On correlation between testing S-duality conjecture and exceptional bundles.

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Introduction

In this paper algebro-geometrical problems are considered, which are possibly interesting for physicists because of relation to testing S-duality conjecture for super-symmetric Yang-Mills quantum field theories in four dimensions. This conjecture asserts that such a theory has a symmetry exchanging strong and week coupling and exchanging also electric and magnetic fields.

In the work [14], with which S-duality conjecture was originated, Montonen and Olive proposed existence of $\mathbb{Z}_2$-symmetry involving only the real coupling constant g. In further development this constant was combined with one more parameter, so-called θ angle[1], in a complex parameter

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2},$$

see [VW] and further references there. For physical reasons it is natural to extend the $\mathbb{Z}_2$ symmetry to $SL(2,\mathbb{Z})$ action on τ

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

The original Montonen-Olive symmetry takes place when $\theta = 0$ for the action by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

One of the ways to test the conjecture deals with some quantities that can be interpreted as correlation functions in the theories. These are power series in $q = e^{2\pi i\tau}$. One has to examine these quantities for $SL(2,\mathbb{Z})$-invariance or more generally for some modular properties under the $SL(2,\mathbb{Z})$-action.

In this paper such power series are considered not for QFT itself but for its semiclassical limit i.e. for classical gauge theory, from which the given QFT is obtained by quantization.

It is important to have zero quantum corrections betweem quantum and semiclassical partition functions. For this it is sufficient for the value of β-function on field configuration of the theory to be zero. Then the theory is scaling-invariant, τ is really a constant, and quantum correlation functions can be calculated at semi-classical limit.

We describe all cases of vanishing of beta-function in 1.4. There are six "regular" field configurations labelled as a), b), c), d), e), f), which have zero β-function for arbitrary rank

[1] or vacuum angle

of the gauge group, and some special cases. We present here the power series corresponding to the regular cases only.

Let S be del Pezzo surface over $\mathbb{C}$, K be the canonical class of S. Denote by $\mathcal{M}(r,c_1,k)$ the moduli space of Mumford-Takemoto stable vector bundles over S w.r. to $-K$ with the rank r, the Chern classes c_1 and c_2, where

$$k = c_2 - \frac{1}{2}c_1 \cdot (c_1 - K) - r \; .$$

The Giesecker-Marujama compactification $\overline{\mathcal{M}}(r,c_1,k)$ of this space is a coarse moduli space of semi-stable sheaves by Giesecker with the same invariants. It is known [13] that $\overline{\mathcal{M}}(r,c_1,k)$ is a projective scheme. Besides, if r and $c_1 \cdot K$ are coprime, then $\overline{\mathcal{M}}(r,c_1,k)$ carries a universal family [13, 6.11] and is smooth [13, 6.7] as far as any semi-stable sheaf with these invariants is stable.

Consider the following power series.

a) Put with fixed r and c_1:

$$Z_{\mathbf{a}}(\tau) = \sum_{k \geqslant 0} \chi(\overline{\mathcal{M}}(r,c_1,k))q^k \; . \tag{1}$$

Here χ denotes topological Euler characteristic.

b) Suppose that the following condition holds:

$$-K^2 < \frac{c_1 \cdot (-K)}{r} < 0 \tag{2}$$

Let F be semistable sheaf with the invariants (r,c_1,k); then from (2), Serre duality and properties of stability it follows that $H^0(F) = H^2(F) = 0$, whence we have:

$$\dim H^1(F) = -\chi(F) = c_2 - \frac{1}{2}c_1 \cdot (c_1 - K) - r = k \; . \tag{3}$$

Observe that this dimension depends only on topological invariants of the sheaf F. Therefore, a sheaf $\mathcal{H}_k$ over $\overline{\mathcal{M}}(r,c_1,k)$ with fiber $(\mathcal{H}_k)_F = H^1(F)$ is locally free.

Restrict the invariants by the condition

$$\dim \mathcal{M}(r,c_1,k) = 2r \dim H^1(E) \tag{4}$$

and consider the following power series (with r and c_1 fixed):

$$Z_{\mathbf{b}}(\tau) = \sum_{k \geqslant 0} c_{top}(\mathcal{H}_k)^{2\,r} q^k \; . \tag{5}$$

The equality (4) is necessary for the coefficients of $Z_{\mathbf{b}}(\tau)$ to be a numbers.

For the rest power series from c) to f) we replace (2) by a stronger condition

$$-\frac{1}{2}K^2 < \frac{c_1 \cdot (-K)}{r} < 0 \; . \tag{6}$$

It is shown in Section 3 that for a semi-stable sheaf F with topological invariants satisfying this condition, the dimensions of $H^1(S^2F)$ and $H^1(\Lambda^2 F)$ depend only on the the invariants of F. Let $\mathcal{S}_k$ and $\mathcal{L}_k$ be bundles over $\mathcal{M}(r, c_1, k)$ with fibers respectively $H^1(S^2F)$ and $H^1(\Lambda^2 F)$.

c) Under the equality of dimensions

$$\dim \mathcal{M}(r, c_1, k) = \dim H^1(S^2F) + \dim H^1(\Lambda^2 F) \tag{7}$$

consider the power series

$$Z_{\mathbf{c}}(\tau) = \sum_k c_{top}(\mathcal{S}_k \oplus \mathcal{L}_k)\, q^k\,, \tag{8}$$

where r and c_1 are fixed; the same is assumed for the rest power series.

d) Under the assumption

$$\dim \mathcal{M}(r, c_1, k) = (r+2) \dim H^1(F) + \dim H^1(\Lambda^2 F) \tag{9}$$

consider the power series

$$Z_{\mathbf{d}}(\tau) = \sum_{k \geqslant 0} c_{top}(\mathcal{H}_k)^{r+2} c_{top}(\mathcal{L}_k) q^k\,. \tag{10}$$

e) Under the assumption

$$\dim \mathcal{M}(r, c_1, k) = (r-2) \dim H^1(F) + \dim H^1(\mathrm{S}^2 F) \tag{11}$$

consider the power series

$$Z_{\mathbf{e}}(\tau) = \sum_{k \geqslant 0} c_{top}(\mathcal{H}_k)^{r-2} c_{top}(\mathcal{S}_k) q^k\,. \tag{12}$$

f) Under the assumption

$$\dim \mathcal{M}(r, c_1, k) = 4 \dim H^1(F) + 2 \dim H^1(\Lambda^2 F) \tag{13}$$

consider the power series

$$Z_{\mathbf{f}}(\tau) = \sum_{k \geqslant 0} c_{top}(\mathcal{H}_k)^4 c_{top}(\mathcal{L}_k)^2 q^k\,. \tag{14}$$

For the power series (1) and (5) there are descriptions of gauge theories such that $Z_{\mathbf{a}}(\tau)$ and $Z_{\mathbf{b}}(\tau)$ can be interpreted as quantum correlation functions calculated in semi-classical limit (see [16] for $Z_{\mathbf{a}}(\tau)$, where $r = 2$, and [8] for $Z_{\mathbf{b}}(\tau)$). The rest power series are composed in a similar way from mathematical viewpoint, but I believe that they have analogous physical meaning.

Observe that for all cases except a) one can construct correlation functions in different ways. In particular, for b) there are $2r+1$ sub-cases that correspond to replacing k of $2r$ copies of $H^1(F)$ with $H^1(F^*)$ in the right hand side of (4). However, it is difficult to calculate $\dim H^1(F^*)$, so, we consider only sub-case described above. For the cases c), d), e), f) the situation is similar.

The case c) was considered in [7]; there is a unique solution to the system of (6) and (7): S is the plane with 8 blowed points, $r = 3$ and $c_1 = K$.

In the case b) the restrictions (2) and (4) distinguish a subset in the set of (r, c_1, k), the most interesting elements of which are related to exceptional bundles. It is shown in [8] that any exceptional bundle that satisfies (2) and has zero cohomology groups produces a correlation function for testing S-duality. Namely, each such a bundle E is initial (zero-dimensional) term of the sequence of moduli spaces $\overline{\mathcal{M}}(r(E), c_1(E), k)$, $k \geqslant 0$, that gives a power series of the form (5). We describe this relation in Section 2.

The set of exceptional sheaves has an interesting combinatorial structure: there are some transformations, so called mutations, which permit to obtain an infinite set of exceptional sheaves from a finite exceptional collection. Moreover, it is often possible to choose the initial exceptional collection such that the sheaves in it are locally free, and the mutations give new exceptional bundles. Herewith, if all cohomology groups of exceptional bundles in the initial collection are trivial, then the same is true for the new exceptional bundles obtained by mutations. Thus, there arises the following question: do the mutations induce any transformations of correlation functions? (See Example 2.3.) For the moment there is no any answer. The first step in attempt to answer this question is certainly related to looking for transformations of the above-mentioned moduli spaces $\overline{\mathcal{M}}(r(E), c_1(E), k)$ that generalize the mutations. We discuss an approach to possible constructions of such transformations for moduli spaces over $\mathbb{P}^2$ in Section 4.

In Section 3 we examine the exceptional bundles that give correlation functions for d), e) and f) cases. There are "less" such bundles than in the case b) because the degree (i. e. $c_1 \cdot (-K)$) of such a bundle is a function of its rank (see 3.2). However, this condition can be compatible with mutations as it is shown in Example 3.4.

In Section 6 we give a description of a part of a moduli space $\overline{\mathcal{M}}(r(E), c_1(E), k)$ over $\mathbb{P}^2$ consisting of non-locally-free sheaves, hoping that this can help to construct transformations of moduli spaces.

I am grateful to Max-Planck Institute für Mathematik in Bonn for hospitality, to A. N. Tyurin for attention to this work, and to A. Levin for useful discussions, which helped me to understand the relation between Dynkin index and second Chern class.

I dedicate this work to memory of my grandmother.

1 Beta-function

This is a function of QFT under consideration. For a classical gauge theory one can calculate beta-function as well. It depends on representations of fiber-wise gauge group that arise from the action on fields. Such a collection of representations we call field configuration of the given theory. So, we start with representations.

Let γ be an irreducible finite dimensional complex representation of $su(r)$ (the Lie algebra for the fiber-wise gauge group), can be the canonical representation of $su(r)$ in $\mathbb{C}^r$. Consider the invariant form

$$B_\gamma(X;Y) = Tr(\gamma(X)\,\gamma(Y))\,.$$

It is well-known that all such forms are proportional to each other. Denote by $c_2(\gamma)$ the coefficient between B_γ and the trace form

$$(X;Y) := Tr(X\,Y) = B_{can}(X;Y)\,. \tag{15}$$

We have:

$$B_\gamma(X;Y) = c_2(\gamma)(X;Y)\,.$$

The number $c_2(\gamma)$ is positive integer, it is called Dynkin index of the representation γ.

One can define c_2 for an irreducible $u(r)$-representation $\widetilde{\gamma}$ as $c_2(\gamma)$, where γ is the restriction of $\widetilde{\gamma}$ on the subalgebra $su(r) \subset u(r)$. All the results of this section hold also for $u(r)$-representations.

For the dual representation we have: $B_{\gamma^*}(X;Y) = B_\gamma(X;Y)$. Therefore, we obtain:

$$c_2(\gamma^*) = c_2(\gamma)\,. \tag{16}$$

Any complex representation of $su(r)$ can be extended to its complexification $sl(r,\mathbb{C})$ without changing c_2, so we consider γ as an irreducible finite dimensional representation of $sl(r,\mathbb{C})$. There is a well-known formula that expresses $c_2(\gamma)$ in terms of the higher weight.

1.1 Proposition. *Let R be the root system for $sl(r,\mathbb{C})$, λ be the highest weight of γ with respect to a base of R, ρ be the half-sum of the positive roots. Then*

$$c_2(\gamma) = \frac{(\lambda;\lambda+2\rho)}{r^2-1}\dim\Gamma = \frac{(\lambda;\lambda+2\rho)}{r^2-1}\prod_{\alpha\in R_+}\frac{(\lambda+\rho;\alpha)}{(\rho;\alpha)}\,. \tag{17}$$

Here Γ is the representation space for γ and $(\,;\,)$ is the dual to the trace form (15).

1.2 Corollary. ([8])

$$c_2(\Lambda^k(can)) = \binom{r-2}{k-1}\,,\quad c_2(\mathrm{S}^k(can)) = \binom{r+k}{k-1}\quad\text{and}\quad c_2(ad) = 2\,r\,. \tag{18}$$

Consider a finite collection of representations $\{\gamma_1,\dots,\gamma_{N_a}\}$, among which there are some isomorphic to each other[2]. The beta-function takes the collection $\{\gamma_1,\dots,\gamma_{N_a}\}$ to the value

$$\beta(\{\gamma_i\}) = -c_2(ad) + \sum_i c_2(\gamma_i)\,, \tag{19}$$

(see [1]). We describe here all cases when $\beta(\{\gamma_i\}) = 0$. The positiveness of $c_2(\gamma)$ imposes an essential restriction on representations included in field configuration with zero beta-function (19):

$$c_2(\gamma) \leqslant 2\,r = c_2(ad)\,. \tag{20}$$

The above-mentioned values of c_2 for $\Lambda^k(can)$ and $\mathrm{S}^k(can)$ show that there is "a small amount" of such representations, since the c_2's grow as polynomials of r. Here is the exact statement.

1.3 Theorem. **([9])** *For $3\leqslant r\leqslant 5$ or $r\geqslant 9$ the list of all irreducible representations γ of $su(r)$ such that $c_2(\gamma)\leqslant c_2(ad)$, is as follows:*

$$ad\,,\ can\,,\ can^*\cong\Lambda^{r-1}(can)\,,\ \mathrm{S}^2(can)\,,\ \mathrm{S}^2(can^*)\,,\ \Lambda^2(can)\ \textit{and}$$
$$\Lambda^2(can^*)\cong\Lambda^{r-2}(can)\,.$$

For $r = 6,7,8$ there are additional representations

$$\Lambda^3(can)\ \textit{and}\ \Lambda^3(can^*)\cong\Lambda^{r-3}(can)\,.$$

[2]From physical viewpoint this collection describes configuration of fields for QFT under consideration. A subcollection consisting of isomorphic to each other representations corresponds to a particle, the number of representations in the subcollection is the number N_f of flavours of the particle, and the dimension r of the canonical representation of the gauge group is the number of colors. So, N_a is the joint number for the flavours of particles in our theory.

Now it is clear which form can have a collection of representations with zero beta-function. By the definition, $c_2(can) = 1$, so any collection $\{\gamma_i\}$ with $\beta(\{\gamma_i\}) < 0$ can be completed by an appropriate number of can's or can^*'s to a collection with $\beta = 0$. All such collections are described in the following statement.

1.4 Proposition. ([9]) *Let $\{\gamma_1, \ldots, \gamma_{N_a}\}$ be a collection of irreducible (possibly equivalent) representations of $su(r)$ such that $\beta(\{\gamma_i\}) = 0$. Then $\{\gamma_i\}$ is one of the following collections.*

1. *For $r = 2$ the cases* a) *and* b) *below; for $r = 3$ —* a),b),c)*; for $r = 4, 5$ and $r \geqslant 9$ — the cases* a)$, \ldots,$f)*.*

a) *$N_a = 1$ and $\gamma_1 = ad$.*
b) *$N_a = 2r$ and each of γ_i is either can or $can^* \cong \Lambda^{r-1}(can)$.*
c) *$N_a = 2$, one of γ_i is either $\Lambda^2(can)$, or $\Lambda^2(can^*)$ and the other is either $\mathrm{S}^2(can)$ or $\mathrm{S}^2(can^*)$.*
d) *$N_a = r + 3$, one of γ_i is either $\Lambda^2(can)$ or $\Lambda^2(can^*)$ and the other $r + 2$ representations are either can or can^*.*
e) *$N_a = r - 1$, one of γ_i is either $\mathrm{S}^2(can)$, or $\mathrm{S}^2(can^*)$ and the other $r - 2$ representations are either can or can^*.*
f) *$N_a = 6$, two of γ_i are either $\Lambda^2(can)$, or $\Lambda^2(can^*)$ and the rest four are either can or can^*.*

2. *For $r = 6$ the following cases add to above-mentioned cases:*

g_6) $N_a = 7$, one of γ_i is either $\Lambda^3(can)$ or $\Lambda^3(can^*)$ and the rest six are either can or can^*.
h_6) $N_a = 4$, one of γ_i is either $\Lambda^3(can)$ or $\Lambda^3(can^*)$, another one is either $\Lambda^2(can)$ or $\Lambda^2(can^*)$ and the rest two are either can or can^*.
i_6) $N_a = 2$, each representation is either $\Lambda^3(can)$ or $\Lambda^3(can^*)$.

3. *For $r = 7$ the following case adds to* a)$, \ldots,$f)*:*

g_7) $N_a = 5$, one of γ_i is either $\Lambda^3(can)$ or $\Lambda^3(can^*)$ and the rest four are either can or can^*.

4. *For $r = 8$ the following case adds to* a)$, \ldots,$f)*:*

g_8) $N_a = 2$, one of γ_i is either $\Lambda^3(can)$, or $\Lambda^3(can^*)$ and the other is either can or can^*.

For $su(2)$ and $su(3)$ in some of the cases above the algebro-geometric problem is been studied by method of Vafa and Witten. For instance, for $\mathbf{P}^2$ and $r = 2$ the case a) is investigated in [16], the case b) in [5]. Geometric realization of the case b) for del Pezzo surfaces is mystically connected with exceptional bundles ([8], [9]). We briefly describe this connection in the next section.

1.5 Dynkin index and second Chern class.

The relation between Dynkin indices and second Chern classes is, of course, known for experts. We describe it here because this fact explains why the conditions on dimensions (4), (7), (9), (11) and (13) expressed via topological invariants are independent of the second Chern class.

Let E be a vector bundle with structure group G. We restrict our attention to the cases when G is one of the following groups:

$$SU(r), \quad SL(r,\mathbb{C}), \quad U(r), \quad GL(r,\mathbb{C}).$$

Suppose the bundle E is given by trivializations $\left\{U_\alpha,\ \varphi_\alpha : E|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times \mathbb{C}^r\right\}$ and transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$.

For a complex representation $\gamma : G \to GL(\Gamma)$ define a new vector bundle γE to have trivializations $\gamma E|_{U_\alpha} \cong U_\alpha \times \Gamma$ and transition functions $\gamma \circ g_{\alpha\beta}$. The correspondent representation of Lie algebra $\mathfrak{g}$ of the group G we denote also by γ. Then the relation between Dynkin index $c_2(\gamma)$ and second Chern classes $c_2(E)$, $c_2(\gamma E)$ is as follows

$$c_2(\gamma E) = c_2(\gamma)\, c_2(E) \quad \text{for} \quad G = SU(r),\ SL(r,\mathbb{C});$$

$$c_2(\gamma E) = c_2(\gamma)\, c_2(E) + N_\gamma\, c_1^2(E) \quad \text{for} \quad G = U(r),\ GL(r,\mathbb{C}),$$

where N_γ is some coefficient. In other words, the coefficient at $c_2(E)$ in the expression of $c_2(\gamma E)$ in terms of the Chern classes of the bundle E is just the Dynkin index $c_2(\gamma)$.

To explain these relations we use the definition of Chern classes via differential geometry [6]. Let a be a connection in the bundle E with the curvature Θ. If the curvature is given locally by collection $\{\Theta_\alpha\}$ of matrices of two-forms, then over $U_\alpha \cap U_\beta$ we have

$$\Theta_\alpha = g_{\alpha\beta} \Theta_\beta {g_{\alpha\beta}}^{-1}.$$

Therefore, if P is an invariant polynomial of degree m on $\mathfrak{g}$, i.e. such that $P(A) = P(gAg^{-1})$, $g \in G$; then $P(\Theta)$ is a differential $2m$-form. Moreover, $dP(\Theta) = 0$ and the cohomology class $[P(\Theta)] \in H^{2m}(M)$ does not depend on Θ (see [6, 3.3]). The Chern classes of E are defined as

$$c_k(E) = \left[P_k(\tfrac{i}{2\pi}\Theta)\right] \in H^{2k}(M),$$

where P_k are the basic invariant polynomials such that

$$det(A + tI) = \sum_{k=0}^{r} P_k(A) t^{r-k}.$$

We have: $P_0 \equiv 1$, $P_1(A) = \operatorname{tr}(A)$, and $P_r(A) = \det(A)$. More generally, $P_k(A) = \operatorname{tr}(\wedge^k A)$, where elements of matrix $\wedge^k A$ are minors of k-th order of A. Observe that the basic invariant polynomials are nothing else as elementary symmetric functions of the eigenvalues.

It is clear now that the coefficients is the expression of $c_2(\gamma E)$ via $c_2(E)$ and $c_1^2(E)$ are exactly the same as the ones in expression of the invariant polynomial $P_2(\gamma(A))$ via $P_2(A)$ and $P_1^2(A)$. For our purposes it is sufficient to determine the coefficient at $P_2(A)$, so we can consider the restriction to the set of traceless matrices. From the identity

$$\operatorname{tr}(A^2) = (\operatorname{tr} A)^2 - 2P_2(A)$$

it follows that

$$\operatorname{tr}(A^2) = -2P_2(A) \quad \text{whenever} \quad \operatorname{tr} A = 0.$$

From the other hand, by the definition of Dynkin index, we have

$$\operatorname{tr}(\gamma(A)^2) = c_2(\gamma)\operatorname{tr}(A^2) \quad \text{for traceless matrices } A.$$

This implies our assertions.

2 Exceptional bundles in realization of the case b)

For a coherent sheaves E, F over del Pezzo surface S we use the following notation:

$$\mu(E) := \frac{c_1(E)\cdot(-K)}{r(E)} \text{ the slope;} \quad d(E) := c_1(E)\cdot(-K) \text{ the degree;}$$

$$h^i(E) := \dim H^i(E); \quad h^i(E,F) := \dim \mathrm{Ext}^i(E,F); \quad \chi(E,F) := \sum_i (-1)^i h^i(E,F)\,.$$

The image of E in $K_0(S)$ we represent by the vector

$$v(E) = (r(E), c_1(E), -\chi(E))\,.$$

The last component is related with $c_2(E)$ by the Riemann – Roch formula

$$-\chi(E) = c_2(E) - \frac{c_1(E)^2}{2} + \frac{c_1(E)\cdot K}{2} - r(E) = -\operatorname{ch}_2(E) + \frac{c_1(E)\cdot K}{2} - r(E)\,. \tag{21}$$

Now it is easy to express $\chi(E,F)$ calculated by the Riemann – Roch formula

$$\chi(E,F) = r(E)r(F) + \frac{r(E)d(F)}{2} - \frac{r(F)d(E)}{2} + r(E)\operatorname{ch}_2(F) + r(F)\operatorname{ch}_2(E) - c_1(E)\cdot c_1(F) \tag{22}$$

in terms of $-\chi(E)$ and $-\chi(F)$. We have:

$$\chi(E,F) = -r(E)r(F) + r(F)c_1(E)\cdot K - c_1(E)\cdot c_1(F) + r(E)\chi(F) + r(F)\chi(E) \tag{23}$$

In this section we interpret $\mathcal{M}(r,c_1,k)$ as the space of orbits of solutions to fields equations by gauge group action (see details in [8]). We consider correlation function (5) under assumptions (2) and (4) made in the introduction.

For a stable sheaf F with $(r(F), c_1(F), -\chi(F)) = (r, c_1, k)$ satisfying (2) the equality (4) is equivalent to

$$c_1^2 - r\,c_1\cdot K + r^2 + 1 = 0\,. \tag{24}$$

In particular, this condition does not depend of k and hence of $c_2(F)$.

Indeed, we have the formula (3). Besides, $\dim \mathrm{Hom}(F,F) = 1$ by the properties of stability. Using Serre duality, we get: $\dim \mathrm{Ext}^2(F,F) = \dim \mathrm{Hom}(F, F\otimes K) = 0$. It follows that

$$\dim \mathcal{M}(r,c_1,k) = \dim \mathrm{Ext}^1(F,F) = 1 - \chi(F,F) = 2rc_2 - (r-1)c_1^2 - r^2 + 1 \tag{25}$$

the last equality holds by Riemann – Roch theorem. Substituting this expression and (3) to (4), we obtain (24).

Recall that a sheaf E is exceptional if $\dim \mathrm{Hom}(E,E) = 1$ and $\mathrm{Ext}^i(E,E) = 0$ for $i > 0$. It is proved in [10] that exceptional sheaf over del Pezzo surface is either locally free or torsion sheaf of the form $\mathcal{O}_\ell(d)$, where ℓ is an exceptional curve. Consider the full subcategory

$$\mathcal{O}^\perp = \{F :\ \mathrm{Ext}^i(\mathcal{O}, F) = 0\}$$

in the category of coherent sheaves over S.

2.1 Theorem. ([9]) *Suppose that the invariants* (r, c_1) *of exceptional bundle* E *satisfy the equality (2); then the following statements are equivalent:*
(i) $E \in \mathcal{O}^{\perp}$
(ii) *For any semistable sheaf* F *with the invariants* (r, c_1, k), $k \geqslant 0$ *the following equality holds:*

$$\dim \overline{\mathcal{M}}(r, c_1, k) = 2r \dim H^1(F)$$

(This is exactly (4) for the sheaf F *.)*

Proof. It is known from [10] that exceptional bundles over S are stable in the sense of Mumford – Takemoto w.r.t. $-K_S$. This implies that $H^0(E) = H^2(E) = 0$. Therefore, E is in $\mathcal{O}^{\perp}$ if and only if $H^1(E) = 0$. This is equivalent to (4) and to (24) for E because $\dim \overline{\mathcal{M}}(r, c_1, 0) = 0$ by exceptionality of E. Since the equality (24) is independent of k, we obtain the required. Observe that there are no semistable sheaves F with the invariants (r, c_1, l) for $l < 0$ because in this case formula (25) gives negative dimension of $\mathrm{Ext}^1(F, F)$. This concludes the proof. □

The theorem means that each exceptional bundle from $\mathcal{O}^{\perp}$ satisfying (2) gives a correlation function for testing S-duality for the field configuration $(can, \ldots, can)$ of the type b).

2.2 Remark. Under the conditions of the theorem all the moduli spaces $\overline{\mathcal{M}}(r, c_1, k)$, $k \geqslant 0$, are smooth, projective, and carry universal families. Indeed, for any exceptional bundle E the integers $r(E)$ and $c_1(E) \cdot K$ are coprime [10].

2.3 Example. $S = \mathbb{P}^2$. Over the projective plane the exceptional bundles belonging to the subcategory $\mathcal{O}^{\perp}$ form a sequence E_n such that $E_{-1} = \mathcal{O}_{\mathbb{P}^2}(-2)$, $E_0 = \mathcal{O}_{\mathbb{P}^2}(-1)$ and the other E_n can be defined from the exact triples

$$0 \longrightarrow E_{n-1} \longrightarrow \mathrm{Hom}(E_n, E_{n+1}) \otimes E_n \longrightarrow E_{n+1} \longrightarrow 0 . \tag{26}$$

There is a canonical identification $\mathrm{Hom}(E_n, E_{n+1}) \cong \mathrm{Hom}(E_{n-1}, E_n)^*$ and the both morphisms of bundles are canonical. Observe that $E_1 \cong \Omega\mathbb{P}^2(1) \cong \mathrm{T}\mathbb{P}^2(-2)$ and for $n = 0$ the exact triple (26) is the Euler sequence twisted by (-2).

The equality (24) that is equivalent to (4) has the following interpretation: the pair (r, c_1) satisfies(24) iff the triple $(r, -c_1, 1)$ is a solution of Markov equation

$$x^2 + y^2 + z^2 = 3\,xyz\ , \tag{27}$$

as long as $c_1 \cdot K = -3\,c_1$ for $\mathbb{P}^2$.

The solution set of (27) is well-known [3]. Namely, any solution can be obtained from $(1, 1, 1)$ by a finite number of mutations. A mutation is changing one variable to another root of the quadratic equation while the rest variables are fixed. So, for $z = 1$ we have a sequence of mutations that gives the set of all solutions (r, c_1) to (24) for $\mathbb{P}^2$. Here are several first solutions such that $r > -c_1$:

$$(2, -1)\,,\ (5, -2)\,,\ (13, -5)\,,\ (34, -13)\,,\ (89, -34)\,,\ (233, -89)\,,\ \ldots \tag{28}$$

The pair after (r, c_1) in this sequence is $(3r + c_1, -r)$.

It is known also that the collection of ranks of exceptional bundles from a helix coil[3] is a solution to (27). Conversely, for any solution of (27) there is an infinitely many helix coils (because of twisting) having the rank collection just the same as the given solution. The set of solutions of (27) of the form $(x, y, 1)$ corresponds up to twisting to the set of helix coils $(E_n, E_{n+1}, \mathcal{O}_{\mathbb{P}^2})$. Herewith, if $r(E_n) = r_n$, then $c_1(E_n) = -r_{n-1}$.

The bundles $E_n = \overline{\mathcal{M}}(r_n, -r_{n-1}, 0)$ are the initial points for the sequences of moduli spaces $\overline{\mathcal{M}}(r_n, -r_{n-1}, k)$, where $k \geqslant 0$. Therefore, we have the diagram:

$$\begin{array}{ccccl}
\vdots & \vdots & \vdots & & \\
\mathcal{O}_{\mathbb{P}^2}(-2) & \mathrm{Hilb}^1(\mathbb{P}^2) & \mathrm{Hilb}^2(\mathbb{P}^2) & \dots & Z_{-1} \text{ is a polynomial of degree } \leqslant 4 \\
 & \| & & & \\
\mathcal{O}_{\mathbb{P}^2}(-1) & \mathbb{P}^2 & \mathrm{Hilb}^2(\mathbb{P}^2) & \dots & Z_0 = 1 + q \\
\Omega_{\mathbb{P}^2}(1) & \overline{\mathcal{M}}(2,-1,1) & \overline{\mathcal{M}}(2,-1,2) & \dots & Z_1 = q + 13q^4 + 729q^5 + 85025q^6 + \\
 & & & & \qquad +15650066q^7 + \cdots \\
E_2 & \overline{\mathcal{M}}(5,-2,1) & \overline{\mathcal{M}}(5,-2,2) & \dots & \\
E_3 & \overline{\mathcal{M}}(13,-5,1) & \overline{\mathcal{M}}(13,-5,2) & \dots & \\
\vdots & \vdots & \vdots & \ddots &
\end{array}$$

Each row of this diagram produces a partition function Z_n. All what is known about these functions for the moment is written in the right hand side. On the other hand, there are mutations (26) for the terms of the first column.

Question. Are there transformations between the whole rows of the diagram that generalize the mutations (26)?

Conjecture. If such a transformation exists, then one can calculate all the partition functions knowing only two of them.

Observe that the first seven coefficients of Z_1 are calculated in [5] with the aid of the Bott's formula. Unfortunately, the authors of this paper don't receive a general formula or recurrent relation for the coefficients because the combinatorial structure of fixed components of the torus action on $\overline{\mathcal{M}}(2, -1, k)$ is very complicated. The calculations have been performed on computer using MAPLE V.3. In the first part of [5] there is given an explicit description of $\overline{\mathcal{M}}(2, -1, k)$. Similar description for all $\overline{\mathcal{M}}(r_n, -r_{n-1}, k)$ and for some series of moduli spaces over $\mathbb{P}^1 \times \mathbb{P}^1$, which start with exceptional bundles satisfying the conditions of 2.1, is given in [12].

2.4 Remark. 1. For the case $S = \mathbf{P}^2$ it can be easily checked that the inequality (2) holds for all bundles E_n from $\mathcal{O}^\perp$. However, it is not so in general. For example, the line bundles $\mathcal{O}(-1, m)$ over $\mathbb{P}^1 \times \mathbb{P}^1$ belong to $\mathcal{O}^\perp$ and have $c_1 \cdot (-K_{\mathbb{P}^1 \times \mathbb{P}^1}) = 2\,(m-1)$.

2. The sequence of bundles E_n from Example 2.3 is a particular case of system of bundles generated by exceptional pair. Such a system is defined as follows: the pair $(E_i,\, E_{i+1})$ is

[3] the same as full exceptional collection for $\mathbb{P}^2$

the right mutation of (E_{i-1}, E_i) and the left mutation of (E_{i+1}, E_{i+2}) (see [4] or [10] for mutations). Herewith, if an exceptional pair belongs to $\mathcal{O}^\perp$, then all the system generated by this pair also belongs to $\mathcal{O}^\perp$. Therefore, there appears a diagram of moduli spaces similar to the above-mentioned one (because of the restriction (2) the diagram can be bounded). For such a diagram we put the similar question and the similar conjecture as above.

3 Exceptional bundles in realization of d), e) and f) cases.

In this section we deal with power series (10),(12) and (14) under the conditions (9),(11) and (13) respectively. Studying modular properties of the series $Z_{\mathbf{d}}(\tau)$, $Z_{\mathbf{e}}(\tau)$ and $Z_{\mathbf{f}}(\tau)$ is a problem related, as we believe, to mathematical testing of S-duality. However, we don't examine this here. Our interest in this section is in exceptional bundles that generate these power series.

3.1 Proposition. *Let F be a semi-stable sheaf over del Pezzo surface with the invariants (r, c_1, k) such that $r \geqslant 3$ and*

$$-\frac{1}{2}K^2 < -\frac{c_1 \cdot K}{r} < 0\,. \tag{6}$$

Then the spaces $H^i(\mathrm{S}^2F)$ and $H^i(\Lambda^2F)$ for $i \neq 1$ are zero.

Proof. The spaces $H^0(\mathrm{S}^2F)$ and $H^0(\Lambda^2F)$ are naturally identified with subspaces of $H^0(F\otimes F)$, which is contained in $H^0(F^{**}\otimes F)$ because of the canonical embedding $F \hookrightarrow F^{**}$. The last space is isomorphic to $\mathrm{Hom}(\tilde{F}, F)$, where $\tilde{F}$ is the dual bundle to F^{**}. The singularity set of F is at least 2-codimensional since F has no torsion, whence $c_1(F) = c_1(F^{**}) = -c_1(\tilde{F})$. From Lemma 6.1 it follows that F^{**} is a semistable bundle, then by the equality $\mu(F) < 0 < \mu(\tilde{F})$ we have: $\mathrm{Hom}(\tilde{F}, F) = 0$. Thus, we get: $H^0(\mathrm{S}^2F) = H^0(\Lambda^2F) = 0$.

The spaces $H^2(\mathrm{S}^2F) = \mathrm{Hom}(\mathrm{S}^2F, K)^*$ and $H^2(\Lambda^2F) = \mathrm{Hom}(\Lambda^2F, K)^*$ are naturally identified with subspaces in $\mathrm{Hom}(F\otimes F, K)^*$. Applying the functor $\mathrm{Hom}(-, K)$ to the exact sequence

$$0 \longrightarrow F\otimes F \longrightarrow F^{**}\otimes F^{**} \longrightarrow Q \longrightarrow 0\,,$$

we have: $\mathrm{Hom}(F\otimes F, K) \cong \mathrm{Hom}(F^{**}\otimes F^{**}, K)$ because $r(Q) = 0$, $c_1(Q) = 0$, and thus $\mathrm{Hom}(Q, K) = \mathrm{Ext}^1(Q, K) = 0$. The sheaf F^{**} is locally free, whence

$$\mathrm{Hom}(F^{**}\otimes F^{**}, K) = H^0(\tilde{F}\otimes\tilde{F}\otimes K) = \mathrm{Hom}(F^{**}, \tilde{F}\otimes K)\,.$$

Since $0 < \mu(\tilde{F}) < \frac{1}{2}K^2$, we have:

$$\mu(\tilde{F}\otimes K) < -\frac{1}{2}K^2 < \mu(F) = \mu(F^{**})\,.$$

It follows that $\mathrm{Hom}(F^{**}, \tilde{F}\otimes K) = 0$ and thus $H^2(\mathrm{S}^2F) = H^2(\Lambda^2F) = 0$. The statement is proved. $\square$

The proposition implies that the dimensions of H^1 of the symmetric and exterior square of F can be expressed in terms of the topological invariants via Riemann – Roch formula:

$$\dim H^1(\mathrm{S}^2F) = -\chi(\mathrm{S}^2F) = (r+2)k - \frac{r+3}{2}c_1^2 + \frac{r+1}{2}c_1\cdot K - \frac{r(r+1)}{2}\,,$$

$$\dim H^1(\Lambda^2 F) = -\chi(\Lambda^2 F) = (r-2)k - \frac{r-1}{2}c_1^2 + \frac{r-1}{2}c_1\cdot K - \frac{r(r-1)}{2}\,.$$

Therefore, the equalities (9) and (11) can be expressed by the following way:

$$\textit{case d)}: \qquad (9) \iff \frac{3}{2}c_1^2 - \left(r+\frac{1}{2}\right) c_1\cdot K + \frac{1}{2}(r^2+3r+2) = 0\,. \tag{29}$$

$$\textit{case e)}: \qquad (11) \iff \frac{3}{2}c_1^2 - \left(r-\frac{1}{2}\right) c_1\cdot K + \frac{1}{2}(r^2-3r+2) = 0\,. \tag{30}$$

As well as (24), these conditions don't depend on k and, hence, on c_2.

Let now E be exceptional bundle. We say that E *generates correlation function for the case* d), e) *or* f) if for any semistable sheaf $F \in \overline{\mathcal{M}}(r, c_1, k)$, where $r = r(E)$, $c_1 = c_1(E)$, $k \geqslant 0$, the equality (9), (11) or respectively (13) holds. In other words, the correspondent power series (10), (12) or (14) is well-defined.

3.2 Theorem. *Suppose exceptional bundle E satisfies $-K^2/2 < \mu(E) < 0$. Then the following statements hold:*

1. *E generates correlation function for the case* d) *iff $E \in \mathcal{O}^\perp$ and $d(E) = 1 - 2r(E)$.*
2. *E generates correlation function for the case* e) *iff $E \in \mathcal{O}^\perp$ and $d(E) = -1 - 2r(E)$.*
3. *E generates correlation function for the case* f) *iff $E \in \mathcal{O}^\perp$ and $d(E) = 1 - 2r(E)$.*

Proof. E generates correlation function for the case d) iff the invariants (r, c_1) of E satisfy (29) or equivalently E satisfies (9). The moduli space $\overline{\mathcal{M}}(r, c_1, 0)$ is zero-dimensional, whence we have

$$\dim H^1(E) = 0 \qquad \text{and} \qquad \dim H^1(\Lambda^2 E) = 0\,. \tag{31}$$

Thus, E generates correlation function for the case d) iff the invariants of E satisfy the both (24) and (29). Eliminating from these equalities the terms with c_1^2, we have:

$$\frac{1-r}{2}c_1\cdot K + r^2 - \frac{3}{2}r + \frac{1}{2} = 0\,,$$

whence after dividing by $\frac{1}{2}(1-r)$ we obtain: $c_1\cdot K = 2r - 1$.

From the proof of 2.1 it follows that (24) is equivalent to $H^1(E) = 0$ that under (6) means that $E \in \mathcal{O}^\perp$. This proves the first statement, the other statements are analogous. Observe that an exceptional bundle E generates correlation functions for d) and for f) cases equivalently, since each condition of (9) and (13) for E means that $H^1(E) = H^1(\Lambda^2 E) = 0$.
$\square$

3.3 Remark. For any exceptional bundle E the equality $d(E) = \pm 1 - 2r(E)$ is equivalent to $\mu(E) = \pm\frac{1}{r(E)} - 2$ as far as $d(E)$ and $r(E)$ are coprime.

It is not hard to check that there is no nontrivial examples of exceptional bundles over $\mathbb{P}^2$ that generate correlation functions for cases d)–f). Indeed, the exceptional bundles in subcategory $\mathcal{O}^\perp$ are E_n, $n \in \mathbb{Z}$ described in Example 2.3. It is readily seen that the sequence $\{\mu(E_n)\}$ is increasing. Consider two cases.

1. $n \geqslant 1$. The sequence $\{r(E_n)\}$ is increasing and the sequence $\{-2 + 1/r(E_n)\}$ is decreasing; besides, $\mu(E_n) = -2 + 1/r(E_n)$ for $n = 1$, $E_1 = \Omega\mathbb{P}^2(1)$. It is clear that $\{\mu(E_n)\} \neq \{-2 + 1/r(E_n)\}$ for $n > 1$. The equality $\mu(E_n) = -2 - 1/r(E_n)$ is impossible because $\mu(E_n) > -2$ for $n \geqslant 1$.

2. $n \leqslant 0$. In this case the equality $\mu(E_n) = -2+1/r(E_n)$ is impossible because $\mu(E_n) < -2$. The another equality $\mu(E_n) = -2 - 1/r(E_n)$ holds only for $n = 0$, $E_0 = \mathcal{O}_{\mathbb{P}^2}(-1)$, since $\mu(E_n) < -3 < -2 - 1/r(E_n)$ for $n < 0$.

Obviously, there is no exceptional bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ generating correlation function for the cases d)–f) because $c_1 \cdot K_{\mathbb{P}^1\times\mathbb{P}^1}$ is even.

In the rest of this section we give an example of exceptional bundles over the plane with five blowed points that generate correlation functions for d) and f) cases.

Denote by X_m the plane with m blowed points in general position, $1 \leqslant m \leqslant 8$. We have:

$$\operatorname{Pic} X_m = \mathbb{Z}\ell_0 \oplus \mathbb{Z}\ell_1 \oplus \mathbb{Z}\ell_2 \oplus \cdots \oplus \mathbb{Z}\ell_m ,$$

where ℓ_0 is the preimage of the line in $\mathbb{P}^2$ and ℓ_i, $1 \leqslant i \leqslant m$ are exceptional curves i.e. preimages of the blowed points. The canonical class of X_m is

$$K = -3\ell_0 + \sum_{i=1}^{m} \ell_i ,$$

the intersection form is given by the relations

$$\ell_0^2 = 1\,,\ \ell_i^2 = -1\,,\ 1 \leqslant i \leqslant m; \quad \ell_i \cdot \ell_j = 0\,,\ i \neq j\,.$$

Put $c_1 = a\ell_0 + \sum_{i=1}^{m} b_i\ell_i$, then we have: $c_1 \cdot K = -3a - \sum_{i=1}^{m} b_i$; $c_1^2 = a^2 - \sum_{i=1}^{m} b_i^2$; $K^2 = \mu(-K) = 9 - m$.

Consider a standard full exceptional collection over X_m:

$$\mathcal{O}_{\ell_1}(-1), \ldots, \mathcal{O}_{\ell_m}(-1),\ \mathcal{O}_{X_m}(-\ell_0),\ \mathcal{O}_{X_m},\ \mathcal{O}_{X_m}(\ell_0).$$

Shifting the torsion sheaves $\mathcal{O}_{\ell_i}(-1) = \mathcal{O}_{\ell_i}(-\ell_0 + \ell_i)$ to the right over $\mathcal{O}_{X_m}(-\ell_0)$, we have the exceptional collection

$$\mathcal{O}_{X_m}(-\ell_0),\ \mathcal{O}_{X_m}(-\ell_0+\ell_1), \ldots, \mathcal{O}_{X_m}(-\ell_0+\ell_m),\ \mathcal{O}_{X_m},\ \mathcal{O}_{X_m}(\ell_0).$$

Shifting $\mathcal{O}_{X_m}(\ell_0)$ to the left over all the other sheaves, we obtain full exceptional collection

$$\mathcal{O}_{X_m}(\ell_0+K),\ \mathcal{O}_{X_m}(-\ell_0),\ \mathcal{O}_{X_m}(-\ell_0+\ell_1), \ldots, \mathcal{O}_{X_m}(-\ell_0+\ell_m),\ \mathcal{O}_{X_m}.$$

All the line bundles except $\mathcal{O}_{X_m}$ in this collection generate the subcategory $\mathcal{O}_{X_m}^{\perp}$ and all mutations of these bundles don't lead outside $\mathcal{O}_{X_m}^{\perp}$.

3.4 Example. Put $m = 5$. We have: $\mu(\mathcal{O}_{X_5}(\ell_0 + K)) = -1$ and $\mu(\mathcal{O}_{X_5}(-\ell_0)) = -3$, so the exceptional pair $(\mathcal{O}_{X_5}(\ell_0+K), \mathcal{O}_{X_5}(-\ell_0))$ is an ext-pair. Put $V = \operatorname{Ext}^1(\mathcal{O}_{X_5}(\ell_0 + K), \mathcal{O}_{X_5}(-\ell_0))$, $\dim V = 2$.

Consider the system of bundles $\{E_n\}_{n\in\mathbb{Z}}$ generated by the pair $(\mathcal{O}_{X_5}(\ell_0+K), \mathcal{O}_{X_5}(-\ell_0))$. By the definition, we have : $E_0 = \mathcal{O}_{X_5}(\ell_0+K)$, $E_1 = \mathcal{O}_{X_5}(-\ell_0)$, the bundles E_2 and E_{-1} are the universal extensions

$$0 \longrightarrow \mathcal{O}_{X_5}(-\ell_0) \otimes V^* \longrightarrow E_2 \longrightarrow \mathcal{O}_{X_5}(\ell_0+K) \longrightarrow 0\,,$$

$$0 \longrightarrow \mathcal{O}_{X_5}(-\ell_0) \longrightarrow E_{-1} \longrightarrow \mathcal{O}_{X_5}(\ell_0+K) \otimes V \longrightarrow 0\,,$$

the bundles E_{i-1} for $i \leqslant -1$ are included into exact triples

$$0 \longrightarrow E_{i-1} \longrightarrow E_i \otimes \mathrm{Hom}(E_i, E_{i+1}) \longrightarrow E_{i+1} \longrightarrow 0\,,$$

and the bundles E_{i+1}, $i \geqslant 2$ are included into the same exact triples with $\mathrm{Hom}(E_i, E_{i+1})$ replaced by canonically isomorphic to it $\mathrm{Hom}^*(E_{i-1}, E_i)$; herewith the both spaces are two-dimensional. It is easily checked that the bundles E_i, $i \leqslant -1$ have the invariants

$$r(E_i) = 2\cdot(-i)+1 \quad \text{and} \quad c_1(E_i) = 4i-1 = -2r(E_i)+1\,,$$

thus, by Theorem 3.2, these bundles generate correlation functions for the cases d) and f). Therefore, we have a diagram similar to the diagram from page 10, for which we put the same question and formulate the same conjecture as in page 10.

4 An approach to transformations of moduli spaces

In this section we look for approaches on possible constructions of transformations between the moduli spaces

$$\overline{\mathcal{M}}_n(\delta) = \overline{\mathcal{M}}(r_n, -r_{n-1}, \delta)$$

described in 2.3, which generalize the mutations (26) of the exceptional bundles $E_n = \overline{\mathcal{M}}_n(0)$.

We consider here only the case $n \geqslant 1$, for which $r_n > r_{n-1}$, and use the recurrent relation of minimal degree on ranks

$$r_{n+1} + r_{n-1} = 3r_n\,. \tag{32}$$

Remind that the triple of positive integers $(r_n, r_{n-1}, 1)$ satisfies Markov equation (27), whence

$$\begin{vmatrix} r_{n-1} & r_n \\ r_n & r_{n+1} \end{vmatrix} = r_{n-1}(3r_n - r_{n-1}) - 3r_n^2 = 1 \tag{33}$$

Denote by X_n semi-stable sheaf with $v(X_n) = (r_n, -r_{n-1}, \delta)$. Observe that X_n is stable as far as r_n and r_{n-1} are coprime.

Since $\dim \overline{\mathcal{M}}_n(\delta) = 2\, r_n\, \delta$, we easily see from (32) that

$$\dim \overline{\mathcal{M}}_{n-1}(\delta) + \dim \overline{\mathcal{M}}_{n+1}(\delta) = 3\, \dim \overline{\mathcal{M}}_n(\delta)\,. \tag{34}$$

Then the naive move is to take for the left hand side an extension of arbitrary X_{n+1} via arbitrary X_{n-1} and for the right hand side to look for a filtration of this extension with the factors (X_n, X_n', X_n''), where X_n' and X_n'' are stable sheaves with the same collection of invariants v as $v(X_n)$. But then there will be no equality of the vectors v because the third component

$$-\chi(X_{n-1}) = \delta, \quad -\chi(X_{n+1}) = \delta; \qquad -\chi(X_n) = -\chi(X_n') = -\chi(X_n'') = \delta\,.$$

So one needs to increase the $-\chi$-component in the left hand side. The sheaves are taken according to (34). In order to not destroy this equality one has to take again the same X_{n+1} or the same X_{n-1}; then the other components of the vectors v are balanced by adding appropriate number of exceptional bundles. Thus, possible construction would be based on (34) and one of the following equalities

$$v(E_{n-1}) + v(X_{n-1}) + 2v(X_{n+1}) = 3v(E_n) + 3v(X_n)\,, \tag{35}$$

or

$$2v(E) + v(X_{n+1}) + v(E_{n+1}) = 3v(E_n) + 3v(X_n)\,. \tag{36}$$

We formulate here more concrete assumptions on transformations of moduli spaces.

Conjectures. 1. Suppose X_{n-1} and X_{n+1} are given; then there is a way to construct a sheaf F having two filtrations with

$$\mathrm{Gr}^{(1)}(F) = E_{n-1} \oplus X_{n-1} \oplus X_{n+1} \oplus X_{n+1} \quad \text{and} \quad \mathrm{Gr}^{(2)}(F) = E_n \oplus E_n \oplus E_n \oplus X_n \oplus X'_n \oplus X''_n$$

where the last two sheaves have the same invariants as X_n.

2. This construction gives a "good" map (finite or birational isomorphism) between $\overline{\mathcal{M}}_{n-1}(\delta) \times \overline{\mathcal{M}}_{n+1}(\delta)$ and $\overline{\mathcal{M}}_n(\delta) \times \overline{\mathcal{M}}_n(\delta) \times \overline{\mathcal{M}}_n(\delta)$.

Let us analyze our conjectures. The following statement gives some necessary conditions for possible construction to exist.

4.1 Proposition. *Suppose there is an exact sequence*

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

of coherent sheaves with $\mu(F') = \mu(F) = \mu(F'')$. Then the following statements hold:
1) F is semistable if and only if F' and F'' are semistable.
2) suppose that F is semistable and has a semistable subsheaf E such that $\mu(E) < \mu(F)$ and the quotient $G = F/E$ is semistable; then $\mathrm{Hom}(E, F'') \neq 0$ and $\mathrm{Hom}(F', G) \neq 0$.

Proof. The first statement is well-known (see, for example, similar statement in [13, 1.4]). Let us prove the second one.

Observe that $\mu(F) < \mu(G)$. There is commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
0 & \longrightarrow & V & \longrightarrow & E & \xrightarrow{g} & F'' & \longrightarrow & W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & & & \\
0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0 & & \\
 & & {\scriptstyle f}\downarrow & & \downarrow & & & & & & \\
 & & G & = & G & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & W & & 0 & & & & & & \\
 & & \downarrow & & & & & & & & \\
 & & 0 & & & & & & & &
\end{array}$$

Here $V = \ker f = \ker g$ and $W = \operatorname{coker} f = \operatorname{coker} g$. Let us show that the maps g and f are non-zero. Indeed, if we had $g = 0$, then we would have $W = F''$ as a quotient-sheaf of G, that contradicts semi-stability of G. Similarly, if we had $f = 0$, then we would have $F' = V$ as a subsheaf of E, which violates semi-stability of E. □

Suppose that a construction conjectured above exists. Then by the first statement of the proposition, F must be semi-stable as far as the second filtration has stable quotients with the slope equal to $\mu(E_n)$. Hence, there is only one direction to make extensions, namely, E_{n-1} and X_{n-1} in $\mathrm{Gr}^{(1)}(F)$ must be sub-sheaves, but not quotient-sheaves of F, and X_{n+1}'s must be quotient-sheaves, but not sub-sheaves of F.

The second statement leads to the following problem: to describe the cases when the spaces $\mathrm{Hom}(X_{n-1}, X_n)$ and $\mathrm{Hom}(E_{n-1}, X_n)$ are non-trivial. Some moves in this direction are made in the next section.

4.2 Theorem. *Suppose X_{n-1} and X_{n+1} are as above but with arbitrary $-\chi(X_{n-1})$ and $-\chi(X_{n+1})$; then any non-trivial extension*

$$0 \longrightarrow X_{n-1} \longrightarrow F \longrightarrow X_{n+1} \longrightarrow 0. \tag{37}$$

has the following properties:
(i) *if X_{n-1} is locally free, then F is semistable;*
(ii) *for X_{n-1} non-locally-free, F is semi-stable if and only if the corresponding to (37) co-cycle e doesn't belong to the image of the map $\mathrm{Hom}(X_{n+1}, X^{**}_{n-1}/X_{n-1}) \to \mathrm{Ext}^1(X_{n+1}, X_{n-1})$ induced by the canonical embedding $X_{n-1} \hookrightarrow X^{**}_{n-1}$;*
(iii) *if F is non-stable, then the Harder – Narasimhan filtration of F has only two terms $0 = F_0 \subset F_1 \subset F_2 = F$, where $(r(F_1), c_1(F_1)) = (r_{n+1}, -r_n)$ and $(r(F/F_1), c_1(F/F_1)) = (r_{n-1}, -r_{n-2})$.*

Proof. Observe that F has no torsion as an extension of torsion-free sheaves. Let $E \subset F$ be a subsheaf such that $0 < r(E) < r(F)$. Without loss of generality we can assume that the quotient $G = F/E$ is without torsion (see [15, Ch. II, 1.2.2]). Consider the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & E & \xrightarrow{id} & E & & \\
 & & & & \downarrow & & \downarrow\psi & & \\
 & & 0 \longrightarrow & X_{n-1} & \longrightarrow & F & \longrightarrow X_{n+1} & \longrightarrow 0 & \\
 & & & \mathrm{id}\downarrow & & \downarrow & \downarrow & & \\
0 \longrightarrow & I & \longrightarrow & X_{n-1} & \longrightarrow & G & \longrightarrow Q & \longrightarrow 0 & \\
 & & & & & \downarrow & \downarrow & & \\
 & & & & & 0 & 0 & &
\end{array} \tag{38}$$

where $I = \ker\psi$ and $Q = \operatorname{coker}\psi$. Observe that $r(F) = 3r_n$ and $c_1(F) = -3r_{n-1}$, so $\mu(F) = \mu(E_n)$.

Suppose $\mu(E) > \mu(F) > \mu(G)$; then $\psi \neq 0$, otherwise E is isomorphic to a subsheaf of X_{n-1} and has the slope less than the slope of F.

Consider the real (x, y)-plane. For a coherent sheaf E we denote by the same symbol E the point $(r(E), c_1(E))$. The equality (33) implies that the vectors $\overrightarrow{OE_n}$ and $\overrightarrow{OX_{n+1}}$ form a basis in the integral lattice of the plane (see Fig. 1, where for clearness different scales are chosen on the axes). If we had $\mu(E) \leqslant \mu(X_{n+1})$, then the point E would belong to the triangle OFA, where $A = OX_{n+1} \cap \{x = 3r_n\}$. But it is easily seen that the only integral points of this triangle are E_n, $B = (2r_n, -2r_{n-1})$, F and G because of the inequality $r_{n+1} + r_n > 3r_n$, which follows from $r_n > r_{n-1}$ and $r_{n+1} + r_{n-1} = 3r_n$. Thus, there are the following possibilities: $\mu(E) > \mu(X_{n+1})$ or $r(E) = r_{n+1}$ and $c_1(E) = -r_n$.

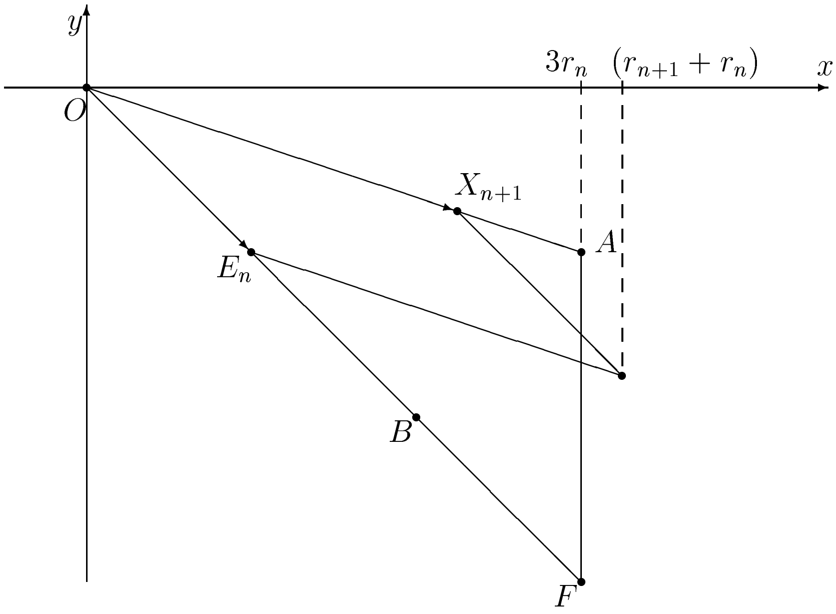

Fig. 1.

The case $\mu(E) > \mu(X_{n+1})$ is impossible because in this case $I \neq 0$ and from the stability of X_{n-1} it follows that $\mu(I) < \mu(X_{n-1}) < \mu(E)$, hence $\mu(I) < \mu(E) < \mu(J)$, where $J = \operatorname{im}\psi$ is a subsheaf of torsion-free sheaf X_{n+1}. We obtain: $\mu(J) > \mu(X_{n+1})$, that contradicts the stability.

Thus, we conclude that $r(E) = r_{n+1}$ and $c_1(E) = -r_n$. This implies immediately that $r(G) = r_{n-1}$ and $c_1(G) = -r_{n-2}$. It is impossible to have $I \neq 0$ because in this case $\mu(I) < \mu(X_{n-1}) < \mu(E) = \mu(X_{n+1}) < \mu(J)$, that contradicts the stability of X_{n+1}. Hence, $I = 0$, $r(J) = r_{n+1}$ and $c_1(J) = -r_n$. It follows that $r(Q) = 0$ and $c_1(Q) = 0$. Therefore, E is stable.

Let X_{n-1} be locally free, then for $Q \neq 0$ the lowest row of (38) is a trivial extension because $\operatorname{Ext}^1(Q, X_{n-1}) = 0$ (remind that $I = 0$). This contradicts the choice of G to be torsion-free. Thus, we have: $Q = 0$, $X_{n-1} \cong G$ and the extension (37) is trivial. Therefore, our assumption $\mu(E) > \mu(F) > \mu(G)$ is impossible and F is semistable. This proves (i).

Now suppose that X_{n-1} isn't locally free. Applying the functor $\operatorname{Hom}(X_{n+1}, -)$ to the exact triple

$$0 \longrightarrow X_{n-1} \longrightarrow X_{n-1}^{**} \longrightarrow X_{n-1}^{**}/X_{n-1} \longrightarrow 0\,,$$

and taking into account stability of X_{n-1}^{**}, which follows from 6.1, we have the exact sequence

$$0 \longrightarrow \operatorname{Hom}(X_{n+1}, X_{n-1}^{**}/X_{n-1}) \longrightarrow \operatorname{Ext}^1(X_{n+1}, X_{n-1}) \xrightarrow{\alpha} \operatorname{Ext}^1(X_{n+1}, X_{n-1}^{**})\,.$$

Suppose that the extension (37) is given by cocycle $e \in \operatorname{Ext}^1(X_{n+1}, X_{n-1})$, then there is the following diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X_{n-1} & \longrightarrow & F & \longrightarrow & X_{n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & X_{n-1}^{**} & \longrightarrow & \tilde{F} & \longrightarrow & X_{n+1} & \longrightarrow & 0 \\
 & & {\scriptstyle \psi''}\downarrow & & \downarrow{\scriptstyle \psi} & & & & \\
 & & Q & \xrightarrow{\mathrm{id}} & Q & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{39}
$$

Here $Q = X_{n-1}^{**}/X_{n-1}$ and the middle row is given by $\alpha(e)$. Clearly, e isn't in the image of $\mathrm{Hom}(X_{n+1}, X_{n-1}^{**}/X_{n-1})$ iff $\alpha(e) \neq 0$, in this case $\tilde{F}$ is semi-stable by (i), hence F is semi-stable by 6.1.

Suppose $\alpha(e) = 0$; then $\tilde{F} = X_{n+1} \oplus X_{n-1}^{**}$ and the map ψ from the diagram (39) is given by the pair of maps (ψ', ψ''), where ψ'' is contained in (39). Denote by Q' the image of ψ' and consider the following diagram, which has the middle column the same as in the diagram (39)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & X_{n+1} & \longrightarrow & X_{n+1} \oplus X_{n-1}^{**} & \longrightarrow & X_{n-1}^{**} & \longrightarrow & 0 \\
 & & {\scriptstyle \psi'}\downarrow & & \downarrow{\scriptstyle \psi} & & \downarrow & & \\
0 & \longrightarrow & Q' & \longrightarrow & Q & \longrightarrow & Q'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Here $E = \ker \psi'$, $(r(E), c_1(E)) = (r_{n+1}, -r_n)$, whence $\mu(E) > \mu(F)$ and F is non-stable. This proves (ii).

Let F be non-stable, then the first term F_1 of the Harder-Narasimhan filtration of F is the subsheaf of maximal rank among sub-sheaves of maximal Gieseker slope. This description determines F_1 uniquely up to isomorphism. In particular, $\mu(F_1) > \mu(F)$, the equality is impossible, otherwise F is semi-stable. But we have proved that any subsheaf of F with the slope greater than $\mu(F)$ has the rank r_{n+1} and the first Chern class $-r_n$. Moreover, there exists the diagram (38) with $E = F_1$, $I = 0$ and $r(Q) = 0$, $c_1(Q) = 0$. By 6.1, $G = F/F_1$ is stable, hence F/F_1 is Gieseker-stable. This implies that F/F_1 has maximal slope among sub-sheaves of F/F_1, so the second term of the Harder – Narasimhan filtration of F coincides with F. This concludes the proof. $\square$

4.3 Remark. In the conditions of the theorem, the set of cocycles $e \in \mathrm{Ext}^1(X_{n+1}, X_{n-1})$ such that the corresponding extension gives a semistable sheaf F is open.

Indeed, by standard stability arguments $h^0(X_{n+1}, X_{n-1}) = h^2(X_{n+1}, X_{n-1}) = 0$, whence

$$h^1(X_{n+1}, X_{n-1}) = -\chi(X_{n+1}, X_{n-1}) = 1 + h^1(X_{n-1})r_{n+1} + h^1(X_{n+1})r_{n-1}\,,$$

so the difference

$$h^1(X_{n+1},X_{n-1})-h^0(X_{n+1},Q)=1+h^1(X_{n+1})r_{n-1}+(h^1(X_{n-1})-h^0(Q))r_{n+1}$$

is always positive as long as $h^0(Q)\leqslant h^1(X_{n-1})$.

Thus, one can take two extensions: (37) and some extension of the form $0\to E_{n-1}\to F'\to X_{n+1}\to 0$ and look for an extension of F with F' (or of F' with F) that has a filtration similar to $\mathrm{Gr}^{(2)}$ in the conjecture. We shall investigate this in subsequent articles.

5 Morphisms between stable sheaves

Here, as in the previous section, we deal with moduli spaces $\overline{\mathcal{M}}_n(\delta)=\overline{\mathcal{M}}(r_n,-r_{n-1},\delta)$ over $\mathbb{P}^2$, described in 2.3. We have:

$$\dim\overline{\mathcal{M}}_n(\delta)=2\,r_n\,\delta\,.$$

Let E_n be exceptional bundle with rank r_n and $c_1(E_n)=-r_{n-1}$, X_n be stable sheaf with

$$v(X_n)=(r_n,-r_{n-1},\delta)\,.$$

By the Riemann — Roch theorem we have: $\chi(X_{n-1},X_n)=3-\delta(r_n+r_{n-1})$, this value is negative for almost all δ and n; so, by the semi-continuity theorem for general (X_{n-1},X_n) one has

$$\mathrm{Hom}(X_{n-1},X_n)=0\,,\qquad \dim\mathrm{Ext}^1(X_{n-1},X_n)=-\chi(X_{n-1},X_n)\,.$$

Here we are interesting in a special cases when $\mathrm{Hom}(X_{n-1},X_n)\neq 0$. It is proved in [12] that any sheaf X_n can be represented as the cohomology of the monad

$$M(X_n):\qquad 0\longrightarrow U_0\otimes E_{n-1}\stackrel{\beta}{\longrightarrow}U_1\otimes E_n\stackrel{\alpha}{\longrightarrow}U_2\otimes\mathcal{O}\longrightarrow 0\,,\tag{40}$$

where

$$U_0=\mathrm{Ext}^1(E_{n-1}(3),X_n)\,,\quad U_1=\mathrm{Ext}^1(E_{n-2}(3),X_n)\,,\quad\text{and}\quad U_2=H^1(X_n)\,.$$

The dimensions of the spaces are

$$\dim U_0=\delta\,r_{n-1}\,,\quad \dim U_1=\delta\,r_{n-2}+1\,,\quad\text{and}\quad \dim U_2=\delta\,.$$

Observe that the monad $M(X_n)$ is nothing else than the first term of Gorodentsev – Beilinson spectral sequence constructed via the full exceptional collection $(E_{n-1},E_n,\mathcal{O})$ and the sheaf X_n (see [12, 3.3]).

Herewith, the sheaf

$$Y_n=\ker\alpha$$

is locally free and stable ([12]) and the Kronecker's $\mathrm{Hom}(E_n,\mathcal{O})^*$-module

$$t_\alpha:U_1\otimes\mathrm{Hom}(E_n,\mathcal{O})\longrightarrow U_2$$

corresponding to α in the monad, is also stable (see [2] on Kronecker's moduli).

Applying the functor $\mathrm{Hom}(E_{n-1},-)$ to β in (40) and taking into account exceptionality of E_{n-1}, we have an inclusion of vector spaces $U_0\hookrightarrow U_1\otimes\mathrm{Hom}(E_{n-1},E_n)$. So, we regard U_0 as a subspace of $U_1\otimes\mathrm{Hom}(E_{n-1},E_n)$.

5.1 Proposition. *Suppose $\delta > 1$ and $V_1 \subset U_1$ is a hyperplane such that*
1) the restriction $t' : V_1 \otimes \mathrm{Hom}(E_n, \mathcal{O})^* \longrightarrow U_2$ of t_α is stable Kronecker's module;
2) the subspace U_0 isn't contained in $V_1 \otimes \mathrm{Hom}(E_{n-1}, E_n)$.
Then there exists a μ-stable sheaf X_{n-1} and a monomorphism $X_{n-1} \longrightarrow X_n$ with the quotient isomorphic to E_n/E_{n-1} with respect to some embedding $E_{n-1} \hookrightarrow E_n$.

Proof. Let us show that the corresponding to t' morphism of bundles

$$\alpha' : V_1 \otimes E_n \longrightarrow U_2 \otimes \mathcal{O}$$

is an epimorphism. We apply here Drezet's result [2, 27] to the dual morphism

$$\alpha'^* : U_2^* \otimes \mathcal{O} \longrightarrow V_1^* \otimes E_n^* .$$

The triad (E, G, F) and the bundle S used in [2] correspond to the full exceptional collection $(\mathcal{O}, E_n^*, E_{n-1}^*)$ and E_{n-2}^* in our case. We see that

$$\frac{\dim U_2^*}{\dim V_1^*} = \frac{\delta}{\delta\, r_{n-2}} = \frac{1}{r_{n-2}} = \frac{r(\mathcal{O})}{r(E_{n-2})} = \frac{r(E)}{r(S)} .$$

From the proof of [2, 27(ii)] it follows that if α'^* is not injective at a point x, then there is a submodule $t' : H_0' \otimes \mathrm{Hom}(E_n, \mathcal{O})^* \longrightarrow H_1'$ of the module t' with $\dim H_0' = 1$ and $\dim H_1' = r_{n-2}$. This contradicts stability of t' because by our assumptions $\delta > 1$. Thus, α'^* is injective and α' is surjective as morphisms of bundles, whence it follows that α' is sheaf epimorphism.

From the assumption $U_0 \not\subset V_1 \otimes \mathrm{Hom}(E_{n-1}, E_n)$ it follows that β in (40) gives a non-zero map from $U_0 \otimes E_{n-1}$ to $(U_1/V_1) \otimes E_n \cong E_n$. Moreover, one can choose a line $L \subset U_0$ such that the restriction φ of this map to $L \otimes E_{n-1} \cong E_{n-1}$ is non-zero. Fixing any projection $U_0 \twoheadrightarrow L$ with the kernel V_0, we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_0 \otimes E_{n-1} & \xrightarrow{i} & U_0 \otimes E_{n-1} & \longrightarrow & E_{n-1} & \longrightarrow & 0 \\ & & {\scriptstyle \beta'}\downarrow & & {\scriptstyle \beta}\downarrow & & {\scriptstyle \varphi}\downarrow & & \\ 0 & \longrightarrow & V_1 \otimes E_n & \longrightarrow & U_1 \otimes E_n & \xrightarrow{j} & E_n & \longrightarrow & 0 \end{array} \tag{41}$$

where $\varphi \neq 0$ and the map β' is induced by $\beta \circ i$ as far as $j \circ \beta \circ i = 0$.

The morphism β is a sheaf embedding, φ is also sheaf monomorphism by [12, 3.5]. Thus, the restriction of β to $V_0 \otimes E_{n-1}$ is also a sheaf embedding. Therefore, we have a morphism of monads

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_0 \otimes E_{n-1} & \xrightarrow{\beta'} & V_1 \otimes E_n & \xrightarrow{\alpha'} & U_2 \otimes \mathcal{O} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \\ 0 & \longrightarrow & U_0 \otimes E_{n-1} & \xrightarrow{\beta} & U_1 \otimes E_n & \xrightarrow{\alpha} & U_2 \otimes \mathcal{O} & \longrightarrow & 0 \end{array} \tag{42}$$

where all vertical arrows are sheaf monomorphisms. Denote by X_{n-1} the cohomology of the upper monad, then we have a monomorphism $X_{n-1} \to X_n$.

It remains to prove that X_{n-1} is stable and $X_n/X_{n-1} \cong E_n/E_{n-1}$. Observe that X_{n-1} is torsion free as a subsheaf of the stable sheaf X_n. By [2, 27], the locally free sheaf $\mathrm{coker}\, \alpha'^*$ is stable by Gieseker, hence the bundle $Y_{n-1} = (\mathrm{coker}\, \alpha'^*)^* = \ker \alpha'$ is μ-semistable. Dividing the upper monad from (42) into exact triples, we have one of them:

$$0 \longrightarrow V_0 \otimes E_{n-1} \xrightarrow{\beta'} Y_{n-1} \longrightarrow X_{n-1} \longrightarrow 0 . \tag{43}$$

Calculating the topological invariants of Y_{n-1}, we obtain with the aid of the recurrent relation and Markov equation

$$c_1(Y_{n-1}) = -r_{n-1}\,\dim V_1 = -\delta r_{n-2}r_{n-1}\,,$$

$$r(Y_{n-1}) = r_n\,\dim V_1 - \delta = \delta(r_{n-2}r_n - 1) = \delta(3r_{n-1}r_n - r_n^2 - 1) = \delta r_{n-1}^2\,.$$

Thus, we conclude that $\mu(Y_{n-1}) = \mu(E_{n-1}) = \mu(X_{n-1})$. Any quotient of X_{n-1} is also a quotient of Y_{n-1}, so X_{n-1} is semistable. The stability follows from the fact that $c_1(X_{n-1})$ and $r(X_{n-1})$ are coprime numbers.

Display of a monad gives two ways to divide the monad into exact triples [15, 3.1]. One way is used in (43); to show that $X_n/X_{n-1} \cong E_n/E_{n-1}$ we need the other. Put $Q' = \operatorname{coker}\beta'$, $Q = \operatorname{coker}\beta$, and regard the diagram (41) as an exact triple of complexes; then the long cohomology sequence has the form

$$0 \longrightarrow Q' \longrightarrow Q \longrightarrow \operatorname{coker}\varphi \longrightarrow 0$$

as long as the vertical arrows are embeddings. Therefore, there is the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & X_{n-1} \arrow[r] \arrow[d] & Q' \arrow[r] \arrow[d] & U_2\otimes\mathcal{O} \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\
0 \arrow[r] & X_n \arrow[r] \arrow[d] & Q \arrow[r, "\alpha"] \arrow[d] & U_2\otimes\mathcal{O} \arrow[r] & 0 \\
 & \operatorname{coker}\varphi \arrow[r, "\mathrm{id}"] \arrow[d] & \operatorname{coker}\varphi \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \tag{44}$$

This completes the proof. □

Unfortunately, for the moment there is no information on the set of hyper-planes of U_1 that satisfy the conditions of the proposition. Nevertheless, the following statement shows that for any hyperplane $V_1 \subset U_1$ the submodule t' of t_α is semistable.

5.2 Proposition. *Let r and δ be positive integers. Suppose $t : U_1 \otimes L \to U_2$ is stable Kronecker's L-module with $\dim U_1 = \delta r + 1$ and $\dim U_2 = \delta$, $t' : U_1' \otimes L \to U_2'$ is a submodule of t such that $U_1' \subset U_1$ is a hyperplane. Then $U_2' = U_2$ and t' is semi-stable.*

Proof. We shall use the following criterion of stability [2, Prop.15]: a non-zero Kronecker's module t is semi-stable (resp. stable) if and only if for any submodule t' of t with $U_1' \neq 0$

$$\frac{\dim U_2'}{\dim U_1'} \geqslant \frac{\dim U_2}{\dim U_1} \text{ (resp. } >\text{) or equivalently } \begin{vmatrix} \dim U_1' & \dim U_1 \\ \dim U_2' & \dim U_2 \end{vmatrix} \leqslant 0 \text{ (resp. } >\text{)}.$$

If we had $\dim U_2' < \delta$, then we would have

$$\begin{vmatrix} \dim U_1' & \dim U_1 \\ \dim U_2' & \dim U_2 \end{vmatrix} = \begin{vmatrix} \delta r & \delta r + 1 \\ \dim U_2' & \delta \end{vmatrix} =$$

$$= \delta r(\delta - \dim U_2') - \dim U_2' > \delta r - \dim U_2' > 0\,.$$

Hence, the submodule t' would violate the semi-stability criterion. Thus, $\dim U_2' = \delta$ and $U_2' = U_2$.

Now consider a submodule $\tau : V_1 \otimes L \to V_2$ of t'. From the stability criterion applied to τ as a submodule of t we have

$$\begin{vmatrix} \dim V_1 & \delta r + 1 \\ \dim V_2 & \delta \end{vmatrix} = \delta(\dim V_1 - r \dim V_2) - \dim V_2 < 0\,.$$

From the inequality $\dim V_2 \leqslant \delta$ it follows that

$$\delta(\dim V_1 - r \dim V_2) = \begin{vmatrix} \dim V_1 & \delta r \\ \dim V_2 & \delta \end{vmatrix} = \begin{vmatrix} \dim V_1 & \dim U_1' \\ \dim V_2 & \dim U_2' \end{vmatrix} \leqslant 0\,.$$

Therefore, t' is semistable by the criterion. This completes the proof. □

In the rest of this section we discuss another sufficient condition for non-trivial morphism $X_{n-1} \to X_n$ to exist.

Fix a sheaf X_n, then the monad (40) defines the exact triple

$$0 \longrightarrow Y_n \longrightarrow U_1 \otimes E_n \xrightarrow{\alpha} U_2 \otimes \mathcal{O} \longrightarrow 0\,, \tag{45}$$

Our claim is to show that any epimorphism $Y_n \twoheadrightarrow E_n$ defines a sheaf X_{n-1} such that $\mathrm{Hom}(X_{n-1}, X_n) \neq 0$. Observe that the data, in terms of which the sheaf X_{n-1} is determined, is the same as in Proposition 5.1 because there is the canonical identification

$$\mathrm{Hom}(Y_n, E_n) \cong U_1^*\,.$$

obtained by applying the functor $\mathrm{Hom}(-, E_n)$ to the triple (45), so hyper-planes in U_1 correspond to non-zero elements of $\mathrm{Hom}(Y_n, E_n)$ up to scaling.

5.3 Proposition. *Let $\varphi : Y_n \to E_n$ be nontrivial morphism; then $\mathrm{im}\,\varphi$ is a stable sheaf with $v(\mathrm{im}\,\varphi) = (r_n, -r_{n-1}, \delta')$ and $\ker\varphi$ is semistable with $v(\ker\varphi) = (\delta r_{n-1}^2, -\delta r_{n-2} r_{n-1}, \delta'')$, where δ' and δ'' are non-negative and $\delta' + \delta'' = \delta$.*

Proof. Consider the real (x,y)-plane. As above, for a coherent sheaf E we denote by the same symbol E the point $(r(E), c_1(E))$. The inequality $\mu(E_{n-1}) < \mu(Y_n) < \mu(E_n)$ implies that the ray OY_n lies between the rays OE_{n-1} and OE_n (see Fig. 2). From stability of Y_n and E_n it follows that $\mu(Y_n) \leqslant \mu(\mathrm{im}\,\varphi) \leqslant \mu(E_n)$; besides, $r(\mathrm{im}\,\varphi) \leqslant r_n$. Thus, the point $\mathrm{im}\,\varphi$ belongs to the triangle OE_nA, where $A = OY_n \cap \{x = r_n\}$. From the equality $\begin{vmatrix} -r_{n-2} & -r_{n-1} \\ r_{n-1} & r_n \end{vmatrix} = -1$ (see (33)) it follows that the parallelogram $OE_{n-1}BE_n$ has no points with integer coordinates except the vertices. The triangle OE_nA lies in the strip between the parallel lines OE_{n-1} and E_nB, which also doesn't contain interior integer points. Therefore, the only possibility for $(r(\mathrm{im}\,\varphi), c_1(\mathrm{im}\,\varphi))$ is $(r_n, -r_{n-1})$. Clearly, $\mathrm{im}\,\varphi$ is stable.

Now let us show that $\ker\varphi$ is semistable. Consider the exact triple

$$0 \longrightarrow \ker\varphi \longrightarrow Y_n \longrightarrow \mathrm{im}\,\varphi \longrightarrow 0\,. \tag{46}$$

Since $\mathrm{im}\,\varphi$ has the rank and c_1 to be equal to $(r_n, -r_{n-1})$, the rank and c_1 of $\ker\varphi$ coincide with $(r(Y_{n-1}), c_1(Y_{n-1}))$ from the proof of 5.1:

$$r(\ker\varphi) = \delta r_{n-1}^2\,; \qquad c_1(\ker\varphi) = -\delta r_{n-2} r_{n-1}\,.$$

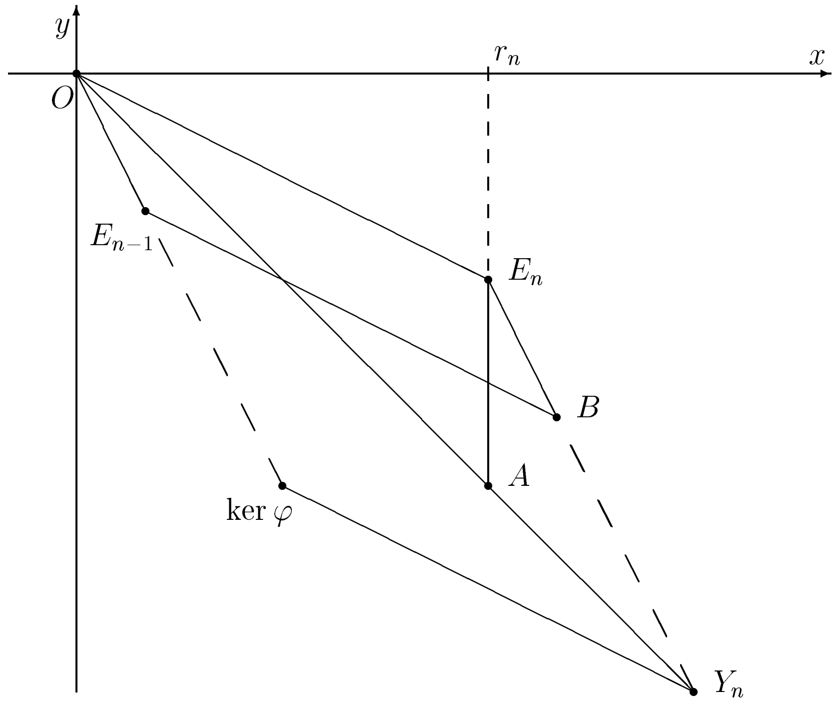

Fig. 2.

We see that $(r(\ker\varphi), c_1(\ker\varphi)) = \delta r_{n-1}(r_{n-1}, -r_{n-2})$; in other words, the point $\ker\varphi$ lies on the ray OE_{n-1}, hence $Y_n \in E_nB$.

Suppose $Q \subset \ker\varphi$ is a subsheaf such that $0 < r(Q) < r(\ker\varphi)$ and $\mu(Q) \geqslant \mu(\ker\varphi)$; then $\mu(Q) < \mu(Y_n)$ as far as Q is a subsheaf of Y_n. It follows that the point Q belongs to the triangle $O\ker\varphi Y_n$. By the above arguments, all the points of this triangle with integer coordinates lie on the side $O\ker\varphi$. Therefore, $\mu(Q) = \mu(\ker\varphi)$ and $\ker\varphi$ is semistable.

By standard arguments, the H^0 and H^2 groups of the sheaves $\ker\varphi$ and $\operatorname{im}\varphi$ are trivial, therefore, $\dim H^1 = -\chi$ for these sheaves. This implies that $\delta', \delta'' \geqslant 0$. The equality $\delta' + \delta'' = \delta$ follows immediately from (46). This concludes the proof. □

5.4 Lemma. *Let Y_{n-1} be semistable subsheaf of Y_n with the invariants*

$$v(Y_{n-1}) = (\delta\, r_{n-1}^2, -\delta\, r_{n-2}r_{n-1}, \delta)\,.$$

Then $Y_n/Y_{n-1} \cong E_n$.

Proof. Denote the quotient Y_n/Y_{n-1} by Q and consider the Gorodentsev – Beilinson spectral sequences (see [12] or [4]) constructed via full exceptional collection $(E_{n-1}, E_n, \mathcal{O})$ that converge to Y_{n-1} and Y_n. Each sequence gives an exact triple and the embedding $Y_{n-1} \overset{i}{\hookrightarrow} Y_n$ induces the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y_{n-1} & \longrightarrow & V_1\otimes E_n & \longrightarrow & H^1(Y_{n-1})\otimes\mathcal{O} & \longrightarrow & 0 \\ & & i\downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y_n & \longrightarrow & U_1\otimes E_n & \longrightarrow & H^1(Y_n)\otimes\mathcal{O} & \longrightarrow & 0 \end{array} \tag{47}$$

Here

$$V_1 = \operatorname{Ext}^1(E_{n-2}(3), Y_{n-1})\,, \qquad U_1 = \operatorname{Ext}^1(E_{n-2}(3), Y_n)\,,$$

$$\dim V_1 = \dim U_1 - 1\,, \qquad \dim H^1(Y_{n-1}) = \dim H^1(Y_n) = \delta\,.$$

Regarding (47) as an exact triple of complexes, we have the long cohomology sequence

$$0 \longrightarrow V \otimes E_n \longrightarrow W \otimes \mathcal{O} \xrightarrow{\varphi} Q \xrightarrow{\psi} U \otimes E_n \longrightarrow W' \otimes \mathcal{O} \longrightarrow 0\,,$$

where $\dim V = \dim U - 1$ and $\dim W = \dim W'$. The sheaf $I = \operatorname{im}\varphi = \ker\psi$ is a quotient-sheaf of $W \otimes \mathcal{O}$ and at the same time a subsheaf of $U \otimes E_n$, whence $I = 0$. Thus, $V \otimes E_n$ is isomorphic to $W \otimes \mathcal{O}$, that is possible only for $\dim V = \dim W = 0$. Therefore, we get: $Q \cong U \otimes E_n$ with U one-dimensional. □

In particular, from two previous statements it follows that epimorphisms $Y_n \twoheadrightarrow E_n$ are in one-to-one correspondence with semistable sub-sheaves Y_{n-1} of Y_n having the invariants as in the previous lemma.

5.5 Proposition. *Suppose X_n is a stable sheaf with the invariants $v(X_n) = (r_n, -r_{n-1}, \delta)$, Y_n is the kernel of the map α in the monad (40), and $\varphi : Y_n \twoheadrightarrow E_n$ is an epimorphism; then there is a hyperplane $V_0 \subset U_0$ such that β (see (40)) induces an embedding of $V_0 \otimes E_{n-1} \hookrightarrow$ $\ker\varphi$, the quotient $X_{n-1} = \ker\varphi / V_0 \otimes E_{n-1}$ is a stable sheaf with $v(X_{n-1}) = (r_{n-1}, -r_{n-2}, \delta)$, and the embedding $\ker\varphi \hookrightarrow Y_n$ induces an embedding $X_{n-1} \hookrightarrow X_n$.*

Proof. According to 5.3, $Y_{n-1} = \ker\varphi$ is stable. Consider the composition map

$$\varphi \circ \beta : U_0 \otimes E_{n-1} \xrightarrow{\beta} Y_n \xrightarrow{\varphi} E_n\,.$$

Observe that the invariants (r, c_1) of the sheaves $U_0 \otimes E_{n-1}$ and Y_{n-1} coincide. If we had $\varphi \circ \beta = 0$; then β would induce an embedding $U_0 \otimes E_{n-1} \hookrightarrow Y_{n-1}$, cokernel of which would be an Artinian sheaf with Euler characteristics equal to $\chi(Y_{n-1}) < 0$. But this is impossible, hence, $\varphi \circ \beta \neq 0$. As in the proof of 5.1, we can choose a line $L \subset U_0$ such that the restriction ψ of $\varphi \circ \beta$ to $L \otimes E_{n-1} \cong E_{n-1}$ is non-zero. Fixing any projection $U_0 \twoheadrightarrow L$ with the kernel V_0, we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \longrightarrow & V_0 \otimes E_{n-1} & \xrightarrow{i} & U_0 \otimes E_{n-1} & \longrightarrow & E_{n-1} & \longrightarrow 0 \\
& {\scriptstyle \beta'}\downarrow & & {\scriptstyle \beta}\downarrow & & {\scriptstyle \psi}\downarrow & \\
0 \longrightarrow & Y_{n-1} & \longrightarrow & Y_n & \xrightarrow{\varphi} & E_n & \longrightarrow 0
\end{array}$$

where, as in the proof of 5.1, β' is induced by $\beta \circ i$ as far as $\varphi \circ \beta \circ i = 0$. Clearly, β is a sheaf embedding by construction of the monad, β' is an embedding as a restriction of β, therefore, ψ is a sheaf embedding[4]. Denote by X_{n-1} the sheaf $\operatorname{coker}\beta$, we have:

$$v(X_{n-1}) = v(Y_{n-1}) - (\dim U_0 - 1)v(E_{n-1}) = (r_{n-1}, -r_{n-2}, \delta)\,.$$

Regarding the last diagram as an exact triple of complexes, we have the long cohomology sequence

$$0 \longrightarrow X_{n-1} \longrightarrow X_n \longrightarrow \operatorname{coker}\psi \longrightarrow 0\,.$$

It follows that X_{n-1} is torsion-free. Besides, $\mu(X_{n-1}) = \mu(Y_{n-1}) = \mu(E_{n-1})$, whence X_{n-1} is semistable by 4.1 and stable because it has the rank and c_1 to be coprime. This completes the proof. □

[4] Remark that any non-zero map $E_{n-1} \to E_n$ is an embedding by [12, 3.5].

6 Non locally free part of $\mathcal{M}_n(\delta)$.

We describe here the subset of $\mathcal{M}_n(\delta)$ parametrizing non locally free sheaves. For such a sheaf X consider the reflexive span X^{**}, which is locally free. The sheaf X has no torsion, so the singularity set of X is at least 2-codimensional and the quotient X^{**}/X is an Artinian sheaf i.e. has zero rank and c_1.

The following statement implies that X^{**} is stable.

6.1 Lemma. *Suppose there is an exact triple of sheaves*

$$0 \longrightarrow F \longrightarrow \tilde{F} \longrightarrow Q \longrightarrow 0\,, \tag{48}$$

where $r(Q) = 0$ *and* $c_1(Q) = 0$. *Then the following statements hold:*
a) if $\tilde{F}$ *is (semi-)stable, then* F *is (semi-)stable;*
b) if $\tilde{F}$ *is torsion-free and* F *is (semi-)stable, then* $\tilde{F}$ *is (semi-)stable.*

Proof. We have: $r(F) = r(\tilde{F})$ and $c_1(F) = c_1(\tilde{F})$. The statement a) is trivial because F has no torsion as a subsheaf of a torsion-free sheaf and any subsheaf of F is also a subsheaf of $\tilde{F}$. Let us prove b).

Suppose E is a subsheaf of $\tilde{F}$ such that $0 < r(E) < r(\tilde{F})$, then there is the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
 & & & E \arrow[r, "\mathrm{id}"] \arrow[d] & E \arrow[d, "\psi"] & \\
 & 0 \arrow[r] & F \arrow[r] \arrow[d, "\mathrm{id}"'] & \tilde{F} \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker\psi \arrow[r] & F \arrow[r] & G \arrow[r] \arrow[d] & \tilde{Q} \arrow[r] \arrow[d] & 0 \\
 & & & 0 & 0 &
\end{tikzcd} \tag{49}$$

Here $\tilde{Q} = \operatorname{coker}\psi$. We have: $r(\operatorname{im}\psi) = 0$ and $c_1(\operatorname{im}\psi) = 0$ because $\operatorname{im}\psi$ is a subsheaf of Q; whence $r(\ker\psi) = r(E)$, $c_1(\ker\psi) = c_1(E)$, and $\mu(\ker\psi) = \mu(E)$. But $\ker\psi$ is a subsheaf of F, so $\mu(\ker\psi) < \mu(F)$ (respectively $\leqslant$) as far as F is stable (resp. semistable). Therefore, $\mu(E) < \mu(\tilde{F})$ (resp. $\leqslant$) that means stability (semi-stability) of $\tilde{F}$. □

The following statement is very close to [15, 1.2.8], it describes the spaces of homomorphisms between two stable sheaves with the same rank and c_1, one of them being locally free.

6.2 Proposition. *Suppose* X, E *are stable sheaves with* $r(X) = r(E)$ *and* $c_1(X) = c_1(E)$, *and* E *is locally free. Then the following statements are true:*
a) $\operatorname{Hom}(X, E)$ *can be only zero- or one-dimensional, in the latter case* $E \cong X^{**}$;
b) $\operatorname{Hom}(E, X) = 0$.

Proof. From the proof of [15, 1.2.8] it follows that any non-zero morphism φ from one of these sheaves to the other is a monomorphism and generically an epimorphism. Therefore, $Q = \operatorname{coker}\varphi$ is an Artinian sheaf.

Suppose $0 \neq \varphi \in \mathrm{Hom}(X, E)$; then applying the functor $\mathrm{Hom}(-, E)$ to the exact triple

$$0 \longrightarrow X \xrightarrow{\varphi} E \xrightarrow{\psi} Q \longrightarrow 0\,, \tag{50}$$

we obtain $\mathrm{Hom}(E, E) \cong \mathrm{Hom}(X, E) \cong \mathbb{C}$ as far as $\mathrm{Hom}(Q, E) = \mathrm{Ext}^1(Q, E) = 0$ and stable bundle E is simple. The embedding φ induces a nontrivial map $X^{**} \to E^{**} = E$, which is an isomorphism as far as X^{**} and E are stable bundles with the same rank and c_1. This proves a). To prove b), let's apply the functor $\mathrm{Hom}(E, -)$ to the triple (50). We obtain an exact sequence $0 \to \mathrm{Hom}(E, X) \to \mathbb{C}\cdot \mathrm{id}_E \to \mathrm{Hom}(E, Q)$, where id_E comes to non-zero map ψ. This implies that $\mathrm{Hom}(E, X) = 0$. $\square$

We are now ready to describe the strata of $\mathcal{M}_n(\delta)$. Put by definition

$$\mathcal{M}_n^{\delta'}(\delta) = \{X \in \mathcal{M}_n(\delta) |\; h^1(X^{**}) = \delta - \delta'\}$$

For each $X \in \mathcal{M}_n^{\delta'}(\delta)$ we have the exact triple

$$0 \longrightarrow X \longrightarrow X^{**} \longrightarrow Q \longrightarrow 0\,, \tag{51}$$

where $h^0(Q) = h^1(X) - h^1(X^{**}) = \delta'$. Clearly, we have: $0 \leqslant \delta' \leqslant \delta$, and $\mathcal{M}_n^0(\delta)$ consists of locally free sheaves. Thus, we get:

$$X \in \mathcal{M}_n^{\delta'}(\delta) \iff X^{**} \in \mathcal{M}_n^0(\delta'')\,, \text{ where } \delta' + \delta'' = \delta\,.$$

Denote by $A(\delta')$ the set of Artinian sheaves Q such that $h^0(Q) = \delta'$. We have a map

$$\pi : \mathcal{M}_n^{\delta'}(\delta) \longrightarrow \mathcal{M}_n^0(\delta'') \times A(\delta')$$

which takes X to the pair (X^{**}, Q) according to (51). Obviously, this map is well-defined. Let us describe the preimage $\pi^{-1}(X^{**}, Q)$. By Lemma 6.1, any epimorphism $\varphi : X^{**} \twoheadrightarrow Q$ gives a stable sheaf $X = \ker\varphi$. Another such an epimorphism φ' has $\ker\varphi' \cong X$ if and only if $\varphi' = g\circ\varphi$ for some $g \in \mathrm{Aut}(Q)$. Indeed, suppose that the rows are given of the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i} & X^{**} & \xrightarrow{\varphi} & Q & \longrightarrow & 0 \\ & & \psi\downarrow & & \mathrm{id}\downarrow & & g\downarrow & & \\ 0 & \longrightarrow & X' & \xrightarrow{i'} & X^{**} & \xrightarrow{\varphi'} & Q & \longrightarrow & 0 \end{array}$$

If there is an isomorphism ψ, take $i = i' \circ \psi$, then the left square is commutative, whence there is an isomorphism $g : Q \to Q$. And vice versa, if $\varphi' = g \circ \varphi$, then the right square is commutative, hence there is an isomorphism ψ closing the diagram. Therefore, we conclude that

$$\pi^{-1}(X^{**}, Q) = \Big(\text{set of epimorphisms in } \mathrm{Hom}(X^{**}, Q)\Big)/\,\mathrm{Aut}(Q)\,.$$

The set $A(\delta')$ has rather complicated description, we give only idea of this. Let $A_k(\delta')$ be the set of Artinian sheaves $Q \in A(\delta')$ such that there exists an epimorphism $\mathcal{O}^{\oplus k} \twoheadrightarrow Q$, but there is no epimorphism $\mathcal{O}^{\oplus(k-1)} \twoheadrightarrow Q$. In other words, any sheaf from $A_k(\delta')$ can be generated by k sections, but can't be generated by $k-1$ sections. Obviously, we have: $A_1(\delta') = \mathrm{Hilb}^{\delta'}(\mathbb{P}^2)$, and there exists a map

$$\bigcup_{j=1}^{\delta'-1} Z_j \longrightarrow A_2(\delta')\,,$$

where Z_j is a cycle in $\mathrm{Hilb}^j(\mathbb{P}^2) \times \mathrm{Hilb}^{\delta'-j}(\mathbb{P}^2)$ consisting of (ξ_1, ξ_2) such that $\operatorname{supp}\xi_1 \cap \operatorname{supp}\xi_2 \neq \emptyset$, and the map takes (ξ_1, ξ_2) to the sheaf $\mathcal{O}_{\xi_1} \oplus \mathcal{O}_{\xi_2}$. This map is, of course, surjective but non-injective. Observe that each Z_j has positive codimension in $\mathrm{Hilb}^j(\mathbb{P}^2) \times \mathrm{Hilb}^{\delta'-j}(\mathbb{P}^2)$. Clearly, there are similar maps for the other $A_k(\delta')$.

In the rest part of this subsection we assume that $Q = \mathcal{O}_\xi \in \mathrm{Hilb}^{\delta'}(\mathbb{P}^2)$ i.e. investigate the map π over the "biggest part" of $A(\delta')$. Let us show that the corresponding restriction of π is surjective, this is equivalent to existence of an epimorphism $E \twoheadrightarrow \mathcal{O}_\xi$ for all $E \in \mathcal{M}_n^0(\delta'')$ and $\xi \in \mathrm{Hilb}^{\delta'}(\mathbb{P}^2)$.

6.3 Lemma. *Suppose E is a vector bundle and the canonical map*

$$E \otimes \mathrm{Hom}(E, \mathcal{O}) \longrightarrow \mathcal{O} \tag{52}$$

is epimorphic. Then for any finite collection of points $\{x_i\}_{i=1}^{\delta}$ there is $\varphi \in \mathrm{Hom}(E, \mathcal{O})$ such that the fiber map at each point $\varphi_{x_i} : E_{x_i} \to \mathcal{O}_{x_i}$ is surjective. Moreover, the set of such maps is open in $\mathrm{Hom}(E, \mathcal{O})$.

Proof. By the lemma of Nakayama we can work out with fibers as the fibers of vector bundles. We don't change the notation for simplicity. Then φ_x is surjective if and only if $\varphi_x \neq 0$. The proof is by induction on δ.

It is clear that for any point x there is $\varphi \in \mathrm{Hom}(E, \mathcal{O})$ having $\varphi_x \neq 0$; otherwise the canonical map (52) would be non-epimorphic at x. Thus, the statement is true for $\delta = 1$.

Suppose $\delta = k$; then by induction hypothesis there is $\psi : E \to \mathcal{O}$ such that all ψ_{x_i}, $i = 1, \ldots, k$ are surjective. Consider one more point, x_{k+1}. Suppose $\psi_{x_{k+1}} = 0$ and take $\psi' \in \mathrm{Hom}(E, \mathcal{O})$ such that $\psi'_{x_{k+1}} \neq 0$. If ψ'_{x_i} is not proportional to ψ_{x_i}, then $\psi'_{x_i} + \psi_{x_i} \neq 0$. Therefore, choosing $\lambda \in \mathbb{C}^* \setminus \{\lambda_i \mid \psi'_{x_i} = \lambda_i \psi_{x_i}, i = 1, \ldots, k\}$, we obtain that $\psi' - \lambda\psi$ is epimorphic at all points x_i, $i = 1, \ldots, k+1$. The last statement follows easily from the fact that the set of $\varphi \in \mathrm{Hom}(E, \mathcal{O})$ such that $\varphi_x = 0$ at a given point x is closed. The lemma is proved. □

6.4 Proposition. *Let E be locally free sheaf, $\xi \subset \mathbb{P}^2$ be zero-dimensional subscheme; then there exists an epimorphism of sheaves $E \twoheadrightarrow \mathcal{O}_\xi$. Besides, the set of epimorphisms is open in $\mathrm{Hom}(E, \mathcal{O}_\xi)$.*

Proof. Observe that $\mathrm{Hom}(E, \mathcal{O}_\xi) = \mathrm{Hom}(E(k), \mathcal{O}_\xi)$ for any $k \in \mathbb{Z}$ because the sheaf $\mathcal{O}_\xi$ is invariant under tensoring by invertible sheaf. So, without loss of generality we can assume that E^* is generated by global sections. Suppose the map of stalks at some point x induced by (52) is non-epimorphic; then by lemma of Nakayama the corresponding map of fibers of bundles $E_x \otimes \mathrm{Hom}(E, \mathcal{O}) \to \mathcal{O}_x$ is also non-epimorphic i.e. vanishes. Thus, for any $e \in E_x$ and $f \in H^0(E^*)$ one has $\langle e, f_x \rangle = 0$. This implies that $f_x = 0$ for all $f \in H^0(E^*)$. This contradicts our assumption that E^* is generated by global sections. Hence, under this assumption the canonical map (52) is epimorphic.

From the lemma it follows that there exist $\varphi \in \mathrm{Hom}(E, \mathcal{O})$ having epimorphic maps of fibers φ_x for all $x \in \operatorname{supp}(\xi)$. Therefore, one can obtain the required map as composition of φ with the standard epimorphism $\mathcal{O} \twoheadrightarrow \mathcal{O}_\xi$.

In addition, tensoring E by an invertible sheaf we can assume that $H^1(E^* \otimes J_\xi) = 0$. Thus, we have a surjective linear map $\mathrm{Hom}(E, \mathcal{O}) \overset{j}{\twoheadrightarrow} \mathrm{Hom}(E, \mathcal{O}_\xi)$. Denote by U the set of epimorphisms in $\mathrm{Hom}(E, \mathcal{O}_\xi)$, by V the set of $\varphi \in \mathrm{Hom}(E, \mathcal{O})$ such that $\varphi_x : E_x \to$

$\mathcal{O}_x$ is epimorphic for all $x \in \operatorname{supp}\xi$. It readily seen that $V = j^{-1}(U)$. Choose a linear subspace $H \subset \operatorname{Hom}(E, \mathcal{O})$ such that $\operatorname{Hom}(E, \mathcal{O}) = H \oplus \ker j$, then the restriction $j|_H$ is an isomorphism that identifies the rings of polynomial functions on H and on $\operatorname{Hom}(E, \mathcal{O}_\xi)$. Besides, $j|_H$ identifies U and $V \cap H$. Thus, for any polynomial f on $\operatorname{Hom}(E, \mathcal{O})$ such that the zero-set $(f)_0$ of f contains $\operatorname{Hom}(E, \mathcal{O}) \setminus V$ the restriction $f|_H$ is identified with a polynomial on $\operatorname{Hom}(E, \mathcal{O}_\xi)$ such that $(f|_H)_0 \supset \operatorname{Hom}(E, \mathcal{O}_\xi) \setminus U$. Therefore, U is open. This concludes the proof. $\square$

Denote by $\mathcal{M}_n^{\delta'}(\delta)_0$ the set of $X \in \mathcal{M}_n^{\delta'}(\delta)$ such that $X^{**}/X = \bigoplus_{i=1}^{\delta'} \mathcal{O}_{p_i}$, where $p_i \neq p_j$ for $i \neq j$. Clearly,

$$\operatorname{Aut}\left(\bigoplus_{i=1}^{\delta'} \mathcal{O}_{p_i}\right) \cong \underbrace{\mathbb{C}^* \times \cdots \times \mathbb{C}^*}_{\delta'}$$

whence $\dim \operatorname{Aut}(\bigoplus_{i=1}^{\delta'} \mathcal{O}_{p_i}) = \delta'$. Therefore, we have:

$$\dim(\mathcal{M}_n^{\delta'}(\delta)_0) = \dim(\mathcal{M}_n^0(\delta'')) + \dim \operatorname{Hilb}^{\delta'}(\mathbb{P}^2) + h^0(X^{**}, \mathcal{O}_\xi) - \delta'$$

Substituting to this the values

$$\dim(\mathcal{M}_n^0(\delta'')) = 2r_n\delta'', \quad \dim \operatorname{Hilb}^{\delta'}(\mathbb{P}^2) = 2\delta', \quad \text{and} \quad h^0(X^{**}, \mathcal{O}_\xi) = r_n\delta',$$

we obtain the following statement.

6.5 Proposition.

$$\dim(\mathcal{M}_n^{\delta'}(\delta)_0) = 2r_n\delta'' + (r_n + 1)\delta'; \qquad \operatorname{codim}(\mathcal{M}_n^{\delta'}(\delta)_0) = (r_n - 1)\delta'.$$

$\square$

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