# INJECTIVE SIMPLICIAL MAPS OF THE ARC COMPLEX

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Abstract. In this paper, we prove that each injective simplicial map of the arc complex of a compact, connected, orientable surface with nonempty boundary is induced by a homeomorphism of the surface. We deduce, from this result, that the group of automorphisms of the arc complex is naturally isomorphic to the extended mapping class group of the surface, provided the surface is not a disc, an annulus, a pair of pants, or a torus with one hole. We also show, for each of these special exceptions, that the group of automorphisms of the arc complex is naturally isomorphic to the quotient of the extended mapping class group of the surface by its center.

## 1. Introduction

In this paper, $R = R_{g,b}$ will denote a compact, connected, oriented surface of genus $g$ with $b$ boundary components, where $b \geq 1$. Let $\partial R$ be the boundary of $R$ and $\partial_i$, $1 \leq i \leq b$ be the components of $\partial R$. We say that $R$ is *a surface of genus $g$ with $b$ holes*. Note that $R_{0,1}$ is a *disc*; $R_{0,2}$ is an *annulus*; $R_{0,3}$ is a *pair of pants*; $R_{0,b}$ is a *sphere with $b$ holes*; and $R_{1,b}$ is a *torus with $b$ holes*.

The *extended mapping class group of $R$* is the group of isotopy classes $\Gamma^*(R)$ of self-homeomorphisms of $R$. The *mapping class group of $R$* is the group of isotopy classes $\Gamma(R)$ of orientation preserving self-homeomorphisms of $R$. Note that $\Gamma(R)$ is a subgroup of index 2 in $\Gamma^*(R)$.

The arc complex $\mathcal{A}(R)$ is the abstract simplicial complex whose simplices are collections of isotopy classes of properly embedded essential

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arcs on $R$ which can be represented by disjoint arcs. $\Gamma^*(R)$ acts naturally on $\mathcal{A}(R)$ by simplicial automorphisms of $\mathcal{A}(R)$, yielding a natural simplicial representation $\rho : \Gamma^*(R) \to Aut((A)(R))$ from $Gamma^*(R)$ to the group of simplicial automorphisms $Aut(\mathcal{A}(R))$ of $\mathcal{A}(R)$.

In this paper, we prove that each injective simplicial map $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ is *geometric* (i.e. induced by a homeomorphism). More precisely, we prove, for each such map, $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$, that there exists a homeomorphism $H : R \to R$ such that the value of $\lambda$ on the isotopy class $[A]$ of any properly embedded essential arc $A$ on $R$ is equal to $[H(A)]$.

As an immediate consequence of this result, it follows that $\rho : \Gamma^*(R) \to Aut(\mathcal{A}(R))$ is surjective with kernel $ker(\rho)$ equal to the subgroup of $\Gamma^*(R)$ consisting of isotopy classes of homeomorphisms $R \to R$ which preserve the isotopy class of every properly embedded essential arc on $R$.

Studying $ker(\rho)$ we show that it is trivial when $R$ is not a disc, an annulus, a pair of pants, or a torus with one hole. When $R$ is either a disc, an annulus, a pair of pants, or a torus with one hole, we show that $ker(\rho)$ is equal to the center $Z(\Gamma^*(R)$ of $\Gamma^*(R)$, and compute explicitly $Z(\Gamma^*(R))$ for each of these special examples.

Here is an outline of the paper.

In Section 2, we review some basic facts about arcs on surfaces used in the following sections.

In Sections 3 and 4, we define and discuss the notions of a *triangle on* $R$ and a *triangulation of* $R$ used in this paper, where we have modified the standard notions of an ideal triangle on a punctured surface and an ideal triangulation of a punctured surface, adapting these notions to our setting of compact surfaces with nonempty boundary. The reader will note that our notions here are equivalent to these standard notions for punctured surfaces. In particular, in Section 3, we define the notions of an *embedded* triangle on $R$ and a *non-embedded* triangle on $R$.

In Section 5, we define and discuss the notion of a *quadrilateral on* $R$ used in this paper, where a quadrilateral on $R$ corresponds, roughly speaking, to the union of two triangles on $R$ along a common side.

In Section 6, we define the notion of an elementary move on a triangulation, which roughly speaking, corresponds to replacing one diagonal of a quadrilateral with the other diagonal.

In Section 7, we define the notion of the *complex of arcs* $\mathcal{A}(R)$ of $R$.

In Section 8, we define the notion of an elementary move on a top-dimensional simplex of $\mathcal{A}(R)$. In particular, in this section, we recall the "Connectivity Theorem for Elementary Moves" [M], reformulated for our setting of compact surfaces with boundary, and we prove two results about elements of a triangulation which distinguish those elements of a triangulation which correspond to sides of an embedded triangle of the triangulation from those which do not.

In Section 9, we give explicit descriptions of the arc complex for each of the following special examples of $R$; a disc, an annulus, a pair of pants, and a torus with one hole. In particular, we compute explicitly the groups of interest, $Z(\Gamma^*(R))$, $\Gamma^*(R)$, and $Aut(\mathcal{A}(R))$, and establish the natural isomorphism $\Gamma^*(R)/Z(\Gamma^*(R)) \to Aut(\mathcal{A}(R))$ corresponding to the natural simplicial action of $\Gamma^*(R)$ on $\mathcal{A}(R)$. The results of this section establish, for these special surfaces, the two main results of this paper mentioned above.

In Section 10, we begin the proof of the first main result of this paper for the remaining surfaces, those which are not a disc, an annulus, a pair of pants, or a torus with one hole. In particular, in this section, we prove, that injective simplicial maps of $\mathcal{A}(R)$ respect isotopy classes of triangulations on $R$. This is the first step towards proving that such maps are geometric.

In Section 11, we prove that injective simplicial maps of $\mathcal{A}(R)$ respect elementary moves on isotopy classes of triangulations on $R$.

In Section 12, we prove that injective simplicial maps of $\mathcal{A}(R)$ respect embedded triangles on $R$.

In Section 13, we prove that injective simplicial maps of $\mathcal{A}(R)$ respect nonembedded triangles on $R$.

In Section 14, we observe that injective simplicial maps of $\mathcal{A}(R)$ respect triangles on $R$, an immediate corollary of the results of Sections 12 and 13.

In Section 15, we prove that injective simplicial maps of $\mathcal{A}(R)$ respect quadrilaterals on $R$.

In Section 16, we prove that injective simplicial maps of $\mathcal{A}(R)$ respect the topological type of ordered triangulations on $R$, deducing this result from the results of Sections 12, 13, 14, and 15.

In Section 17, we prove the first main result of this paper, that injective simplicial maps of $\mathcal{A}(R)$ are geometric, deducing this from the result of Section 16 by a well-known argument involving the Connectivity Theorem for Elementary Moves [M]. This argument was previously used, in particular, in Ivanov's proof of his theorem that automorphisms of the comlex of curves are geometric [I1], the seminal result which motivates the second main result of this paper.

In Section 18, we prove the second main result of this paper, which gives a complete description of $Aut(\mathcal{A}(R))$. In particular, we prove that the natural representation $\rho : \Gamma^*(R) \to Aut(\mathcal{A}(R))$ is an isomorphism when $R$ is not a disc, an annulus, a pair of pants, or a torus with one hole.

## 2. Preliminaries

Throughout this paper, all isotopies between subspaces of $R$ will be ambient isotopies. More precisely, if $X$ and $Y$ are subspaces of $R$, an *isotopy from $X$ to $Y$* is a map $H : R \times [0,1] \to R$ such that the maps $H_t : R \to R$, $0 \leq t \leq 1$, defined by the rule $H_t(x) = H(x,t)$, $x \in R$, are homeomorphisms of $R$, $H_0 = id_R : R \to R$, and $H_1(X) = Y$.

An *arc* $A$ on $R$ is a subspace of $R$ which is homeomorphic to the interval $[0,1]$. Let $A$ be an arc on $R$. The *endpoints of* $A$ are the images of 0 and 1 under a homeomorphism from $[0,1]$ to $A$. We say that an arc $A$ on $R$ is *properly embedded on* $R$ if $A$ intersects $\partial R$ precisely at the endpoints of $A$.

An arc $A$ on $R$ is *essential on* $R$ if it is properly embedded on $R$ and there does not exist an embedded closed disk on $R$ whose boundary is equal to the union of $A$ with an arc contained in $\partial R$.

Note that there are no essential arcs on any disc $R_{0,1}$; and there is a unique isotopy class of an essential arc on any annulus $R_{0,2}$. This isotopy class is represented by any arc on $R_{0,2}$ which joins the two distinct components of $\partial R_{0,2}$.

If $b \geq 2$, then any arc on $R$ joining two distinct components of $\partial R$ is essential on $R$. Likewise, if $g > 0$ and $b = 1$, then any arc on $R$ which intersects a simple closed curve on $R$ transversely and at exactly one point is essential on $R$.

A *system of arcs on* $R$ is a family of pairwise disjoint and nonisotopic properly embedded arcs on $R$. Note that any subset of a system of arcs on $R$ is itself a system of arcs on $R$.

Let $T$ be a system of arcs on $R$. We say that $T$ is *maximal* if $T$ is not a proper subset of any system of arcs on $R$.

By the previous observations, there are no systems of essential arcs on any disc $R_{0,1}$, and there is a unique isotopy class of systems of essential arcs on any annulus $R_{0,2}$, represented by any arc on $R_{0,2}$ which joins the two distinct components of $\partial R_{0,2}$.

Unless otherwise indicated, all arcs will be assumed to be essential arcs on $R$.

We shall denote arcs by capital letters and their isotopy classes by the corresponding lower case letters (e.g. $A$ and $a = [A] \in \mathcal{A}(R)$).

**Definition 2.1.** *Let $a$ and $b$ be isotopy classes of properly embedded essential arcs on $R$. The geometric intersection number $i(a,b)$ of $a$ and*

*$b$ is the minimum number of points in $A \cap B$ where $A$ and $B$ are arcs on $R$ which represent $a$ and $b$.*

**Definition 2.2.** *Let $a$ and $b$ be isotopy classes of properly embedded essential arcs on $R$. Let $A$ and $B$ be arcs on $R$ representing $a$ and $b$. We say that $A$ and $B$ are in minimal position on $R$ if the number of points of intersection of $A$ and $B$ is equal to $i(a, b)$.*

The following proposition is a standard characterization of minimal position for properly embedded essential arcs, similar to the standard characterization of minimal position for curves [FLP].

**Proposition 2.3.** *Let $a$ and $b$ be isotopy classes of properly embedded essential arcs on $R$. Let $A$ and $B$ be arcs on $R$ representing $a$ and $b$. Then $A$ and $B$ are in minimal position on $R$ if and only if there does not exist a disc $D$ on $R$ such that $\partial D$ is equal to either (i) the union of an arc of $A$ with an arc of $B$ or (ii) the union of an arc of $A$, an arc of $B$, and an arc of $\partial R$.*

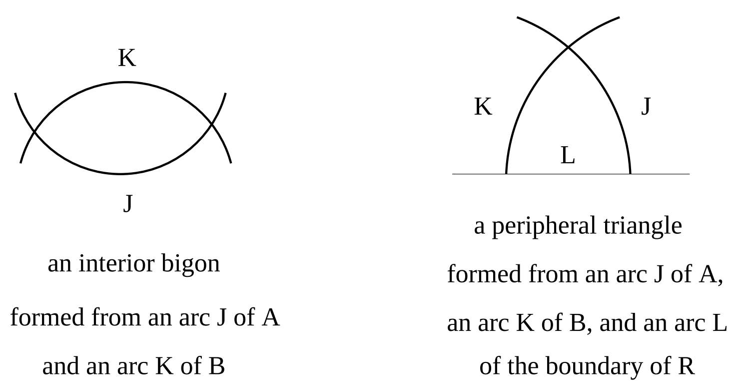


FIGURE 1. Arcs which are not in minimal position

## 3. TRIANGLES

In this section, we assume that $R$ is neither a disc or an annulus. Since $\partial R$ has at least one component, it follows that the Euler characteristic $\chi(R)$ of $R$ is negative. Indeed, $\chi(R) = 2 - 2g - b < 0$, since either (i) $g = 0$ and $b \geq 3$ or (ii) $g \geq 1$ and $b \geq 1$.

Let $T$ be a system of arcs on $R$, $R_T$ denote the surface obtained from $R$ by cutting $R$ along $T$, and $q : R_T \to R$ be the natural quotient map. Suppose that $A$ is an element of $T$. Then the preimage $q^{-1}(A)$ is a disjoint union of two arcs $J$ and $K$ contained in $\partial R_T$. We say that *$J$ and $K$ correspond to the element $A$ of $T$* and *$J$ and $K$ are the sides* *of $A$ in $R_T$.* Note that the restrictions $q| : J \to A$ and $q| : K \to A$ are both homeomorphisms.

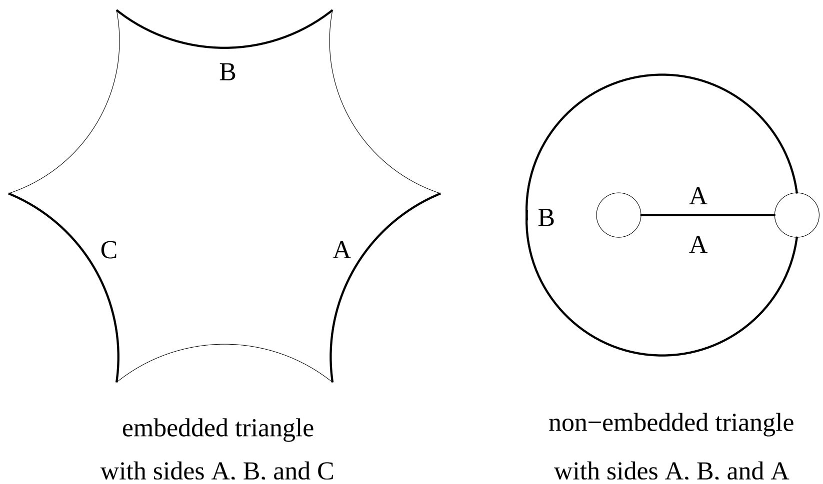


FIGURE 2. The two types of triangles on $R$

**Definition 3.1.** *Let $\{A, B, C\}$ be a system of arcs on $R$, $\Delta$ be a component of $R_{\{A,B,C\}}$, and $q : R_{\{A,B,C\}} \to R$ be the natural quotient map. We say that $\Delta$ is a triangle on $R$ with essential sides corresponding to $A$, $B$, and $C$ if $\Delta$ is a disc such that $\partial\Delta$ is the union of three disjoint arcs, $J$, $K$, and $L$, such that $q(J) = A$, $q(K) = B$, $q(L) = C$, and three disjoint arcs, $X$, $Y$, and $Z$, such that $q(X \cup Y \cup Z) \subset \partial R$; $X$ joins an endpoint of $J$ to an endpoint of $K$, $Y$ joins an endpoint of $K$ to an endpoint of $L$, and $Z$ joins an endpoint of $L$ to an endpoint of $J$.*

Suppose that $\Delta$ is a triangle on $R$ with sides corresponding to $A$, $B$, and $C$. We also say that *$A$, $B$, and $C$ cut off the triangle $\Delta$ from $R$.*

Let $(\Delta, J, K, L, X, Y, Z)$ be as in Definition 3.1. We call $J$, $K$, and $L$ the *essential sides of $\Delta$ corresponding to $A$, $B$, and $C$* and $X$, $Y$, and $Z$ the *peripheral sides of $\Delta$.*

Note, on the one hand, that the restrictions $q| : J \to A$, $q| : K \to B$, and $q| : L \to C$ are homeomorphisms.

Note, on the other hand, that each of the restrictions $q| : X \to q(X)$, $q| : Y \to q(Y)$, and $q| : Z \to q(Z)$ is either a homeomorphism or a quotient map identifying the endpoints of its domain, an arc, to a point. Hence, each of $q(X)$, $q(Y)$, and $q(Z)$ is either an arc of $\partial R$ or a component of $\partial R$.

From hereon, unless otherwise specified, the phrase "side of a triangle of $R$" will be assumed to refer to an essential side of a triangle of $R$.

Since $\{A, B, C\}$ is a system of arcs on $R$, either $A$, $B$, and $C$ are disjoint nonisotopic arcs or at least two of the arcs $A$, $B$, and $C$ are equal. In the former case, we say that $\Delta$ is *embedded*. In the latter case, we say that $\Delta$ is *non-embedded*.

Let $Q$ be the image of $\Delta$ under $q : R_{\{A,B,C\}} \to R$. Suppose, on the one hand, that $\Delta$ is an embedded triangle. Then the restriction $q| : (\Delta, J, K, L, X, Y, Z) \to (Q, A, B, C, q(X), q(Y), q(Z))$ is a homeomorphism. This is the motivation for saying that $\Delta$ is an embedded triangle. In particular, if $\Delta$ is an embedded triangle, then $Q$ is a disc on $R$ such that $\partial Q$ is equal to the union of three disjoint nonisotopic essential arcs, $A$, $B$, and $C$, on $R$, and three arcs $q(X)$, $q(Y)$, and $q(Z)$ in $\partial R$. Note that, in this situation, the six arcs, $A$, $B$, $C$, $q(X)$, $q(Y)$, and $q(Z)$ are determined from the disc $Q$ on $R$. Indeed, $q(X)$, $q(Y)$, and $q(Z)$ are the components of $Q \cap \partial R$ and $A$, $B$, and $C$ are the closures of the components of $\partial Q \setminus (Q \cap \partial R)$. We say that $Q$ *is an embedded triangle on $R$ with essential sides $A$, $B$, and $C$ and peripheral sides $q(X)$, $q(Y)$, and $q(Z)$.*

Suppose, on the other hand, that $\Delta$ is a non-embedded triangle. Then either $A = B$ or $B = C$ or $C = A$. Suppose, for instance, that $C = A$. Then the restriction

$$q : (\Delta, J, K, L, X, Y, Z) \to (Q, A, B, A, q(X), q(Y), q(Z))$$

exhibits $Q$ as a quotient of the disc $\Delta$ obtained by identifying the arcs $J$ and $L$ in $\partial\Delta$ by a homeomorphism in such a way that the resulting quotient $Q$ is orientable. It follows that $Q$ is an annulus on $R$, $q(Z)$ is a component of $\partial R$, $B$ is a properly embedded essential arc on $R$ joining the unique component of $\partial R$ containing $q(X) \cup q(Y)$ to itself, and $A$ is a properly embedded arc on $R$ joining this component of $\partial R$ to the other component $q(Z)$ of $\partial R$. We say that $\Delta$ is a *non-embedded triangle on $R$ with sides corresponding to $A$, $B$, and $A$*; and $(A, B)$ *cuts off the non-embedded triangle $\Delta$ from $R$, with $A$ joining two different components of $\partial R$, and $B$ joining a component of $\partial R$ to itself.* Note that, in this situation, the arc $B$ and the component $q(Z)$ of $\partial R$ are determined from the annulus $Q$ on $R$. Indeed, $q(Z)$ is the unique component of $\partial Q$ which is a component of $\partial R$ and $B$ is the closure of $\partial Q \setminus (Q \cap \partial R)$. On the other hand, $A$, $q(X)$, and $q(Y)$ are not determined from the annulus $Q$ on $R$. In order to determine $A$, $q(X)$, and $q(Y)$, it suffices to specify the arc $A$, which is a properly embedded arc in $Q$ disjoint from $B$ and joining two distinct components of $\partial R$. Once $A$ is specified, we also say that $(Q, A)$ *is a non-embedded triangle on* $R$ *with essential sides* $A$ *and* $B$, $A$ *joining two different components of* $\partial R$, *and* $B$ *joining a component of* $\partial R$ *to itself.*

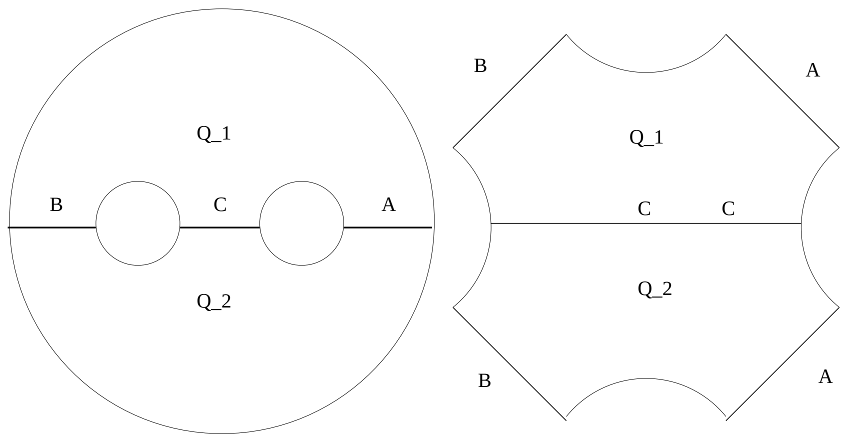


FIGURE 3. Two embedded triangles with the same sides on a pair of pants

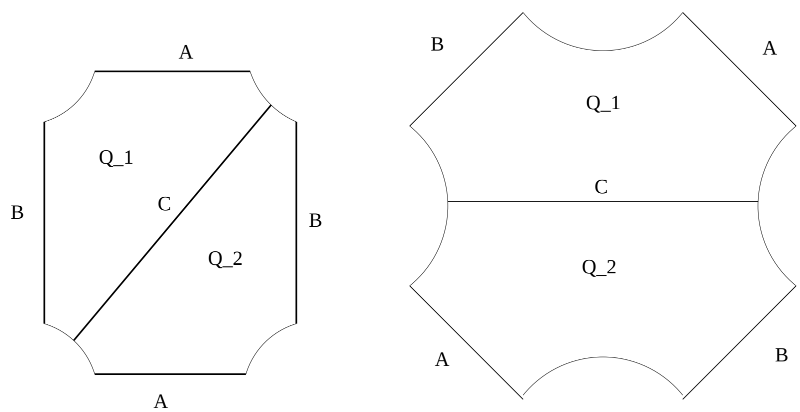


FIGURE 4. Two embedded triangles with the same sides on a torus with one hole

**Proposition 3.2.** *Suppose that* $R$ *is not a disc or an annulus. Let* $\{A, B, C\}$ *be a system of arcs on* $R$. *Suppose that* $\Delta_1$ *and* $\Delta_2$ *are triangles on* $R$ *with sides corresponding to* $A$, $B$, *and* $C$. *Let* $Q_i = q(\Delta_i)$, $i = 1, 2$. *Then either:*

*(1)* $\Delta_1 = \Delta_2$ *or*
*(2)* $R$ *is a pair of pants (i.e. a sphere with three holes);* $\Delta_1$ *and* $\Delta_2$ *are embedded triangles on* $R$*;* $Q_1$ *and* $Q_2$ *have disjoint interiors on* $R$*;* $A \cup B \cup C = \partial Q_1 = Q_1 \cap Q_2 = \partial Q_2$*; and* $R = Q_1 \cup Q_2$ *or*

(3) *$R$ is a torus with one hole; $\Delta_1$ and $\Delta_2$ are embedded triangles on $R$; $Q_1$ and $Q_2$ have disjoint interiors on $R$; $A \cup B \cup C = \partial Q_1 = Q_1 \cap Q_2 = \partial Q_2$; and $R = Q_1 \cup Q_2$.*

*Proof.* Let $i$ be an integer with $1 \leq i \leq 2$. Since $\Delta_i$ is a triangle on $R$ with sides corresponding to $A$, $B$, and $C$, it follows from Definition 3.1 that $\Delta_i$ is a component of $R_{\{A,B,C\}}$; $\Delta_i$ is a disc; and $\partial\Delta_i$ is the union of three disjoint arcs $J_i$, $K_i$, and $L_i$, such that $q(J_i) = A$, $q(K_i) = B$, $q(L_i) = C$, and three disjoint arcs $X_i$, $Y_i$, and $Z_i$, such that $q(X_i \cup Y_i \cup Z_i) \subset \partial R$, $X_i$ joins an endpoint of $J_i$ to an endpoint of $K_i$, $Y_i$ joins an endpoint of $K_i$ to an endpoint of $L_i$, and $Z_i$ joins an endpoint of $L_i$ to an endpoint of $J_i$.

Suppose that $\Delta_1$ is not equal to $\Delta_2$. Then, since $\Delta_1$ and $\Delta_2$ are distinct components of $R_{\{A,B,C\}}$, it follows that the restriction $q| : \Delta_1 \cup \Delta_2 \to Q_1 \cup Q_2$ exhibits $Q_1 \cup Q_2$ as a quotient of the disjoint union $\Delta_1 \cup \Delta_2$ of the discs $\Delta_1$ and $\Delta_2$ obtained by identifying $J_1$ to $J_2$, $K_1$ to $K_2$, and $L_1$ to $L_2$ by homeomorphisms.

It follows that $\Delta_1$ and $\Delta_2$ are embedded triangles on $R$; $Q_1$ and $Q_2$ are discs on $R$ with disjoint interiors on $R$; and $A \cup B \cup C = \partial Q_1 = Q_1 \cap Q_2 = \partial Q_2$. Hence, $Q_1 \cup Q_2$ is a compact surface with $\partial(Q_1 \cup Q_2)$ equal to $q(X_1 \cup Y_1 \cup Z_1) \cup q(X_2 \cup Y_2 \cup Z_2)$. Moreover, since $q(X_i \cup Y_i \cup Z_i) \subset \partial R$, $i = 1, 2$, it follows that $\partial(Q_1 \cup Q_2)$ is contained in $\partial R$. Since $R$ is connected, it follows that $Q_1 \cup Q_2 = R$.

Thus, $R$ is the union of two embedded triangles, $Q_1$ and $Q_2$, meeting along their common sides, $A$, $B$, and $C$. From the definition of an embedded triangle on $R$, $Q_1$ and $Q_2$ are discs, each of whose boundaries is a union of six arcs meeting only at their endpoints. Note that $A$, $B$, and $C$ constitute three of the six arcs on $\partial Q_1$ and three of the six arcs on $\partial Q_2$. Since $Q_1$ is not equal to $Q_2$, we conclude that $\partial Q_1 \cup \partial Q_2$ is a union of exactly 9 (i.e. $6 + 6 - 3$) arcs meeting only at their endpoints. Hence, we have a cell decomposition of $R$ with exactly six 0-cells, the endpoints of the disjoint arcs, $A$, $B$, and $C$; nine open 1-cells, the disjoint interiors of the nine arcs of $\partial Q_1 \cup \partial Q_2$; and two open 2-cells, the disjoint interiors of the discs $Q_1$ and $Q_2$. It follows that the Euler characteristic $\chi(R)$ of $R$ is given by $\chi(R) = v - e + f = 6 - 9 + 2 = -1$. Since $\chi(R) = 2 - 2g - b$, we conclude that $2 - 2g - b = -1$ and, hence, $2g + b = 3$. This implies that either $g = 0$ and $b = 3$ or $g = 1$ and $b = 1$. That is to say, $R$ is either a pair of pants or a torus with one hole. This completes the proof. □

Suppose that $A$, $B$ and $C$ cut off distinct triangles $\Delta_1$ and $\Delta_2$ from $R$. Then, by Proposition 3.2, $\Delta_1$ and $\Delta_2$ are embedded triangles on $R$, and $R$ is either a pair of pants or a torus with one hole. Hence, if $R$ is not a pair of pants or a torus with one hole, then $A$, $B$, and $C$ cut off at most one triangle from $R$.

Moreover, if $R$ is not a pair of pants or a torus with one hole, and $A$, $B$, and $C$ cut off a triangle $\Delta$ from $R$, then $\Delta$ is the unique triangle on $R$ which has sides corresponding to $A$, $B$, and $C$.

## 4. Triangulations

In this section, we assume that $R$ is not a disc or an annulus.

Let $T$ be a system of arcs on $R$ and $\{A, B, C\} \subset T$. Suppose that $\Delta$ is a triangle on $R$ with sides corresponding to $A$, $B$, and $C$. We say that $\Delta$ *is a triangle of* $T$ *on* $R$.

If $\Delta$ is a non-embedded triangle on $R$ with $A$ joining two distinct components of $\partial R$ and $B$ joining a component of $\partial R$ to itself, then $\Delta$ is the unique triangle on $R$ which is cut off from $R$ by $A$ and $B$, and $\Delta$ is the unique triangle of $T$ on $R$ having a side corresponding to $A$ as one of its sides.

Let $R_0 = R_{\{A,B,C\}}$ and $R_1 = R_T$. Suppose that $q_0 : R_0 \to R$ and $q_1 : R_1 \to R$ are the natural quotient maps. Since $\{A, B, C\} \subset T$, there exists a natural quotient map $q_{10} : R_1 \to R_0$ such that $q_1 = q_0 \circ q_{10} : R_1 \to R$.

Note that there exists a unique component $\Delta_1$ of $R_1$ such that $q_{10}(\Delta_1) = \Delta$. Moreover, the restriction $q_{10}| : \Delta_1 \to \Delta$ is a homeomorphism. We may use this restriction $q_{10}| : \Delta_1 \to \Delta$ to identify $\Delta_1$ with $\Delta$. In this way, we canonically identify the triangle $\Delta$ on $R$, which is a component of $R_0$, with a component of $R_1$.

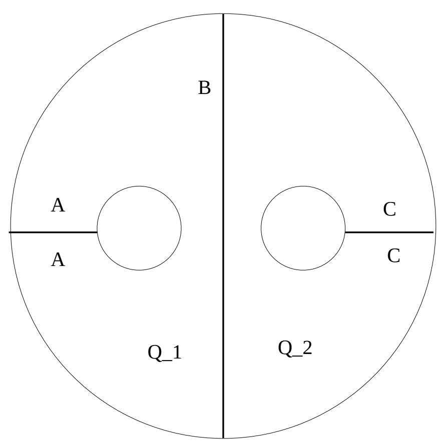


FIGURE 5. Two non-embedded triangles with a common side on a pair of pants

Suppose that $T'$ is a system of arcs on $R$ such that $T \subset T'$ and, hence, $\{A, B, C\} \subset T'$. Let $R_2 = R_{T'}$. Suppose that $q_2 : R_2 \to R$, $q_{21} : R_2 \to R_1$, and $q_{20} : R_2 \to R_0$ are the natural quotient maps. Let $\Delta_2$ be the unique component of $R_2$ such that $q_{20}(\Delta_2) = \Delta$. Then $q_2 = q_1 \circ q_{21} : R_2 \to R$ and $q_{21}(\Delta_2) = \Delta_1$. Hence, the canonical identifications of the triangle $\Delta$ on $R$, which is a component of $R_0$, with components $\Delta_1$ and $\Delta_2$ of $R_1$ and $R_2$ are compatible identifications.

With these canonical identifications in mind, we have the following definition of a triangulation of $R$.

**Definition 4.1.** *Let $T$ be a system of arcs on $R$. We say that $T$ is a triangulation of $R$ if each component of $R_T$ is a triangle on $R$.*

Let $T$ be a triangulation of $R$ and $\Delta$ be a component of $R_T$. Then there exists a unique subset $\{A, B, C\}$ of $T$ such that $\Delta$ is a triangle on $R$ with sides corresponding to $A$, $B$, and $C$.

If $\Delta$ is a non-embedded triangle with $A = C$, then $A$ joins two different components of $\partial R$, and $B$ joins a component of $\partial R$ to itself. Note that, in this situation, $\Delta$ is the unique triangle of $T$ on $R$ having a side corresponding to $A$ as one of its sides, $\Delta$ is a triangle of $T$ on $R$ which has a side corresponding to $B$ as one of its sides, and there is exactly one other triangle $\Delta'$ of $T$ on $R$ having a side corresponding to $B$ as one of its sides. If $\Delta'$ is also non-embedded, then $R$ is a pair of pants. Hence, if $R$ is not a pair of pants, then any two distinct elements of a triangulation $T$ of $R$ cut off at most one non-embedded triangle of $T$ from $R$.

Let $Q = q(\Delta)$. If $D$ is an element of $T$, then either $D \in \{A, B, C\}$ or $D \cap Q = \emptyset$.

In general, some of the triangles of a triangulation of $R$ will be embedded, while others are non-embedded.

**Proposition 4.2.** *Suppose that $R$ is not a disc or an annulus. Let $T$ be a system of arcs on $R$. Then the following are equivalent.*

*(1) $T$ is a maximal system of arcs on $R$.*
*(2) $T$ is a triangulation of $R$.*
*(3) $T$ has exactly $6g + 3b - 6$ elements.*

*Proof.* Since any system of arcs on $R$ is a subset of some triangulation of $R$, any maximal system of arcs on $R$ is a triangulation of $R$. This proves that (1) implies (2).

Suppose that $T$ is a triangulation of $R$. Let $a$ be the number of elements of $T$ and $t$ be the number of components of $R_T$. Let $U$ be a regular neighborhood on $R$ of the union $|T|$ of the elements of $T$ and $V$ be the complement of $|T|$ in $R$. Then $R = U \cup V$, $U$ is a disjoint union of $a$ contractible open sets, $V$ is a disjoint union of $t$ contractible open sets, and $U \cap V$ is a disjoint union of $2a$ contractible open sets. It follows that the Euler characteristic $\chi(R)$ of $R$ satisfies the formula:

$$\chi(R) = \chi(U) + \chi(V) - \chi(U \cap V) = a + t - 2a = t - a. \tag{4.1}$$

Since each triangle of $T$ has exactly three sides and each arc in $T$ has exactly two sides:

$$2a = 3t. \tag{4.2}$$

Since $\chi(R) = 2-2g-b$, we conclude that $2-2g-b = (2a/3)-a = -a/3$ and, hence, $a = 6g+3b-6$. This proves that (2) implies (3).

Suppose that $T$ has $6g + 3b - 6$ vertices. Let $T'$ be a triangulation containing $T$. Since (1) implies (2), it follows that $T'$ has $6g + 3b - 6$ vertices. Since $T$ is contained in $T'$, we conclude that $T$ is equal to $T'$ and, hence, $T$ is a triangulation of $R$. This proves that (3) implies (1), completing the proof. □

**Proposition 4.3.** *Suppose that $R$ is not a disc or an annulus. Let $T$ be a triangulation of $R$. Then $R_T$ has $4g + 2b - 4$ elements.*

*Proof.* Let $a$ be the number of elements of $T$ and $t$ be the number of components of $R_T$. By Proposition 4.2, $a = 6g + 3b - 6$. Then, as in the proof of Proposition 4.2, we conclude that $3t = 2a = 12g + 6b - 12$ and, hence, $t = 4g + 2b - 4$, completing the proof. □

## 5. Quadrilaterals

In this section, we assume that $R$ is not a disc or an annulus.

**Definition 5.1.** *Let $\{A, B, C, D\}$ be a system of arcs on $R$, $\Omega$ be a component of $R_{\{A,B,C,D\}}$, and $q : R_{\{A,B,C,D\}} \to R$ be the natural quotient map. We say that $\Omega$ is a quadrilateral on $R$ with essential sides corresponding to $A$, $B$, $C$, and $D$ if $\Omega$ is a disc such that $\partial\Omega$ is a union of four arcs, $J$, $K$, $L$, and $M$ such that $q(J) = A$, $q(K) = B$, $q(L) = C$, and $Q(M) = D$, and four disjoint arcs $X$, $Y$, $Z$, and $W$ such that $q(X \cup Y \cup Z \cup W) \subset \partial R$; $X$ joins an endpoint of $J$ to an endpoint of $K$; $Y$ joins an endpoint of $K$ to an endpoint of $L$; $Z$ joins an endpoint of $L$ to an endpoint of $M$; and $W$ joins an endpoint of $M$ to an endpoint of $J$.*

Suppose that $\Omega$ is a quadrilateral on $R$ with sides corresponding to $A$, $B$, $C$, and $D$. We also say that $A$, $B$, $C$, and $D$ cut off the quadrilateral $\Omega$ from $R$.

Let $(\Omega, J, X, K, Y, L, Z, M, W)$ be as in Definition 5.1. We call $J$, $K$, $L$, and $M$ the *essential sides of* $\Omega$ *corresponding to* $A$*,* $B$*,* $C$*, and* $D$ and $X$, $Y$, $Z$, and $W$ the *peripheral sides of* $\Omega$.

Note that the restrictions $q| : J \to A$, $q : |K \to B$, $q| : L \to C$, and $q| : M \to D$ are homeomorphisms.

Let $X$, $Y$, $Z$. and $W$ be the peripheral sides of $\Omega$. Note that each of the restrictions $q| : X \to q(X)$, $q| : Y \to q(Y)$, $q| : Z \to q(Z)$, and $q| : W \to q(W)$ is either a homeomorphism or a quotient map identifying the two endpoints of its domain, an arc, to a point. Hence, each of $q(X)$, $q(Y)$, $q(Z)$, and $q(W)$ is either an arc of $\partial R$ or a component of $\partial R$.

From hereon, unless otherwise specified, the phrase "side of a quadrilateral of $R$"' will be assumed to refer to an essential side of a quadrilateral of $R$.

Since $\{A, B, C, D\}$ is a system of arcs on $R$, either $A$, $B$, $C$, and $D$ are disjoint nonisotopic arcs on $R$ or at least two of the arcs $A$, $B$, $C$, and $D$ are equal. In the former case, we say that $\Omega$ is *embedded*. In the latter case, we say that $\Omega$ is *non-embedded*. We leave it to the reader to enumerate the various possibilities for the corresponding restriction of $q| : \Omega \to q(\Omega)$ when $\Omega$ is non-embedded.

**Definition 5.2.** *Let* $\Omega$ *be a quadrilateral on* $R$ *with sides corresponding to* $A$*,* $B$*,* $C$*, and* $D$*. Let* $(\Omega, J, X, K, Y, L, Z, M, W)$ *be as in Definition*

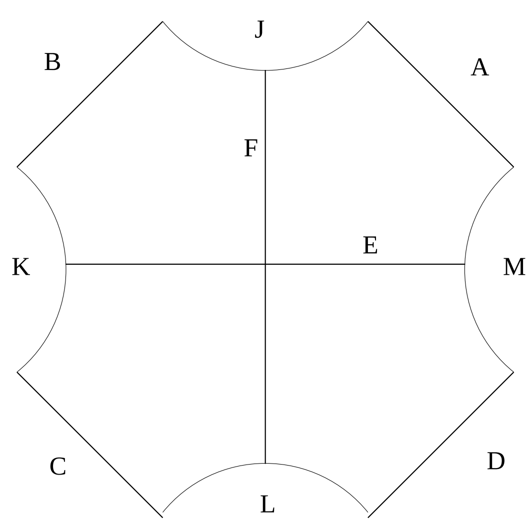


FIGURE 6. An embedded quadrilateral with opposite sides, A and C; opposite sides, B and D; and a pair of diagonals, E and F

*5.1, so that $J$ and $L$ are in different components of $\partial\Omega \setminus (K \cup M)$. We say that $J$ and $L$ are opposite sides of $\Omega$.*

Let $\Omega$ be a quadrilateral on $R$ with sides corresponding to $A$, $B$, $C$, and $D$. Note that $A$ and $C$ correspond to opposite sides of $\Omega$ if and only $B$ and $D$ correspond to opposite sides of $\Omega$.

**Definition 5.3.** *Suppose that $\Omega$ is a quadrilateral on $R$ with sides corresponding to $A$, $B$, $C$, and $D$. Let $(\Omega, J, X, K, Y, L, Z, M, W)$ be as in Definition 5.1, so that $A$ and $C$ correspond to opposite sides of $\Omega$. Let $U$ be a properly embedded arc in $\Omega$ such that $J \cup K$ and $L \cup M$ are in different components of $\Omega \setminus U$ and $E = q(U)$. We say that $E$ is a diagonal of $\Omega$ separating $\{A, B\}$ from $\{C, D\}$.*

Let $Q = q(\Omega)$. Suppose that $E_1$ and $E_2$ are diagonals of $\Omega$ separating $\{A, B\}$ from $\{C, D\}$. Then $E_1$ and $E_2$ are properly embedded arcs on $Q$; $E_1$ and $E_2$ are contained in the subset $(Q \cap \partial R) \cup A \cup B \cup C \cup D$ of $Q$; and $E_1$ and $E_2$ are isotopic on $Q$ by an isotopy on $Q$ which fixes $(Q \cap \partial R) \cup A \cup B \cup C \cup D$ pointwise. Note that any such isotopy on $Q$ extends to an isotopy on $R$ which is supported on $Q$.

Suppose that $E$ is a diagonal of $\Omega$ separating $\{A, B\}$ from $\{C, D\}$. Then there exists a diagonal $F$ of $\Omega$ separating $\{B, C\}$ from $\{D, A\}$ such that $E$ and $F$ intersect transversely and there is exactly one point in $E \cap F$.

**Definition 5.4.** *Suppose that $\Omega$ is a quadrilateral on $R$ with sides corresponding to $A$, $B$, $C$, and $D$; $A$ and $C$ correspond to opposite sides of $\Omega$; $E$ is a diagonal of $\Omega$ separating $\{A, B\}$ from $\{C, D\}$; $F$ is a diagonal of $\Omega$ separating $\{B, C\}$ from $\{D, A\}$; $E$ and $F$ intersect transversely; and there is exactly one point in $E \cap F$. We say that $\{E, F\}$ is a pair of diagonals of $\Omega$.*

Note that a pair of diagonals $\{E, F\}$ of $\Omega$ is uniquely determined from $\Omega$ up to isotopies on $Q$ which fix the subset $(Q \cap \partial R) \cup A \cup B \cup C \cup D$ of $Q$ pointwise. Again, note that any such isotopy on $Q$ extends to an isotopy on $R$ which is supported on $Q$.

Suppose that $\{E, F\}$ is a pair of diagonals of $\Omega$ and $G$ is an arc on $R$ such that $\{A, B, C, D, G\}$ is a system of arcs on $R$. Then either $G \in \{A, B, C, D\}$, $G$ is isotopic on $R$ to $E$, $G$ is isotopic on $R$ to $F$, or $G \cap Q = \emptyset$.

**Proposition 5.5.** *Suppose that $R$ is not a disc or an annulus. Let $\Omega$ be a quadrilateral on $R$ with sides corresponding to $A$, $B$, $C$, and $D$; opposite sides corresponding to $A$ and $B$; and a pair of diagonals $\{E, F\}$. Then the following hold:*

*(1) $E$ and $F$ are disjoint from each of the essential arcs $A$, $B$, $C$, and $D$ on $R$.*
*(2) $E$ and $F$ are essential arcs on $R$.*
*(3) $E$ and $F$ are in minimal position on $R$ with $i([E],[F]) = 1$ and, hence, $E$ and $F$ are not isotopic to each other on $R$.*
*(4) $E$ and $F$ are not isotopic to any of the arcs $A$, $B$, $C$, and $D$ on $R$.*
*(5) $\{A, B, C, D, E\}$ is a system of arcs on $R$.*
*(6) $\{A, B, C, D, F\}$ is a system of arcs on $R$.*
*(7) There exist triangles on $R$, $\Delta_1$ with sides corresponding to $A$, $B$, and $E$; $\Delta_2$ with sides corresponding to $B$, $C$, and $F$; $\Delta_3$ with sides corresponding to $C$, $D$, and $E$; and $\Delta_4$ with sides corresponding to $D$, $A$, and $F$ such that, if $Q = q(\Omega)$ and $Q_i = q(\Delta_i)$, $i = 1, 2, 3, 4$, then $Q_1 \cup Q_3 = Q = Q_2 \cup Q_4$.*
*(8) $\Delta_1$ and $\Delta_3$ are the unique triangles of $\{A, B, C, D, E\}$ on $R$ which have a side corresponding to $E$ as a side.*
*(9) $\Delta_2$ and $\Delta_4$ are the unique triangles of $\{A, B, C, D, F\}$ on $R$ which have a side corresponding to $F$ as a side.*

*Proof.* By construction, $E$ and $F$ are disjoint from the essential arcs $A$, $B$, $C$, and $D$ on $R$. This proves (1).

Suppose that $E$ is not an essential arc on $R$. Then there exists a disc $P$ on $R$ whose boundary is the union of $E$ with an arc $N$ in $\partial R$. It follows that either $A \cup J \cup B$ is contained in $P$ or $C \cup L \cup D$ is contained in $P$. Hence, either $A$ and $B$ are both inessential arcs on $R$ or $C$ and $D$ are both inessential arcs on $R$. Since $A$, $B$, $C$, and $D$ are essential arcs on $D$, this is a contradiction. Hence, $E$ is an essential arc on $R$. Likewise, $F$ is an essential arc on $R$. This proves (2).

Let $a$, $b$, $c$, $d$, $e$, and $f$ be the isotopy classes of the essential arcs $A$, $B$, $C$, $D$, $E$ and $F$ on $R$.

Suppose that $E$ and $F$ are not in minimal position on $R$. Then, since $E$ and $F$ meet at only one point, it follows from Proposition 2.3, that there exists a disc $P$ on $R$ whose boundary is the union of an arc $E_0$ of $E$, an arc $F_0$ of $F$, and an arc $G$ in $\partial R$. It follows that either $A$ is contained in $P$, $B$ is contained in $P$, $C$ is contained in $P$, or $D$ is contained in $P$. This implies that either $A$, $B$, $C$, or $D$ is inessential, which is a contradiction. Hence, $E$ and $F$ are in minimal position on $R$. Since $E$ and $F$ are in minimal position on $R$ and intersect at exactly one point, $i(e, f) = 1$ and, hence, $E$ and $F$ are not isotopic to each other on $R$. This proves (3).

On the other hand, since $E$ and $F$ are disjoint from $A$, $B$, $C$, and $D$, $i(x, y) = 0$, $x \in \{e, f\}$, $y \in \{a, b, c, d\}$. Since $i(e, f) = 1$, it follows that $e$ and $f$ are not equal to any of the isotopy classes, $a$, $b$, $c$, and $d$, of essential arcs on $R$. In other words, $E$ and $F$ are not isotopic to any of the arcs $A$, $B$, $C$ and $D$ on $R$. This proves (4).

Since $\{A, B, C, D\}$ is a system of arcs on $R$, it follows from (1), (2), and (4) that $\{A, B, C, D, E\}$ and $\{A, B, C, D, F\}$ are systems of arcs on $R$. This proves (5) and (6).

Let $T_0 = \{A, B, C, D\}$, $T_1 = T_0 \cup \{E\}$, and $T_2 = T_0 \cup \{F\}$. Suppose that $q_0 : R_0 \to R$, $q_1 : R_1 \to R$, $q_2 : R_2 \to R$, $q_{10} : R_1 \to R_0$, and $q_{20} : R_2 \to R_0$ are the natural quotient maps corresponding to cutting $R$ along $T_0$, $T_1$ and $T_2$.

Since $T_1$ and $T_2$ are systems of arcs on $R$, it follows that $\{A, B, E\}$, $\{B, C, F\}$, $\{C, D, E\}$, and $\{D, A, F\}$ are systems of arcs on $R$.

Let $U$ and $V$ be the unique arcs in $\Omega$ such that $q_0(U) = E$ and $q_0(V) = F$. Note that $U$ and $V$ are properly embedded arcs in the disc $\Omega$ which intersect essentially once and are disjoint from the arcs in $\Omega$ which correspond to $A$, $B$, $C$, and $D$.

Note that $\Omega$ is the union of two discs $\Delta_1$ and $\Delta_3$ such that $\Delta_1 \cap \Delta_3 = U$; $J \cup K \cup U$ is contained in $\partial\Delta_1$; and $L \cup M \cup U$ is contained in $\partial\Delta_3$. Hence $\Delta_1$ is a triangle on $R$ with sides corresponding to $A$, $B$, and $E$ and $\Delta_3$ is a triangle on $R$ with sides corresponding to $C$, $D$, and $F$.

Likewise $\Omega$ is the union of two discs $\Delta_2$ and $\Delta_4$ such that $\Delta_2 \cap \Delta_4 = V$; $K \cup L \cup V$ is contained in $\partial\Delta_2$; and $M \cup J \cup V$ is contained in $\partial\Delta_4$. Hence $\Delta_2$ is a triangle on $R$ with sides corresponding to $B$, $C$, and $F$ and $\Delta_4$ is a triangle on $R$ with sides corresponding to $D$, $A$, and $F$. This proves (7).

Since $\{A, B, E\}$ and $\{C, D, E\}$ are both contained in the system of arcs $\{A, B, C, D, E\}$, the triangles $\Delta_1$ and $\Delta_3$ on $R$ are both triangles of $\{A, B, C, D, E\}$ on $R$ having $E$ as a side. Note that $U_1$ and $U_3$ are the unique arcs in $R_1$ which map onto $E$ by $q_1 : R_1 \to R$.

Suppose that $\Delta$ is a triangle of $T_1$ on $R$ having a side $U$ corresponding to $E$. Then $U$ is an arc in $R_1$ such that $q(U) = E$. Since $U_1$ and $U_3$ are the unique arcs in $R_1$ which map onto $E$ by $q_1 : R_1 \to R$, it follows that $U$ is equal to either $U_1$ or $U_3$. Hence, $\Delta$ is equal to either $\Delta_1$ (i.e. the unique component of $R_1$ which contains $U_1$) or $\Delta_3$ (i.e. the unique component of $R_1$ which contains $U_3$). This proves (8).

A similar argument proves (9).

□

## 6. Elementary moves on triangulations

In this section, we assume that $R$ is not a disc or an annulus.

**Definition 6.1.** *Let $T_1$ be a triangulation of $R$; $\Omega$ be a quadrilateral of $T$ on $R$ with sides corresponding to $A$, $B$, $C$, and $D$; and $\{E, F\}$ be a pair of diagonals of $\Omega$. By Proposition 5.5, $T_2 = (T_1 \setminus \{E\}) \cup \{F\}$ is a triangulation of $R$. We say that $T_2$ is obtained from $T_1$ by an elementary move replacing $E$ with $F$.*

Note that if $\Delta$ is an embedded triangle of a triangulation $T$ on $R$, then there is a unique triangle $\Delta_A$ of $T$ on $R$ which is different from $\Delta$ and has a side corresponding to $A$ as one of its sides.

**Proposition 6.2.** *Suppose that $R$ is not a disc, an annulus, a pair of pants, or a torus with one hole. Suppose that $\Delta$ is an embedded triangle on $R$ with sides corresponding to $A$, $B$, and $C$. Then there exists a triangulation $T$ on $R$ containing $\{A, B, C\}$ such that the unique triangles $\Delta_A$, $\Delta_B$, and $\Delta_C$ of $T$ on $R$ which are different from $\Delta$ and have, respectively, a side corresponding to $A$, a side corresponding to $B$, and a side corresponding to $C$, are distinct triangles of $T$ on $R$.*

*Proof.* Since $\Delta$ is an embedded triangle on $R$ with sides corresponding to $A$, $B$, and $C$, $\{A, B, C\}$ is a system of arcs on $R$. Since $R$ is not a disc or an annulus, it follows from Proposition 4.2 that there exists a triangulation $T$ of $R$ such that $\{A, B, C\}$ is contained in $T$. Since $\Delta$ is an embedded triangle on $R$, it follows that there exist unique triangles $\Delta_A$, $\Delta_B$, and $\Delta_C$ of $T$ on $R$ which are different from $\Delta$ and have, respectively, a side corresponding to $A$, a side corresponding to $B$, and a side corresponding to $C$.

Suppose that $\Delta_A$, $\Delta_B$, and $\Delta_C$ are the same triangle on $R$. Then $\Delta$ and $\Delta_A$ are two distinct triangles on $R$ with the same sides, $A$, $B$, and $C$. Since $R$ is not a disc or an annulus, it follows from Proposition 3.2 that $R$ is either a pair of pants or an annulus, which is a contradiction. Hence, $\Delta_A$, $\Delta_B$, and $\Delta_C$ are not the same triangle on $R$.

Suppose that $\Delta_A = \Delta_B$. Then $\Delta_A$ has a side corresponding to $A$ and another side corresponding to $B$.

Suppose that $\Delta_A$ is a non-embedded triangle on $R$. Then, since $\Delta_A$ has sides corresponding to $A$ and $B$, either $\Delta_A$ is the unique triangle of $T$ on $R$ having a side corresponding to $A$ or $\Delta_A$ is the unique triangle of $T$ on $R$ having a side corresponding to $B$, which is a contradiction, as $\Delta$ is a triangle different from $\Delta_A$ having a side corresponding to $A$ and a side corresponding to $B$. It follows that $\Delta_A$ is an embedded triangle on $R$.

It follows that $\Delta_A$ has a side corresponding to an element $D$ of $T$, where $D$ is not equal to $A$ or $B$.

Suppose that $D = C$. Then $\Delta_A$ is a triangle of $T$ on $R$ different from $\Delta$ having a side corresponding to $C$. In other words, $\Delta_A = \Delta_C$

and, hence, $\Delta_A$, $\Delta_B$, and $\Delta_C$ are the same triangle on $R$, which is a contradiction. Hence, $D$ is not equal to $C$.

Since $\Delta_A$ is an embedded triangle of $T$ on $R$ with sides corresponding to $A$, $B$, and $D$, there exists a unique triangle $\Delta_D$ of $T$ on $R$ which is different from $\Delta_A$ and has a side corresponding to $D$.

Note that there is exactly one side of $\Delta_D$ corresponding to $D$. Suppose that the other two sides of $\Delta_D$ correspond to elements $E$ and $F$ of $T$.

Suppose, on the one hand, that $C$ is not equal to $E$ or $F$. Since the sides of $\Delta_D$ correspond to $D$, $E$, and $F$, none of which are equal to $C$, $\Delta_D$ has no side corresponding to $C$. Since $\Delta_C$ has a side corresponding to $C$, it follows that $\Delta_C$ and $\Delta_D$ are distinct triangles of $T$ on $R$. Since $\Delta_A$ and $\Delta_D$ are distinct triangles of $T$ on $R$ having a side corresponding to $D$, there is a quadrilateral $\Omega$ on $R$ with sides corresponding to $A$, $B$, $E$, and $F$, and diagonal $D$. Let $D'$ be a diagonal of $Q$ such that $\{D, D'\}$ is a pair of diagonals of $\Omega$. Let $T' = (T \setminus \{D\}) \cup \{D'\}$ be the triangulation on $R$ which is obtained from the triangulation $T$ on $R$ by an elementary move replacing $D$ with $D'$. It follows that the unique triangles $\Delta'_A$, $\Delta'_B$, and $\Delta'_C$ of $T'$ on $R$ which are distinct from the triangle $\Delta$ of $T'$ on $R$ and have, respectively, a side corresponding to $A$, a side corresponding to $B$, and a side corresponding to $C$ are distinct triangles of $T'$ on $R$ (see Figure 7).

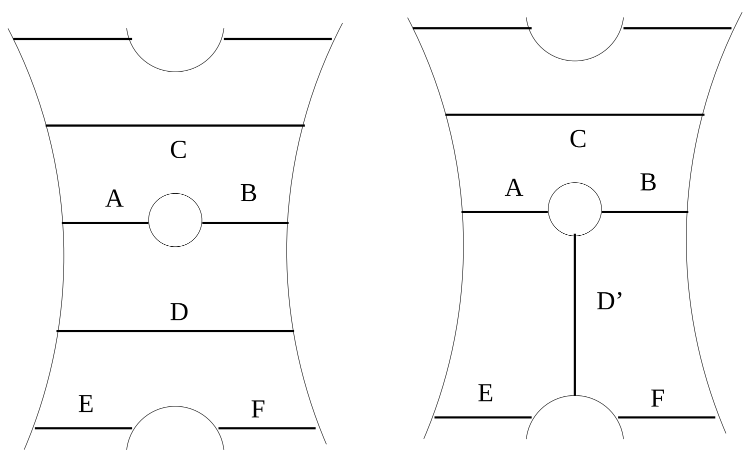


FIGURE 7. Obtaining four triangles by one elementary move

Suppose, on the other hand, that $C$ is equal to either $E$ or $F$. We may assume that $C = E$. It follows, by arguments similar to those given above, that $\Delta_D$ is an embedded triangle of $T$ on $R$ with sides corresponding to $C$, $D$, and $F$, where $F$ is some element of $T$ which is not equal to $A$, $B$, $C$, or $D$.

Since $F$ is a side of the embedded triangle $\Delta_D$ of $T$ on $R$, there is a unique triangle $\Delta_F$ of $T$ on $R$ which is distinct from $\Delta_D$ and has a side corresponding to $F$. By arguments similar to those given above, there is exactly one side of $\Delta_F$ which corresponds to $F$. Let the other two sides of $\Delta_F$ correspond to elements $G$ and $H$ of $T$.

Let $T'$ be the triangulation obtained from $T$ by an elementary move replacing the element $F$ of $T$ by an element $F'$ of $T$. Then let $T''$ be the triangulation obtained from $T'$ by an elementary move replacing the element $D$ of $T'$ by an element $D''$ of $T''$. It follows that the unique triangles $\Delta''_A$, $\Delta''_B$, and $\Delta''_C$ of $T''$ on $R$ which are distinct from the triangle $\Delta$ of $T''$ on $R$ and have, respectively, a side corresponding to $A$, a side corresponding to $B$, and a side corresponding to $C$, are distinct triangles of $T''$ on $R$ (see Figure 7).

This shows, in any case, that there exists a triangulation of $R$ with the desired properties, completing the proof.

□

## 7. The complex of arcs

The *complex of arcs*, $\mathcal{A}(R)$, on $R$ is an abstract simplicial complex. Its vertices are the isotopy classes of properly embedded essential arcs on $R$. A set of vertices of $\mathcal{A}(R)$ forms a simplex of $\mathcal{A}(R)$ if these vertices can be represented by pairwise disjoint arcs on $R$. We denote the group of simplicial automorphisms of $\mathcal{A}(R)$ by $Aut(\mathcal{A}(R))$.

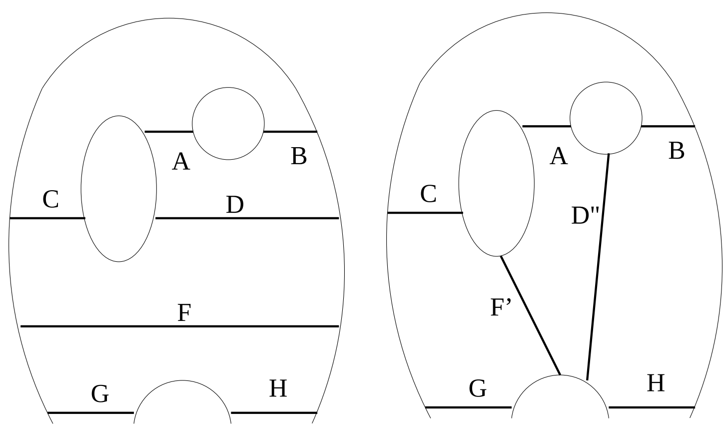


Figure 8. Obtaining four triangles by two elementary moves

The following result is an immediate consequence of preceding observations and Proposition 4.2.

**Proposition 7.1.** *If $R$ is a disc, then $\mathcal{A}(R)$ is empty. If $R$ is an annulus, then $\mathcal{A}(R)$ is a single vertex. If $R$ is not a disc or an annulus, then the dimension of $\mathcal{A}(R)$ is equal to $6g+3b-7$. In any case, every simplex of $\mathcal{A}(R)$ is contained in a top-dimensional simplex of $\mathcal{A}(R)$.*

## 8. Elementary moves on maximal simplices

**Definition 8.1.** *Let $T_1$ be a triangulation of $R$; $\Omega$ be a quadrilateral of $T$ on $R$ with sides corresponding to $A$, $B$, $C$, and $D$; and $\{E, F\}$ be a pair of diagonals of $\Omega$. Let $e$ and $f$ be the vertices of $\mathcal{A}(R)$ corresponding to $E$ and $F$. By Proposition 5.5, $T_2 = (T_1 \setminus \{E\}) \cup \{F\}$ is a triangulation of $R$. Let $\sigma_i$ be the simplex of $\mathcal{A}(R)$ corresponding to $T_i$, $i = 1, 2$. Then we say that $\sigma_2$ is obtained from $\sigma_1$ by an elementary move replacing $e$ with $f$.*

Suppose that a maximal simplex $\sigma_2$ of $\mathcal{A}(R)$ is obtained from a maximal simplex $\sigma_1$ of $\mathcal{A}(R)$ by an elementary move replacing $e$ with $f$. Note that $\sigma_1$ is obtained from $\sigma_2$ by an elementary move replacing $f$ with $e$; $\sigma_1 \setminus \sigma_2 = \{e\}$; $\sigma_2 \setminus \sigma_1 = \{f\}$; and $\sigma_1 \setminus \{e\} = \sigma_1 \cap \sigma_2 = \sigma_2 \setminus \{f\}$.

Let $\sigma_0 = \sigma_1 \cap \sigma_2$. Note that $\sigma_0$ is a codimension one face of $\mathcal{A}(R)$ and $\sigma_1$ and $\sigma_2$ are the unique codimension zero faces of $\mathcal{A}(R)$ containing $\sigma_0$.

We shall need the following strong form of connectivity for $\mathcal{A}(R)$ stated as the "Connectivity Theorem for Elementary Moves" in Mosher [M]. See also Corollary 5.5.B in Ivanov's survey article on Mapping Class Groups [I2].

**Theorem 8.2.** *(Connectivity Theorem for Elementary Moves,* [M]*) Suppose that $R$ is not a disc or an annulus. Then any two triangulations of $R$ are related by a finite sequence of elementary moves. More precisely, if $T$ and $T'$ are triangulations of $R$ and $\sigma$ and $\sigma'$ are the simplices of $\mathcal{A}(R)$ corresponding to $T$ and $T'$, then there exists a sequence of simplices $\sigma_i$, $1 \leq i \leq N$, such that $\sigma_1 = \sigma$, $\sigma_N = \sigma'$, and for each integer $i$ with $1 \leq i < N$, $\sigma_{i+1}$ is obtained from $\sigma_i$ by an elementary move.*

**Remark 8.3.** *The statement of Theorem 8.2 in Mosher* [M] *is in terms of ideal triangulations of a punctured surface $(S, P)$ rather than triangulations of $R$. For our purposes here, we let $S$ be the closed surface of genus $g$ obtained from $R$ by attaching a disc $D_i$ to each component $\partial_i$ of $\partial R$, $1 \leq i \leq b$, and $P$ be a set of points, $x_i$, $1 \leq i \leq p$, with $x_i$ in the interior of $D_i$, $1 \leq i \leq b$. Then we may relate triangulations of*

*$R$ as defined in this paper to ideal triangulations of $(S, P)$ as defined in Mosher* [M] *by "coning off" arcs on $R$ to arcs or loops on $S$ joining points in $P$ to points in $P$. In this way, we obtain the above restatement of the Connectivity Theorem for Elementary Moves in a form suitable for our purposes in this paper.*

We now describe how elementary moves on maximal simplices arise from considering elements of triangulations of $R$.

**Proposition 8.4.** *Suppose that $R$ is not a disc or an annulus. Let $T$ be a triangulation of $R$; $E$ be an element of $T$; $\sigma$ be the simplex of $\mathcal{A}(R)$ corresponding to $T$; $e$ be the vertex of $\mathcal{A}(R)$ corresponding to $E$; and $\sigma_0 = \sigma \setminus \{e\}$. Then the following are equivalent:*

1. *There is a unique triangle of $T$ having a side corresponding to $E$ as a side.*
2. *There exists a non-embedded triangle of $T$ on $R$ with sides corresponding to $E$, $B$, and $E$, with $E$ joining two different components of $\partial R$, and $B$ joining a component of $\partial R$ to itself.*
3. *$\sigma$ is the unique maximal simplex of $\mathcal{A}(R)$ containing $\sigma_0$.*

*Proof.* Note that, by assumption, $\sigma$ is a maximal simplex of $\mathcal{A}(R)$ containing $\sigma_0$.

Suppose that there is a unique triangle $\Delta$ of $T$ having a side corresponding to $E$ as a side. Let $J$ and $K$ be the two arcs in $R_T$ which map via the natural quotient map $q : R_T \to R$ onto $E$. Let $\Delta_1$ be the unique component of $R_T$ containing the arc $J$ and $\Delta_2$ be the unique component of $R_T$ containing the arc $K$. Then $\Delta_1$ and $\Delta_2$ are both triangles of $T$ having a side corresponding to $E$ as a side. Since $\Delta$ is the unique triangle of $T$ having $E$ as a side, we conclude that $\Delta_1 = \Delta = \Delta_2$. It follows that $J$ and $K$ are two of the sides in the component $\Delta$ of $R_T$. Let $L$ be the remaining side of $\Delta$ and $B = q(L)$. Since $q(J) = E = q(K)$, it follows that $\Delta$ is a non-embedded triangle of $T$ on $R$ with sides corresponding to $E$, $B$, and $E$, with $E$ joining two different components of $\partial R$, and $B$ joining a component of $\partial R$ to itself. This proves that (1) implies (2).

Suppose that there exists a non-embedded triangle $\Delta$ of $T$ on $R$ with sides corresponding to $E$, $B$, and $E$, with $E$ joining two different components of $\partial R$, and $B$ joining a component of $\partial R$ to itself. Let $\sigma'$ be a maximal simplex of $\mathcal{A}(R)$ containing $\sigma_0$. Since $\sigma_0 = \sigma \setminus \{e\}$ and $T \setminus \{E\}$ is a system of arcs representing the simplex $\sigma_0$, there exists a triangulation $T'$ of $R$ such that $T \setminus \{E\} \subset T'$. Hence, there exists a unique element $E'$ of $T'$ such that $T' \setminus \{E'\} = T \setminus \{E\}$.

Let $Q_1 = q(\Delta)$. Since $E'$ is disjoint from and not isotopic to any element of $T \setminus \{E\}$ and $\Delta$ is a non-embedded triangle of $T$ on $R$ with sides corresponding to $E$, $B$, and $E$, with $E$ joining two different components of $\partial R$, and $B$ joining a component of $\partial R$ to itself, it follows that $Q_1$ is an annulus and $E'$ is a properly embedded arc in $Q_1$ which is disjoint from $B$ and joins the two distinct components of $\partial Q_1$ to each other. Since $E$ is also a properly embedded arc in $Q_1$ which is disjoint from $B$ and joins the two different components of $\partial Q_1$, it follows that $E'$ is isotopic to $E$. This implies that $\sigma' = \sigma_0 \cup \{[E']\} = \sigma_0 \cup \{[E]\} = \sigma_0 \cup \{e\} = \sigma$. This proves that (2) implies (3).

It remains to prove that (3) implies (1). Since $E$ is contained in exactly one or exactly two triangles of $\mathcal{A}(R)$, it suffices to prove the following proposition, using that (1) implies (3), which we have already established here.

□

**Proposition 8.5.** *Suppose that $R$ is not a disc or an annulus. Let $T_1$ be a triangulation of $R$; $E$ be an element of $T_1$; $T_0 = T_1 \setminus \{E\}$; $\sigma_1$ be the simplex of $\mathcal{A}(R)$ corresponding to $T_1$; $e$ be the vertex of $\mathcal{A}(R)$ corresponding to $E$; and $\sigma_0 = \sigma_1 \setminus \{e\}$. Then the following are equivalent:*

*(1) There are exactly two triangles of $T_1$ on $R$ having a side corresponding to $E$ as a side.*
*(2) There exists a quadrilateral of $T_0$ on $R$ having $E$ as one of its diagonals.*
*(3) There are exactly two maximal simplices of $\mathcal{A}(R)$ containing $\sigma_0$, $\sigma_1$ and a simplex $\sigma_2$ which is obtained from $\sigma_1$ by an elementary move replacing $e$ by a vertex $f$ of $\sigma_2$.*

*Proof.* Let $q_0 : R_0 \to R$, $q_1 : R_1 \to R$, and $q_{10} : R_1 \to R_0$ be the natural quotient maps corresponding to cutting $R$ along $T_0$ and $T_1$.

Suppose that there are exactly two triangles of $T_1$ on $R$, $\Delta_1$ and $\Delta_2$, having a side corresponding to $E$ as a side.

Suppose that the sides of $\Delta_1$ correspond to $A$, $B$, and $E$; and the sides of $\Delta_2$ correspond to $C$, $D$, and $E$. Since $\Delta_1$ and $\Delta_2$ are triangles of $T_1$ on $R$, $\{A, B, C, D, E\} \subset T_1$. Since $\Delta_1$ and $\Delta_2$ are distinct triangles of $T_1$ on $R$ having a side corresponding to $E$ as a side, it follows that $E$ is not equal to $A$, $B$, $C$, or $D$. Hence, $\{A, B, C, D\} \subset T_0 = T_1 \setminus \{E\}$.

It follows that there is a quadrilateral $\Omega$ of $T_O$ on $R$ with sides corresponding to $A$, $B$, $C$, and $D$ such that $\Omega = q_{10}(\Delta_1) \cup q_{10}(\Delta_2)$. Since $\Omega = q_{10}(\Delta_1) \cup q_{10}(\Delta_2)$, it follows that $E$ is a diagonal of $\Omega$. Hence, $\Omega$ is a quadrilateral of $T_0$ on $R$ having $E$ as one of its diagonals. This proves that (1) implies (2).

Suppose that there exists a quadrilateral $\Omega$ of $T_0$ having $E$ as one of its diagonals. Let $Q = q_0(\Omega)$. Note that there exists a diagonal $F$ of $\Omega$ such that $\{E, F\}$ is a pair of diagonals of $\Omega$. Let $T_2 = (T \setminus \{E\}) \cup \{F\}$ be the triangulation of $R$ which is obtained from the triangulation $T_1$ of $R$ by an elementary move replacing $E$ with $F$. Let $f$ be the vertex of $\mathcal{A}(R)$ represented by $F$ and $\sigma_2$ be the simplex of $\mathcal{A}(R)$ corresponding to $T_2$. Then $\sigma_1$ and $\sigma_2$ are both maximal simplices of $\mathcal{A}(R)$ containing $\sigma_0$ and $\sigma_2$ is obtained from $\sigma_1$ by an elementary move replacing $e$ with $f$.

Suppose that $\sigma$ is a maximal simplex of $\mathcal{A}(R)$ containing $\sigma_0$. Since $T_0$ is a system of arcs representing $\sigma_0$, there exists a triangulation $T$ of $R$ and an element $G$ of $T$ such that $T_0 = T \setminus \{G\}$. Since $T$ is a triangulation of $R$ and $\Omega$ is a quadrilateral of $T_0$ on $R$, it follows that $G$ is contained in $Q$ and is disjoint from $A$, $B$, $C$, and $D$. This implies that $G$ is a diagonal of $\Omega$. Since any diagonal of $\Omega$ is isotopic to one of the diagonals, $E$ or $F$, of a pair of diagonals, $\{E, F\}$, of $\Omega$, it follows that $G$ is isotopic to either $E$ or $F$. Let $g$ be the isotopy class of $G$ on $R$. Then either $g = e$ or $g = f$. Hence, either $\sigma = \sigma_0 \cup \{g\} = \sigma_0 \cup \{e\} = \sigma_1$ or $\sigma = \sigma_0 \cup \{g\} = \sigma_0 \cup \{f\} = \sigma_2$. This proves that (2) implies (3).

Suppose that there are exactly two maximal simplices of $\mathcal{A}(R)$ containing $\sigma_0$, $\sigma_1$ and a simplex $\sigma_2$ which is obtained from $\sigma_1$ by an elementary move replacing $e$ by a vertex $f$ of $\sigma_2$.

Since $E$ is an element of $T$, there are either exactly one or exactly two triangles of $T$ having $E$ as a side. Suppose that there is exactly one triangle of $T$ having $E$ as a side. It follows from the proof of Proposition 8.4 above, where it is proved that the condition (1) in Proposition 8.4 implies the condition (3) in Proposition 8.4, that $\sigma_1$ is the unique maximal simplex of $\mathcal{A}(R)$ containing $E$, which is a contradiction. Hence, there are exactly two triangles of $T$ on $R$ having $E$ as a side. This proves that (3) implies (1).

This completes the proof of Proposition 8.5 and, hence, as previously observed, the proof of Proposition 8.4. $\square$

## 9. Examples

There is a natural action of the extended mapping class group $\Gamma^*(R)$ of $R$ on $\mathcal{A}(R)$ given by the rule $h_*(j) = [H(J)]$, where $H : R \to R$ represents $h \in \Gamma^*(R)$ and $J \subset R$ represents $j \in \mathcal{A}(R)$. We denote the corresponding representation of $\Gamma^*(R)$ as a group of automorphisms of $\mathcal{A}(R)$ by $\rho : \Gamma^*(R) \to Aut(\mathcal{A}(R))$ so that $\rho(h) = h_*$.

The automorphisms of $\mathcal{A}(R)$ in the image of the natural representation $\rho : \Gamma^*(R) \to Aut(\mathcal{A}(R))$ are exactly the *geometric* automorphisms of $\mathcal{A}(R)$ (i.e. the automorphisms of $\mathcal{A}(R)$ which are induced by homeomorphisms $R \to R$). Our goal is to determine whether or not all injective simplicial maps $\mathcal{A}(R) \to \mathcal{A}(R)$ are geometric. We begin our study of this question with a few illustrative examples.

9.1. $g = 0$, $b = 1$. If $R$ is a disc (i.e. a sphere with one hole), then no arc on $R$ is essential. Hence, $\mathcal{A}(R) = \emptyset$; every injective simplicial map $\mathcal{A}(R) \to \mathcal{A}(R)$ is an automorphism of $\mathcal{A}(R)$; and $Aut(\mathcal{A}(R))$ is a trivial group.

On the other hand, $\Gamma^*(R)$ is a cyclic group of order two. It follows that $Z(\Gamma^*(R)) = \Gamma^*(R)$ and, hence, $\Gamma^*(R)/Z(\Gamma^*(R))$ is also a trivial group. Hence, $\Gamma^*(R)/Z(\Gamma^*(R))$ is isomorphic to $Aut(\mathcal{A}(R))$.

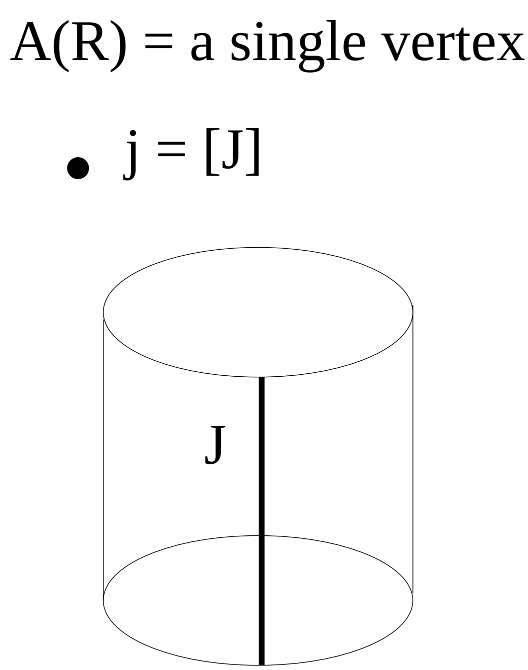


Figure 9. The arc complex of a sphere with two holes and an arc representing its unique vertex

9.2. $g = 0$, $b = 2$. If $R$ is an annulus (i.e. a sphere with two holes), then an essential arc must join the two components of $\partial R$ and all such arcs are isotopic on $R$. It follows that $\mathcal{A}(R)$ consists of a single vertex $j$ as illustrated in Figure 9; every injective simplicial map $\mathcal{A}(R) \to \mathcal{A}(R)$ is an automorphism of $\mathcal{A}(R)$; and $Aut(\mathcal{A}(R))$ is a trivial group.

Note, on the other hand, that the action of $\Gamma^*(R)$ on $\pi_0(\partial R)$ yields a short exact sequence:

$$1 \to Z_2 \to \Gamma^*(R) \to \Sigma(\pi_0(\partial R) \to 1 \tag{9.1}$$

where $\Sigma(\pi_0(\partial R)) \cong \Sigma_2$ is the group of permutations of $\pi_0(\partial R)$, $\Gamma^*(R) \to \Sigma(\pi_0(\partial R))$ is the corresponding representation, and the kernel $Z_2$ of $\Gamma^*(R) \to \Sigma(\pi_0(\partial R))$ is the cyclic group of order 2 generated by the isotopy class of any orientation reversing involution of $R$ which preserves each component of $\partial R$.

The natural representation $\Gamma^*(R) \to \Sigma(\pi_0(\partial R))$ restricts to an isomorphism $\Gamma(R) \to \Sigma(\pi_0(\partial R))$. It follows that the above exact sequence (9.1) is a split short exact sequence. It follows that $\Gamma^*(R)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. This implies that the center $Z(\Gamma^*(R))$ of $\Gamma^*(R)$ is equal to $\Gamma^*(R)$; $\Gamma^*(R)/Z(\Gamma^*(R))$ is a trivial group; and, hence, $\Gamma^*(R)/Z(\Gamma^*(R))$ is isomorphic to $Aut(\mathcal{A}(R))$.

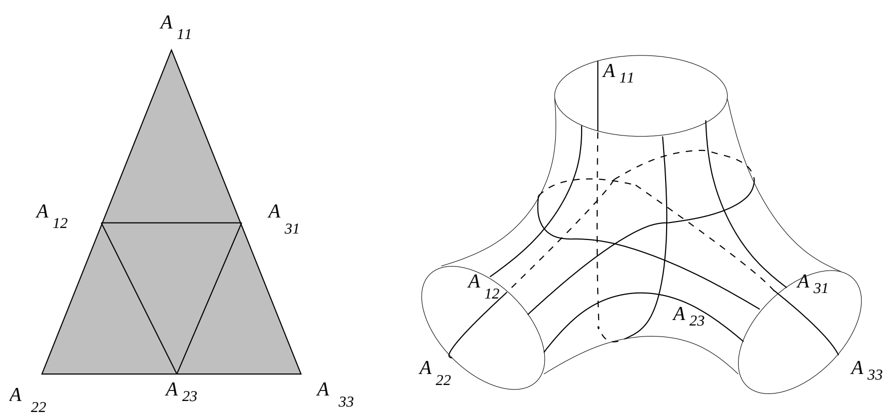


FIGURE 10. The arc complex of a sphere with three holes and arcs representing its six vertices

9.3. $g = 0$, $b = 3$. If $R$ is a pair of pants (i.e. a sphere with three holes), then there are exactly six isotopy classes of essential arcs on $R$ and $\mathcal{A}(R)$ is a two-complex represented by a regular tesselation of a triangle by four triangles as illustrated in Figure 10. Note that each maximal simplex of $\mathcal{A}(R)$ corresponds to a triangulation of $R$; each maximal simplex of $\mathcal{A}(R)$ is a top-dimensional simplex of $\mathcal{A}(R)$; and each codimension one simplex of $\mathcal{A}(R)$ is a face of one or two codimension zero faces of $\mathcal{A}(R)$.

The latter case, where the codimension one simplex of $\mathcal{A}(R)$ has two sides in $\mathcal{A}(R)$, corresponds to removing an element of a triangulation corresponding to a side of an embedded triangle of this triangulation and replacing it by the corresponding diagonal of the associated quadrilateral. The former case, where the codimension one simplex of $\mathcal{A}(R)$ has one side in $\mathcal{A}(R)$, corresponds to removing the double-sided edge of a non-embedded triangle.

Note, furthermore, that any two distinct codimension zero faces are connected by a sequence of codimension zero faces such that any two consecutive codimension zero faces in this sequence have a common codimension one face.

Since $\mathcal{A}(R)$ is a finite simplicial complex, every injective simplicial map $\mathcal{A}(R) \to \mathcal{A}(R)$ is an automorphism of $\mathcal{A}(R)$.

Note, on the one hand, that $Aut(\mathcal{A}(R))$ is isomorphic to the symmetric group $\Sigma_3$ on three letters. Indeed, $Aut(\mathcal{A}(R))$ is naturally isomorphic to the group of permutations $\Sigma(\pi_0(\partial R))$ of the set of components $\pi_0(\partial R)$ of $\partial R$.

Note, on the other hand, that the action of $\Gamma^*(R)$ on $\pi_0(\partial R)$ yields a short exact sequence:

$$1 \to Z_2 \to \Gamma^*(R) \to \Sigma(\pi_0(\partial R) \to 1 \tag{9.2}$$

where $\Sigma(\pi_0(\partial R)) \cong \Sigma_3$ is the group of permutations of $\pi_0(\partial R)$, $\Gamma^*(R) \to \Sigma(\pi_0(\partial R))$ is the corresponding representation, and the kernel $Z_2$ of $\Gamma^*(R) \to \Sigma(\pi_0(\partial R))$ is the cyclic group of order 2 generated by the isotopy class of any orientation reversing involution of $R$ which preserves each component of $\partial R$.

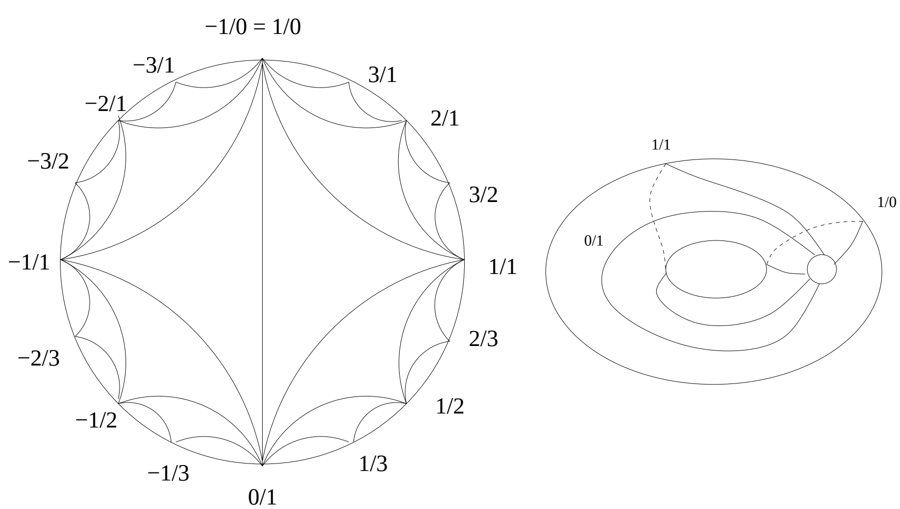


Figure 11. The arc complex of a torus with one hole and three arcs representing one of its triangles

The natural representation $\Gamma^*(R) \to \Sigma(\pi_0(\partial R))$ restricts to an isomorphism $\Gamma(R) \to \Sigma(\pi_0(\partial R))$. It follows that the above exact sequence (9.2) is a split short exact sequence. Since $\Sigma_3$ has trivial center, it follows that $Z_2$ is equal to the center $Z(\Gamma^*(R))$ of $\Gamma^*(R)$; $\Gamma^*(R)/Z(\Gamma^*(R))$ is also naturally isomorphic to $\Sigma(\pi_0(\partial R))$; and, hence, $\Gamma^*(R)/Z(\Gamma^*(R))$ is naturally isomorphic to $Aut(\mathcal{A}(R))$.

9.4. $g = 1$**,** $b = 1$**.** If $R$ is a torus with one hole, then $\mathcal{A}(R)$ is represented by the decomposition of the hyperbolic plane $\mathbb{H}$ into ideal triangles by the familiar *Farey graph*, $\mathfrak{F}$.

More precisely, let $S$ be the torus obtained by attaching a disc $D$ to $\partial R$ and $P$ be a point in the interior of $D$. Choose an identification of $(S, P)$ with the standard torus, $(S^1 \times S^1, (1, 1))$. Then the isotopy classes of arcs on $R$ correspond naturally to the rational points on the circle at infinity $S_\infty = \mathbb{R}^* = \mathbb{R} \cup \infty$, where the arc $A$ on $R$ corresponds to the rational point $p/q$ if and only if the extension of the arc $A$ on $R$ to a closed curve on $S$ by "coning off" the endpoints of $A$ in $\partial D$ to the "center" $P$ of $D$ represents $\pm(p, q) \in \mathbb{Z} \oplus \mathbb{Z} = \pi_1(S^1 \times S^1, (1, 1))$.

The ideal triangles of the decomposition of the hyperbolic plane $\mathbb{H}$ by the *Farey graph* correspond to ideal triangulations of $(S, P)$, which correspond to maximal simplices of $\mathcal{A}(R)$, as in Remark 8.3.

Again, in this example, any injective simplicial map $\mathcal{A}(R) \to \mathcal{A}(R)$ is an automorphism of $\mathcal{A}(R)$.

In this case, as is well-known, $(\Gamma^*(R), Z(\Gamma^*(R)) \cong (GL(2, \mathbb{Z}), \{\pm I\})$, and, hence, $\Gamma^*(R)/Z(\Gamma^*(R)) \cong PGL(2, \mathbb{Z}) \cong Aut(\mathfrak{F}) \cong Aut(\mathcal{A}(R))$.

## 10. Preservation of triangulations

In this section, we assume that $R$ is not a disc or an annulus.

**Definition 10.1.** *Let $\sigma$ be a simplex of $\mathcal{A}(R)$. Let $T$ be a triangulation of $R$. We say that $\sigma$ is the simplex of $\mathcal{A}(R)$ corresponding to $T$ if $\sigma$ is the set of isotopy classes of the elements of $T$.*

**Proposition 10.2.** *Suppose that $R$ is not a disc or an annulus. Let $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ be an injective simplicial map. Let $\sigma$ be a simplex of $\mathcal{A}(R)$ corresponding to a triangulation $T$ of $R$. Then there exists a triangulation $T'$ of $R$ such that $\lambda(\sigma)$ is the simplex of $\mathcal{A}(R)$ corresponding to $T'$.*

*Proof.* By Proposition 4.2, $T$ has exactly $6g + 3b - 6$ elements. Since $T$ is a system of arcs on $S$, it follows that $\sigma$ has exactly $6g + 3b - 6$ vertices. Since $\lambda$ is injective, $\lambda(\sigma)$ has exactly $6g + 3b - 6$ vertices. Let $T'$ be a system of arcs on $R$ representing the simplex $\lambda(\sigma)$ of $\mathcal{A}(R)$.

Then $T'$ has exactly $6g + 3b - 6$ elements. It follows from Proposition 4.2 that $T'$ is a triangulation of $R$. $\square$

## 11. Preservation of elementary moves

In this section, we assume that $R$ is not a disc or an annulus.

**Proposition 11.1.** *Suppose that $R$ is not a disc or an annulus. Let $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ be an injective simplicial map of $\mathcal{A}(R)$. Suppose that $a$ and $b$ are vertices of $\mathcal{A}(R)$ such that $i(a, b) = 1$. Then $i(\lambda(a), \lambda(b)) = 1$.*

*Proof.* Let $A$ and $B$ be representatives of $a$ and $b$ intersecting once. Note that we may complete $A$ to a triangulation $T_1$ of $R$ such that $(T_1 \setminus \{A\}) \cup \{B\}$ is also a triangulation of $R$. Let $T_2 = (T_1 \setminus \{A\}) \cup \{B\}$.

Let $\sigma_i$ be the simplex of $\mathcal{A}(R)$ corresponding to the triangulation $T_i$ of $R$, $i = 1, 2$, and $\sigma_0 = \sigma_1 \cap \sigma_2$. Note that $\sigma_2 \setminus \{b\} = \sigma_0 = \sigma_1 \setminus \{a\}$, and $\sigma_2$ is obtained from $\sigma_1$ by an elementary move replacing $a$ with $b$.

Let $\sigma'_i = \lambda(\sigma_i)$, $i = 0, 1, 2$, $a' = \lambda(a)$, and $b' = \lambda(b)$. Since $\lambda$ is injective, it follows from Proposition 10.2 that there exists a triangulation $T'_i$ corresponding to $\sigma'_i$, $i = 1, 2$.

Since $i(a, b) \neq 0$, there does not exist a simplex of $\mathcal{A}(R)$ having both $a$ and $b$ as vertices. Since $a \in \sigma_1$ and $b \in \sigma_2$, it follows that $\sigma_1 \neq \sigma_2$. Since $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ is an injective simplicial map, it follows that $\sigma'_1 \neq \sigma'_2$.

Let $A'$ be the representative of $a'$ in $T'_1$.

Since $\sigma_2 \setminus \{b\} = \sigma_0 = \sigma_1 \setminus \{a\}$ and $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ is an injective simplicial map, $\sigma'_2 \setminus \{b'\} = \sigma'_0 = \sigma'_1 \setminus \{a'\}$.

Note that we may choose a representative $B'$ of $b'$ such that $B'$ is disjoint from and not isotopic to each element of $T'_1 \setminus \{A'\}$. Let $T'_2 = (T'_1 \setminus \{A'\}) \cup \{B'\}$. Then $T'_2$ is a triangulation of $R$ and $\sigma'_2$ is the simplex of $\mathcal{A}(R)$ corresponding to $T'_2$. Since $\sigma'_1$ and $\sigma'_2$ are distinct maximal simplices of $\mathcal{A}(R)$ containing $\sigma'_0$, it follows from Proposition 8.5 that $\sigma'_2$ is obtained from $\sigma'_1$ by an elementary move replacing $a'$ with $b'$. It follows from Proposition 5.5 that $i(a', b') = 1$, completing the proof. $\square$

## 12. Preservation of embedded triangles

In this section, we assume that $R$ is not a disc, an annulus, a pair of pants, or a torus with one hole.

**Definition 12.1.** *Let $\{a, b, c\}$ be a 2-simplex of $\mathcal{A}(R)$. We say that $\{a, b, c\}$ corresponds to an embedded triangle on $R$ if there exists an*

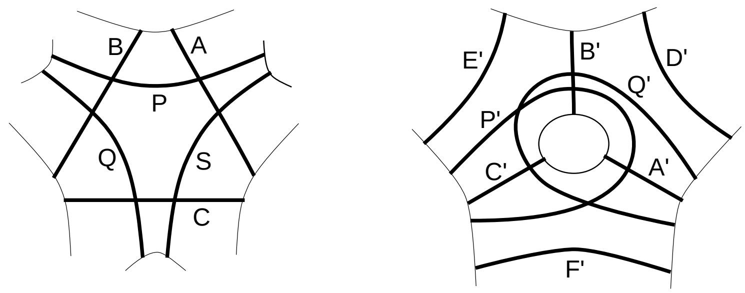


FIGURE 12

*embedded triangle $\Delta$ on $R$ with sides corresponding to $A$, $B$, and $C$ representing $a$, $b$, and $c$.*

**Proposition 12.2.** *Suppose that $R$ is not a disc, an annulus, a pair of pants, or a torus with one hole. Let $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ be an injective simplicial map and $\{a, b, c\}$ be a 2-simplex of $\mathcal{A}(R)$. If $\{a, b, c\}$ corresponds to an embedded triangle on $R$, then $\{\lambda(a), \lambda(b), \lambda(c)\}$ corresponds to an embedded triangle on $R$.*

*Proof.* Let $\Delta$ be an embedded triangle on $R$ with sides corresponding to $A$, $B$, and $C$ representing $a$, $b$, and $c$. Let $T_0 = \{A, B, C\}$. Since $R$ is not a disc, an annulus, a pair of pants, or a torus with one hole, it follows from Proposition 6.2 that we can complete the system of arcs $T_0$ on $R$ to a triangulation $T_1$ of $R$ such that if $\Delta_A$ is the unique triangle of $T_1$ on $R$ different from $\Delta$ having a side corresponding to $A$, $\Delta_B$ is the unique triangle of $T_1$ on $R$ different from $\Delta$ having a side corresponding to $B$, and $\Delta_C$ is the unique triangle of $T_1$ on $R$ different from $\Delta$ having a side corresponding to $C$, then $\Delta$, $\Delta_A$, $\Delta_B$, and $\Delta_C$ are four distinct triangles of $T_1$ on $R$.

Let $q_0 : R_0 \to R$, $q_1 : R_1 \to R$, and $q_{10} : R_1 \to R_0$ be the natural quotient maps corresponding to cutting $R$ along $T_0$ and $T_1$.

Note that $\partial\Delta_A$ is equal to a union of arcs, $A_1$, $X_1$, $B_1$, $Y_1$, $C_1$, and $Z_1$, where $A_1$, $B_1$, and $C_1$ correspond to elements of $T_1$, and each of $X_1$, $Y_1$, and $Z_1$ corresponds to an arc in $\partial R$ or a component of $\partial R$. Without loss of generality, we assume that $A_1$ corresponds to $A$, and $Y_1$ is disjoint from $A_1$.

Similarly, $\partial\Delta_B$ is equal to a union of arcs, $A_2$, $X_2$, $B_2$, $Y_2$, $C_2$, and $Z_2$, where $A_2$, $B_2$, and $C_2$ correspond to elements of $T_1$, and each of $X_2$, $Y_2$, and $Z_2$ corresponds to an arc in $\partial R$ or a component of $\partial R$. Without loss of generality, we assume that $B_2$ corresponds to $B$, and $Z_2$ is disjoint from $B_2$.

Likewise, $\partial\Delta_C$ is equal to a union of arcs, $A_3$, $X_3$, $B_3$, $Y_3$, $C_3$, and $Z_3$, where $A_3$, $B_3$, and $C_3$ correspond to elements of $T_1$, and each of $X_3$, $Y_3$, and $Z_3$ corresponds to an arc in $\partial R$ or a component of $\partial R$. Without loss of generality, we assume that $C_3$ corresponds to $C$, and $X_3$ is disjoint from $C_3$.

Let $T_2 = \{q_1(B_1), q_1(C_1), q_1(C_2), q_1(A_2), q_1(A_3), q_1(B_3)\}$. Let $q_2 : R_2 \to R$ and $q_{12} : R_1 \to R_2$ be the natural quotient maps corresponding to cutting $R$ along the systems of arcs $T_1$ and $T_2$ on $R$.

Note that $q_{12}(\Delta \cup \Delta_A \cup \Delta_B \cup \Delta_C)$ is a component $D_2$ of $R_2$. Moreover, $D_2$ is a disc which is a union of four discs $q_{12}(\Delta)$, $q_{12}(\Delta_A)$, $q_{12}(\Delta_B)$, and $q_{12}(\Delta_C)$, which may be identified with the distinct components $\Delta$, $\Delta_A$, $\Delta_B$, and $\Delta_C$ of $R_1$ via the appropriate restrictions of $q_{12} : R_1 \to R_2$.

With these identifications in mind, note that there exists three properly embedded disjoint arcs, $P_2$, $Q_2$, and $S_2$, on $D_2$ such that $P_2$ joins $Y_1$ to $Z_2$; $Q_2$ joins $Z_2$ to $X_3$; $S_2$ joins $X_3$ to $Y_1$; $P_2$ intersects each of $A_1$ and $B_2$ once essentially and is disjoint from $C_3$; $Q_2$ intersects each of $B_2$ and $C_3$ once essentially and is disjoint from $A_1$; and $S_2$ intersects each of $C_3$ and $A_1$ once essentially and is disjoint from $B_2$.

Let $P = q_2(P_2)$, $Q = q_2(Q_2)$, and $S = q_2(S_2)$. Then $P$, $Q$, and $S$ are disjoint essential arcs on $R$; $P$ intersects each of $A$ and $B$ once essentially and is disjoint from the other elements of $T$; $Q$ intersects each of $B$ and $C$ once essentially and is disjoint from the other elements of $T$; and $S$ intersects each of $C$ and $A$ once essentially and is disjoint from the other elements of $T$; and $P$, $Q$, and $S$ cut off an embedded triangle from $R$, as shown in Figure 12.

Let $\sigma$ be the simplex of $\mathcal{A}(R)$ corresponding to $T$. Since $R$ is not a disc or an annulus and $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ is an injective simplicial map, it follows from Proposition 10.2 that there exists a triangulation $T'$ on $R$ such that $\lambda(\sigma)$ is the simplex of $\mathcal{A}(R)$ corresponding to $T'$.

Let $A'$, $B'$, and $C'$ be, respectively, the unique representatives of $\lambda(a)$, $\lambda(b)$, and $\lambda(c)$ in $T'$. Since $\Delta$ is an embedded triangle on $R$ with sides corresponding to $A$, $B$, and $C$, it follows that $A$, $B$, and $C$ are disjoint nonisotopic arcs on $R$. Since $A$, $B$, and $C$ represent the vertices $a$, $b$, and $c$ of $\mathcal{A}(R)$, these vertices are distinct. Since $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ is an injective simplicial map, this implies that $\lambda(a)$, $\lambda(b)$, and $\lambda(c)$ are distinct vertices of $\mathcal{A}(R)$. It follows that the elements $A'$, $B'$, and $C'$ of the triangulation $T'$ of $R$ are distinct, and, hence, disjoint and nonisotopic.

Let $p$, $q$, and $r$ be the vertices of $\mathcal{A}(R)$ which are represented by the essential arcs $P$, $Q$, and $S$ on $R$.

Note that $i(p, x) = 0$ for every vertex $x$ of $\sigma$ other than $a$ and $b$, $i(p, a) = 1$, and $i(p, b) = 1$. Since $R$ is not a disc or an annulus and $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ is an injective simplicial map, it follows from Proposition 11.1 that $i(\lambda(p), y) = 0$ for every vertex $y$ of $\lambda(\sigma)$ other than $\lambda(a)$ and $\lambda(b)$; $i(\lambda(p), \lambda(a)) = 1$, and $i(\lambda(p), \lambda(b)) = 1$. Hence, there exists an arc $P'$ on $R$ representing $\lambda(p)$ such that $P'$ intersects $A'$ and $B'$ essentially once and is disjoint from the other elements of $T'$. Likewise, there exists an arc $Q'$ on $R$ representing $\lambda(q)$ such that $Q'$ intersects $B'$ and $C'$ essentially once and is disjoint from the other elements of $T'$; and there exists an arc $S'$ on $R$ representing $\lambda(s)$ such that $S'$ intersects $C'$ and $A'$ essentially once and is disjoint from the other elements of $T'$.

Since the essential arc $P'$ on $R$ intersects $A'$ and $B'$ essentially once and is disjoint from the other elements of the triangulation $T'$ of $R$, there exists a triangle $\Delta_1$ of $T'$ on $R$ having sides corresponding to $A'$ and $B'$. Similarly, there exists a triangle $\Delta_2$ of $T'$ on $R$ having sides corresponding to $B'$ and $C'$, and a triangle $\Delta_3$ of $T'$ on $R$ having sides corresponding to $C'$ and $A'$. Let the third side of $\Delta_1$ correspond to the element $D'$ of $T'$; the third side of $\Delta_2$ correspond to the element $E'$ of $T'$; and the third side of $\Delta_3$ correspond to the element $F'$ of $T'$.

Suppose, on the one hand, that $D' = C'$. Then $\Delta_1$ is a triangle of $T'$ on $R$ with sides corresponding to $A'$, $B'$ and $C'$. Since $A'$, $B'$, and $C'$ are disjoint and nonisotopic essential arcs on $R$, it follows that $\Delta_1$ is an embedded triangle of $T'$ on $R$ with sides corresponding to arcs $A'$, $B'$, and $C'$ on $R$ representing the vertices $\lambda(a)$, $\lambda(b)$, and $\lambda(c)$ of $\mathcal{A}(R)$. Hence, $\{\lambda(a), \lambda(b), \lambda(c)\}$ corresponds to an embedded triangle on $R$. Thus, if $D' = C'$, we are done. Likewise, if $E' = A'$ or $F' = B'$, then we are done.

Hence, we may assume that $D' \neq C'$, $E' \neq A'$, and $F' \neq B'$. Note that, since $A'$, $B'$, and $C'$ are distinct arcs on $R$, $\Delta_1$ has no side corresponding to $C'$. Since $\Delta_2$ has a side corresponding to $C'$, it follows that $\Delta_1 \neq \Delta_2$. Likewise, $\Delta_2 \neq \Delta_3$ and $\Delta_3 \neq \Delta_1$. Hence, $\Delta_1$, $\Delta_2$, and $\Delta_3$ are three distinct components of $R_{T'}$.

Since $P$, $Q$, and $R$ are disjoint, $i(p, q) = i(q, r) = i(r, p) = 0$. Since $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ is a simplicial map, it follows that $i(\lambda(p), \lambda(q)) = i(\lambda(q), \lambda(r)) = i(\lambda(r), \lambda(p)) = 0$. Hence, we may assume that $P'$, $Q'$, and $R'$ are disjoint arcs on $R$.

There are three cases to consider, depending on the placement of the arcs corresponding to $C'$, $E'$, and $F'$ on $\partial\Delta_2$ and $\partial\Delta_3$. These cases are shown in Figures 12 and 13.

Case (i): Assume $A', B', C', D', E', F'$ are as shown in Figure 12. Note that the arc $P'$ on $R$ representing $\lambda(p)$ intersects $B'$ and $A'$ once essentially and is disjoint from $E'$, $D'$, $F'$, and $C'$; and the arc $Q'$ on $R$ representing $\lambda(q)$ intersects $B'$ and $C'$ once essentially and is disjoint from $E'$, $D'$, $F'$, and $A'$. But then we see that $P'$ and $Q'$ intersect essentially (see Figure 12), which gives a contradiction, since $i(\lambda(p), \lambda(q)) = 0$.

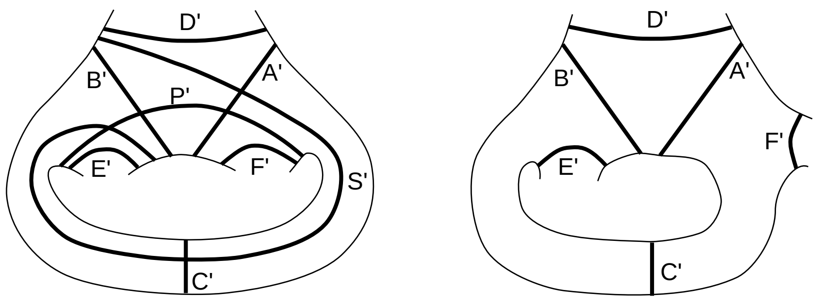


FIGURE 13

Case (ii): Assume $A', B', C', D', E', F'$ are as shown in the first part of Figure 13. As before, it follows that the arc $P'$ on $R$ representing $\lambda(p)$ intersects $B'$ and $A'$ once essentially and is disjoint from $E'$, $D'$, $F'$, and $C'$; and the arc $S'$ on $R$ representing $\lambda(s)$ intersects $A'$ and $C'$ once essentially and is disjoint from $E'$, $D'$, $F'$, $B'$. But then we see that $P'$ and $S'$ intersect essentially (see Figure 13), which gives a contradiction, since $i(\lambda(p), \lambda(s)) = 0$.

The proof for the third case is similar to the proof for Case (ii), (see the second part of Figure 13).

Hence, we see that either $D' = C'$ or $E' = A'$ or $F' = B'$ and, hence, as argued above, we are done.

$\square$

## 13. Preservation of non-embedded triangles

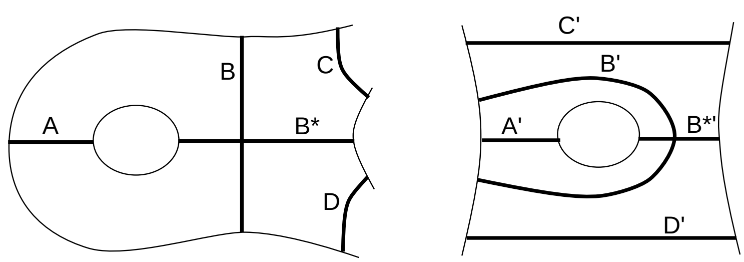


FIGURE 14

In this section, we assume that $R$ is not a disc, an annulus, a pair of pants, or a torus with one hole.

**Definition 13.1.** *Let $(a, b)$ be an ordered 1-simplex of $\mathcal{A}(R)$. We say that $(a, b)$ corresponds to a non-embedded triangle on $R$ if there exists a non-embedded triangle $\Delta$ on $R$ with sides corresponding to $A$, $B$, and $A$, where $A$ and $B$ represent $a$ and $b$ and $A$ joins two different components of $\partial R$.*

**Proposition 13.2.** *Suppose that $R$ is not a disc, an annulus, a pair of pants, or a torus with one hole. Let $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ be an injective simplicial map and $(a, b)$ be an oriented edge of $\mathcal{A}(R)$. If $(a, b)$ corresponds to a non-embedded triangle on $R$, then $(\lambda(a), \lambda(b))$ corresponds to a non-embedded triangle on $R$.*

*Proof.* Let $\Delta$ be a non-embedded triangle on $R$ with sides corresponding to $A$, $B$, and $A$, where $A$ and $B$ represent $a$ and $b$, and $A$ joins two different components of $\partial R$.

Since $R$ is not a pair of pants, there is an embedded triangle $\Delta'$ of $T$ on $R$ having a side corresponding to $B$. Note that $\Delta$ and $\Delta'$ are on different sides of $B$ (i.e. the sides of $B$ in $\Delta$ and $\Delta'$ are different sides of $B$). Since $\Delta'$ is an embedded triangle having a side corresponding to $B$, there is exactly one side of $\Delta$ which corresponds to $B$. Suppose that the other sides of $\Delta'$ correspond to $C$ and $D$ as shown in Figure 14.

Let $B^*$ be as shown in Figure 14, $b^*$ be the vertex of $\mathcal{A}(R)$ corresponding to $B^*$, $c$ be the vertex of $\mathcal{A}(R)$ corresponding to $C$, and $d$ be the vertex of $\mathcal{A}(R)$ corresponding to $D$. Note that there is an embedded triangle $\Delta_1$ on $R$ with sides corresponding to $A$, $B^*$, and $C$, and an embedded triangle $\Delta_2$ on $R$ with sides corresponding to $A$, $B^*$, and $D$. Since $R$ is not a disc, an annulus, a pair of pants, or a torus with one hole, and $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ is an injective simplicial map, it follows from Proposition 12.2 that there are embedded triangles $\Delta'_1$ and $\Delta'_2$ on $R$ such that $\Delta'_1$ has sides corresponding to $A'$, $B^{*\prime}$, and $C'$, and $\Delta'_2$ has sides corresponding to $A'$, $B^{*\prime}$, and $D'$, where $A'$, $B^{*\prime}$, $C'$, and $D'$ represent $\lambda(a)$, $\lambda(b^*)$, $\lambda(c)$, and $\lambda(d)$.

Since $R$ is not a disc or an annulus, $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ is an injective simplicial map and $i(b, b^*) = 1$, it follows from Proposition 11.1 that $i(\lambda(b), \lambda(b^*)) = 1$. Since $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ is an injective simplicial map, we see that the arc $B'$ representing $\lambda(b)$ can be chosen so that it is disjoint from $A'$, $C'$, and $D'$ and intersects $B^{*\prime}$ once. But then, this implies that $A'$ and $B'$ are the sides of a non-embedded triangle on $R$, and $A'$ connects two different components of $\partial R$, (see Figure 14).

Since $A'$ and $B'$ represent $\lambda(a)$ and $\lambda(b)$, it follows that $(\lambda(a), \lambda(b))$ corresponds to a non-embedded triangle on $R$.

This shows, in any case, that $(\lambda(a), \lambda(b))$ corresponds to a non-embedded triangle on $R$, which completes the proof. $\square$

## 14. Preservation of triangles

In this section, we assume that $R$ is not a disc, an annulus, a pair of pants, or a torus with one hole.

**Definition 14.1.** *Let $\{a, b, c\}$ be a simplex of $\mathcal{A}(R)$. We say that $\{a, b, c\}$ corresponds to a triangle on $R$ if there exists a triangle $\Delta$ on $R$ with sides corresponding to $A$, $B$, and $C$ such that $\{A, B, C\}$ is a system of arcs on $R$ representing $\{a, b, c\}$.*

The following proposition is a corollary of Propositions 12.2 and 13.2.

**Proposition 14.2.** *Suppose that $R$ is not a disc, an annulus, a pair of pants, or a torus with one hole. Let $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ be an injective simplicial map and $\{a, b, c\}$ be a simplex of $\mathcal{A}(R)$. If $\{a, b, c\}$ corresponds to a triangle on $R$, then $\{\lambda(a), \lambda(b), \lambda(c)\}$ corresponds to a triangle on $R$.*

## 15. Preservation of triangulated quadrilaterals

In this section, we assume that $R$ is not a disc, an annulus, a pair of pants, or a torus with one hole.

**Definition 15.1.** *Let $(a, b, c, d, e)$ be an ordered 5-tuple of vertices of $\mathcal{A}(R)$. We say that $(a, b, c, d, e)$ corresponds to a triangulated quadrilateral on $R$ if there exists a quadrilateral $\Omega$ on $R$ with sides corresponding to $A$, $B$, $C$, and $D$, and a diagonal $E$ of $\Omega$ such that $A$, $B$, $C$, $D$, and $E$ represent $a$, $b$, $c$, $d$, and $e$, and $A$ and $C$ correspond to opposite sides of $\Omega$.*

**Proposition 15.2.** *Suppose that $R$ is not a disc, an annulus, a pair of pants, or a torus with one hole. Let $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ be an injective simplicial map and $(a, b, c, d, e)$ be an ordered 5-tuple of vertices of $\mathcal{A}(R)$. If $(a, b, c, d, e)$ corresponds to a triangulated quadrilateral on $R$, then $(\lambda(a), \lambda(b), \lambda(c), \lambda(d), \lambda(e))$ corresponds to a triangulated quadrilateral on $R$.*

*Proof.* Let $\Omega$ be a quadrilateral on $R$ with sides corresponding to $A$, $B$, $C$, and $D$ and a diagonal $E$, such that $A$, $B$, $C$, $D$, and $E$ represent $a$, $b$, $c$, $d$, and $e$, and $A$ and $C$ correspond to opposite sides of $\Omega$.

Let $(E, F)$ be a pair of diagonals of $\Omega$ such that $E$ separates $\{A, B\}$ from $\{C, D\}$, and $F$ separates $\{B, C\}$ from $\{D, A\}$. By Proposition 5.5, $E$ and $F$ are disjoint from each of the essential arcs, $A$, $B$, $C$, and $D$; $E$ and $F$ are essential arcs on $R$; and $E$ and $F$ are in minimal position with $i([E], [F]) = 1$. Hence, $i(e, f) = 1$, where $f$ is the vertex of $\mathcal{A}(R)$ represented by $F$.

By Proposition 5.5, $\{A, B, C, D, E\}$ and $\{A, B, C, D, F\}$ are systems of arcs on $R$. Let $T_0 = \{A, B, C, D\}$, $T_1 = T_0 \cup \{E\}$, and $T_2 = T_0 \cup \{F\}$. Let $q_0 : R_0 \to R$, $q_1 : R_1 \to R$, $q_2 : R_2 \to R$, $q_{10} : R_1 \to R_0$, and $q_{20} : R_2 \to R_0$ be the natural quotient maps corresponding to cutting $R$ along $T_0$, $T_1$, and $T_2$.

Furthermore, by Proposition 5.5, there exist unique triangles on $R$, $\Delta_1$ with sides corresponding to $A$, $B$, and $E$; $\Delta_2$ with sides corresponding to $B$, $C$, and $F$; $\Delta_3$ with sides corresponding to $C$, $D$, and $E$; and $\Delta_4$ with sides corresponding to $D$, $A$, and $F$; such that $q_{10}(\Delta_1) \cup q_{10}(\Delta_3) = \Omega = q_{20}(\Delta_2) \cup q_{20}(\Delta_4)$.

It follows that $\{a, b, c, d, e\}$ and $\{a, b, c, d, f\}$ are simplices of $\mathcal{A}(R)$, and $\{a, b, e\}$, $\{b, c, f\}$, $\{c, d, e\}$, and $\{d, a, f\}$ correspond to triangles on $R$.

Let $a' = \lambda(a)$, $b' = \lambda(b)$, $c' = \lambda(c)$, $d' = \lambda(d)$, $e' = \lambda(e)$, and $f' = \lambda(f)$. Since $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ is an injective simplicial map, it follows from Proposition 14.2 that $\{a', b', c', d', e'\}$ and $\{a', b', c', d', f'\}$ are simplices of $\mathcal{A}(R)$, and $\{a', b', e'\}$, $\{b', c', f'\}$, $\{c', d', e'\}$, and $\{d', a', f'\}$ correspond to triangles on $R$. Moreover, since $R$ is not a disc or an annulus and $i(e, f) = 1$, it follows from Proposition 11.1, that $i(e', f') = 1$.

Since $\{a', b', c', d', e'\}$ and $\{a', b', c', d', f'\}$ are simplices of $\mathcal{A}(R)$, there exist arcs $A'$, $B'$, $C'$, $D'$, $E'$, and $F'$ such that $\{A', B', C', D', E'\}$ and $\{A', B', C', D', F'\}$ are systems of arcs representing $\{a', b', c', d', e'\}$ and $\{a', b', c', d', f'\}$. We may assume that $E'$ and $F'$ are in minimal position. Then, since $i(e', f') = 1$, it follows that $E'$ and $F'$ intersect once essentially, and $E'$ and $F'$ are not isotopic to any of the arcs $A'$, $B'$, $C'$, and $D'$.

Let $T_3 = \{A', B', C', D'\}$, $T_4 = T_3 \cup \{E'\}$, and $T_5 = T_3 \cup \{F'\}$. Let $q_3 : R_3 \to R$, $q_4 : R_4 \to R$, $q_5 : R_5 \to R$, $q_{43} : R_4 \to R_3$, and $q_{53} : R_5 \to R_3$ be the natural quotient maps corresponding to cutting $R$ along $T_3$, $T_4$, and $T_5$.

Since $\{a', b', e'\}$, $\{b', c', f'\}$, $\{c', d', e'\}$, and $\{d', a', f'\}$ correspond to triangles on $R$, it follows that there exist triangles $\Delta_1'$ on $R$ with sides corresponding to $A'$, $B'$, and $E'$; $\Delta_2'$ on $R$ with sides corresponding to $B'$, $C'$, and $F'$; $\Delta_3'$ on $R$ with sides corresponding to $C'$, $D'$, and $E'$; and $\Delta_4'$ on $R$ with sides corresponding to $D'$, $A'$, and $F'$.

Suppose that $\Delta'_1 = \Delta'_3$. Since the sides of $\Delta'_1$ correspond to $A'$, $B'$, and $E'$ and the sides of $\Delta'_3$ correspond to $C'$, $D'$ and $E$', it follows that $\{A', B', E'\} = \{C', D', E'\}$. This implies that $\{a', b', e'\} = \{c', d', e'\}$. Since $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ is injective, it follows that $\{a, b, e\} = \{c, d, e\}$. By Proposition 5.5, $e$ is not equal to $a$, $b$, $c$, or $d$. It follows that $\{a, b\} = \{c, d\}$. Since $\{A, B, C, D\}$ is a system of arcs on $R$ representing $\{a, b, c, d\}$, we conclude that $\{A, B\} = \{C, D\}$. Since $q_{10}(\Delta_1) \cup q_{10}(\Delta_3) = \Omega$, it follows that $\Delta_1$ is not equal to $\Delta_3$. Since the sides of $\Delta_1$ correspond to $A$, $B$, and $E$ and the sides of $\Delta_3$ correspond to $C$, $D$, and $E$, it follows that $\Delta_1$ and $\Delta_3$ are distinct triangles on $R$ with sides corresponding to $A$, $B$, and $E$. Since $R$ is not a disc or an annulus, it follows from Proposition 3.2, that $R$ is either a pair of pants or a torus with one hole, which is a contradiction. Hence, $\Delta'_1$ is not equal to $\Delta'_3$. Likewise, $\Delta'_2$ is not equal to $\Delta'_4$.

Since $\{A', B', C', D', E'\}$ is a system of arcs on $R$ and $\Delta'_1$ and $\Delta'_3$ are distinct triangles of $\{A', B', C', D', E'\}$ on $R$ both with sides corresponding to $E'$, it follows that there is a quadrilateral $\Omega'$ of $T_3$ on $R$ with sides corresponding to $A'$, $B'$, $C'$, and $D'$ and diagonal $E'$ where $q_{43}(\Delta'_1) \cup q_{43}(\Delta'_3) = \Omega'$.

Hence, either $A'$ and $C'$ correspond to opposite sides of $\Omega'$ or $A'$ and $D'$ correspond to opposite sides of $\Omega'$. Since $\{A', B', C', D', E'\}$ and $\{A', B', C', D', F'\}$ are systems of arcs and $E'$ and $F'$ are in minimal position, with $E'$ and $F'$ intersecting once essentially, it follows that $\{E', F'\}$ is a pair of diagonals of $\Omega'$.

Suppose that $A'$ and $D'$ correspond to opposite sides of $\Omega'$. Then, by Proposition 5.5, there exist triangles on $R$, $\Delta''_1$ with sides corresponding to $A'$, $B'$, and $E'$; $\Delta''_2$ with sides corresponding to $B'$, $D'$, and $F'$; $\Delta''_3$ with sides corresponding to $D'$, $C'$, and $E'$; and $\Delta''_4$ with sides corresponding to $C'$, $A'$, and $F'$; such that $q_{43}(\Delta''_1) \cup q_{43}(\Delta''_3) = \Omega' = q_{53}(\Delta''_2) \cup q_{53}(\Delta''_4)$.

Moreover, by Proposition 5.5, $\Delta''_2$ and $\Delta''_4$ are the unique triangles of $\{A', B', C', D', F'\}$ on $R$ with a side corresponding to $F'$ as a side.

Since $\Delta'_2$ and $\Delta'_4$ are distinct triangles of $\{A', B', C', D', F'\}$ with a side corresponding to $F'$ as a side, it follows that either $(\Delta'_2, \Delta'_4) = (\Delta''_2, \Delta''_4)$ or $(\Delta'_2, \Delta'_4) = (\Delta''_4, \Delta''_2)$.

Suppose, on the one hand, that $(\Delta'_2, \Delta'_4) = (\Delta''_2, \Delta''_4)$. Since $\Delta'_2$ has sides corresponding to $B'$, $C'$, and $F'$ and $\Delta''_2$ has sides corresponding to $B'$, $D'$, and $F'$ and $\Delta'_2 = \Delta''_2$, it follows that $\{B', C', F'\} = \{B', D', F'\}$. Since $F'$ is not isotopic to $B'$, $C'$, and $D'$, this implies that $\{B', C'\} = \{B', D'\}$ and, hence, $C' = D'$. Since $A'$ and $D'$ correspond to opposite sides of $\Omega'$, this implies that $A'$ and $C'$ correspond to opposite sides of $\Omega'$.

Suppose, on the other hand, that $(\Delta_2', \Delta_4') = (\Delta_4'', \Delta_2'')$. Since $\Delta_2'$ has sides corresponding to $B'$, $C'$, and $F'$ and $\Delta_4''$ has sides corresponding to $C'$, $A'$, and $F'$ and $\Delta_2' = \Delta_4''$, it follows that $\{B', C', F'\} = \{C', A', F'\}$. Since $F'$ is not isotopic to $A'$, $B'$, and $C'$, this implies that $\{B', C'\} = \{C', A'\}$ and, hence, $A' = B'$. Since $A'$ and $D'$ correspond to opposite sides of $\Omega'$, this implies that $B'$ and $D'$ correspond to opposite sides of $\Omega'$ and, hence, $A'$ and $C'$ correspond to opposite sides of $\Omega'$.

It follows that, in any case, $A'$ and $C'$ correspond to opposite sides of $\Omega'$. Thus, $\Omega'$ is a quadrilateral on $R$ with sides $A'$, $B'$, $C'$, and $D'$ and diagonal $E'$ representing $a'$, $b'$, $c'$, $d'$, and $e'$, with $A'$ and $C'$ corresponding to opposite sides of $\Omega'$. This implies that $(a', b', c', d', e')$ corresponds to a triangulated quadrilateral on $R$, completing the proof. $\square$

## 16. Preservation of topological type of triangulations

In this section, we assume that $R$ is not a disc, an annulus, a pair of pants, or a torus with one hole.

**Proposition 16.1.** *Suppose that $R$ is not a disc, an annulus, a pair of pants, or a torus with one hole. Let $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ be an injective simplicial map. Let $\sigma$ be a simplex of $\mathcal{A}(R)$ corresponding to a triangulation $T$ of $R$. Let $T'$ be a triangulation of $R$ such that $\lambda(\sigma)$ is the simplex of $\mathcal{A}(R)$ corresponding to $T'$. For each arc $J$ of $T$ let $J'$ be the unique arc of $T'$ such that $\lambda([J]) = [J']$. Then there exist a homeomorphism $H : R \to R$ such that $H(J) = J'$ for each arc $J$ of $T$.*

*Proof.* Let $R_1$ denote the surface obtained from $R$ by cutting $R$ along $T$ and $R_2$ be the surface obtained from $R$ by cutting $R$ along $T'$.

By Proposition 4.3, $R_1$ and $R_2$ both have $N$ components, where $N = 4g + 2b - 4$. Let $\{\Delta_i | 1 \leq i \leq N\}$ be the $N$ distinct components of $R_1$.

Since $R$ is not a pair of pants or a torus with two holes, no two distinct components of $R_1$ can have sides corresponding to the same elements of $T$. Likewise, no two distinct components of $R_2$ can have sides corresponding to the same elements of $T'$.

Let $i$ be an integer with $1 \leq i \leq N$. Since $\Delta_i$ is a component of $R_1$, $\Delta_i$ is a triangle of $T$ with sides corresponding to elements $A_i$, $B_i$, and $C_i$ of $T$. Let $a_i$, $b_i$, and $c_i$ be the vertices of $\mathcal{A}(R)$ represented by $A_i$, $B_i$ and $C_i$. Then, $\{a_i, b_i, c_i\}$ corresponds to a triangle on $R$. Hence, by Proposition 14.2, $\{a_i', b_i', c_i'\}$ corresponds to a triangle on $R$, where $a_i' = \lambda(a_i)$, $b_i' = \lambda(b_i)$, and $c_i' = \lambda(c_i)$. Let $A_i'$, $B_i'$, and $C_i'$ be the unique elements of $T'$ which represent $a_i'$, $b_i'$, and $c_i'$. It follows that there exists a unique triangle $\Delta_i'$ of $T'$ on $R$ with sides corresponding to $A_i'$, $B_i'$, and $C_i'$.

Moreover, the correspondence $\Delta_i \mapsto \Delta_i'$ establishes a bijection from the set of exactly $N$ distinct components $\{\Delta_i | 1 \leq i \leq N\}$ of $R_1$ to the set of exactly $N$ distinct components $\{\Delta_i' | 1 \leq i \leq N\}$ of $R_2$.

Suppose, on the one hand, that $\Delta_i$ is embedded. Then, by Proposition 12.2, $\Delta_i'$ is embedded. Let $J_i$, $K_i$, and $L_i$ be the arcs in $\Delta_i$ corresponding to $A_i$, $B_i$, and $C_i$, and $J_i'$, $K_i'$, and $L_i'$ be the arcs in $\Delta_i'$ corresponding to $A_i'$, $B_i'$, and $C_i'$. Note that there exists a homeomorphism $F_i : (\Delta_i, J_i, K_i, L_i) \to (\Delta_i', J_i', K_i', L_i')$ which is well-defined up to relative isotopies. In particular, the orientation type of $F_i : (\Delta_i, J_i, K_i, L_i) \to (\Delta_i', J_i', K_i', L_i')$ (i.e. whether it is orientation-reversing or orientation-preserving) is fixed.

Suppose, on the other hand, that $\Delta_i$ is non-embedded. Let $J_i$, $K_i$, and $L_i$ be the arcs in $\Delta_i$ corresponding to $A_i$, $B_i$, and $C_i$, and $J_i'$, $K_i'$, and $L_i'$ be the arcs in $\Delta_i'$ corresponding to $A_i'$, $B_i'$, and $C_i'$. We may assume that $A_i = C_i$, so that $A_i$ joins two different components of $\partial R$, and $B_i$ joins a component of $\partial R$ to itself. Then, by Proposition 13.2, $\Delta_i'$ is non-embedded, $A_i' = C_i'$, $A_i'$ joins two different boundary components of $\partial R$, and $B_i'$ joins a component of $\partial R$ to itself.

Note, in this situation, that there is an ambiguity in the choice of $J_i$ and $L_i$. After all, $J_i$ and $L_i$ both correspond to $A_i$ (i.e. $J_i$ and $L_i$ both correspond to $C_i$). Likewise, there is an ambiguity in the choice of $J_i'$ and $L_i'$. Suppose that $(J_i, L_i, J_i', L_i')$ has been specified. Then there exists a homeomorphism $F_i : (\Delta_i, J_i, K_i, L_i) \to (\Delta_i', J_i', K_i', L_i')$ which is well-defined up to relative isotopies and a homeomorphism $F_i^* : (\Delta_i, J_i, K_i, L_i) \to (\Delta_i', L_i', K_i', J_i')$ which is well-defined up to isotopies. In particular, in this situation, the orientation type of $F_i : (\Delta_i, J_i, K_i, L_i) \to (\Delta_i', J_i', K_i', L_i')$ is fixed; the orientation type of $F_i^* : (\Delta_i, J_i, K_i, L_i) \to (\Delta_i', L_i', K_i', J_i')$ is fixed; and $F_i : (\Delta_i, J_i, K_i, L_i) \to (\Delta_i', J_i', K_i', L_i')$ and $F_i^* : (\Delta_i, J_i, K_i, L_i) \to (\Delta_i', L_i', K_i', J_i')$ have opposite orientation types.

Proposition 15.2 now ensures that we can choose homeomorphisms $G_i : \Delta_i \to \Delta_i'$, $1 \leq i \leq N$, where $G_i$ is isotopic to $F_i$, if $\Delta_i$ is embedded, and $G_i$ is isotopic to either $F_i$ or $F_i^*$, if $\Delta_i$ is non-embedded, so that the unique homeomorphism $G : R_1 \to R_2$ whose restriction to $\Delta_i$ is equal to $G_i$, $1 \leq i \leq N$, covers a homeomorphism $H : R \to R$.

Roughly speaking, Proposition 15.2 ensures that the homeomorphisms $F_i : \Delta_i \to \Delta_i'$ and $F_j : \Delta_j \to \Delta_j'$ associated to embedded triangles $\Delta_i$ and $\Delta_j$ which have sides corresponding to the same element of $T$, can be isotoped by a relative isotopy to agree, relative to the natural quotient maps, $q_1 : R_1 \to R$ and $q_2 : R_2 \to R$. In other words, the restrictions of $F_i$ and $F_j$ to pairs of sides which correspond to the same element of $T$, which restrictions may be identified, via $q_1$ and $q_2$, to homeomorphisms from a fixed element of $T$ to a fixed element of $T'$, have the same orientation type as such homeomorphisms between fixed elements of $T$ and $T'$.

When $\Delta_i$ is nonembedded, this condition on compatibility of orientation types of restrictions on pairs of sides which correspond to the same element of $T$ can be realized on all such pairs by making the appropriate choice of either $F_i$ or $F_i^*$, $1 \leq i \leq N$.

Once the correct choices are made so that this compatibility of orientations is realized, we may isotope the chosen homeomorphisms, $F_i$ or $F_i^*$, to homeomorphisms $G_i$ which agree, as homeomorphisms between fixed elements of $T$ and $T'$, on all pairs of sides which correspond to the same element of $T$.

It follows that $H : R \to R$ is a homeomorphism which maps each element $J$ of $T$ to the corresponding element $J'$ of $T'$, completing the proof. □

## 17. Injective simplicial maps

**Theorem 17.1.** *Let $R$ be a compact, connected, oriented surface of genus $g$ with $b$ boundary components, where $b \geq 1$. Let $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ be an injective simplicial map. Then $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ is geometric (i.e. there exists a homeomorphism $H : R \to R$ such that for every essential arc $A$ on $R$, $\lambda([A]) = [H(A)]$).*

*Proof.* Let $\lambda : \mathcal{A}(R) \to \mathcal{A}(R)$ be an injective simplicial map.

If $R$ is a disc, an annulus, a pair of pants, or a torus with one hole, then the result follows from the discussion of special examples in Section 9.

Suppose, therefore, that $R$ is not a disc, an annulus, a pair of pants, or a torus with one hole. Let $\tau$ be a maximal simplex of $\mathcal{A}(R)$. Let $\tau'$, $T$, $T'$, and $H : R \to R$ be as in Proposition 16.1. Let $\psi = H_*^{-1} \circ \lambda : \mathcal{A}(R) \to \mathcal{A}(R)$. Note that $\psi : \mathcal{A}(R) \to \mathcal{A}(R)$ is an automorphism of $\mathcal{A}(R)$ such that $\psi(x) = x$ for each vertex $x$ of $\tau$. Recall that (i) each vertex of $\mathcal{A}(R)$ is contained in a codimension zero face of $\mathcal{A}(R)$, (ii) each codimension one face of $\mathcal{A}(R)$ is contained in one or two codimension zero faces of $\mathcal{A}(R)$, and (iii) Theorem 8.2, Mosher's "Connectivity by Elementary Moves" holds. It follows from these facts that $\psi = id_{\mathcal{A}(R)} : \mathcal{A}(R) \to \mathcal{A}(R)$. Hence, $\lambda = H_* : \mathcal{A}(R) \to \mathcal{A}(R)$. That is to say, $\lambda$ is geometric, being induced by the self-homeomorphism $H : R \to R$. □

**Remark 17.2.** *Once Proposition 16.1 has been established and applied at the beginning of the above proof of Theorem 17.1, the final step in this* *proof is standard. This final step is a standard argument for proving Lemma 8.4.A in Ivanov's survey article on Mapping Class Groups* [I2].

## 18. Automorphisms

The next proposition gives an explicit description of the kernel of the natural representation $\rho : \Gamma^*(R) \to Aut(\mathcal{A}(R))$.

**Proposition 18.1.** *Let $R$ be a compact, connected, oriented surface of genus $g$ with $b$ boundary components, where $b \geq 1$. Let $Z(\Gamma^*(R))$ be the center of the extended mapping class group $\Gamma^*(R)$ of $R$. Let $\rho : \Gamma^*(R) \to Aut(\mathcal{A}(R))$ be the natural representation corresponding to the natural action of $\Gamma^*(R)$ on the complex of arcs $\mathcal{A}(R)$ of $R$. Let $ker(\rho)$ be the kernel of $\rho : \Gamma^*(R) \to Aut(\mathcal{A}(R))$. Then the following hold:*

1. *If $g = 0$ and $b = 1$, then $ker(\rho) = \Gamma^*(R) = Z(\Gamma^*(R))$, which is a cyclic group of order two generated by the isotopy class of any orientation-reversing involution $H : R \to R$ of $R$.*
2. *If $g = 0$ and $b = 2$, then $ker(\rho) = \Gamma^*(R) = Z(\Gamma^*(R))$, which is a direct sum of two cyclic groups of order two, one generated by the isotopy class of any orientation-preversing involution $H_1 : R \to R$ of $R$ interchanging the two components of $\partial R$, and the other generated by the isotopy class of any orientation-reversing involution $H : R \to R$ of $R$ preserving each of the components of $\partial R$.*
3. *If $g = 0$ and $b = 3$, then $ker(\rho) = Z(\Gamma^*(R))$, which is a cyclic group of order two generated by the isotopy class of any orientation-reversing involution $H : R \to R$ of $R$ preserving each of the components of $\partial R$.*
4. *If $g = 1$ and $b = 1$, then $ker(\rho) = Z(\Gamma(R)) = Z(\Gamma^*(R))$, which is a cyclic group of order two generated by the isotopy class of any hyperelliptic involution of $R$.*
5. *Otherwise, $ker(\rho)$ is equal to the trivial subgroup $\{[id_R : R \to R]\}$ of $\Gamma^*(R)$.*

*Proof.* (1), (2), (3), and (4) follow from our discussion of special examples in Section 9.

Suppose now that $R$ is not a disc, an annulus, a pair of pants, or a torus with one hole. Let $h$ be an element of $ker(\rho)$ and $H : R \to R$ be a homeomorphism of $R$ representing $h$.

By the definition of the action of $\Gamma^*(R)$ on $\mathcal{A}(R)$ and of the corresponding representation $\rho : \Gamma^*(R) \to \mathcal{A}(R)$, $H : R \to R$ preserves the isotopy class of every essential arc on $R$.

Since $R$ is not a disc or an annulus, there exists a triangulation $T$ of $R$. Since $H : R \to R$ preserves the isotopy class of every essential arc on $R$, we may isotope $H : R \to R$ to a homeomorphism $H_0 : R \to R$ such that, for each element $J$ of $T$, $H_0(J) = J$.

Since $R$ is not a pair of pants, there exists an embedded triangle $\Delta$ of $T$ on $R$ with sides corresponding to $A$, $B$, and $C$. Let $Q = q(\Delta)$. Since $A$, $B$, and $C$ are elements of $T$, it follows that there exists an embedded triangle $\Delta'$ of $T$ on $R$ with sides corresponding to $H_0(A)$, $H_0(B)$, and $H_0(C)$ such that $q(\Delta') = H_0(Q)$.

Since $A$, $B$, and $C$ are elements of $T$, $H_0(A) = A$, $H_0(B) = B$, and $H_0(C) = C$. It follows that $\Delta'$ and $\Delta$ are triangles on $S$ with sides corresponding to $A$, $B$, and $C$.

Since $R$ is not a disc, an annulus, a pair of pants, or a torus with one hole, it follows from Proposition 3.2 that $\Delta' = \Delta$. It follows that $H_0(Q)$ is equal to $Q$ and, hence, the homeomorphism $H_0 : R \to R$ restricts to a homeomorphism $H_0| : (Q, A, B, C) \to (Q, A, B, C)$. It follows that we may isotope $H_0 : R \to R$ relative to the union $|T|$ of the elements of $T$, to a homeomorphism $H_1 : R \to R$ which restricts to the identity map $H_1| = id_Q : Q \to Q$ of $Q$.

Note that any other triangle $\Delta'$ of the triangulation $T$ of $R$ is connected to the triangle $\Delta$ of the triangulation $T$ of $R$ by a sequence of triangles which have sides corresponding to the same element of $T$. Since $H : R \to R$ is orientation-preserving, it follows, by a finite induction argument, that we may construct a sequence of homeomorphisms, $H_i : R \to R$, $0 \leq i \leq N$, with $N$ equal to the number of triangles of $T$ on $R$, such that $H_0$ preserves each element of $T$ and is isotopic on $R$ to $H$; $H_1$ preserves each element of $T$, fixes each point of at least one triangle of $T$ on $R$, and is isotopic on $R$ to $H_0$ relative to $|T|$; and for each integer $i$ with $2 \leq i \leq N$, $H_i$ preserves each element of $T$, fixes each point of at least $i$ triangles of $T$, and is isotopic on $R$ to $H_{i-1}$ relative to the union of $|T|$ with $i-1$ triangles of $T$ fixed pointwise by $H_{i-1}$.

Since $N$ is equal to the number of triangles of $T$ on $R$, it follows that $H_N = id_R : R \to R$. Since $H : R \to R$ is, by induction, isotopic to $H_N : R \to R$, it follows that $H : R \to R$ is isotopic to $id_R : R \to R$, which completes the proof.

□

**Theorem 18.2.** *Let $R$ be a compact, connected, oriented surface of genus $g$ with $b$ boundary components, where $b \geq 1$. Then the following hold:*

(1) *If $R$ is a disc, an annulus, a pair of pants, or a torus with one hole, then $Aut(\mathcal{A}(R))$ is naturally isomorphic to the quotient of the extended mapping class group $\Gamma^*(R)$ by its center $Z(\Gamma^*(R))$. More precisely, we have a natural short exact sequence corresponding to the natural action of $\Gamma^*(R)$ on $\mathcal{A}(R)$:*

$$1 \to Z(\Gamma^*(R)) \to \Gamma^*(R) \to Aut(\mathcal{A}(R)) \to 1.$$

(2) *If $R$ is not a disc, an annulus, a pair of pants, or a torus with one hole, then $Aut(\mathcal{A}(R))$ is naturally isomorphic to the extended mapping class group $\Gamma^*(R)$. More precisely, we have an isomorphism corresponding to the natural action of $\Gamma^*(R)$ on $\mathcal{A}(R)$:*

$$1 \to \Gamma^*(R) \to Aut(\mathcal{A}(R)) \to 1.$$

*Proof.* It follows from Theorem 17.1 that the natural representation $\rho : \Gamma^*(R) \to Aut(\mathcal{A}(R))$ is surjective. Hence, Theorem 18.2 follows from the description in Proposition 18.1 of the kernel of the natural representation $\rho : \Gamma^*(R) \to Aut(\mathcal{A}(R))$ corresponding to the natural action of $\Gamma^*(R)$ on $\mathcal{A}(R)$.

□

**Remark 18.3.** *It follows from Proposition 18.1 that, for the surfaces listed in (1) of Theorem 18.2, $Z(\Gamma^*(R))$ is either cyclic of order two or a direct sum of two cyclic groups of order two. In particular, therefore, (1) of Theorem 18.2 implies that, for these surfaces, the natural representation $\rho : \Gamma^*(R) \to Aut(\mathcal{A}(R))$ exhibits $\Gamma^*(R)$ as an explicit finite central extension of $Aut(\mathcal{A}(R))$. Otherwise, for the remaining surfaces listed in (2) of Theorem 18.2, the natural representation $\rho : \Gamma^*(R) \to Aut(\mathcal{A}(R))$ is an isomorphism.*

**Remark 18.4.** *The main consequence of Theorem 18.2 is that automorphisms of arc complexes are geometric (i.e. are induced by self-homeomorphisms of the underlying surface). Ivanov used the arc complex in his proof of his theorem on automorphisms of the complex of curves* [I1]. *In this context, he showed that automorphisms of arc complexes which are induced from automorphisms of the complex of curves are geometric. Our result, Theorem 18.2, does not assume that our automorphisms of arc complexes are induced from automorphisms of curve complexes.*

## References

[FLP] A. Fathi, F. Laudenbach. and V. Poénaru, *Travaux de Thurston sur les surfaces*, Séminaire Orsay, Astérisque, Vol. 66-67, Soc. Math. de France, 1979.

[I1] N. V. Ivanov, Automorphisms of complexes of curves and of Teichmüller spaces, Preprint IHES/M/89/60, 1989, 13 pp.; Also in: *Progress in knot theory and related topics,* Travaux en Cours, V. 56, *Hermann, Paris,* 1997, 113-120.

[I2] N. V. Ivanov, Mapping class groups. *Handbook of geometric topology*, 523–633, North-Holland, Amsterdam, 2002.

[M] L. Mosher, Tiling the projective foliation space of a punctured surface, Trans. Am. Math. Soc. 306 (1) (1988) 1-70

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