is spanned by a number of series of generators; the Virasoro generators are amoung them. We will say that $\mathfrak{g}$ has $(m|n)$-series of generators, if it is spanned by m series of even elements and n series of odd elements.

The main series of superconformal algebras are $W(N)$ $(N \geq 0)$, $S'(N, \alpha)$ $(N \geq 2)$, and $K'(N)$, $(N \geq 1)$. The corresponding central extensions were classified in [KL]. Recall the necessary definitions.

The superalgebras $W(N)$. Consider the superalgebra $\mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$, where $\Lambda(N)$ is the Grassmann algebra in N variables $\theta_1, \ldots, \theta_N$. Let p be the parity of a homogeneous element. Let $p(t) = \bar{0}$, $p(\theta_i) = \bar{1}$, $i = 1, \ldots, N$. By definition $W(N)$ is the superalgebra of all derivations of $\mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$. Let ∂_i stand for $\partial/\partial\theta_i$, and let ∂_t stand for $\partial/\partial t$. Every $D \in W(N)$ is represented by a differential operator

$$D = f\partial_t + \sum_{i=1}^{N} f_i \partial_i, \tag{1.3}$$

where $f, f_i \in \mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$.

A non-trivial central extension of $W(1)$ is isomorphic to *the* $N = 2$ *superconformal algebra*. Recall that this superalgebra is spanned by the Virasoro generators $\mathfrak{L}_n$, Heisenberg generators H_n, two fermionic fields $G_r^{\pm}$, and a central element C, where $n \in \mathbb{Z}, r \in \mathbb{Z} + 1/2$, and where the nonvanishing commutation relations are as follows [FST]:

$$\begin{aligned}
&[\mathfrak{L}_n, \mathfrak{L}_m] = (n-m)\mathfrak{L}_{n+m} + \frac{\mathsf{C}}{12}(n^3 - n)\delta_{n,-m}, \\
&[\mathfrak{L}_n, H_m] = -mH_{n+m}, \\
&[\mathfrak{L}_n, G_r^{\pm}] = (\frac{n}{2} - r)G_{n+r}^{\pm}, \\
&[G_r^+, G_s^-] = 2\mathfrak{L}_{r+s} + (r-s)H_{r+s} + \frac{\mathsf{C}}{3}(r^2 - \frac{1}{4})\delta_{r,-s}, \\
&[H_n, H_m] = \frac{\mathsf{C}}{3} n\delta_{n,-m}, \\
&[H_n, G_r^{\pm}] = \pm G_{n+r}^{\pm}.
\end{aligned} \tag{1.4}$$

We can change the Virasoro generators by setting

$$L_n = -\mathfrak{L}_{-n} + \frac{1-n}{2} H_{-n}. \tag{1.5}$$

Then the element L_0 defines a $\mathbb{Z}$-grading on the $N=2$ superconformal algebras as requires the condition (1.2).

The superalgebras $S(N,\alpha)$. The superalgebra $W(N)$ contains a one-parameter family of superalgebras $S(N,\alpha)$. By definition

$$S(N,\alpha)=\{D\in W(N)\mid Div(t^{\alpha}D)=0\} \text{ for } \alpha\in\mathbb{C}. \tag{1.6}$$

Recall that

$$Div(f\partial_t+\sum_{i=1}^{N}f_i\partial_i)=\partial_t f+\sum_{i=1}^{N}(-1)^{p(f_i)}\partial_i f_i, \tag{1.7}$$

where $f, f_i\in\mathbb{C}[t,t^{-1}]\otimes\Lambda(N)$, and

$$Div(fD)=Df+fDivD, \tag{1.8}$$

where f is an even function. Let $S'(N,\alpha)=[S(N,\alpha),S(N,\alpha)]$ be the derived superalgebra. Assume that $N>1$. If $\alpha\notin\mathbb{Z}$, then $S(N,\alpha)$ is simple, and if $\alpha\in\mathbb{Z}$, then $S'(N,\alpha)$ is a simple ideal of $S(N,\alpha)$ of codimension 1:

$$0\to S'(N,\alpha)\to S(N,\alpha)\to \mathbb{C}t^{-\alpha}\theta_1\cdots\theta_N\partial_t\to 0. \tag{1.9}$$

Notice that

$$S(N,\alpha)\cong S(N,\alpha+n) \text{ for } n\in\mathbb{Z}. \tag{1.10}$$

The superalgebra $S(N,\alpha)$ has a non-trivial central extension if and only if $N=2$, see [KL]. Let

$$\{\mathfrak{L}_n^{\alpha}, E_n, H_n, F_n, \mathfrak{h}_n^{\alpha}, \mathfrak{p}_n, \mathfrak{x}_n, \mathfrak{y}_n^{\alpha}\}_{n\in\mathbb{Z}} \tag{1.11}$$

be a basis of $S'(2,\alpha)$ defined as follows:

$$\begin{aligned}
\mathfrak{L}_n^{\alpha}&=-t^n(t\partial_t+\frac{1}{2}(n+\alpha+1)(\theta_1\partial_1+\theta_2\partial_2)),\\
E_n&=t^n\theta_2\partial_1,\\
H_n&=t^n(\theta_2\partial_2-\theta_1\partial_1),\\
F_n&=t^n\theta_1\partial_2,\\
\mathfrak{h}_n^{\alpha}&=t^n\theta_2\partial_t-(n+\alpha)t^{n-1}\theta_1\theta_2\partial_1,\\
\mathfrak{p}_n&=-t^{n+1}\partial_2,\\
\mathfrak{x}_n&=t^{n+1}\partial_1,\\
\mathfrak{y}_n^{\alpha}&=t^n\theta_1\partial_t+(n+\alpha)t^{n-1}\theta_1\theta_2\partial_2.
\end{aligned} \tag{1.12}$$

The non-vanishing commutation relations between the above elements are

$$
\begin{aligned}
&[\mathcal{L}_n^\alpha, \mathcal{L}_k^\alpha] = (n-k)\mathcal{L}_{n+k}^\alpha, \qquad (1.13)\\
&[E_n, F_k] = H_{n+k}, [H_n, E_k] = 2E_{n+k}, [H_n, F_k] = -2F_{n+k},\\
&[\mathcal{L}_n^\alpha, E_k] = -kE_{n+k}, [\mathcal{L}_n^\alpha, H_k] = -kH_{n+k}, [\mathcal{L}_n^\alpha, F_k] = -kF_{n+k},\\
&[\mathcal{L}_n^\alpha, \mathfrak{h}_k^\alpha] = \frac{1}{2}(n-2k+1-\alpha)\mathfrak{h}_{n+k}^\alpha, [\mathcal{L}_n^\alpha, \mathfrak{p}_k] = \frac{1}{2}(n-2k-1+\alpha)\mathfrak{p}_{n+k},\\
&[\mathcal{L}_n^\alpha, \mathfrak{x}_k] = \frac{1}{2}(n-2k-1+\alpha)\mathfrak{x}_{n+k}, [\mathcal{L}_n^\alpha, \mathfrak{y}_k^\alpha] = \frac{1}{2}(n-2k+1-\alpha)\mathfrak{y}_{n+k}^\alpha,\\
&[E_n, \mathfrak{y}_k^\alpha] = \mathfrak{h}_{n+k}^\alpha, [F_n, \mathfrak{h}_k^\alpha] = \mathfrak{y}_{n+k}^\alpha,\\
&[E_n, \mathfrak{p}_k] = \mathfrak{x}_{n+k}, [F_n, \mathfrak{x}_k] = \mathfrak{p}_{n+k},\\
&[H_n, \mathfrak{h}_k^\alpha] = \mathfrak{h}_{n+k}^\alpha, [H_n, \mathfrak{y}_k^\alpha] = -\mathfrak{y}_{n+k}^\alpha,\\
&[H_n, \mathfrak{x}_k] = \mathfrak{x}_{n+k}, [H_n, \mathfrak{p}_k] = -\mathfrak{p}_{n+k},\\
&[\mathfrak{h}_n^\alpha, \mathfrak{x}_k] = (k+1-n-\alpha)E_{n+k}, [\mathfrak{p}_n, \mathfrak{y}_k^\alpha] = (k-n-1+\alpha)F_{n+k},\\
&[\mathfrak{h}_n^\alpha, \mathfrak{p}_k] = \mathcal{L}_{n+k}^\alpha - \frac{1}{2}(k-n+1-\alpha)H_{n+k}, [\mathfrak{x}_n, \mathfrak{y}_k^\alpha] = -\mathcal{L}_{n+k}^\alpha + \frac{1}{2}(k-n-1+\alpha)H_{n+k}.
\end{aligned}
$$

Let

$$\hat{S}'(2,\alpha) = S'(2,\alpha) \oplus \langle \mathsf{C} \rangle \qquad (1.14)$$

be a non-trivial central extension of $S'(2,\alpha)$. The corresponding 2-cocycle is

$$
\begin{aligned}
c(\mathcal{L}_n^\alpha, \mathcal{L}_k^\alpha) &= \frac{\mathsf{C}}{12}n(n^2-1)\delta_{n,-k}, \qquad (1.15)\\
c(E_n, F_k) &= \frac{\mathsf{C}}{6}n\delta_{n,-k},\\
c(H_n, H_k) &= \frac{\mathsf{C}}{3}n\delta_{n,-k},\\
c(\mathfrak{h}_n^\alpha, \mathfrak{p}_k) &= \frac{\mathsf{C}}{6}((n-1+\frac{\alpha+1}{2})^2 - \frac{1}{4})\delta_{n,-k},\\
c(\mathfrak{x}_n, \mathfrak{y}_k^\alpha) &= -\frac{\mathsf{C}}{6}((-n-1+\frac{\alpha+1}{2})^2 - \frac{1}{4})\delta_{n,-k},
\end{aligned}
$$

see [KL].

Remark 1.1. Notice that

$$S'(2,\alpha) = S'(2,\alpha)_{\bar{0}} \oplus S'(2,\alpha)_{\bar{1}}, \qquad (1.16)$$

where the even part is

$$
\begin{aligned}
&S'(2,\alpha)_{\bar{0}} = Witt \ltimes \tilde{\mathfrak{sl}}(2), \qquad (1.17)\\
&Witt = \langle \mathcal{L}_n^\alpha \rangle_{n\in\mathbb{Z}},\\
&\tilde{\mathfrak{sl}}(2) = \langle E_n, H_n, F_n \rangle_{n\in\mathbb{Z}},
\end{aligned}
$$

and the odd part

$$S'(2,\alpha)_{\bar{1}} = \langle \mathsf{h}^\alpha_n, \mathsf{y}^\alpha_n \rangle_{n\in\mathbb{Z}} \oplus \langle \mathsf{p}_n, \mathsf{x}_n \rangle_{n\in\mathbb{Z}} \tag{1.18}$$

is a direct sum of two standard (odd) $\tilde{\mathfrak{sl}}(2)$-modules.

Remark 1.2. One can think of the $N = 2$ superconformal algebra as a one-parameter family of isomorphic superalgebras related by the spectral flow transformation, see [FST]. This family can be included in the family $\hat{S}'(2,\alpha)$.

For any $\alpha \in \mathbb{C}$ one can consider the subalgebra of $\hat{S}'(2,\alpha)$, spanned by $\mathcal{L}^\alpha_n, H_n, \mathsf{h}^\alpha_n, \mathsf{p}_n$, and C. Thus we obtain a one-parameter family of isomorphic superalgebras. The isomorphism

$$\phi : \langle \mathcal{L}^\alpha_n, H_n, \mathsf{h}^\alpha_n, \mathsf{p}_n, \mathsf{C} \rangle \longrightarrow \langle \mathcal{L}^0_n, H_n, \mathsf{h}^0_n, \mathsf{p}_n, \mathsf{C} \rangle \tag{1.19}$$

is given as follows:

$$\begin{aligned}
&\phi(\mathcal{L}^\alpha_n) = \mathcal{L}^0_n - \frac{\alpha}{2} H_n + \frac{\alpha^2}{24}\delta_{n,0}\mathsf{C}, \\
&\phi(H_n) = H_n - \frac{\alpha}{6}\delta_{n,0}\mathsf{C}, \\
&\phi(\mathsf{h}^\alpha_n) = \mathsf{h}^0_n,\ \ \phi(\mathsf{p}_n) = \mathsf{p}_n, \phi(\mathsf{C}) = \mathsf{C}.
\end{aligned} \tag{1.20}$$

The superalgebras of this family are isomorphic to the $N = 2$ superconformal algebra defined in (1.4). For $\alpha = 0$ the isomorphism is given by

$$\mathcal{L}^0_n \to \mathcal{L}_n, H_n \to H_n, \mathsf{h}^0_n \to \frac{1}{\sqrt{2}} G^+_{n-\frac{1}{2}}, \mathsf{p}_n \to \frac{1}{\sqrt{2}} G^-_{n+\frac{1}{2}}, \mathsf{C} \to \mathsf{C}. \tag{1.21}$$

Let $DerS'(2,\alpha)$ be the Lie superalgebra of all derivations of $S'(2,\alpha)$, and $Der_{ext}S'(2,\alpha)$ be the Lie superalgebra of the exterior derivations of $S'(2,\alpha)$.

Proposition 1.1. cf. [P1], [P2].

1) If $\alpha \in \mathbb{Z}$, then $Der_{ext}S'(2,\alpha) \cong \mathfrak{sl}(2) = \langle \boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{F}} \rangle$, where

$$[\boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{E}}] = 2\boldsymbol{\mathcal{E}}, [\boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{F}}] = -2\boldsymbol{\mathcal{F}}, [\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{F}}] = \boldsymbol{\mathcal{H}}. \tag{1.22}$$

The action of $\mathfrak{sl}(2)$ on $S'(2,\alpha)_{\bar{0}}$ is trivial and on $S'(2,\alpha)_{\bar{1}}$ is given as follows:

$$\begin{aligned}
&[\boldsymbol{\mathcal{E}}, \mathsf{h}^\alpha_k] = \mathsf{x}_{k-1+\alpha}, [\boldsymbol{\mathcal{E}}, \mathsf{y}^\alpha_k] = \mathsf{p}_{k-1+\alpha}; \\
&[\boldsymbol{\mathcal{F}}, \mathsf{x}_k] = \mathsf{h}^\alpha_{k+1-\alpha}, [\boldsymbol{\mathcal{F}}, \mathsf{p}_k] = \mathsf{y}^\alpha_{k+1-\alpha}; \\
&[\boldsymbol{\mathcal{H}}, \mathsf{x}_k] = \mathsf{x}_k, [\boldsymbol{\mathcal{H}}, \mathsf{h}^\alpha_k] = -\mathsf{h}^\alpha_k, \\
&[\boldsymbol{\mathcal{H}}, \mathsf{p}_k] = \mathsf{p}_k, [\boldsymbol{\mathcal{H}}, \mathsf{y}^\alpha_k] = -\mathsf{y}^\alpha_k.
\end{aligned} \tag{1.23}$$

2) If $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, then $Der_{ext} S(2,\alpha) = \langle \boldsymbol{\mathcal{H}} \rangle$.

Proof. Recall that the exterior derivations of a Lie (super) algebra can be identified with its first cohomology with coefficients in the adjoint representation, see [F]. Thus

$$Der_{ext} S'(2,\alpha) \cong H^1(S'(2,\alpha), S'(2,\alpha)), \text{ where } \alpha \in \mathbb{C}. \tag{1.24}$$

Notice that the superalgebra $S'(2,\alpha)$ has the following $\mathbb{Z} \pm \alpha$-grading deg:

$$\begin{aligned} &deg \mathcal{L}_n^\alpha = n, deg E_n = n+1-\alpha, deg F_n = n-1+\alpha, deg H_n = n, \\ &deg \mathsf{h}_n^\alpha = n, deg \mathsf{p}_n = n, deg \mathsf{x}_n = n+1-\alpha, deg \mathsf{y}_n^\alpha = n-1+\alpha. \end{aligned} \tag{1.25}$$

Then, accordingly, there is the grading deg in $Der_{ext} S'(2,\alpha)$. Let

$$L_0^\alpha = -\mathcal{L}_0^\alpha + \frac{1}{2}(1-\alpha) H_0. \tag{1.26}$$

Notice that for a homogeneous element $s \in S'(2,\alpha)$

$$[L_0^\alpha, s] = (deg s) s. \tag{1.27}$$

Let us consider the natural action of $S'(2,\alpha)$ on $Der_{ext} S'(2,\alpha)$. Then for a homogeneous element $D \in Der_{ext} S'(2,\alpha)$

$$L_0^\alpha(D) = (deg D) D. \tag{1.28}$$

On the other hand, since the action of a Lie superalgebra on its cohomology is trivial, then for any $D \in H^1(S'(2,\alpha), S'(2,\alpha))$ one must have

$$L_0^\alpha(D) = 0. \tag{1.29}$$

Hence the nonzero elements of $Der_{ext} S'(2,\alpha)$ must have $deg = 0$.

Let $\alpha \in \mathbb{Z}$. Notice that the degree zero part of the grading deg of $S'(2,\alpha)$ is the Lie superalgebra $S'(2,\alpha)_{deg=0}$ spanned by the elements

$$\{\mathcal{L}_0^\alpha, E_{\alpha-1}, F_{1-\alpha}, H_0, \mathsf{h}_0^\alpha, \mathsf{p}_0, \mathsf{x}_{\alpha-1}, \mathsf{y}_{1-\alpha}^\alpha\}. \tag{1.30}$$

Since every $D \in Der_{ext} S'(2,\alpha)$ preserves $S'(2,\alpha)_{deg=0}$, then it defines a derivation of $S'(2,\alpha)_{deg=0}$. One can then check that the space of the exterior derivations of $S'(2,\alpha)_{deg=0}$ is spanned by the derivations $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{F}}$ acting on the basis (1.30) as follows:

$$\begin{aligned} &[\boldsymbol{\mathcal{E}}, \mathsf{h}_0^\alpha] = \mathsf{x}_{\alpha-1}, [\boldsymbol{\mathcal{E}}, \mathsf{y}_{1-\alpha}^\alpha] = \mathsf{p}_0; \\ &[\boldsymbol{\mathcal{F}}, \mathsf{x}_{\alpha-1}] = \mathsf{h}_0^\alpha, [\boldsymbol{\mathcal{F}}, \mathsf{p}_0] = \mathsf{y}_{1-\alpha}^\alpha; \\ &[\boldsymbol{\mathcal{H}}, \mathsf{x}_{\alpha-1}] = \mathsf{x}_{\alpha-1}, [\boldsymbol{\mathcal{H}}, \mathsf{h}_0^\alpha] = -\mathsf{h}_0^\alpha, \\ &[\boldsymbol{\mathcal{H}}, \mathsf{p}_0] = \mathsf{p}_0, [\boldsymbol{\mathcal{H}}, \mathsf{y}_{1-\alpha}^\alpha] = -\mathsf{y}_{1-\alpha}^\alpha. \end{aligned} \tag{1.31}$$

Then one can extend $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{F}}$ to exterior derivations of $S'(2,\alpha)$ as in (1.23). One should note that if the restriction of a derivation of $S'(2,\alpha)$ to $S'(2,\alpha)_{deg=0}$ is zero, then this derivation is inner. Finally, notice that $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{F}}$ interchange $\mathfrak{h}$'s with $\mathfrak{x}$'s and $\mathfrak{y}$'s with $\mathfrak{p}$'s. If $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, then $deg\mathfrak{h}_n^\alpha - deg\mathfrak{x}_k \notin \mathbb{Z}$ and $deg\mathfrak{y}_n^\alpha - deg\mathfrak{p}_k \notin \mathbb{Z}$ for any $n, k \in \mathbb{Z}$. Thus we cannot obtain $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{F}}$ of $deg = 0$ in this case. By this reason, $Der_{ext}S(2,\alpha) = \langle \boldsymbol{\mathcal{H}} \rangle$. $\square$

Remark 1.3. Let $\alpha \in \mathbb{Z}$. Then one can identify

$$\boldsymbol{\mathcal{F}} \leftrightarrow -t^{-\alpha}\theta_1\theta_2\partial_t, \tag{1.32}$$

see (1.9). However we cannot interpret $\boldsymbol{\mathcal{E}}$ as a regular differential operator on the circle or its superanalogue. To find a realization for $\mathfrak{sl}(2) = \langle \boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{F}} \rangle$, we have to consider a more general superalgebra.

2. The superalgebra of pseudodifferential symbols

Recall that the ring R of pseudodifferential symbols is the ring of the formal series

$$A(t,\xi) = \Sigma_{-\infty}^{n} a_i(t)\xi^i, \tag{2.1}$$

where $a_i(t) \in \mathbb{C}[t, t^{-1}]$, and the variable ξ corresponds to $\partial/\partial t$, see [KK]. The multiplication rule in R is determined as follows:

$$A(t,\xi) \circ B(t,\xi) = \Sigma_{n\geq 0} \frac{1}{n!} \partial_\xi^n A(t,\xi) \partial_t^n B(t,\xi). \tag{2.2}$$

Notice that R is a generalization of the associative algebra of the regular differential operators on the circle, and the multiplication rule in R, when restricted to the polynomials in ξ, coincides with the multiplication rule for the differential operators. The Lie algebra structure on R is given by

$$[A, B] = A \circ B - B \circ A, \tag{2.3}$$

where $A, B \in R$.

To construct a superanalogue of R, consider a superalgebra $\Theta(N)$ with generators $\theta_1, \ldots, \theta_N, \partial_1, \ldots, \partial_N$ and relations

$$\begin{aligned} \theta_i\theta_j &= -\theta_j\theta_i, \\ \partial_i\partial_j &= -\partial_j\partial_i, \\ \partial_i\theta_j &= \delta_{i,j} - \theta_j\partial_i. \end{aligned} \tag{2.4}$$

Define an associative superalgebra

$$R(N) = R \otimes \Theta(N). \tag{2.5}$$

The product in $R(N)$ determines the natural Lie superalgebra structure on this space:

$$[A \otimes X, B \otimes Y] = (A \circ B) \otimes (XY) - (-1)^{p(X)p(Y)}(B \circ A) \otimes (YX), \tag{2.6}$$

where $A, B \in R$, and $X, Y \in \Theta(N)$. There exists an inclusion

$$W(N) \subset R(N), \tag{2.7}$$

so that the commutation relations in $R(N)$, when restricted to $W(N)$, coincide with commutation relations in $W(N)$. Recall that the Lie algebra R has two independent central extensions, see [KZ]. Analogously, one can define two 2-cocycles φ_ξ and φ_t on $R(N)$. Let $A, B \in R$, and $X, Y \in \Theta(N)$. Then

$$\begin{aligned}\varphi_\xi(A \otimes X, B \otimes Y) = {} & \text{ the coefficient of } t^{-1}\xi^{-1}\theta_1 \dots \theta_N \partial_1 \dots \partial_N \\ & \text{in } ([\log \xi, A] \circ B) \otimes (XY),\end{aligned} \tag{2.8}$$

where

$$[\log \xi, A(t, \xi)] = \Sigma_{k \geq 1} \frac{(-1)^{k+1}}{k} \partial_t^k A(t, \xi) \xi^{-k}, \tag{2.9}$$

cf. [KK], [KZ], and

$$\begin{aligned}\varphi_t(A \otimes X, B \otimes Y) = {} & \text{ the coefficient of } t^{-1}\xi^{-1}\theta_1 \dots \theta_N \partial_1 \dots \partial_N \\ & \text{in } ([\log t, A] \circ B) \otimes (XY),\end{aligned} \tag{2.10}$$

where

$$[\log t, A(t, \xi)] = \Sigma_{k \geq 1} \frac{(-1)^{k+1}}{k} t^{-k} \partial_\xi^k A(t, \xi). \tag{2.11}$$

Proposition 2.1. The exterior derivations $Der_{ext}S'(2, \alpha)$ for all $\alpha \in \mathbb{Z}$ form the loop algebra

$$\tilde{\mathcal{SL}}(2) \subset R(2). \tag{2.12}$$

Proof. Let $\alpha = 1$. Then $\mathcal{SL}(2) = \langle \boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{E}} \rangle$ defined by (1.23) is isomorphic to a subalgebra of $R(2)$. In fact, set

$$\begin{aligned}\boldsymbol{\mathcal{F}} &= -t^{-1}\xi\theta_1\theta_2, \\ \boldsymbol{\mathcal{H}} &= 1 - \theta_1\partial_1 - \theta_2\partial_2, \\ \boldsymbol{\mathcal{E}} &= (t\xi^{-1} - \xi^{-2})\partial_1\partial_2.\end{aligned} \tag{2.13}$$

One can then construct the loop algebra of $\mathcal{SL}(2)$ as follows:

$$\tilde{\mathcal{SL}}(2) = \langle \boldsymbol{\mathcal{F}}_n, \boldsymbol{\mathcal{H}}_n, \boldsymbol{\mathcal{E}}_n \rangle_{n \in \mathbb{Z}}, \tag{2.14}$$

where

$$\begin{aligned} \boldsymbol{\mathcal{F}}_n &= -t^{n-1}\xi\theta_1\theta_2, \\ \boldsymbol{\mathcal{H}}_n &= (\xi^{-1} \circ t^n \xi)(1 - \theta_1\partial_1 - \theta_2\partial_2 - \theta_1\theta_2\partial_1\partial_2) + t^n\theta_1\theta_2\partial_1\partial_2, \\ \boldsymbol{\mathcal{E}}_n &= (\xi^{-1} \circ t^{n+1})\partial_1\partial_2. \end{aligned} \tag{2.15}$$

The non-vanishing commutation relations are

$$\begin{aligned} [\boldsymbol{\mathcal{H}}_n, \boldsymbol{\mathcal{E}}_k] &= 2\boldsymbol{\mathcal{E}}_{n+k}, \\ [\boldsymbol{\mathcal{H}}_n, \boldsymbol{\mathcal{F}}_k] &= -2\boldsymbol{\mathcal{F}}_{n+k}, \\ [\boldsymbol{\mathcal{E}}_n, \boldsymbol{\mathcal{F}}_k] &= \boldsymbol{\mathcal{H}}_{n+k}. \end{aligned} \tag{2.16}$$

Let $\alpha \in \mathbb{Z}$. Then

$$Der_{ext}S'(2, \alpha) \cong \langle \boldsymbol{\mathcal{F}}_{-\alpha+1}, \boldsymbol{\mathcal{H}}_0, \boldsymbol{\mathcal{E}}_{\alpha-1} \rangle. \tag{2.17}$$

□

Remark 2.1. Let $g(t) \in t^\beta\mathbb{C}[t, t^{-1}]$, where $\beta \in \mathbb{R} \setminus \mathbb{Z}$. One can think of ξ^{-1} as the anti-derivative

$$\xi^{-1}g(t) = \int g(t)dt. \tag{2.18}$$

Let $f(t) \in \mathbb{C}[t, t^{-1}]$. According to (2.2)

$$\xi^{-1} \circ f = \sum_{n=0}^{\infty} (-1)^n(\xi^n f)\xi^{-n-1}. \tag{2.19}$$

Notice that this formula, when applied to a function g, corresponds to the integration by parts.

Proposition 2.2. The superalgebras $S'(2, \alpha)$, where $\alpha \in \mathbb{Z}$, generate $W(2)$.

Proof. Recall that $W(2)$ is spanned by 6 series of even generators and 6 series of odd generators. Define

$$\begin{aligned} \mathfrak{r}_n &= t^{n-1}\theta_1\theta_2\partial_1, \\ \mathfrak{s}_n &= t^{n-1}\theta_1\theta_2\partial_2, \\ I_n &= t^n(\theta_1\partial_1 + \theta_2\partial_2). \end{aligned} \tag{2.20}$$

Notice that

$$\mathcal{L}_n^\alpha = \mathcal{L}_n^0 - \frac{1}{2}\alpha I_n, \tag{2.21}$$
$$\mathfrak{h}_n^\alpha = \mathfrak{h}_n^0 - \alpha \mathfrak{r}_n,$$
$$\mathfrak{y}_n^\alpha = \mathfrak{y}_n^0 + \alpha \mathfrak{s}_n.$$

Let us denote

$$\mathcal{L}_n = \mathcal{L}_n^0, \mathfrak{h}_n = \mathfrak{h}_n^0, \mathfrak{y}_n = \mathfrak{y}_n^0. \tag{2.22}$$

Recall that the following 12 series of elements are contained in $W(2)$:

$$\{\mathcal{L}_n, E_n, H_n, F_n, I_n, \boldsymbol{\mathcal{F}}_n, \mathfrak{h}_n, \mathfrak{p}_n, \mathfrak{x}_n, \mathfrak{y}_n, \mathfrak{r}_n, \mathfrak{s}_n\}. \tag{2.23}$$

It is easy to see that they span $W(2)$. According to (1.13) the non-vanishing commutation relations between $\mathcal{L}_n, E_n, H_n, F_n, \mathfrak{h}_n, \mathfrak{p}_n, \mathfrak{x}_n, \mathfrak{y}_n$ are

$$\begin{aligned}
&[\mathcal{L}_n, \mathcal{L}_k] = (n-k)\mathcal{L}_{n+k}, &(2.24)\\
&[E_n, F_k] = H_{n+k}, [H_n, E_k] = 2E_{n+k}, [H_n, F_k] = -2F_{n+k},\\
&[\mathcal{L}_n, E_k] = -kE_{n+k}, [\mathcal{L}_n, H_k] = -kH_{n+k}, [\mathcal{L}_n, F_k] = -kF_{n+k},\\
&[\mathcal{L}_n, \mathfrak{h}_k] = \frac{1}{2}(n-2k+1)\mathfrak{h}_{n+k}, [\mathcal{L}_n, \mathfrak{p}_k] = \frac{1}{2}(n-2k-1)\mathfrak{p}_{n+k},\\
&[\mathcal{L}_n, \mathfrak{x}_k] = \frac{1}{2}(n-2k-1)\mathfrak{x}_{n+k}, [\mathcal{L}_n, \mathfrak{y}_k] = \frac{1}{2}(n-2k+1)\mathfrak{y}_{n+k},\\
&[E_n, \mathfrak{y}_k] = \mathfrak{h}_{n+k}, [F_n, \mathfrak{h}_k] = \mathfrak{y}_{n+k},\\
&[E_n, \mathfrak{p}_k] = \mathfrak{x}_{n+k}, [F_n, \mathfrak{x}_k] = \mathfrak{p}_{n+k},\\
&[H_n, \mathfrak{h}_k] = \mathfrak{h}_{n+k}, [H_n, \mathfrak{y}_k] = -\mathfrak{y}_{n+k},\\
&[H_n, \mathfrak{x}_k] = \mathfrak{x}_{n+k}, [H_n, \mathfrak{p}_k] = -\mathfrak{p}_{n+k},\\
&[\mathfrak{h}_n, \mathfrak{x}_k] = (k+1-n)E_{n+k}, [\mathfrak{p}_n, \mathfrak{y}_k] = (k-n-1)F_{n+k},\\
&[\mathfrak{h}_n, \mathfrak{p}_k] = \mathcal{L}_{n+k} - \frac{1}{2}(k-n+1)H_{n+k}, [\mathfrak{x}_n, \mathfrak{y}_k] = -\mathcal{L}_{n+k} + \frac{1}{2}(k-n-1)H_{n+k}.
\end{aligned}$$

The additional non-vanishing commutation relations between the elements (2.23), i.e. those that involve $I_n, \boldsymbol{\mathcal{F}}_n, \mathfrak{r}_n$, or $\mathfrak{s}_n$, are:

$$\begin{aligned}
&[\mathcal{L}_n, \mathfrak{r}_k] = -\frac{1}{2}(n+2k-1)\mathfrak{r}_{n+k}, [\mathcal{L}_n, \mathfrak{s}_k] = -\frac{1}{2}(n+2k-1)\mathfrak{s}_{n+k}, &(2.25)\\
&[\mathcal{L}_n, I_k] = -kI_{n+k}, [\mathcal{L}_n, \boldsymbol{\mathcal{F}}_k] = (1-k)\boldsymbol{\mathcal{F}}_{n+k},\\
&[E_n, \mathfrak{s}_k] = -\mathfrak{r}_{n+k}, [H_n, \mathfrak{r}_k] = \mathfrak{r}_{n+k}, [H_n, \mathfrak{s}_k] = -\mathfrak{s}_{n+k},
\end{aligned}$$

$$[F_n, \mathfrak{r}_k] = -\mathfrak{s}_{n+k}, [\mathfrak{h}_n, \mathfrak{s}_k] = -\boldsymbol{\mathcal{F}}_{n+k}, [\mathfrak{h}_n, I_k] = -\mathfrak{h}_{n+k} - 2k\mathfrak{r}_{n+k},$$
$$[\mathfrak{p}_n, \mathfrak{r}_k] = -\frac{1}{2}H_{n+k} + \frac{1}{2}I_{n+k}, [\mathfrak{p}_n, \mathfrak{s}_k] = F_{n+k}, [\mathfrak{p}_n, I_k] = \mathfrak{p}_{n+k},$$
$$[\boldsymbol{\mathcal{F}}_k, \mathfrak{p}_n] = \mathfrak{y}_{n+k} + (1-k)\mathfrak{s}_{n+k}, [\mathfrak{x}_n, \mathfrak{r}_k] = E_{n+k}, [\mathfrak{x}_n, \mathfrak{s}_k] = \frac{1}{2}H_{n+k} + \frac{1}{2}I_{n+k},$$
$$[I_k, \mathfrak{x}_n] = -\mathfrak{x}_{n+k}, [\boldsymbol{\mathcal{F}}_k, \mathfrak{x}_n] = \mathfrak{h}_{n+k} + (k-1)\mathfrak{r}_{n+k},$$
$$[\mathfrak{y}_n, \mathfrak{r}_k] = -\boldsymbol{\mathcal{F}}_{n+k}, [I_k, \mathfrak{y}_n] = \mathfrak{y}_{n+k} - 2k\mathfrak{s}_{n+k},$$
$$[I_n, \mathfrak{r}_k] = \mathfrak{r}_{n+k}, [I_n, \mathfrak{s}_k] = \mathfrak{s}_{n+k}, [I_n, \boldsymbol{\mathcal{F}}_k] = 2\boldsymbol{\mathcal{F}}_{n+k}.$$

□

In the next section we will prove that the family $S'(2,\alpha)$, where $\alpha \in \mathbb{Z}$, together with $\tilde{\mathfrak{SL}}(2)$ generate a contact Lie superalgebra.

3. The superalgebra $K(4)$

The superalgebras $K(N)$. By definition

$$K(N) = \{D \in W(N) \mid D\omega = f\omega \text{ for some } f \in \mathbb{C}[t, t^{-1}] \otimes \Lambda(N)\}, \tag{3.1}$$

where

$$\omega = dt - \sum_{i=1}^{N} \theta_i d\theta_i \tag{3.2}$$

is a contact 1-form, see [KL]. Every differential operator $D \in K(N)$ can be represented by a single function

$$f \in \mathbb{C}[t, t^{-1}] \otimes \Lambda(N) : f \to D_f. \tag{3.3}$$

Let

$$\Delta(f) = 2f - \sum_{i=1}^{N} \theta_i \partial_i(f). \tag{3.4}$$

Then

$$D_f = \Delta(f)\partial_t + \partial_t(f)\sum_{i=1}^{N} \theta_i\partial_i + (-1)^{p(f)}\sum_{i=1}^{N} \partial_i(f)\partial_i. \tag{3.5}$$

Notice that

$$D_{f+g} = D_f + D_g, \tag{3.6}$$
$$[D_f, D_g] = D_{\{f,g\}},$$

where

$$\{f,g\} = \Delta(f)\partial_t(g) - \partial_t(f)\Delta(g) + (-1)^{p(f)} \sum_{i=1}^{N} \partial_i(f)\partial_i(g), \tag{3.7}$$

see [KL].

The superalgebras $K(N)$ are simple, except when $N=4$. If $N=4$, then the derived superalgebra $K'(4) = [K(4), K(4)]$ is a simple ideal in $K(4)$ of codimension one:

$$0 \to K'(4) \to K(4) \to \mathbb{C}D_{t^{-1}\theta_1\theta_2\theta_3\theta_4} \to 0. \tag{3.8}$$

Proposition 3.1. The superalgebras $S'(2,\alpha)$, where $\alpha \in \mathbb{Z}$, and $\tilde{\mathfrak{SL}}(2)$ generate a Lie superalgebra isomorphic to a central extension of $K'(4)$.

Proof. According to Proposition 2.2, the family $S'(2,\alpha)$ generates $W(2)$, which has $(6|6)$-series of elements. If we also consider $\tilde{\mathfrak{SL}}(2)$, then we include two additional series of even elements, $\boldsymbol{\mathcal{E}}_n$ and $\boldsymbol{\mathcal{H}}_n$, into the picture. From the commutation relations we also obtain two additional series of odd elements:

$$\begin{aligned} \mathfrak{q}_n &= (\xi^{-1} \circ t^n)(\partial_1 + \theta_2\partial_1\partial_2), \\ \mathfrak{t}_n &= (\xi^{-1} \circ t^n)(\partial_2 - \theta_1\partial_1\partial_2). \end{aligned} \tag{3.9}$$

Let $\mathfrak{g} \subset R(2)$ be the superalgebra spanned by the $(8|8)$-series of elements

$$\{\mathfrak{L}_n, E_n, H_n, F_n, I_n, \boldsymbol{\mathcal{F}}_n, \boldsymbol{\mathcal{E}}_n, \boldsymbol{\mathcal{H}}_n, \mathfrak{h}_n, \mathfrak{p}_n, \mathfrak{x}_n, \mathfrak{y}_n, \mathfrak{r}_n, \mathfrak{s}_n, \mathfrak{q}_n, \mathfrak{t}_n\}. \tag{3.10}$$

To obtain the commutation relations for $\mathfrak{g}$ one can make use of its representation in the space

$$t^\beta\mathbb{C}[t,t^{-1}] \otimes \Lambda(2) = t^\beta\mathbb{C}[t,t^{-1}] \otimes \langle 1, \theta_1, \theta_2, \theta_1\theta_2\rangle, \text{ where } \beta \in \mathbb{R} \setminus \mathbb{Z}, \tag{3.11}$$

see Remark 2.1. We obtain that the commutation relations for $\mathfrak{g}$ are that for $W(2)$ and for $\tilde{\mathfrak{SL}}(2)$, see (2.23), (2.24), and the additional non-vanishing commutation relations, i.e. those that involve $\boldsymbol{\mathcal{E}}_n, \boldsymbol{\mathcal{H}}_n, \mathfrak{q}_n$ and $\mathfrak{t}_n$, are:

$$\begin{aligned} &[\mathfrak{L}_n, \mathfrak{q}_k] = -\frac{1}{2}(n+2k+1)\mathfrak{q}_{n+k}, [\mathfrak{L}_n, \mathfrak{t}_k] = -\frac{1}{2}(n+2k+1)\mathfrak{t}_{n+k}, \\ &[\mathfrak{L}_n, \boldsymbol{\mathcal{E}}_k] = -(k+1)\boldsymbol{\mathcal{E}}_{n+k}, [\mathfrak{L}_n, \boldsymbol{\mathcal{H}}_k] = -k\boldsymbol{\mathcal{H}}_{n+k}, \\ &[E_n, \mathfrak{t}_k] = -\mathfrak{q}_{n+k}, [H_n, \mathfrak{q}_k] = \mathfrak{q}_{n+k}, [H_n, \mathfrak{t}_k] = -\mathfrak{t}_{n+k}, [F_n, \mathfrak{q}_k] = -\mathfrak{t}_{n+k}, \end{aligned} \tag{3.12}$$

$$[\mathfrak{h}_n, \mathfrak{q}_k] = E_{n+k}, [\mathfrak{h}_n, \mathfrak{t}_k] = \boldsymbol{\mathcal{H}}_{n+k} + \frac{1}{2}I_{n+k} + \frac{1}{2}H_{n+k},$$
$$[\boldsymbol{\mathcal{E}}_k, \mathfrak{h}_n] = \mathfrak{x}_{n+k} - (k+1)\mathfrak{q}_{n+k}, [\boldsymbol{\mathcal{H}}_k, \mathfrak{h}_n,] = -\mathfrak{h}_{n+k} - k\mathfrak{r}_{n+k},$$
$$[\mathfrak{p}_n, \mathfrak{q}_k] = -\boldsymbol{\mathcal{E}}_{n+k}, [\boldsymbol{\mathcal{H}}_k, \mathfrak{p}_n] = \mathfrak{p}_{n+k} + k\mathfrak{t}_{n+k}, [\mathfrak{x}_n, \mathfrak{t}_k] = -\boldsymbol{\mathcal{E}}_{n+k},$$
$$[\boldsymbol{\mathcal{H}}_k, \mathfrak{x}_n] = \mathfrak{x}_{n+k} - k\mathfrak{q}_{n+k}, [\mathfrak{y}_n, \mathfrak{q}_k] = \boldsymbol{\mathcal{H}}_{n+k} + \frac{1}{2}I_{n+k} - \frac{1}{2}H_{n+k}, [\mathfrak{y}_n, \mathfrak{t}_k] = F_{n+k},$$
$$[\boldsymbol{\mathcal{E}}_k, \mathfrak{y}_n] = \mathfrak{p}_{n+k} + (k+1)\mathfrak{t}_{n+k}, [\boldsymbol{\mathcal{H}}_k, \mathfrak{y}_n] = -\mathfrak{y}_{n+k} + k\mathfrak{s}_{n+k},$$
$$[\boldsymbol{\mathcal{E}}_k, \mathfrak{r}_n] = -\mathfrak{q}_{n+k}, [\boldsymbol{\mathcal{H}}_k, \mathfrak{r}_n] = -\mathfrak{r}_{n+k},$$
$$[I_n, \mathfrak{q}_k] = -\mathfrak{q}_{n+k}, [I_n, \mathfrak{t}_k] = -\mathfrak{t}_{n+k}, [I_n, \boldsymbol{\mathcal{E}}_k] = -2\boldsymbol{\mathcal{E}}_{n+k},$$
$$[\boldsymbol{\mathcal{E}}_k, \mathfrak{s}_n] = -\mathfrak{t}_{n+k}, [\boldsymbol{\mathcal{H}}_k, \mathfrak{s}_n] = -\mathfrak{s}_{n+k}.$$
$$[\boldsymbol{\mathcal{F}}_n, \mathfrak{q}_k] = -\mathfrak{r}_{n+k}, [\boldsymbol{\mathcal{F}}_n, \mathfrak{t}_k] = -\mathfrak{s}_{n+k},$$
$$[\boldsymbol{\mathcal{H}}_n, \mathfrak{q}_k] = \mathfrak{q}_{n+k}, [\boldsymbol{\mathcal{H}}_n, \mathfrak{t}_k] = \mathfrak{t}_{n+k},$$
$$[\boldsymbol{\mathcal{H}}_n, \boldsymbol{\mathcal{E}}_k] = 2\boldsymbol{\mathcal{E}}_{n+k}, [\boldsymbol{\mathcal{H}}_n, \boldsymbol{\mathcal{F}}_k] = -2\boldsymbol{\mathcal{F}}_{n+k}, [\boldsymbol{\mathcal{E}}_n, \boldsymbol{\mathcal{F}}_k] = \boldsymbol{\mathcal{H}}_{n+k},$$

Let

$$\begin{aligned} \mathfrak{L}'_n &= \mathfrak{L}_n + \boldsymbol{\mathcal{H}}_n + \frac{1}{2}I_n, \\ I'_n &= I_n + \boldsymbol{\mathcal{H}}_n, \\ \mathfrak{p}'_n &= \mathfrak{p}_n + \mathfrak{t}_n, \\ \mathfrak{x}'_n &= \mathfrak{x}_n - \mathfrak{q}_n. \end{aligned} \tag{3.13}$$

Notice that I'_0 is a central element in $\mathfrak{g}$, and $I'_0 = \imath$, where $\imath \in R(2)$ is the identity element.

We will explicitly describe the isomorphism between certain central extension $\hat{K}'(4)$ and $\mathfrak{g}$:

$$\phi : \hat{K}'(4) \longrightarrow \mathfrak{g}. \tag{3.14}$$

Let

$$\check{\theta}_1 = \theta_2\theta_3\theta_4, \check{\theta}_2 = \theta_1\theta_3\theta_4, \check{\theta}_3 = \theta_1\theta_2\theta_4, \check{\theta}_4 = \theta_1\theta_2\theta_3. \tag{3.15}$$

The following 16 series of functions span $\mathbb{C}[t, t^{-1}] \otimes \Lambda(4)$:

$$\begin{aligned} f_n^1 &= -\frac{1}{2}t^{n+1} + \frac{1}{2}it^n(\theta_2\theta_3 - \theta_1\theta_4) - \frac{1}{2}n(n+1)t^{n-1}\theta_1\theta_2\theta_3\theta_4, \\ f_n^2 &= \frac{1}{2}t^{n-1}(\theta_1\theta_2 - \theta_3\theta_4 - i\theta_1\theta_3 - i\theta_2\theta_4), \\ f_n^3 &= \frac{1}{2}t^{n+1}(-\theta_1\theta_2 + \theta_3\theta_4 - i\theta_1\theta_3 - i\theta_2\theta_4), \\ f_n^4 &= it^n(\theta_1\theta_4 - \theta_2\theta_3), \end{aligned} \tag{3.16}$$

$$
\begin{aligned}
f_n^5 &= \frac{i}{\sqrt{8}}\big(t^n(\theta_1 - i\theta_2 - \theta_3 + i\theta_4) - nt^{n-1}(\check{\theta}_1 + i\check{\theta}_2 - \check{\theta}_3 - i\check{\theta}_4)\big),\\
f_n^6 &= \frac{1}{\sqrt{8}}\big(t^n(\theta_1 + i\theta_2 + \theta_3 + i\theta_4) - nt^{n-1}(\check{\theta}_1 - i\check{\theta}_2 + \check{\theta}_3 - i\check{\theta}_4)\big),\\
f_n^7 &= \frac{i}{\sqrt{8}}\big(t^{n+1}(\theta_1 + i\theta_2 - \theta_3 - i\theta_4) - (n+1)t^n(\check{\theta}_1 - i\check{\theta}_2 - \check{\theta}_3 + i\check{\theta}_4)\big),\\
f_n^8 &= \frac{1}{\sqrt{8}}\big(t^{n+1}(\theta_1 - i\theta_2 + \theta_3 - i\theta_4) - (n+1)t^n(\check{\theta}_1 + i\check{\theta}_2 + \check{\theta}_3 + i\check{\theta}_4)\big),\\
f_n^9 &= \frac{-i}{\sqrt{2}}t^{n-1}(\check{\theta}_1 + i\check{\theta}_2 - \check{\theta}_3 - i\check{\theta}_4),\\
f_n^{10} &= \frac{1}{\sqrt{2}}t^{n-1}(\check{\theta}_1 - i\check{\theta}_2 + \check{\theta}_3 - i\check{\theta}_4),\\
f_n^{11} &= \frac{-i}{\sqrt{2}}t^{n}(\check{\theta}_1 - i\check{\theta}_2 - \check{\theta}_3 + i\check{\theta}_4),\\
f_n^{12} &= \frac{1}{\sqrt{2}}t^{n}(\check{\theta}_1 + i\check{\theta}_2 + \check{\theta}_3 + i\check{\theta}_4),\\
f_n^{13} &= \frac{1}{2}t^n(-\theta_1\theta_4 - \theta_2\theta_3 + i\theta_2\theta_4 - i\theta_1\theta_3),\\
f_n^{14} &= \frac{1}{2}t^n(\theta_1\theta_4 + \theta_2\theta_3 - i\theta_1\theta_3 + i\theta_2\theta_4),\\
f_n^{15} &= -it^n(\theta_1\theta_2 + \theta_3\theta_4),\\
f_n^{16} &= 2nt^{n-1}\theta_1\theta_2\theta_3\theta_4.
\end{aligned}
$$

The 16 series of the corresponding differential operators $\{D_{f_n^i}\}_{i=1,\dots,16}$ span $K'(4)$. Notice that $f_n^{16} = 0$, if $n = 0$. This corresponds to the fact that $D_{t^{-1}\theta_1\theta_2\theta_3\theta_4} \notin K'(4)$. Let

$$
\begin{aligned}
&\phi(D_{f_n^1}) = \mathfrak{L}'_n, \phi(D_{f_n^2}) = E_n, \phi(D_{f_n^3}) = F_n, \phi(D_{f_n^4}) = H_n, \qquad (3.17)\\
&\phi(D_{f_n^5}) = \mathfrak{x}'_n, \phi(D_{f_n^6}) = \mathfrak{h}_n, \phi(D_{f_n^7}) = \mathfrak{y}_n, \phi(D_{f_n^8}) = \mathfrak{p}'_n,\\
&\phi(D_{f_n^9}) = \mathfrak{q}_n, \phi(D_{f_n^{10}}) = \mathfrak{r}_n, \phi(D_{f_n^{11}}) = \mathfrak{s}_n, \phi(D_{f_n^{12}}) = \mathfrak{t}_n,\\
&\phi(D_{f_n^{13}}) = \boldsymbol{\mathcal{E}}_n, \phi(D_{f_n^{14}}) = \boldsymbol{\mathcal{F}}_n, \phi(D_{f_n^{15}}) = \boldsymbol{\mathcal{H}}_n, \phi(D_{f_n^{16}}) = I'_n.
\end{aligned}
$$

One can verify that ϕ is an isomorphism from $\hat{K}'(4)$ onto $\mathfrak{g}$, where the central element of $\mathfrak{g}$ is $I'_0 = \imath$ and the corresponding 2-cocycle is

$$
\begin{aligned}
\varphi(\mathfrak{p}'_n, \mathfrak{r}_k) &= \frac{1}{2}\delta_{n,-k}, \qquad (3.18)\\
\varphi(\mathfrak{x}'_n, \mathfrak{s}_k) &= \frac{1}{2}\delta_{n,-k},\\
\varphi(\mathfrak{h}_n, \mathfrak{t}_k) &= \frac{1}{2}\delta_{n,-k},
\end{aligned}
$$

$$\varphi(\mathfrak{y}_n, \mathfrak{q}_k) = \frac{1}{2}\delta_{n,-k},$$
$$\varphi(\mathcal{L}'_n, I'_k) = n\delta_{n,-k}.$$

□

Corollary 3.1. We have obtained an inclusion of Lie superalgebras, which have $(2|2)$, $(4|4)$, and $(8|8)$-series of generators, correspondingly:

$$\text{the centerless } N=2 \text{ SCA } \subset S'(2,0) \subset K'(4). \tag{3.19}$$

Remark 3.1. Recall that the superalgebra $K'(4)$ has 3 independent central extensions ([KL]). One of them is given by (3.18).

The restriction of the cocycle φ_ξ to $K'(4)$ gives another 2-cocycle:

$$\begin{aligned}
&\varphi_\xi(\mathcal{L}'_n, \mathcal{L}'_k) = (\frac{1}{2}n^3 - 2n)\delta_{n,-k}, \\
&\varphi_\xi(E_n, F_k) = n\delta_{n,-k}, \\
&\varphi_\xi(H_n, H_k) = 2n\delta_{n,-k}, \\
&\varphi_\xi(\boldsymbol{\mathcal{E}}_n, \boldsymbol{\mathcal{F}}_k) = -(n+1)\delta_{n,-k}, \\
&\varphi_\xi(\boldsymbol{\mathcal{H}}_n, \boldsymbol{\mathcal{H}}_k) = -2n\delta_{n,-k}, \\
&\varphi_\xi(I'_n, I'_k) = -4n\delta_{n,-k}, \\
&\varphi_\xi(\mathcal{L}'_n, I'_k) = n(n-2)\delta_{n,-k}, \\
&\varphi_\xi(\mathcal{L}'_n, \boldsymbol{\mathcal{H}}_k) = -2n\delta_{n,-k}, \\
&\varphi_\xi(\boldsymbol{\mathcal{H}}_n, I'_k) = -2n\delta_{n,-k}, \\
&\varphi_\xi(\mathfrak{h}_n, \mathfrak{p}_k) = n(n-1)\delta_{n,-k}, \\
&\varphi_\xi(\mathfrak{h}_n, \mathfrak{t}_k) = n\delta_{n,-k}, \\
&\varphi_\xi(\mathfrak{x}_n, \mathfrak{y}_k) = -n(n+1)\delta_{n,-k}, \\
&\varphi_\xi(\mathfrak{x}_n, \mathfrak{s}_k) = (n+1)\delta_{n,-k}, \\
&\varphi_\xi(\mathfrak{p}_n, \mathfrak{r}_k) = (n+1)\delta_{n,-k}, \\
&\varphi_\xi(\mathfrak{y}_n, \mathfrak{q}_k) = n\delta_{n,-k}.
\end{aligned} \tag{3.20}$$

Notice that the restriction of φ_ξ to $S'(2,\alpha)$ defines the 2-cocycle (1.15). Finally, the restriction of the cocycle φ_t gives the third 2-cocycle for $K'(4)$:

$$\varphi_t(\mathcal{L}'_n, \mathcal{L}'_k) = 2n\delta_{n,-k}, \tag{3.21}$$

$$\varphi_t(\mathcal{L}'_n, \boldsymbol{\mathcal{H}}_k) = n\delta_{n,-k},$$
$$\varphi_t(\mathcal{L}'_n, I'_k) = n\delta_{n,-k},$$
$$\varphi_t(\boldsymbol{\mathcal{E}}_n, \boldsymbol{\mathcal{F}}_k) = \delta_{n,-k},$$
$$\varphi_t(\mathsf{h}_n, \mathsf{t}_k) = \delta_{n,-k},$$
$$\varphi_t(\mathfrak{y}_n, \mathfrak{q}_k) = \delta_{n,-k}.$$

In the next section we will observe a correspondence between the superconformal algebras (3.19) and Lie superalgebras of the Hodge theory.

4. Lie superalgebras of the Hodge theory

The Kähler case. Let M be a compact Kähler manifold with associated $(1,1)$-form ω, let $dim_{\mathbb{C}} M = n$. There exists a number of operators on the space $A^*(M)$ of differential forms on M such as $\partial, \bar\partial, d, d_c$, their corresponding adjoint operators and the associated Laplacians, see [GH]. Recall that

$$\begin{aligned} &\partial : A^{p,q}(M) \to A^{p+1,q}(M), \\ &\bar\partial : A^{p,q}(M) \to A^{p,q+1}(M), \\ &d = \partial + \bar\partial, \\ &d_c = i(\partial - \bar\partial), \\ &\triangle = dd^* + d^*d = 2\triangle_{\partial} = 2\triangle_{\bar\partial}. \end{aligned} \tag{4.1}$$

The Hodge $\star$-operator maps

$$\star : A^{p,q}(M) \longrightarrow A^{n-q,n-p}(M), \tag{4.2}$$

so that $\star^2 = (-1)^{p+q}$ on $A^{p,q}(M)$. Correspondingly, the Hodge inner product is defined on each of $A^{p,q}(M)$:

$$(\varphi, \psi) = \int_M \varphi \wedge \star\bar\psi. \tag{4.3}$$

In addition, $A^*(M)$ admits an $\mathfrak{sl}(2)$-module structure. Namely, $\mathfrak{sl}(2) = \langle L, H, \Lambda\rangle$, where

$$[L, \Lambda] = H, [H, L] = 2L, [H, \Lambda] = -2\Lambda. \tag{4.4}$$

The operator

$$L : A^{p,q}(M) \to A^{p+1,q+1}(M), \tag{4.5}$$

is defined by

$$L(\varphi) = \varphi \wedge \omega. \tag{4.6}$$

Let $\Lambda = L^*$ be its adjoint operator:

$$\Lambda : A^{p,q}(M) \to A^{p-1,q-1}(M), \tag{4.7}$$

and

$$H \mid_{A^{p,q}(M)} = p + q - n. \tag{4.8}$$

According to the Lefschetz theorem, there exists the corresponding action of $\mathfrak{sl}(2)$ on $H^*(M)$. These operators satisfy a series of identities, known as the Hodge identities, see [GH]:

$$\begin{aligned}
&[L, \partial] = 0, [L, \partial^*] = i\bar{\partial}, \\
&[L, \bar{\partial}] = 0, [L, \bar{\partial}^*] = -i\partial, \\
&[\Lambda, \partial] = i\bar{\partial}^*, [\Lambda, \partial^*] = 0, \\
&[\Lambda, \bar{\partial}] = -i\partial^*, [\Lambda, \bar{\partial}^*] = 0.
\end{aligned} \tag{4.9}$$

Let

$$\mathcal{S} = \langle \triangle, L, H, \Lambda, d, d^*, d_c, d_c^* \rangle. \tag{4.10}$$

The commutation relations in $\mathcal{S}$ are as follows:

$$\begin{aligned}
&[L, \Lambda] = H, [H, L] = 2L, [H, \Lambda] = -2\Lambda, \\
&[d, d^*] = dd^* + d^*d = \triangle, \\
&[d_c, d_c^*] = d_c d_c^* + d_c^* d_c = \triangle, \\
&[H, d] = d, [H, d^*] = -d^*, \\
&[H, d_c] = d_c, [H, d_c^*] = -d_c^*, \\
&[L, d^*] = -d_c, [L, d_c^*] = d, \\
&[\Lambda, d] = d_c^*, [\Lambda, d_c] = -d^*,
\end{aligned} \tag{4.11}$$

and they are zero otherwise.

Recall that by definition a superconformal algebra $\mathfrak{g}$ contains the centerless Virasoro algebra $Witt = \oplus_{n\in\mathbb{Z}}\mathbb{C}L_n$ as a subalgebra, and the element L_0 defines a $\mathbb{Z}$-grading on $\mathfrak{g}$.

Proposition 4.1. The degree zero part of the $\mathbb{Z}$-grading deg of $S'(2,0)$ is isomorphic to the Lie superalgebra of classical operators in Kähler geometry.

Proof. Recall that the $\mathbb{Z}$-grading deg on $S'(2,0)$ is defined by the element

$$L_0 = -\mathcal{L}_0 + \frac{1}{2}H_0, \tag{4.12}$$

so that

$$[L_0, s] = (degs)s, \tag{4.13}$$

for a homogeneous element $s \in S'(2,0)$, see (1.27). One can easily see that

$$S'(2,0)_{deg=0} = \langle L_0, E_{-1}, H_0, F_1, \mathfrak{h}_0, \mathfrak{p}_0, \mathfrak{x}_{-1}, \mathfrak{y}_1 \rangle. \tag{4.14}$$

The isomorphism of Lie superalgebras

$$\psi : \mathcal{S} \longrightarrow S'(2,0)_{deg=0} \tag{4.15}$$

is given as follows:

$$\psi(\triangle) = L_0, \psi(H) = H_0, \psi(d) = \mathfrak{h}_0, \psi(d^*) = -\mathfrak{p}_0, \tag{4.16}$$

$$\psi(L) = E_{-1}, \psi(\Lambda) = F_1, \psi(d_c) = \mathfrak{x}_{-1}, \psi(d_c^*) = \mathfrak{y}_1. \tag{4.17}$$

$\square$

Remark 4.1. Given a compact (oriented) Riemannian manifold M, $dimM = n$, there exist four classical operators on the space of differential forms $A^*(M)$: $d, d^*, \triangle$, and H, where d is the de Rham differential, d^* is its adjoint, $\triangle$ is the Laplace operator, and $H\mid_{A^p(M)} = p - n/2$. These operators span the Lie superalgebra $\mathfrak{gl}(1|1)$, which is isomorphic to the $deg = 0$ part of the $\mathbb{Z}$-grading of the centerless $N = 2$ superconformal algebra, defined by the element L_0, see (4.12), (4.16).

The hyperKähler case. Let M be a compact hyperKähler manifold. By definition, M is a Riemannian manifold endowed with three complex structures I, J, and K, such that $I \circ J = -J \circ I = K$ and M is Kähler with respect to the complex structures I, J and K. Correspondingly, we have 3 closed 2-forms: ω_I, ω_j and ω_K. We fix the complex structure I, thus $\omega = \omega_I$. The operators L_i, L_j, and L_k and their adjoint operators Λ_i, Λ_j, and Λ_k

with respect to the Hodge inner product are naturally defined. In particular, the $\mathfrak{sl}(2)$ provided by the Lefschetz theorem is spanned by $\langle L_i, H, \Lambda_i\rangle$. Set

$$\partial^i = \partial, \bar{\partial}^i = \bar{\partial}, d_c^i = d_c, (d_c^i)^* = d_c^*. \tag{4.18}$$

One can define the analogous differentials for the complex structures J and K (see [V2]):

$$\partial^j, \bar{\partial}^j, \partial^k, \bar{\partial}^k, d_c^j, (d_c^j)^*, d_c^k, (d_c^k)^*. \tag{4.19}$$

Remark 4.2. M. Verbitsky proved that in the hyperKähler case there is a natural action of the Lie algebra $\mathfrak{so}(5)$ on $H^*(M)$ ([V1]). The $\mathfrak{so}(5)$ is spanned by the operators $L_i, L_j, L_k, H, \Lambda_i, \Lambda_j, \Lambda_k$ and 3 operators K_{ij}, where

$$K_{ij} = -K_{ji} = [L_i, \Lambda_j], \quad i \neq j. \tag{4.20}$$

Thus we obtain an $(11|8)$-dimensional Lie superalgebra, where the even part is the direct sum of $\mathfrak{so}(5)$ and the Laplace operator $\triangle$, and the odd part is spanned by 8 differentials

$$d, d^*, d_c^i, (d_c^i)^*, d_c^j, (d_c^j)^*, d_c^k, (d_c^k)^*, \tag{4.21}$$

cf. [FKS]. The analogues of the Hodge identities are the commutation relations for $\mathfrak{so}(5)$ ([V1]), and the following identities ([V2]):

$$\begin{aligned}
&[d, d^*] = [d_c^i, (d_c^i)^*] = [d_c^j, (d_c^j)^*] = [d_c^k, (d_c^k)^*] = \triangle, \\
&[L_i, (d_c^j)^*] = -d_c^k, [\Lambda_i, d_c^j] = (d_c^k)^*, \\
&[H, d_c^i] = d_c^i, [H, (d_c^i)^*] = -(d_c^i)^*.
\end{aligned} \tag{4.22}$$

We set

$$d_c^1 = d, (d_c^1)^* = d^*. \tag{4.23}$$

The commutation relations for the indexes $\{i, j, k, 1\}$ satisfy the quaternionic rules. A particular case of the identities (4.22) are the familiar relations in the Kähler case:

$$\begin{aligned}
&[L, d^*] = -d_c, [L, d_c^*] = d, \\
&[\Lambda, d] = d_c^*, [\Lambda, d_c] = -d^*.
\end{aligned} \tag{4.24}$$

Remark 4.3. Consider the $\mathbb{Z}$-grading of $K'(4)$ defined by the element L_0, see (4.12). Notice that $K'(4)_{deg=0}$ is a $(7|8)$-dimensional Lie superalgebra spanned by $S'(2,0)_{deg=0}$

together with 3 even elements, $\boldsymbol{\mathcal{E}}_{-1}, \boldsymbol{\mathcal{H}}_0$ and $\boldsymbol{\mathcal{F}}_1$, and 4 odd elements, $\mathfrak{r}_0, \mathfrak{t}_0, \mathfrak{q}_{-1}$ and $\mathfrak{s}_1$. We have identified $S'(2,0)_{deg=0}$ with the Lie superalgebra $\mathcal{S} = \langle \triangle, L_i, H, \Lambda_i, d, d^*, d_c^i, (d_c^i)^* \rangle$. Notice that $\langle \boldsymbol{\mathcal{E}}_{-1}, \boldsymbol{\mathcal{H}}_0, \boldsymbol{\mathcal{F}}_1 \rangle \cong Der_{ext}\mathcal{S}$, see (2.17). We will show that the elements $\mathfrak{r}_0, \mathfrak{t}_0, \mathfrak{q}_{-1}$ and $\mathfrak{s}_1$ can be identified with the differentials $-d_c^j, (d_c^j)^*, d_c^k$ and $(d_c^k)^*$, respectively.

Lemma 4.1. The Lie superalgebra $K'(4)_{deg=0}$ is isomorphic to a central extension of $\mathfrak{psl}(2|2)$.

Proof. Recall that by definition

$$\mathfrak{psl}(2|2) = \mathfrak{sl}(2|2)/\langle 1_4 \rangle, \tag{4.25}$$

where 1_4 is the identity matrix. Thus $\mathfrak{psl}(2|2)$ is a simple Lie superalgebra of dimension $(6|8)$.

From (2.24), (2.25) and (3.12) we obtain the following non-vanishing commutation relations for $K'(4)_{deg=0} = \langle L_0, E_{-1}, H_0, F_1, \boldsymbol{\mathcal{E}}_{-1}, \boldsymbol{\mathcal{F}}_1, \boldsymbol{\mathcal{H}}_0, \mathfrak{h}_0, \mathfrak{p}_0, \mathfrak{x}_{-1}, \mathfrak{y}_1, \mathfrak{r}_0, \mathfrak{t}_0, \mathfrak{q}_{-1}, \mathfrak{s}_1 \rangle$:

$$\begin{aligned}
&[E_{-1}, F_1] = H_0, [H_0, E_{-1}] = 2E_{-1}, [H_0, F_1] = -2F_1, \\
&[E_{-1}, \mathfrak{y}_1] = \mathfrak{h}_0, [E_{-1}, \mathfrak{p}_0] = \mathfrak{x}_{-1}, [E_{-1}, \mathfrak{s}_1] = -\mathfrak{r}_0, [E_{-1}, \mathfrak{t}_0] = -\mathfrak{q}_{-1}, \\
&[F_1, \mathfrak{h}_0] = \mathfrak{y}_1, [F_1, \mathfrak{x}_{-1}] = \mathfrak{p}_0, [F_1, \mathfrak{r}_0] = -\mathfrak{s}_1, [F_1, \mathfrak{q}_{-1}] = -\mathfrak{t}_0, \\
&[H_0, \mathfrak{h}_0] = \mathfrak{h}_0, [H_0, \mathfrak{y}_1] = -\mathfrak{y}_1, [H_0, \mathfrak{x}_{-1}] = \mathfrak{x}_{-1}, [H_0, \mathfrak{p}_0] = -\mathfrak{p}_0, \\
&[H_0, \mathfrak{r}_0] = \mathfrak{r}_0, [H_0, \mathfrak{s}_1] = -\mathfrak{s}_1, [H_0, \mathfrak{t}_0] = -\mathfrak{t}_0, [H_0, \mathfrak{q}_{-1}] = \mathfrak{q}_{-1}, \\
&[\boldsymbol{\mathcal{H}}_0, \boldsymbol{\mathcal{E}}_{-1}] = 2\boldsymbol{\mathcal{E}}_{-1}, [\boldsymbol{\mathcal{H}}_0, \boldsymbol{\mathcal{F}}_1] = -2\boldsymbol{\mathcal{F}}_1, [\boldsymbol{\mathcal{E}}_{-1}, \boldsymbol{\mathcal{F}}_1] = \boldsymbol{\mathcal{H}}_0, \\
&[\boldsymbol{\mathcal{E}}_{-1}, \mathfrak{h}_0] = \mathfrak{x}_{-1}, [\boldsymbol{\mathcal{E}}_{-1}, \mathfrak{y}_1] = \mathfrak{p}_0, [\boldsymbol{\mathcal{E}}_{-1}, \mathfrak{r}_0] = -\mathfrak{q}_{-1}, [\boldsymbol{\mathcal{E}}_{-1}, \mathfrak{s}_1] = -\mathfrak{t}_0, \\
&[\boldsymbol{\mathcal{F}}_1, \mathfrak{p}_0] = \mathfrak{y}_1, [\boldsymbol{\mathcal{F}}_1, \mathfrak{x}_{-1}] = \mathfrak{h}_0, [\boldsymbol{\mathcal{F}}_1, \mathfrak{q}_{-1}] = -\mathfrak{r}_0, [\boldsymbol{\mathcal{F}}_1, \mathfrak{t}_0] = -\mathfrak{s}_1, \\
&[\boldsymbol{\mathcal{H}}_0, \mathfrak{h}_0] = -\mathfrak{h}_0, [\boldsymbol{\mathcal{H}}_0, \mathfrak{p}_0] = \mathfrak{p}_0, [\boldsymbol{\mathcal{H}}_0, \mathfrak{y}_1] = -\mathfrak{y}_1, [\boldsymbol{\mathcal{H}}_0, \mathfrak{x}_{-1}] = \mathfrak{x}_{-1}, \\
&[\boldsymbol{\mathcal{H}}_0, \mathfrak{r}_0] = -\mathfrak{r}_0, [\boldsymbol{\mathcal{H}}_0, \mathfrak{s}_1] = -\mathfrak{s}_1, [\boldsymbol{\mathcal{H}}_0, \mathfrak{q}_{-1}] = \mathfrak{q}_{-1}, [\boldsymbol{\mathcal{H}}_0, \mathfrak{t}_0] = \mathfrak{t}_0, \\
&[\mathfrak{x}_{-1}, \mathfrak{r}_0] = E_{-1}, [\mathfrak{h}_0, \mathfrak{q}_{-1}] = E_{-1}, [\mathfrak{p}_0, \mathfrak{s}_1] = F_1, [\mathfrak{y}_1, \mathfrak{t}_0] = F_1, \\
&[\mathfrak{p}_0, \mathfrak{q}_{-1}] = -\boldsymbol{\mathcal{E}}_{-1}, [\mathfrak{x}_{-1}, \mathfrak{t}_0] = -\boldsymbol{\mathcal{E}}_{-1}, [\mathfrak{h}_0, \mathfrak{s}_1] = -\boldsymbol{\mathcal{F}}_1, [\mathfrak{y}_1, \mathfrak{r}_0] = -\boldsymbol{\mathcal{F}}_1, \\
&[\mathfrak{p}_0, \mathfrak{r}_0] = -\frac{1}{2}H_0 - \frac{1}{2}\boldsymbol{\mathcal{H}}_0, [\mathfrak{x}_{-1}, \mathfrak{s}_1] = \frac{1}{2}H_0 - \frac{1}{2}\boldsymbol{\mathcal{H}}_0, \\
&[\mathfrak{h}_0, \mathfrak{t}_0] = \frac{1}{2}H_0 + \frac{1}{2}\boldsymbol{\mathcal{H}}_0, [\mathfrak{y}_1, \mathfrak{q}_{-1}] = -\frac{1}{2}H_0 + \frac{1}{2}\boldsymbol{\mathcal{H}}_0, \\
&[\mathfrak{h}_0, \mathfrak{p}_0] = -L_0, [\mathfrak{x}_{-1}, \mathfrak{y}_1] = L_0.
\end{aligned} \tag{4.26}$$

Hence $K'(4)_{deg=0}$ is a central extension of $\mathfrak{psl}(2|2)$, where the central element is L_0, and the corresponding 2-cocycle is

$$c_1(\mathfrak{h}_0, \mathfrak{p}_0) = -1, c_1(\mathfrak{x}_{-1}, \mathfrak{y}_1) = 1. \tag{4.27}$$

$\square$

Remark 4.4. It is known that the exterior derivations of $\mathfrak{psl}(2|2)$ form an $\mathfrak{sl}(2)$ ([S]). Correspondingly, $\mathfrak{psl}(2|2)$ has 3 independent central extensions, and the $\mathfrak{sl}(2)$ naturally acts on 2-cocycles. Content-Length: 20801 X-Lines: 570 Status: RO

One of the 2-cocycles is given by (4.27). The restriction of the cocycle φ (see (3.18)) to $K'(4)_{deg=0}$ is the second 2-cocycle for $\mathfrak{psl}(2|2)$:

$$\begin{aligned} c_2(\mathfrak{h}_0, \mathfrak{t}_0) &= \frac{1}{2}, \\ c_2(\mathfrak{p}_0, \mathfrak{r}_0) &= \frac{1}{2}, \\ c_2(\mathfrak{x}_{-1}, \mathfrak{s}_1) &= \frac{1}{2}, \\ c_2(\mathfrak{y}_1, \mathfrak{q}_{-1}) &= \frac{1}{2}. \end{aligned} \tag{4.28}$$

Notice that the restrictions of the cocycles φ_ξ and φ_t to $K'(4)_{deg=0}$ are 2-cocycles equivalent to c_2.

The element

$$X = -2t^{-1}\theta_1\theta_2\theta_3\theta_4 \in K(4) \tag{4.29}$$

obviously defines an exterior derivation of $K'(4)$, see (3.8). The restriction of this derivation to $K'(4)_{deg=0}$ is an exterior derivation of $\mathfrak{psl}(2|2)$:

$$X(\mathfrak{h}_0) = -\mathfrak{r}_0, X(\mathfrak{p}_0) = -\mathfrak{t}_0, X(\mathfrak{x}_{-1}) = \mathfrak{q}_{-1}, X(\mathfrak{y}_1) = \mathfrak{s}_1. \tag{4.30}$$

We obtain that $Der_{ext}\mathfrak{psl}(2|2) = \langle X, Y, Z\rangle$, where

$$[X, Y] = Z, [Z, X] = 2X, [Z, Y] = -2Y, \tag{4.31}$$

and the derivations Y and Z are defined as follows:

$$\begin{aligned} &Y(\mathfrak{r}_0) = -\mathfrak{h}_0, Y(\mathfrak{t}_0) = -\mathfrak{p}_0, Y(\mathfrak{q}_{-1}) = \mathfrak{x}_{-1}, Y(\mathfrak{s}_1) = \mathfrak{y}_1; \\ &Z(\mathfrak{h}_0) = -\mathfrak{h}_0, Z(\mathfrak{p}_0) = -\mathfrak{p}_0, Z(\mathfrak{x}_{-1}) = -\mathfrak{x}_{-1}, Z(\mathfrak{y}_1) = -\mathfrak{y}_1, \\ &Z(\mathfrak{r}_0) = \mathfrak{r}_0, Z(\mathfrak{s}_1) = \mathfrak{s}_1, Z(\mathfrak{q}_{-1}) = \mathfrak{q}_{-1}, Z(\mathfrak{t}_0) = \mathfrak{t}_0. \end{aligned} \tag{4.32}$$

Let C_1 and C_2 be the central elements of the extensions defined by the cocycles c_1 and c_2. Thus $\mathrm{C}_1 = L_0 \in Witt$, and $\mathrm{C}_2 = 1 \in R(2)$. Notice that

$$X(\mathrm{C}_1) = \mathrm{C}_2. \tag{4.33}$$

Let C_3 be the central element of the third independent central extension of $\mathfrak{psl}(2|2)$, so that

$$X(\mathrm{C}_2) = -2\mathrm{C}_3. \tag{4.34}$$

Then the corresponding 2-cocycle is

$$\begin{aligned} c_3(\mathfrak{t}_0, \mathfrak{r}_0) &= 1, \\ c_3(\mathfrak{s}_1, \mathfrak{q}_{-1}) &= -1. \end{aligned} \tag{4.35}$$

Let

$$\hat{\mathfrak{psl}}(2|2) = \mathfrak{psl}(2|2) \oplus \langle \mathrm{C} \rangle, \tag{4.36}$$

be the central extension of $\mathfrak{psl}(2|2)$ defined by the 2-cocycle $c_1 - c_3$. Let $\mathcal{K}$ be the Lie superalgebra such that $\mathcal{K} = \hat{\mathfrak{psl}}(2|2)$ as a vector space, and

$$\begin{aligned} &[g_1, g_2]_{\mathcal{K}} = [g_1, g_2]_{\hat{\mathfrak{psl}}(2|2)}, \text{ if } g_1 \text{ or } g_2 \text{ is even}, \\ &[g_1, g_2]_{\mathcal{K}} = (c_1 - c_3)(g_1, g_2)\mathrm{C}, \text{ if } g_1 \text{ and } g_2 \text{ are odd}. \end{aligned} \tag{4.37}$$

Proposition 4.2. The Lie superalgebra spanned by the operators

$$\langle \triangle, L_i, H, \Lambda_i, d, d^*, d_c^i, (d_c^i)^*, d_c^j, (d_c^j)^*, d_c^k, (d_c^k)^* \rangle \tag{4.38}$$

is isomorphic to the subalgebra of $\mathcal{K}$ spanned by the elements

$$\{\mathsf{C}, E_{-1}, H_0, F_1, \mathfrak{h}_0, \mathfrak{p}_0, \mathfrak{x}_{-1}, \mathfrak{y}_1, \mathfrak{r}_0, \mathfrak{t}_0, \mathfrak{q}_{-1}, \mathfrak{s}_1\}. \tag{4.39}$$

Proof. Recall that in the Kähler case we obtained the isomorphism ψ of the Lie superalgebra of the classical operators and $S'(2,0)_{deg=0}$, see (4.16), (4.17). We extend the isomorphism ψ as follows:

$$\begin{aligned} &\psi(\triangle) = \mathsf{C}, \psi(L_i) = E_{-1}, \psi(\Lambda_i) = F_1, \psi(H) = H_0, \\ &\psi(d) = \mathfrak{h}_0, \psi(d^*) = -\mathfrak{p}_0, \psi(d_c^i) = \mathfrak{x}_{-1}, \psi((d_c^i)^*) = \mathfrak{y}_1, \\ &\psi(d_c^j) = -\mathfrak{r}_0, \psi((d_c^j)^*) = \mathfrak{t}_0, \psi(d_c^k) = \mathfrak{q}_{-1}, \psi((d_c^k)^*) = \mathfrak{s}_1. \end{aligned} \tag{4.40}$$

□

In the next section we will consider the relative semi-infinite Weil complex of a loop algebra, and describe some structures on it, which are similar to those of Kähler geometry.

5. Relative semi-infinite cohomology and Kähler geometry

The semi-infinite Weil complex of a graded Lie algebra was introduced by B. Feigin and E. Frenkel in [FF]. In particular, let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ be the loop algebra of a complex finite-dimensional Lie algebra $\mathfrak{g}$. Naturally, $\tilde{\mathfrak{g}} = \oplus_{n \in \mathbb{Z}} \tilde{\mathfrak{g}}_n$. Let $\tilde{\mathfrak{g}}' = \oplus_{n \in \mathbb{Z}} \tilde{\mathfrak{g}}'_n$ be the restricted dual of $\tilde{\mathfrak{g}}$. The linear space $\tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}'$ carries non-degenerate skew-symmetric and symmetric bilinear forms: $(\cdot, \cdot)$ and $\{\cdot, \cdot\}$. Let $H(\tilde{\mathfrak{g}})$ and $C(\tilde{\mathfrak{g}})$ be the quotients of the tensor algebra $T^*(\tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}')$ by the ideals generated by the elements of the form $xy - yx - (x, y)$ and $xy + yx - \{x, y\}$, respectively, where $x, y \in \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}'$. Consider the polarization $\tilde{\mathfrak{g}} = \mathfrak{n}_+ \oplus \mathfrak{n}_-$, where $\mathfrak{n}_+ = \oplus_{n>0} \tilde{\mathfrak{g}}_n$, $\mathfrak{n}_- = \oplus_{n \leq 0} \tilde{\mathfrak{g}}_n$. Let $S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$, $\Lambda^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$ be the representations of $H(\tilde{\mathfrak{g}})$ and $C(\tilde{\mathfrak{g}})$ induced from the trivial representations $< 1_S >$ and $< 1_\Lambda >$ of $S^*(\mathfrak{n}_+ \oplus \mathfrak{n}'_-)$ and of $\Lambda^*(\mathfrak{n}_+ \oplus \mathfrak{n}'_-)$, respectively. Denote the actions of $H(\tilde{\mathfrak{g}})$ and $C(\tilde{\mathfrak{g}})$ on these modules by $\beta(x), \gamma(x')$ and $\tau(x), \varepsilon(x')$, respectively, for $x \in \tilde{\mathfrak{g}}$, $x' \in \tilde{\mathfrak{g}}'$. Let $Deg \varepsilon(x') = Deg \gamma(x') = 1$, and $Deg \tau(x) = Deg \beta(x) = -1$. Correspondingly, $S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}) = \oplus_{i \in \mathbb{Z}} S^{\frac{\infty}{2}+i}(\tilde{\mathfrak{g}})$, $\Lambda^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}) = \oplus_{i \in \mathbb{Z}} \Lambda^{\frac{\infty}{2}+i}(\tilde{\mathfrak{g}})$. Let u run through a fixed basis of $\mathfrak{g}$. Then $\{u_m\}_{m \in \mathbb{Z}}$ is a basis of $\tilde{\mathfrak{g}}$.

One can define projective representations ρ and π of $\tilde{\mathfrak{g}}$ in $\Lambda^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$ and $S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$, respectively:

$$\rho(x) = \sum_{i \in \mathbb{Z}} \sum_u : \tau([x, u_i]) \varepsilon(u'_i) :, \tag{5.1}$$

$$\pi(x) = \sum_{i \in \mathbb{Z}} \sum_u : \beta([x, u_i] \gamma(u'_i) :, \tag{5.2}$$

where $x \in \tilde{\mathfrak{g}}$, and where the double colons $: \quad :$ denote a normal ordering operation:

$$: \tau(u_j) \varepsilon(v'_i) := \left\{ \begin{array}{c} \tau(u_j)\varepsilon(v'_i) \text{ if } i \leq 0 \\ -\varepsilon(v'_i)\tau(u_j) \text{ if } i > 0 \end{array} \right\}, \tag{5.3}$$

with the similar formulaes for β and γ, but without the minus sign. Then

$$\begin{aligned} [\rho(x), \rho(y)] &= \rho([x, y]) + c_\Lambda(x, y), \\ [\pi(x), \pi(y)] &= \pi([x, y]) + c_S(x, y), \end{aligned} \tag{5.4}$$

where $x, y \in \tilde{\mathfrak{g}}$ and c_Λ, c_S are 2-cocycles. Notice that $c_\Lambda = -c_S$. Let

$$W^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}) = S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}) \otimes \Lambda^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}). \tag{5.5}$$

Since the cocycles corresponding to the projective representations cancel, the representation $\theta(x) = \rho(x) + \pi(x)$ of $\tilde{\mathfrak{g}}$ in $W^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$ is well-defined. We define a $\mathbb{Z}$-grading on $W^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$ setting

$$W^{\frac{\infty}{2}+i}(\tilde{\mathfrak{g}}) = \oplus_{2l+j=i} S^{\frac{\infty}{2}+l}(\tilde{\mathfrak{g}}) \otimes \Lambda^{\frac{\infty}{2}+j}(\tilde{\mathfrak{g}}). \tag{5.6}$$

We also define two differentials on the space $W^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$:

$$d = \frac{1}{2}\sum_{u,v,i,j} :\tau([u_i, v_j])\varepsilon(v'_j)\varepsilon(u'_i): + \sum_{u,v,i,j} :\beta([u_i, v_j])\gamma(v'_j)\varepsilon(u'_i):, \tag{5.7}$$

$$\mathrm{h} = \sum_{u,i} \gamma(u'_i)\tau(u_i), \tag{5.8}$$

where u, v run through a fixed basis of $\mathfrak{g}$, and $i, j \in \mathbb{Z}$. The differential d is the analogue of the classical differential for the Lie algebra cohomology and h is the (semi-infinite) Koszul differential. We obtain *the semi-infinite Weil complex*

$$\{W^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}),\ d + \mathrm{h}\}. \tag{5.9}$$

Notice that

$$d^2 = 0, \mathrm{h}^2 = 0, [d, \mathrm{h}] = 0, (d + \mathrm{h})^2 = 0. \tag{5.10}$$

Let

$$W_{rel}^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}) = \{w \in W^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}) \mid \tau(x)w = \theta(x)w = 0 \text{ for all } x \in \tilde{\mathfrak{g}}_0\}. \tag{5.11}$$

The differential d preserves the space $W_{rel}^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$ since

$$[d, \tau(x)] = d\tau(x) + \tau(x)d = \theta(x), \tag{5.12}$$

and

$$[d, \theta(x)] = 0. \tag{5.13}$$

for any $x \in \tilde{\mathfrak{g}}$. The complex $\{W_{rel}^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}), d\}$ is called *the relative semi-infinite Weil complex.* Its cohomology is called *the relative semi-infinite cohomology* $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$.

Proposition 5.1. Let $\mathfrak{g}$ be a complex finite-dimensional Lie algebra with a non-degenerate invariant symmetric bilinear form. Then there exist operators on $W_{rel}^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$, which are analogous to the classical operators in Kähler geometry.

Proof. It was shown in [FGZ] that a relative semi-infinite complex $C^*_\infty(\mathfrak{l}, \mathfrak{l}_0, V)$, where $\mathfrak{l} = \oplus_{n\in\mathbb{Z}}\mathfrak{l}_n$ is a complex $\mathbb{Z}$-graded Lie algebra, and V is a graded Hermitian $\mathfrak{l}$-module, admits a structure, which is similar to that of the de Rham complex in Kähler geometry. It is assumed that there exists a 2-cocycle γ on $\mathfrak{l}$ such that $\gamma|_{\mathfrak{l}_n\times\mathfrak{l}_{-n}}$ is non-degenerate if $n\in\mathbb{Z}\backslash 0$ and it is zero otherwise. Then there exist operators on $C^*_\infty(\mathfrak{l},\mathfrak{l}_0,V)$ analogous to the classical ones.

We can define similar operators on $W_{rel}^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$. Using the form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ we obtain the 2-cocycle γ on $\tilde{\mathfrak{g}}$:

$$\gamma(g_1\otimes t^n, g_2\otimes t^m) = n\langle g_1,g_2\rangle\delta_{n,-m}, \text{ for } g_1,g_2\in\mathfrak{g}. \tag{5.14}$$

Notice that $\gamma|_{\tilde{\mathfrak{g}}_n\times\tilde{\mathfrak{g}}_{-n}}$ is non-degenerate if $n\in\mathbb{Z}\backslash 0$ and zero otherwise. Let

$$\Lambda_{rel}^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}) = \oplus_{a,b\geq 0}\Lambda^a(\mathfrak{n}'_+)\wedge\Lambda^b_\infty(\mathfrak{n}'_-). \tag{5.15}$$

For a homogeneous element in $\Lambda^a(\mathfrak{n}'_+)\wedge\Lambda^b_\infty(\mathfrak{n}'_-)$, a is the number of added elements, and b is the number of missing elements with respect to the vacuum vector $1_{\Lambda_{rel}}$. Let

$$C^{a,b}(\tilde{\mathfrak{g}}) = [S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})\otimes\Lambda^a(\mathfrak{n}'_+)\wedge\Lambda^b_\infty(\mathfrak{n}'_-)]^{\mathfrak{g}_0}. \tag{5.16}$$

We obtain a bigrading on the relative semi-infinite Weil complex, such that

$$W_{rel}^{\frac{\infty}{2}+i}(\tilde{\mathfrak{g}}) = \oplus_{a-b=i}C^{a,b}(\tilde{\mathfrak{g}}). \tag{5.17}$$

Notice that

$$d: C^{a,b}(\tilde{\mathfrak{g}}) \longrightarrow C^{a+1,b}(\tilde{\mathfrak{g}})\oplus C^{a,b-1}(\tilde{\mathfrak{g}}). \tag{5.18}$$

Define d_1 and d_2 such that

$$\begin{aligned} &d = d_1+d_2, \\ &d_1: C^{a,b}(\tilde{\mathfrak{g}}) \longrightarrow C^{a+1,b}(\tilde{\mathfrak{g}}), \\ &d_2: C^{a,b}(\tilde{\mathfrak{g}}) \longrightarrow C^{a,b-1}(\tilde{\mathfrak{g}}). \end{aligned} \tag{5.19}$$

Let

$$d_c = i(d_1-d_2). \tag{5.20}$$

To define the adjoint operators, we have to introduce a Hermitian form on $W_{rel}^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$.

It was shown in [FGZ] that if a $\mathbb{Z}$-graded Lie algebra $\mathfrak{l}$ admits an antilinear automorphism σ of order 2 such that $\sigma(\mathfrak{l}_n)=\mathfrak{l}_{-n}$, then there exists a Hermitian form on $\Lambda^{\frac{\infty}{2}+*}(\mathfrak{l})$ such that

$$\varepsilon(x')^* = -\varepsilon(\sigma(x')), \quad \tau(x)^* = -\tau(\sigma(x)), \tag{5.21}$$

where $x \in \mathfrak{l}, x' \in \mathfrak{l}'$.

To define a Hermitian form $\{\cdot,\cdot\}$ on $\Lambda_{rel}^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$, we set $\{1_{\Lambda_{rel}}, 1_{\Lambda_{rel}}\} = 1$. We fix a basis $\{v_i\}$ of $\mathfrak{g}$ so that $\langle v_i, v_j\rangle = \delta_{i,j}$. Let u run through this basis. We define an antilinear automorphism σ of $\tilde{\mathfrak{g}}$ as follows:

$$\sigma(u_n) = iu_{-n}. \tag{5.22}$$

Correspondingly,

$$\sigma(u'_n) = -iu'_{-n}. \tag{5.23}$$

We introduce a Hermitian form on $\Lambda_{rel}^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$ such that the relations (5.21), where $x \in \tilde{\mathfrak{g}}/\tilde{\mathfrak{g}}_0, x' \in (\tilde{\mathfrak{g}}/\tilde{\mathfrak{g}}_0)'$, hold. In the similar way we introduce a Hermitian form on $S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$, such that

$$\gamma(x')^* = -\gamma(\sigma(x')), \quad \beta(x)^* = -\beta(\sigma(x)), \text{ where } x \in \tilde{\mathfrak{g}}/\tilde{\mathfrak{g}}_0, x' \in (\tilde{\mathfrak{g}}/\tilde{\mathfrak{g}}_0)'. \tag{5.24}$$

Then we obtain a Hermitian form $\{\cdot,\cdot\}$ on $W_{rel}^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$ by tensoring these two forms. It gives a pairing: $C^{a,b}(\tilde{\mathfrak{g}}) \longrightarrow C^{b,a}(\tilde{\mathfrak{g}})$. Notice that $d = d^*$. To define a Hermitian form on $C^{a,b}(\tilde{\mathfrak{g}})$ we use the linear map

$$* : C^{a,b}(\tilde{\mathfrak{g}}) \longrightarrow C^{b,a}(\tilde{\mathfrak{g}}), \tag{5.25}$$

defined as follows:

$$* : (v \otimes u'_{m_1} \wedge u'_{m_2} \wedge \dots) = (sign) v \otimes u'_{-n_1} \wedge u'_{-n_2} \wedge \dots, \tag{5.26}$$

where $v \in S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$, $\{n_i\}_{i\geq 1}$ is the complement of $\{m_i\}_{i\geq 1}$ in $\mathbb{Z}\backslash 0$, and $sign$ corresponds to a permutation, such that $\dots n_2, n_1, m_1, m_2, \dots$ has decreasing order. Finally, the Hermitian form on $C^{a,b}(\tilde{\mathfrak{g}})$ is given by $(w_1, w_2) = \{i^{a+b} * w_1, w_2\}$, see [FGZ] for more detail. We introduce the adjoint operators d^*, d_c^* and the Laplace operator $\triangle = dd^* + d^*d$.

It was pointed out in [FGZ] that as in the classical theory ([GH]), there exists an action of $\mathfrak{sl}(2)$ on $H^*_\infty(\mathfrak{l}, \mathfrak{l}_0, V)$. One can identify $\mathfrak{l}'_n$ with $\mathfrak{l}_{-n}$ by means of the cocycle γ. If $\{e_i\}$ is a homogeneous basis in $\mathfrak{l}$, then $\mathfrak{sl}(2) = \langle L, H, \Lambda\rangle$ is defined as follows:

$$\begin{aligned} L &= (i/2) \sum_{m\in\mathbb{Z}\backslash 0} \varepsilon(e_m)\varepsilon(e'_m), \\ \Lambda &= (i/2) \sum_{m\in\mathbb{Z}\backslash 0} \tau(e_m)\tau(e'_m), \\ H &= -\sum_{m\in\mathbb{Z}\backslash 0} :\tau(e_m)\varepsilon(e'_m): . \end{aligned} \tag{5.27}$$

We identify $\tilde{\mathfrak{g}}'_n$ with $\tilde{\mathfrak{g}}_{-n}$ by means of the cocycle γ, see (5.14). Then

$$L = i\sum_u\sum_{m>0} m\varepsilon(u'_{-m})\varepsilon(u'_m), \tag{5.28}$$

$$H = -\sum_u\sum_{m\neq 0} :\tau(u_m)\varepsilon(u'_m):,$$

$$\Lambda = -i\sum_u\sum_{m>0}(1/m)\tau(u_m)\tau(u_{-m}).$$

The operators

$$\{\triangle, L, H, \Lambda, d, d^*, d_c, d_c^*\} \tag{5.29}$$

are analogues of the classical operators (4.10).

□

Proposition 5.2. Let $\mathfrak{g}$ be a complex finite-dimensional Lie algebra endowed with a non-degenerate invariant symmetric bilinear form. Then $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$ is a module over $\hat{D}erS'(2,0)$ with central charge $3dim\mathfrak{g}$.

Proof. This theorem was stated in [P1, P2]. The detailed version is [P3]. Let $\{v_i\}$ be a basis of $\mathfrak{g}$ so that $\langle v_i, v_j\rangle = \delta_{i,j}$. Let u run through this basis, and let $m\in\mathbb{Z}$. We identify the elements of $S'(2,0)$ with the following quadratic expansions:

$$L_n = -\sum_u\sum_m[m:\tau(u_{m+n})\varepsilon(u'_m): + m:\beta(u_{m+n})\gamma(u'_m):], \tag{5.30}$$

$$E_n = -(1/2)i\sum_u\sum_m \gamma(u'_m)\gamma(u'_{1-m+n}),$$

$$H_n = -\sum_u\sum_m :\beta(u_m)\gamma(u'_{m+n}):,$$

$$F_n = -(1/2)i\sum_u\sum_m \beta(u_m)\beta(u_{1-m-n}),$$

$$\mathfrak{h}_n = \sum_u\sum_m \gamma(u'_{m+n})\tau(u_m),$$

$$\mathfrak{p}_n = \sum_u\sum_m m\beta(u_{m-n})\varepsilon(u'_m),$$

$$\mathfrak{y}_n = i\sum_u\sum_m \beta(u_m)\tau(u_{1-m-n}),$$

$$\mathfrak{x}_n = -i\sum_u\sum_m m\gamma(u'_{1-m+n})\varepsilon(u'_m).$$

Set

$$\mathfrak{L}_n = -L_{-n} + \frac{n+1}{2}H_n. \tag{5.31}$$

One can check that the commutation relations (2.24) are fulfiled. Notice that $\mathfrak{h}_0$ is the semi-infinite Koszul differential $\mathfrak{h}$, and $\mathfrak{p}_0$ is the semi-infinite analogue of the homotopy operator, see [F]. Since the form $\langle \cdot, \cdot \rangle$ is invariant, then there is an action of $\hat{S}'(2,0)$ on $W_{rel}^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$. Notice that

$$[S'(2,0), d] = 0, \tag{5.32}$$

thus we obtain a representation of $\hat{S}'(2,0)$ in $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$.

Recall that $Der_{ext}S'(2,0) = \langle \boldsymbol{\mathcal{E}}_{-1}, \boldsymbol{\mathcal{H}}_0, \boldsymbol{\mathcal{F}}_1 \rangle$, see (2.17). Finally, we set

$$\boldsymbol{\mathcal{E}}_{-1} = L, \boldsymbol{\mathcal{H}}_0 = H, \boldsymbol{\mathcal{F}}_1 = \Lambda, \tag{5.33}$$

as in (5.28). Notice that as in the classical case, $[\Lambda, d] \neq 0$. Nevertheless, there exists an action of $\mathfrak{sl}(2)$ on the relative semi-infinite cohomology according to [FGZ].

$\square$

Corollary 5.1. The action of $S'(2,0)_{deg=0}$ gives an additional set of (quadratic) operators on $W_{rel}^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$ (correspondingly, on $H^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_0, S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}))$), which are analogues of the classical ones, and include the semi-infinite Koszul differential and the semi-infinite homotopy operator.

Proof. Follows from Proposition 4.1.

$\square$

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