G_0 and Goldie ranks of polycyclic group algebras

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INTRODUCTION

Let $F[G]$ be the group algebra of a polycyclic-by-finite group G over a field F. Then $F[G]$ is a left and right Noetherian ring and it is well-known that $F[G]$ has an Artinian ring of quotients, $Q(F[G])$, which is obtained from $F[G]$ by inverting the regular elements of $F[G]$. The composition length of $Q(F[G])$ is an interesting invariant of $F[G]$, usually called the <u>Goldie rank</u> of $F[G]$ and written

$$\rho(F[G]) \quad .$$

In general, the explicit determination of $\rho(F[G])$ presents a formidable task. If G is finite, then $Q(F[G]) = F[G]$ and the problem of finding $\rho(F[G])$ belongs to the realm of representation theory. In another direction, a celebrated result due to Farkas, Snider, and Cliff [F-S],[C] asserts that if G is torsion-free then $F[G]$ has no zero divisors or, equivalently, $\rho(F[G]) = 1$. Beyond this, little is known for infinite polycyclic-by-finite groups G. It has been conjectured

([F],[Ro1]) that, in case G has no finite <u>normal</u> subgroups $\neq \langle 1 \rangle$, the Goldie rank of $F[G]$ is given by the formula

$$\rho(F[G]) = \ell.c.m.\ \{ |U| \mid U \text{ is a finite subgroup of } G \} .$$

The relevance of the assumption on G here stems from the fact that it is satisfied precisely when $F[G]$ is a prime ring. The conjecture has been confirmed in a number of special cases ([Ga-Ro],[Lo2],[Pa2],[Ro2]), but in general it is open at present.

The rôle of $G_0(F[G])$, the Grothendieck group of the category of all finitely generated $F[G]$-modules, in this context is as follows. For any finitely generated $F[G]$-module V, the <u>reduced rank</u> of V is defined by

$$\rho(V) = \text{composition length of } V \otimes_{F[G]} Q(F[G]) \text{ over } Q(F[G]) .$$

Thus $\rho(F[G])$ is the reduced rank of the regular $F[G]$-module. Inasmuch as $Q(F[G])$ is flat over $F[G]$, ρ defines an integer-valued function on $G_0(F[G])$. Put

$$\mathcal{F} = \mathcal{F}(G) = \{ U \mid U \text{ is a finite subgroup of } G \}$$

and

$$G_0(F[G])_{\mathcal{F}} = \sum_{U \in \mathcal{F}} \mathrm{Ind}_U^G\, G_0(F[U]) ,$$

where $\mathrm{Ind}_U^G : G_0(F[U]) \longrightarrow G_0(F[G])$ is the usual induction homomorphism. Then $G_0(F[G])_{\mathcal{F}}$ is a subgroup of $G_0(F[G])$ and, under certain special circumstances, one can in fact show that $G_0(F[G]) = G_0(F[G])_{\mathcal{F}}$ (see Proposition 1.4 below). In general, however, the exact relationship between $G_0(F[G])$ and $G_0(F[G])_{\mathcal{F}}$ is quite unclear. Indeed, the above conjectural Goldie rank formula in the situation where $F[G]$ is prime is equivalent to the equality

$$\rho(G_0(F[G])) = \rho(G_0(F[G])_{\mathcal{F}})$$

(Corollary 1.3i below). We mention two partial results towards clarifying the structure of $G_0(F[G])/G_0(F[G])_{\mathcal{F}}$:

(a) If G is finitely generated abelian-by-finite, then $G_0(F[G])/G_0(F[G])_{\mathcal{F}}$ is a torsion group, of exponent dividing $[G:A]^{\operatorname{rank} A}$ for any abelian normal subgroup A having finite index in G ([Br-H-Lo, Theorem A]).

(b) Let G again be an arbitrary polycyclic-by-finite group. Then for any normal subgroup N of G having finite index in G and such that $F[N]$ has finite global dimension, one can define a map $\gamma = \gamma_{F[G],N} : G_0(F[G]) \longrightarrow G_0(F[G/N])$. In Proposition 4.4 below, we show that $\gamma(G_0(F[G])_{\mathcal{F}})$ always has finite index in $\gamma(G_0(F[G]))$.

For further results along these lines see [Qu], in particular [Qu, Corollary 1.5c]. At present, no example seems to be known with $G_0(F[G]) \neq G_0(F[G])_{\mathcal{F}}$.

In this article, we discuss a number of general techniques that are especially useful in dealing with $G_0(F[G])$ for G polycyclic-by-finite, and hence for computing the Goldie rank $\rho(F[G])$. In particular, we derive all major known partial solutions of the Goldie rank problem, as well as a number of new estimates of $\rho(F[G])$ as consequences of more fundamental results on $G_0(F[G])$. The general approach, namely via G_0, has been motivated by the work of Rosset ([Ro1],[Ro2],[Ga-Ro]).

We now outline the contents of this article. Section 1 is devoted to the normalized reduced rank function, χ, which was introduced in [Lo1]. This is a slight modification of the above reduced rank ρ which behaves especially well under restriction and induction. These "functoriality" properties of χ very easily yield all basic divisibility results for $\rho(F[G])$. Most of these are known, but the proofs presented here are new. The central theme in Section 2 is the cokernel of the Cartan map $c : K_0(F[G]) \longrightarrow G_0(F[G])$ in the case where $\operatorname{char} F = p > 0$. We show that the exponent, e, of $G_0(F[G])/cK_0(F[G])$ is finite and divides the p-part of $[G:N]$ for any subgroup N of finite index in G without elements of order p. Moreover, if $F[G]$ is prime, then the p-part of $\rho(F[G])$ divides e (Proposition 2.2). As an application, we offer a slightly extended version of Passman's solution of the Goldie rank problem for "elementary abelian p-tops" in characteristic p (Theorem 2.5). In Section 3, we describe the familiar decomposition map and its basic features in a setting suitable for application to polycyclic group algebras. This tool can be used to shed some light on the effect of varying the coefficient field F on $\rho(F[G])$. It is still unclear, however, that $\rho(F[G])$ is independent of F whenever G has no finite normal subgroups, as it would be implied by the above explicit conjectural formula for $\rho(F[G])$. Finally, in Section 4, we study certain maps $\gamma_{F[G],N} : G_0(F[G]) \longrightarrow G_0(F[G/N])$, where N is normal of finite index in G and without elements of order p in case $\operatorname{char} F = p > 0$ (cf. (b) above). This yields a somewhat technical upper bound for $\rho(F[G])$ in terms of the representations of suitable finite images of G which does at least imply the main result of [Ga-Ro]. The major part of this section elaborates on ideas from [Ga-Ro].

We have opted to work over coefficient <u>fields</u> throughout, although some of the results could easily be transferred to more general (commutative) coefficient rings. We hope that this helps to keep the technicalities in the exposition at a tolerable level.

NOTATIONS AND CONVENTIONS

The following notations will be kept fixed throughout this article:

F	is a commutative field,
G	is a polycyclic-by-finite group,
$\mathcal{F}=\mathcal{F}(G)$	denotes the set of all finite subgroups of G,
$f(G)$	is the least common multiple of the orders $\|U\|$, $U \in \mathcal{F}(G)$,
ωG	is the augmentation ideal of the group algebra $F[G]$,
ρ_R or ρ	denotes Goldie's reduced rank for R-modules (R a given Noetherian ring),
n_p	for a rational prime p, denotes the p-part of $n \in \mathbb{Z}$, i.e. the largest p-power dividing n.

All modules will be left modules. In general, we follow the notation of [Pa1] for groups and group algebras, and of [Ba1] for K_0 and G_0. In particular, $[V]$ denotes the element of $G_0(F[G])$ (resp. $K_0(F[G])$) corresponding to the finitely generated (projective) $F[G]$-module V. Furthermore, $c : K_0(F[G] \longrightarrow G_0(F[G])$ will denote the Cartan map. Following [Sw] we put, for any family $\mathfrak{U}$ of subgroups of G

$$G_0(F[G])_{\mathfrak{U}} = \sum_{U \in \mathfrak{U}} \mathrm{Ind}_U^G (G_0(F[U])) \quad .$$

1. THE NORMALIZED REDUCED RANK

For any finitely generated $F[G]$-module V, define the *normalized reduced rank* of V by

$$\chi(V) = \chi_{F[G]}(V) = \rho(V)/\rho(F[G]) \quad .$$

Here, $\rho = \rho_{F[G]}$ denotes the reduced rank of $F[G]$-modules. Since ρ is additive on short exact sequences, the same is true for χ which can therefore be viewed as a function on $G_0(F[G])$. The following lemma describes some basic properties of χ in the case when $F[G]$ is prime.

LEMMA 1.1. Assume that $F[G]$ is prime, and let $H \leqq G$ be a subgroup of finite index.

i. For any finitely generated $F[G]$-module,

$$\chi_{F[G]}(V) = [G : H]^{-1} \cdot \chi_{F[H]}(V) \quad .$$

ii. Let W be a finitely generated $F[H]$-module. Then

$$\chi_{F[G]}(F[G] \otimes_{F[H]} W) = \chi_{F[H]}(W) \quad .$$

PROOF. (i) is [Lo1, Lemma 7].

(ii). Choose a normal subgroup N of G having finite index in G and such that $N \leqq H$. Then, by part (i), we have

$$\chi_{F[G]}(F[G] \otimes_{F[H]} W) = [G:N]^{-1} \cdot \chi_{F[N]}(\bigoplus_x W^{(x)}) \quad ,$$

where x runs over a set of right coset representatives for H in G and $W^{(x)}$ is the x-conjugate module of W. As $\chi_{F[N]}(W^{(x)}) = \chi_{F[N]}(W)$ for all x, we obtain

$$\chi_{F[G]}(F[G] \otimes_{F[H]} W) = [G:N]^{-1} \cdot [G:H] \cdot \chi_{F[N]}(W) = \chi_{F[H]}(W) \ .$$

Here, the latter equality again follows from part (i). (Note that $F[H]$ is prime.)

■

The above definiton of χ is taken from [Lo1]. Rosset [Ro1] works with a similar "Euler characteristic" which coincides with $\chi_{F[G]}$ in the case when $F[G]$ is prime, but not in general. Let us quickly derive a number of standard facts about the relations between the Goldie ranks of $F[G]$ and $F[H]$ for subgroups $H \leqq G$ (cf. [Lo2, Lemma 1.1]).

<u>COROLLARY 1.2.</u> Assume that $F[G]$ is prime.

i) Let $H \leqq G$ be a subgroup of finite index. Then

$$\rho(F[H]) \,\Big|\, \rho(F[G]) \,\Big|\, [G:H] \cdot \rho(F[H]) \ .$$

In particular, $\rho(F[G])$ divides $[G:H]$ for any torsion-free subgroup $H \leqq G$ of finite index.

ii) Let $N \leqq G$ be a torsion-free normal subgroup of finite index and let $U \leqq G$ be a finite subgroup. Set $H = \langle N,U \rangle \leqq G$. Then

$$\rho(F[H]) = |U| \,\Big|\, \rho(F[G]) \quad .$$

In particular, $f(G) \,\Big|\, \rho(F[G])$.

PROOF. (i). Set $r = \rho(F[G])$ and $s = \rho(F[H])$. If V is a finitely generated $F[G]$-module then, by definition of χ , $s \cdot \chi_{F[H]}(V)$ is an integer. Thus Lemma 1.1(i) implies that $s \cdot [G:H] \cdot \chi_{F[G]}(V)$ is an integer. Taking V with $\rho(V) = 1$ we see that $r \mid s \cdot [G:H]$. Similarly, if W is a finitely generated $F[H]$-module then, using Lemma 1.1(ii), we get that $r \cdot \chi_{F[G]}(F[G] \otimes_{F[H]} W) = r \cdot \chi_{F[H]}(W)$ is an integer, whence $s \mid r$. The last assertion follows from the Farkas-Snider-Cliff theorem [F-S],[C].

(ii). The group algebra $F[N]$ is an $F[H]$-module via $\alpha \cdot \sum_{u \in U} \alpha_u u = \sum_{u \in U} (\alpha\alpha_u)^u$ $(\alpha, \alpha_u \in F[N])$. By Lemma 1.1(i), $\chi_{F[H]}(F[N]) = |U|^{-1} \chi_{F[N]}(F[N]) = |U|^{-1}$. On the other hand, $\chi_{F[H]}(F[N]) = \rho_{F[H]}(F[N])/\rho(F[H])$ and $\rho_{F[H]}(F[N]) = 1$, since $F[N]$ is a domain by the Farkas-Snider-Cliff theorem. Therefore, $\rho(F[H]) = |U|$.

■

We remark that Corollary 1.2i is a special case of much more general "additivity principles" which relate the Goldie ranks of suitable prime factor rings of $F[H]$ and $F[G]$. For example, using the above notation, if Q is a prime ideal of $F[H]$ and $P_1,\ldots,P_s$ are the minimal covering primes of the induced ideal Q^G of $F[G]$ (see [Lo-Pa] for the definition of induced ideals), then there are positive integers $z_1,\ldots,z_s$ with

$$\sum_{i=1}^{s} z_i \cdot \rho(F[G]/P_i) = [G:H] \cdot \rho(F[H]/Q) \quad .$$

This can be proved using results of Warfield [W]. For details and further results along these lines see [Br-H-Lo].

In the next corollary, we give a number of equivalent formulations of the Goldie rank conjecture $\rho(F[G]) = f(G)$ for prime $F[G]$. Recall that

$$G_0(F[G])_{\mathcal{F}} = \sum_{U\in\mathcal{F}} \operatorname{Ind}_U^G G_0(F[U]) \quad ,$$

where $\mathcal{F} = \mathcal{F}(G)$ is the set of finite subgroups of G. Since $\mathcal{F}$ falls into finitely many G-conjugacy classes ([Ma] or [S, Theorem 5 on p. 175]) and since each $G_0(F[U])$, $U \in \mathcal{F}$, is a finitely generated free abelian group, it follows that $G_0(F[G])_{\mathcal{F}}$ is a finitely generated subgroup of $G_0(F[G])$. Note also that the factor group $G_0(F[G])/G_0(F[G])_{\mathcal{F}} + \operatorname{Ker} \chi$ is certainly a finite cyclic group whose order divides $\rho(F[G])$

(because $[F[G]] \in G_0(F[G])_{\mathcal{F}}$) . Part (i) of the next corollary shows that the Goldie rank conjecture, for $F[G]$ prime, holds precisely if the above factor group is trivial.

<u>COROLLARY 1.3</u>. Assume that $F[G]$ is prime.

i) The following are equivalent:

(a) $\rho(F[G]) = f(G)$;

(b) The function $f(G)\cdot\chi_{F[G]}$ on $G_0(F[G])$ is integer-valued;

(c) $G_0(F[G]) = G_0(F[G])_{\mathcal{F}} + \operatorname{Ker}\chi_{F[G]}$.

ii) (Reduction to "p-tops" [Lo2],[Ro2]). Fix a normal subgroup N of G having finite index in G . For each prime p let $G_p \leqq G$ denote the inverse image in G of a Sylow p-subgroup of G/N . Then

$$\rho(F[G]) = \rho(F[N])\cdot\prod_p \rho(F[G_p])/\rho(F[N]) \quad .$$

In particular, if $\rho(F[G_p]) = f(G_p)$ holds for all primes p , then $\rho(F[G]) = f(G)$.

<u>PROOF</u>. (i). <u>(c) ⇒ (b)</u>: Let $U \in \mathcal{F}$, let M be a finitely generated $F[U]$-module, and set $V = F[G]\otimes_{F[U]}M$. We show that $\chi_{F[G]}(V) = |U|^{-1}\cdot\dim_F M$ which will clearly prove (b). Choose a torsion-free normal subgroup N of finite index in G and set $H = \langle N,U\rangle$ and $W = F[H]\otimes_{F[U]}M$. Then, by Lemma 1.1, $\chi_{F[G]}(V) = \chi_{F[H]}(W) = |U|^{-1}\chi_{F[N]}(W)$. But

$$W\big|_{F[N]} \cong F[N]^{\dim_F M}$$

and so $\chi_{F[N]}(W) = \dim_F M$, as required.

<u>(b) ⇒ (a)</u>: (b) says that $\rho(F[G])$ divides $f(G)$, so equality must hold in view of Corollary 1.2ii.

<u>(a) ⇒ (c)</u>: If f_p denotes the p-part of $f = f(G)$, then there exists $U_p \in \mathcal{F}(G)$ with $|U_p| = f_p$. Choose integers z_p with $\sum_p z_p f/f_p = 1$ and set $\alpha = \sum_p z_p \cdot \mathrm{Ind}_{U_p}^G([F]) \in G_0(F[G])_{\mathcal{F}}$, where F denotes the trivial one-dimensional $F[U_p]$-module. Then the formula established in the proof of (c) ⇒ (b) above shows that $\chi_{F[G]}(\alpha) = \sum_p z_p \cdot |U_p|^{-1} = f^{-1}$. Thus (a) implies that $\chi_{F[G]}(G_0(F[G])) = \langle\chi_{F[G]}(\alpha)\rangle = \chi_{F[G]}(G_0(F[G])_{\mathcal{F}})$.

(ii). By Corollary 1.2i, $\rho(F[G])/\rho(F[N])$ and each $\rho(F[G_p])/\rho(F[N])$ are integers, the latter is a p-power, and

$$\frac{\rho([G_p])}{\rho(F[N])} \;\Big|\; \frac{\rho(F[G])}{\rho(F[N])} \;\Big|\; \frac{\rho(F[G_p])}{\rho(F[N])} \cdot [G:G_p] \;.$$

We conclude that $\rho(F[G_p])/\rho(F[N])$ is the p-part of $\rho(F[G])/\rho(F[N])$ which proves the formula for $\rho(F[G])$.

An analogous argument, based on [Lo2, Lemma 1.2] instead of Corollary 1.2i, shows that $f(G) = f(N) \cdot \prod_p f(G_p)/f(N)$. The last assertion follows from this.

∎

Part (ii) of the above corollary together with the Farkas-Snider-Cliff theorem can be used to show that, for $F[G]$ prime, $\rho(F[G])$ and $f(G)$ have the same prime divisors. To see this, fix a torsion-free normal subgroup N of finite index in G and, for each prime p, define $G_p \leq G$ as in the corollary. Then, as we have seen, $\rho(F[G_p])$ is the p-part of $\rho(F[G])$ (this uses the Farkas-Snider-Cliff theorem for $F[N]$) and $f(G_p)$ is the p-part of $f(G)$. But, by the Farkas-Snider-Cliff theorem again, $f(G_p) = 1$ implies $\rho(F[G_p]) = 1$ and so all primes dividing $\rho(F[G])$ must also divide $f(G)$. The converse is clear, by Corollary 1.2ii.

We now show that a much stronger statement than (c) in Corollary 1.3i above holds for the very special class of finite-by-poly-(infinite cyclic or infinite dihedral) groups, i.e. groups G having a finite subnormal series $\langle 1\rangle \trianglelefteq G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_s = G$ with G_0 finite and G_i/G_{i-1} infinite cyclic or infinite dihedral $(i \geq 1)$. By a result of Formanek [Fo], all finite-by-supersolvable groups are of this form. Our proof follows the lines of [Ro2, proof of Theorem (0.3)] and ultimately rests on results of Waldhausen [Wa].

PROPOSITION 1.4. If G is finite-by-poly-(infinite cyclic or infinite dihedral), then $G_0(F[G]) = G_0(F[G])_{\mathcal{F}}$. (In particular, the Goldie rank conjecture holds for prime supersolvable group algebras [Lo2],[Ro2].)

PROOF. We argue by induction on the Hirsch number $h(G)$ of G ($= s$ in the above subnormal series), the assertion being trivial for $h(G) = 0$, i.e. G finite. If G is infinite then G has a normal subgroup N such that G/N is either infinite cyclic or infinite dihedral. In the former case, it follows from [Wa, Proposition 4.1] or [Q, Exercise on p. 122] that $\mathrm{Ind}_N^G : G_0(F[N]) \longrightarrow G_0(F[G])$ is surjective. By induction, we know that $G_0(F[N]) = G_0(F[N])_{\mathcal{F}(N)}$ and so the corresponding equality follows for $F[G]$.

If G/N is infinite dihedral we can write $G = G_1 *_N G_2$ with G_i/N cyclic of order 2 $(i = 1,2)$. The group algebra $F[G]$ is a free product with amalgamation, $F[G] \cong F[G_1] *_{F[N]} F[G_2]$, and [Wa, Proposition 4.1] again implies that the map $\mathrm{Ind} : G_0(F[G_1]) \oplus G_0(F[G_2]) \longrightarrow G_0(F[G])$ is surjective. The induction hypothesis again yields the result.

■

For G finitely generated abelian-by-finite, the main result of [Br-H-Lo] asserts that $G_0(F[G])/G_0(F[G])_{\mathcal{F}}$ is periodic, of exponent dividing $[G:A]^{\mathrm{rank}\,A}$, where A is any abelian normal subgroup of finite index in G.

2. THE RÔLE OF PROJECTIVE MODULES IN CHARACTERISTIC p

In this section, we study the structure of the cokernel of the Cartan map $c : K_0(F[G]) \longrightarrow G_0(F[G])$. Of course, this is of interest only when $\operatorname{char} F$ divides $f(G)$, because otherwise $F[G]$ has finite global dimension and c is an isomorphism. Thus the ground field F will always have characteristic $p > 0$. Furthermore, we will use the following

NOTATION. For any family $\mathfrak{U}$ of subgroups of the group G define $P(\mathfrak{U}) \subseteq G_0(F[G])$ to be the subgroup of $G_0(F[G])$ generated by all $[V]$ with V a finitely generated $F[G]$-module such that $V\big|_{F[U]}$ has finite homological dimension for all $U \in \mathfrak{U}$ or, equivalently, by all $[V]$ with $V\big|_{F[U]}$ projective for all $U \in \mathfrak{U}$. If $\mathfrak{U}$ consists of a single subgroup U , we also write $P(U)$ instead of $P(\mathfrak{U})$.

REMARKS. (1) Clearly, $P(\mathfrak{U})$ contains $cK_0(F[G]) = P(G)$, and if all $U \in \mathfrak{U}$ have finite index in G , then $P(\mathfrak{U})$ is contained in

$$R(\mathfrak{U}) := \text{kernel of } \; G_0(F[G]) \xrightarrow{\text{Res}} \bigoplus_{U\in\mathfrak{U}} G_0(F[U])/cK_0(F[U]) \; .$$

The inclusion $P(\mathfrak{U}) \subseteq R(\mathfrak{U})$ is proper in general. For example, if G is cyclic of order p^n $(n > 1)$ and $U \leq G$ is cyclic of order p , then $P(U) = cK_0(F[G]) = \langle [F[G]] \rangle$, so $G_0(F[G])/P(U) \cong \mathbf{Z}/p^n\mathbf{Z}$, but

$G_0(F[G])/R(U) \cong G_0(F[U])/c\,K_0(F[U]) \cong \mathbf{Z}/p\mathbf{Z}$. Another, perhaps more interesting example is given by

$$H = \langle x,y \mid z := [x,y] \text{ is central}, \quad x^p = y^p = 1\rangle \quad ,$$

the non-abelian group of order p^3 and exponent p . One easily checks that $G_0(F[H])/P(\langle z\rangle) \cong \mathbf{Z}/p^2\mathbf{Z}$, whereas $G_0(F[H])/R(\langle z\rangle) \cong \mathbf{Z}/p\mathbf{Z}$. On the other hand, $G_0(F[H])/P(\langle x\rangle) \cong G_0(F[H])/R(\langle x\rangle) \cong \mathbf{Z}/p\mathbf{Z}$ so that $P(\langle x\rangle) = R(\langle x\rangle)$. Further examples with $P(\mathfrak{U}) = R(\mathfrak{U})$ can be obtained from Lemma 2.3 below.

(2) Let V be an $F[G]$-module. Then, by a theorem of Serre ([Se3], cf. also [Pa2, Proposition 1]), V has finite homological dimension if and only if $V|_{F[U]}$ is projective for all $U \in \mathcal{F} = \mathcal{F}(G)$. Moreover, by Chouinard's theorem [Ch], V is projective over $F[U]$ if and only if $V|_{F[A]}$ is projective for all elementary abelian p-subgroups A of U $(p = \operatorname{char} F)$. Therefore, if $\mathfrak{U}$ is any family of subgroups of G which is closed under taking subgroups and

$$\mathfrak{U}_{fin} = \{U \in \mathfrak{U} \mid U \text{ is finite}\} \quad ,$$

$$\mathfrak{U}_{p\text{-el.ab}} = \{U \in \mathfrak{U}_{fin} \mid U \text{ is an elementary abelian p-group}\} \quad \text{, and}$$

$$\mathfrak{U}^* = \{V \leq G \mid \text{all finite elementary abelian p-subgroups of } V \text{ belong to } \mathfrak{U}\} \quad ,$$

then

$$P(\mathfrak{U}) = P(\mathfrak{U}_{fin}) = P(\mathfrak{U}_{p\text{-el.ab}}) = P(\mathfrak{U}^*) \quad .$$

In particular,

$$c\,K_0(F[G]) = P(\mathcal{F}) = P(\mathcal{F}_{p\text{-el.ab}}) \quad .$$

(3) Let H be a finite group and let $\mathfrak{U}$ be a family of subgroups of H which is closed under taking subgroups and consists of p-groups (WLOG) . If E/F is an algebraic p'-extension (i.e. each finite subextension has p'-degree), then identifying $G_0(F[H])$ with its image in $G_0(E[H])$ under the scalar extension map (cf. [Se1, Sec.14.6]), we have

$$P_F(\mathfrak{U}) = G_0(F[H]) \cap P_E(\mathfrak{U}) \quad ,$$

with the obvious notation $P_F(.)$ and $P_E(.)$. Indeed, we may clearly assume E/F to be Galois. Then each fin. gen. $E[H]$-module V has the form $V = E \otimes_L V_0$ for some <u>finite</u> Galois subextension L/F of E/F and some $L[H]$-module V_0 . Moreover, setting $\hat{U} = \sum_{u \in U} u \in F[U]$ $(U \in \mathfrak{U})$ we have

$$\begin{aligned} V|_{E[U]} \text{ is free } &\iff \dim_E \hat{U} \cdot V = |U|^{-1} \dim_E V \\ &\iff \dim_L \hat{U} \cdot V_0 = |U|^{-1} \dim_L V_0 \\ &\iff V_0|_{L[U]} \text{ is free} \quad . \end{aligned}$$

This shows that each $\alpha \in P_E(\mathfrak{U})$ is contained in $P_L(\mathfrak{U})$ for some finite Galois subextension L/F of E/F , and so we may assume that E/F is <u>finite</u> Galois. Set $\Gamma = \mathrm{Gal}(E/F)$ and, for any fin. gen. $E[H]$-module V set $\bar{V} = \bigoplus_{\sigma \in \Gamma} V^{\sigma}$, where V^{σ} is the σ-conjugate module of V. Then $\bar{V} \cong E \otimes_F V'$ with V' denoting the restriction of V to $F[H]$. If V is free over $E[U]$ $(U \in \mathfrak{U})$, then each V^{σ} is free over $E[U]$ and hence so is $\bar{V}$. As we have just seen, this implies that V' is free over $F[U]$. Now let $\alpha \in G_0(F[H]) \cap P_E(\mathfrak{U})$ be given and write $\alpha = [V_1] - [V_2]$ with fin. gen. $E[H]$-modules V_i which are free over $E[U]$ for all $U \in \mathfrak{U}$. Then, under the action of Γ on $G_0(E[U])$, we have

$$|\Gamma| \cdot \alpha = \sum_{\alpha \in \Gamma} \alpha^{\sigma} = [\bar{V}_1] - [\bar{V}_2] = [E \otimes_F V_1'] - [E \otimes_F V_2'] \in P_F(\mathfrak{U}) \quad .$$

Thus $|\Gamma|$ annihilates the group $X := G_0(F[H]) \cap P_E(\mathfrak{U})/P_F(\mathfrak{U})$ so that X has p'-order. On the other hand, X is a subgroup of the p-group $G_0(F[H])/P_F(\mathfrak{U})$ (cf. Lemma 2.1 below), whence X must be trivial, as we have claimed.

The first lemma is a refinement of Brauer's well-known theorem that, for finite groups, the determinant of the Cartan map is a power of $p = \operatorname{char} F$ (cf. [Se1, Théorème 35], for example).

<u>LEMMA 2.1</u>. Let $\operatorname{char} F = p > 0$, let H be a finite group, and let $\mathfrak{U}$ be a family of subgroups of H which is closed under conjugation and taking subgroups. Then $P(\mathfrak{U})$ is an ideal of $G_0(F[H])$ and $G_0(F[H])/P(\mathfrak{U})$ is a finite p-group. Its exponent satisfies

$$\max_{U\in\mathfrak{U}^*} |U|_p \;\Big|\; \exp(G_0(F[H])/P(\mathfrak{U})) \;\Big|\; [H:N]_p \quad ,$$

where N is any subgroup of H with $p \nmid |N \cap U|$ for all $U \in \mathfrak{U}$.

<u>PROOF</u>. Clearly, $P(\mathfrak{U})$ is an ideal of $G_0(F[H])$ with $c\,K_0(F[H]) \subseteq P(\mathfrak{U})$. The fact that $G_0(F[H])/P(\mathfrak{U})$ is a finite p-group now follows from Brauer's theorem. Moreover, if $U \in \mathfrak{U}$ is a p-group, then restriction yields a surjection $G_0(F[H])/P(\mathfrak{U}) \twoheadrightarrow G_0(F[U])/c\,K_0(F[U]) \cong \mathbf{Z}/|U|\mathbf{Z}$. Inasmuch as $P(\mathfrak{U}) = P(\mathfrak{U}^*)$, this establishes the lower bound for the exponent of $G_0(F[H]/P(\mathfrak{U}))$. It remains to check the upper bound.

Let $\mathfrak{X}(p)$ denote the set of subgroups $X \leq H$ which are semidirect products of the form $X = C \rtimes P$ with C a cyclic p'-group and P a p-group. Then, by [Se1, Théorème 28, cf. also the proof of Théorème 39], $G_0(F[H])/G_0(F[H])_{\mathfrak{X}(p)}$ is a finite p'-group. Since $G_0(F[H])/P(\mathfrak{U})$ is a p-group, it follows that $G_0(F[H])_{\mathfrak{X}(p)}$ maps onto $G_0(F[H])/P(\mathfrak{U})$. Now, for any subgroup $N \leq H$, write

$$\mathfrak{U} \cap N = \{U \cap N \mid U \in \mathfrak{U}\} = \{U \in \mathfrak{U} \mid U \subseteq N\} \quad .$$

Then it is easily seen, using Mackey decomposition, that Ind_N^H maps $P(\mathfrak{U} \cap N)$ to $P(\mathfrak{U})$. Therefore, $G_0(F[H])/P(\mathfrak{U})$ is an image of $\bigoplus_{X\in\mathfrak{X}(p)} G_0(F[X])/P(\mathfrak{U} \cap X)$, and so we may assume that

H has the form $H = C \rtimes P$, where C is a p'-group and P a p-group. But, in this case, $G_0(F[H])/P(\mathfrak{U})$ is a module over $G_0(F[P])/P(\mathfrak{U} \cap P)$ via inflation and $\otimes_F$, because inflation from P to H maps $P(\mathfrak{U} \cap P)$ to $P(\mathfrak{U})$ and $P(\mathfrak{U})$ is an ideal of $G_0(F[H])$. This shows that the exponent of $G_0(F[H])/P(\mathfrak{U})$ divides the exponent of $G_0(F[P])/P(\mathfrak{U} \cap P)$, thereby reducing the problem to the case where H is a p-group. Finally, if $N \leqq H$ satisfies $N \cap \mathfrak{U} = \{<1>\}$, then consider $V = \mathrm{Ind}_N^H(F)$, where F is the trivial $F[N]$-module. We have $[V] \in P(\mathfrak{U})$ and $\dim_F V = [H:N]$. Therefore, $G_0(F[H])/<[V]> \cong \mathbb{Z}/[H:N]\mathbb{Z}$ maps onto $G_0(F[H])/P(\mathfrak{U})$ which completes the proof of the lemma.

$\blacksquare$

We now apply the above to group algebras of polycyclic-by-finite groups G .

PROPOSITION 2.2. Assume that $\mathrm{char}\, F = p > 0$.

i. $G_0(F[G])/c\,K_0(F[G])$ is a torsion group of finite exponent dividing $[G:N]_p$, where N is any subgroup of finite index in G with $p \nmid f(N)$.

ii. If $F[G]$ is prime, then the group $G_0(F[G])/c\,K_0(F[G]) + \mathrm{Ker}\,\chi_{F[G]}$ is cyclic of order $\rho(F[G])_p$.

PROOF. (i) Let M be any normal subgroup of finite index in G , let $\bar{\ } : G \longrightarrow G/M$ denote the canonical map, and let $\mathfrak{U}$ be any family of subgroups of G. Then, viewing $G_0(F[G])$ as a module over $G_0(F[\bar{G}])$ via inflation and $\otimes_F$, it is straightforward to check that

$$P(M \cap \mathfrak{U}) \cdot P(\bar{\mathfrak{U}}) \subseteq P(\mathfrak{U}) \quad .$$

Here, $M \cap \mathfrak{U} = \{M \cap U \mid U \in \mathfrak{U}\}$ and $\bar{\mathfrak{U}} = \{\bar{U} \mid U \in \mathfrak{U}\}$. In particular,

$$\exp(G_0(F[G])/P(\mathfrak{U})) \mid \exp(G_0(F[G])/P(M \cap \mathfrak{U})) \cdot \exp(G_0(F[\bar{G}])/P(\bar{\mathfrak{U}})) \ .$$

Now take N as in part (i) above and set $M = \bigcap_{x \in G} N^x$. Then $G_0(F[G]) = P(M \cap \mathcal{F})$ and $c\,K_0(F[G]) = P(\mathcal{F})$, by Serre's theorem, and so it suffices to quote Lemma 2.1, with $H = \bar{G}$ and $N = \bar{N}$, to finish the proof of (i).

(ii). Let N be a normal torsion-free subgroup of G having finite index in G and, for each prime q, define $G_q \leqq G$ to be the preimage of a Sylow q-subgroup of G/N. Then induction yields an epimorphism

$$\bigoplus_q G_0(F[G_q])/\mathrm{Ker}\,\chi_{F[G_q]} \longrightarrow\!\!\!\!\!\rightarrow G_0(F[G])/\mathrm{Ker}\,\chi_{F[G]} \quad .$$

To prove surjectivity, choose integers z_q with $\sum_q z_q \cdot [G : G_q] = 1$. Then, for each $\alpha \in G_0(F[G])$, Lemma 1.1 implies

$$\chi_{F[G]}(\alpha) = \chi_{F[G]}\Big(\sum_q z_q \cdot \mathrm{Ind}_{G_q}^{G} \mathrm{Res}_{G_q}^{G}(\alpha)\Big)$$

which proves surjectivity. Therefore, induction also yields an epimorphism

$$\bigoplus_q \mathcal{G}_0(F[G_q]) \twoheadrightarrow \mathcal{G}_0(F[G]) \quad ,$$

where we have set $\mathcal{G}_0(F[G]) = G_0(F[G])/c\,K_0(F[G]) + \operatorname{Ker}\chi_{F[G]}$ and similarly for G_q . By our above remarks, $\mathcal{G}_0(F[G_q])$ is trivial for all primes $q \neq p$ so that $\mathcal{G}_0(F[G_p])$ maps onto $\mathcal{G}_0(F[G])$. Moreover, since $\chi_{F[G_p]}: G_0(F[G_p]) \longrightarrow \frac{1}{\rho(F[G_p])}\cdot \mathbf{Z}$ maps $[F[G_p]] \in c\,K_0(F[G_p])$ to 1 , we see that $\mathcal{G}_0(F[G_p])$ is cyclic of order at most $\rho(F[G_p]) = \rho(F[G])_p$. We claim that $\chi_{F[G_p]}(P) \in \mathbf{Z}$ holds for every finitely generated projective $F[G_p]$-module P . Indeed, by Lemma 1.1i and [Lo1, Proposition 8], we have

$$\chi_{F[G_p]}(P) = [G_p : N]^{-1}\cdot \chi_{F[N]}(P) = [G_p : N]^{-1}\cdot \dim_F H_0(N,P) \quad .$$

Here, $H_0(N,P) = P/(\omega N)P$ is projective and hence free over the local ring $F[G_p/N]$. Thus $\dim_F H_0(N,P)$ is divisible by $[G_p : N]$, as required. This proves that $\mathcal{G}_0(F[G_p])$ is cyclic of order precisely $\rho(F[G])_p$. It remains to check that $\mathcal{G}_0(F[G])$ is isomorphic to $\mathcal{G}_0(F[G_p])$. For this, we show that the map $\mathcal{G}_0(F[G]) \longrightarrow \mathcal{G}_0(F[G_p])$ given by restriction is surjective. For $\alpha \in G_0(F[G_p])$ we have, by Lemma 1.1,

$$\chi_{F[G_p]}\left(\operatorname{Res}^G_{G_p} \operatorname{Ind}^G_{G_p}(\alpha)\right) = [G : G_p]\,\chi_{F[G_p]}(\alpha) \quad ,$$

whence

$$[G:G_p]\alpha - \mathrm{Res}_{G_p}^{G}\,\mathrm{Ind}_{G_p}^{G}(\alpha) \in \mathrm{Ker}\,\chi_{F[G_p]}$$

Letting $\bar{\alpha}$ denote the image of α in $\mathcal{G}_0(F[G_p])$ and using the fact that multiplication by $[G:G_p]$ is bijective on $\mathcal{G}_0(F[G_p])$ and $\mathcal{G}_0(F[G])$, we get

$$\bar{\alpha} = [G:G_p]^{-1}\,\mathrm{Res}_{G_p}^{G}\,\mathrm{Ind}_{G_p}^{G}(\bar{\alpha}) = \mathrm{Res}_{G_p}^{G}([G:G_p]^{-1}\cdot\mathrm{Ind}_{G_p}^{G}(\bar{\alpha})) .$$

This proves surjectivity of the map $\mathrm{Res}_{G_p}^{G} : \mathcal{G}_0(F[G]) \longrightarrow \mathcal{G}_0(F[G_p])$ and thus completes the proof of the proposition.

■

<u>COROLLARY 2.3</u>. (of proof). Let $\mathrm{char}\,F = p > 0$ and assume that $F[G]$ is prime. Then $\rho(F[G])_p$ divides the exponent of $G_0(F[\bar{G}])/P(\bar{\mathcal{F}})$, where $\bar{G} = G/N$ is any finite image of G with $p \nmid f(N)$, and $\bar{\mathcal{F}} = \{UN/N \mid U \in \mathcal{F}(G)\}$.

<u>PROOF</u>. By part (ii) above, $\rho(F[G])_p$ divides the exponent of $G_0(F[G])/c\,K_0(F[G])$ which in turn, by the proof of part (i), divides the exponent of $G_0(F[\bar{G}])/P(\bar{\mathcal{F}})$.

■

Presumably the bound $[G:N]_p$ in Proposition 2.2i can be improved to $f(G)_p$. In view of part (ii) of the proposition, this would prove the equality $\rho(F[G])_p = f(G)_p$ for $\operatorname{char} F = p > 0$. It would also prove the Goldie rank conjecture for <u>any</u> prime group algebra $F[G]$ with $\operatorname{char} F = 0$ (Corollary 1.3 ii and Corollary 3.3 ii below). By [Se, Théorème 35], the image of $G_0(F[G])_{\mathcal{F}}$ in $G_0(F[G])/c\,K_0(F[G])$ does indeed have exponent $f(G)_p$.

We now apply the techniques developed so far to derive a slightly polished version of Passman's solution of the Goldie rank problem for "elementary abelian p-tops in characteristic p" [Pa2]. The following lemma is an interpretation in terms of G_0 of Lemma 4 in [Pa2]. For the reader's convenience, we sketch the argument. For brevity, any subgroup of the group H which has a normal complement in H will be called a <u>splitting subgroup</u> of H.

<u>LEMMA 2.4</u>. Let F be a field with $\operatorname{char} F = p > 0$ and let H be a finite group. Let $U \leqq H$ be an abelian p-subgroup of H which is a splitting subgroup of H and let $\mathfrak{U}$ denote the set of all splitting subgroups of H which are isomorphic to U. Then $G_0(F[H]) \cdot |U| \subseteq P(\mathfrak{U})$.

<u>PROOF</u>. By Remark (3) above, we may assume that F is infinite. Note that $\mathcal{H} := \operatorname{Hom}(H, 1 + \omega U)$ has the structure of an affine space over F. Indeed, letting $(H^{ab})_p$ denote the Sylow p-subgroup of $H^{ab} = H/[H,H]$ and writing $(H^{ab})_p = \bigoplus_{i=1}^{s} \langle h_i \rangle$ we get

$$\mathcal{H} \cong \mathrm{Hom}\,((H^{ab})_p\,,\,1+\omega U) \cong \prod_{i=1}^{s} \mathrm{Hom}\,(\langle h_i\rangle\,,\,1+\omega U)$$

$$\cong \prod_{i=1}^{s} (1+\omega U)_{o(h_i)} \cong \prod_{i=1}^{s} (\omega U)_{o(h_i)} \quad \text{(as sets)} \quad .$$

Here, $o(h_i)$ denotes the order of h_i and, for each $n \geqq 0$, $(1+\omega U)_n = \{\alpha \in 1+\omega U \mid \alpha^n = 1\}$, $(\omega U)_n = \{\alpha \in \omega U \mid \alpha^n = 0\}$. As $o(h_i)$ is a p-power, $(\omega U)_{o(h_i)}$ is an F-vector space (even an ideal of $F[U]$), and so $\mathcal{H} \cong F^m$ for some m, by selecting bases for the $(\omega U)_{o(h_i)}$. Viewing $\mathcal{H}$ as a subset of $\mathrm{Hom}_{F\text{-alg}}(F[H], F[U])$, one checks that for each $\alpha \in F[H]$ the set $V(\alpha) = \{\sigma \in \mathcal{H} \mid \sigma(\alpha) = 0\}$ is Zariski-closed in $\mathcal{H}$. Moreover, if $X \in \mathfrak{U}$ then there exists a homomorphism $\sigma : H \longrightarrow U$ which is an isomorphism when restricted to X. Therefore, writing $\hat{X} = \sum_{x \in X} x \in F[X]$ as usual, we see that $\sigma(\hat{X}) = \hat{U} \neq 0$ so that $V(\hat{X}) \neq \mathcal{H}$. Since $F \cdot \hat{X}$ is the unique smallest ideal of $F[X]$, each $\sigma \in \mathcal{H}\setminus V(\hat{X})$ yields a homomorphism $F[H] \longrightarrow F[U]$ whose restriction to $F[X]$ is an isomorphism, thereby making $F[U]$ an $F[H]$-module which is free over $F[X]$. Moreover, in $G_0(F[H])$ we have $[F[U]] = |U| \cdot [F]$, where F is the trivial one-dimension $F[H]$-module. Finally, since F is infinite and there are only finitely many $X \in \mathfrak{U}$, we can select $\sigma \in \mathcal{H}\setminus \bigcup_{X\in\mathfrak{U}} V(\hat{X})$, so that $[F[U]] \in P(\mathfrak{U})$ via σ. The lemma now follows from the fact that $P(\mathfrak{U})$ is an ideal and $[F]$ is the identity of the ring $G_0(F[H])$.

$\blacksquare$

<u>THEOREM 2.5</u>. (cf. [Pa2]). Let F be a field with $\operatorname{char} F = p > 0$ and assume that $F[G]$ is prime. Assume further that G has a normal subgroup N of finite index such that $p \nmid f(N)$ and the Sylow p-subgroup of G/N is abelian, of exponent p^e. If $m = \max_U \operatorname{rank} U$, where U runs over the finite elementary abelian p-subgroups of G, then

$$\rho(F[G])_p \,\Big|\, p^{em} \quad .$$

In particular, if $e = 1$, i.e. if the Sylow p-subgroup of $\overline{G}$ is elementary abelian, then $\rho(F[G])_p = f(G)_p$.

<u>PROOF</u>. Using Corollary 1.3 ii, we immediately reduce to the case where $\overline{G} = G/N$ is a finite abelian p-group of exponent p^e.

Embed $\overline{G}$ into an abelian group of the form $H = (\mathbf{Z}/p^e\mathbf{Z})^r$ for some $r \geqq m$ and let $\mathfrak{U}$ denote the set of all subgroups of H which are isomorphic to $(\mathbf{Z}/p^e\mathbf{Z})^m$. These are all direct summands of H. Moreover, if $\overline{\mathcal{F}} = \{UN/N \mid U$ a finite p-subgroup of $G\}$ then, for all $\overline{U} \in \overline{\mathcal{F}}$, there exists $U \in \mathfrak{U}$ with $\overline{U} \leqq U$. Therefore, $\operatorname{Res}^H_{\overline{G}} : G_0(F[H]) \longrightarrow G_0(F[\overline{G}])$ maps $P(\mathfrak{U})$ to $P(\overline{\mathcal{F}})$. By Lemma 2.3, we conclude that $p^{em} \cdot [F] \in P(\overline{\mathcal{F}})$ and so Corollary 2.3 implies that $\rho(F[G])_p$ divides p^{em}. Finally, if $e = 1$ then clearly $p^m = f(G)_p$, and this divides $\rho(F[G])_p$, by Corollary 1.2 ii.

∎

3. THE DECOMPOSITION MAP

The decomposition map is a basic tool in representation theory of finite (or algebraic) groups (cf. [C-R, §16]). In this section, we describe some of its basic features in a setting suitable for application to polycyclic group algebras. The following notations will be kept fixed:

A is a discrete valuation ring with maximal ideal $\mathfrak{m} = (\pi)$,

$k = A/\mathfrak{m}$ is the residue field of A , and

$K = \mathrm{Fract}(A)$ is the field of fraction of A .

R_1 will be a (left) Noetherian A-algebra which is torsion-free over A ,

$R = K \otimes_A R_1$, so $R_1 \subseteq R$, and

$\tilde{R}_1 = k \otimes_A R_1 = R_1/\mathfrak{m}R_1$.

Consider a finitely generated R-module V . Then V contains a finitely generated R_1-submodule V_1 such that $V = K \cdot V_1$ (e.g., if $V = \Sigma R v_i$ then one can take $V_1 = \Sigma R_1 v_i$). Such a V_1 is called an R_1-form of V . The proof of the first proposition follows traditional lines, except for minor modifications caused by the extra bit of generality in our assumptions.

PROPOSITION 3.1. Let V be a finitely generated R-module and let V_1 be an R_1-form of V . Then the element $[\tilde{V}_1] \in G_0(\tilde{R}_1)$ corresponding to $\tilde{V}_1 = V_1/\mathfrak{m}V_1 \cong k \otimes_A V_1$ does not depend on the choice of the particular R_1-form V_1 of V . The map

$$d : G_0(R) \longrightarrow G_0(\tilde{R}_1) \ , \ d([V]) = [\tilde{V}_1]$$

is a well-defined homomorphism.

PROOF. Suppose that $V_2 = \sum_{j=1}^{m} R_1 w_j$ is another R_1-form of V. Then, since $V_2 \leq V = K \cdot V_1$, there are finitely many $v_\ell \in V_1$ and $\xi_{\ell j} \in K$ with

$$w_j = \sum_\ell \xi_{\ell j} v_\ell \quad (j = 1,\dots,m) \quad .$$

Choose a common denominator $a \in A$ for all $\xi_{\ell j}$'s so that $a \cdot \xi_{\ell j} \in A$ for all ℓ and j. Then $aw_j \in V_1$ for all j and so $a \cdot V_2 \subseteq V_1$. Similarly, one finds $b \in A$ with $b \cdot V_1 \subseteq V_2$. Thus $V_1 \supseteq a \cdot V_2 \supseteq ab \cdot V_1 = m^n V_1$, where n is chosen so that $abA = m^n$. Now V is certainly torsion-free over A, being a K-vector space, and so V_2 and $a \cdot V_2$ are isomorphic R_1-modules. Thus, in order to show that $[\tilde{V}_2] = [\tilde{V}_1]$ holds in $G_0(\tilde{R}_1)$, we can replace V_2 by $a \cdot V_2$ and thus assume that

$$V_1 \supseteq V_2 \supseteq m^n V_1 \quad .$$

We argue by induction on n. First suppose that $n = 1$ and set $T = V_1/V_2$. Then T is a finitely generated $\tilde{R}_1$-module, and we have an exact sequence of finitely generated $\tilde{R}_1$-modules

$$0 \longrightarrow T \xrightarrow{\cdot\pi} \tilde{V}_2 = V_2/\pi \cdot V_2 \longrightarrow \tilde{V}_1 = V_1/\pi \cdot V_1 \longrightarrow T \longrightarrow 0 \ .$$

In $G_0(\tilde{R}_1)$, this yields $[T] - [\tilde{V}_2] + [\tilde{V}_1] - [T] = 0$, whence

$[\tilde{V}_2] = [\tilde{V}_1]$. If $n > 1$, then set $V_3 = m^{n-1}V_1 + V_2$. Then V_3 is an R_1-form of V with $V_1 \supseteq V_3 \supseteq m^{n-1}V_1$ and $V_3 \supseteq V_2 \supseteq mV_3$. By induction, we conclude that $[\tilde{V}_1] = [\tilde{V}_3] = [\tilde{V}_2]$ so that $[\tilde{V}_1] = [\tilde{V}_2]$ follows in general.

Now let $0 \longrightarrow U \longrightarrow V \xrightarrow{\varphi} W \longrightarrow 0$ be an exact sequence of finitely generated R-modules and choose an R_1-form $V_1 \subseteq V$ of V . Then, clearly, $W_1 = \varphi(V_1)$ is an R_1-form of W . Also, $U_1 = U \cap V_1$ is finitely generated over R_1 , since R_1 is Noetherian, and

$$K \cdot U_1 = \{\xi u \mid u \in U \cap V_1 , \xi \in K\} = U \cap K \cdot V_1 = U \quad ,$$

so U_1 is an R_1-form of U . Finally, since W_1 is torsion-free and hence flat over A , the exact sequence $0 \longrightarrow U_1 \longrightarrow V_1 \longrightarrow W_1 \longrightarrow 0$ remains exact under tensoring with $k \otimes_A (.)$. This shows that $[\tilde{V}_1] = [\tilde{U}_1] + [\tilde{W}_1]$ holds in $G_0(\tilde{R}_1)$. Therefore, the map $[V] \longmapsto [\tilde{V}_1]$ defines a homomorphism $G_0(R) \longrightarrow G_0(\tilde{R}_1)$.

∎

The homomorphism $d : G_0(R) \longrightarrow G_0(\tilde{R}_1)$ constructed above is called the <u>decomposition map</u>.

The foregoing applies in particular to group algebras of polycyclic-by-finite groups. It is routine to verify that, for any subgroup H of G , the following diagram commutes:

$$\begin{tikzcd} G_0(K[H]) \arrow[r, "d"] \arrow[d, "\mathrm{Ind}_H^G"] & G_0(k[H]) \arrow[d, "\mathrm{Ind}_H^G"] \\ G_0(K[G]) \arrow[r, "d"] & G_0(k[G]) \end{tikzcd} .$$

Since, for all $U \in \mathcal{F} = \mathcal{F}(G)$, the decomposition map $G_0(K[U]) \longrightarrow G_0(k[U])$ is onto if K is complete [Se, Théorêm 33], we conclude in particular that

$$d(G_0(K[G])_{\mathcal{F}}) = G_0(k[G])_{\mathcal{F}} \quad (K \text{ complete}) .$$

An analogous commutative diagram exists for Res_H^G, $H \leqq G$ a subgroup of finite index.

<u>LEMMA 3.2</u>. Assume that G has no finite normal subgroups $\neq \langle 1 \rangle$. Then the following diagram commutes:

$$\begin{tikzcd} G_0(K[G]) \arrow[rr, "d"] \arrow[dr, "\chi_{K[G]}"'] & & G_0(k[G]) \arrow[dl, "\chi_{k[G]}"] \\ & \mathbf{Q} & \end{tikzcd} .$$

In particular, $\rho(K[G]) \mid \rho(k[G])$.

<u>PROOF</u>. Using the fact that, for $H \leqq G$ of finite index, the decomposition maps commute with Res_H^G in connection with

Lemma 1.1(i), we immediately reduce to the case where G is poly-$\mathbf{Z}$. But then the "twisted Grothendieck theorem" [Q, Exercise on p.122] or [Fa-H, Theorem 27] implies that $G_0(K[G])$ is generated by $[K[G]]$. Finally, $d([K[G]]) = [k[G]]$ and $\chi_{[G]}(k[G]) = 1 = \chi_{K[G]}(K[G])$. This shows that the diagram commutes. The remaining assertion is obvious from the definition of χ.

∎

COROLLARY 3.3. Assume that G has no finite normal subgroups $\neq \langle 1 \rangle$.

i. $\rho(F[G]) \mid \rho(E[G])$ for some finite extension E of the prime subfield of F. Consequently, if the Goldie rank conjecture holds for all $E[G]$, where E is a finite extension of the prime subfield of F, then it also holds for $F[G]$.

ii. Let p be a rational prime and assume that $\operatorname{char} F = 0$. Then $\rho(F[G]) \mid \rho(E[G])$ holds for some finite field E with $\operatorname{char} E = p$. In particular, if the Goldie rank conjecture holds for all $E[G]$, where E is a finite field of char p, then it also holds for all $F[G]$ with $\operatorname{char} F = 0$ [Ro2].

PROOF. (i). We may assume that F is finitely generated over its prime subfield F_0, because $\rho(F[G]) = \rho(F'[G])$ for some finitely generated subextension F'/F_0 of F/F_0. (Consider the F-coefficients of the generators in a direct sum of nonzero right ideals of maximal length in $F[G]$). In order to apply Lemma 3.2, we only have to exhibit a discrete valuation of F whose residue field is a finite extension of F_0. For this, let $F_1 \supseteq F_0$ be a purely transcendental subextension of F/F_0 such that F/F_1 is finite and let $F_0((T))$ be the field of Laurent power series over F_0. Then $F_0((T))$ has infinite transcendence degree over F_0 and so the embedding $F_0 \subseteq F_0((T))$ extends to an embedding of F_1 into $F_0((T))$. Now $F_0((T))$ has a discrete valuation, the "order valuation", with valuation ring $F_0[[T]]$ and maximal ideal $T \cdot F_0[[T]]$. By restriction, this yields a discrete valuation of F_1 with residue field F_0. Finally, v extends to a discrete valuation v' of F with residue field a finite extension of F_0, as required.

(ii). As above, it suffices to show that every finitely generated field extension of $\mathbb{Q}$ has a discrete valuation with residue field a finite field of $\operatorname{char} p$. The existence of such a valuation is well-known [MacL].

∎

It would be interesting to know whether or not the decomposition map $d: G_0(K[G]) \longrightarrow G_0(k[G])$ is surjective for (K,A,k) as above, with K complete of characteristic 0 and $\operatorname{char} k = p > 0$. Since the image of d contains $G_0(k[G])_{\mathcal{F}}$, d is of course surjective whenever $G_0(k[G]) = G_0(k[G])_{\mathcal{F}}$.

We close this section with a characteristic 0 version of Theorem 2.5, also due to D.S.Passman.

<u>THEOREM 3.4</u>. (Passman [Pa2]). Let F be a field of characteristic 0 and assume that $F[G]$ is prime. If G has a normal subgroup N of finite index with $p \nmid f(N)$ and such that the Sylow p-subgroup of G/N is elementary abelian, then $\rho(F[G])_p = f(G)_p$.

<u>PROOF</u>. This follows from Theorem 2.5 and Corollary 3.3ii. ∎

4. FACTORING OUT TORSION-FREE NORMAL SUBGROUPS OF FINITE INDEX

Throughout this section, N denotes a normal subgroup of G such that

$$\operatorname{char} F \nmid f(N) \quad \text{or, equivalently,} \quad \operatorname{gl.dim} F[N] < \infty$$

(cf. [Pa1, Theorem 10.3.13]). The canonical map $F[G] \longrightarrow F[G/N]$ extending the group homomorphism $G \longrightarrow G/N$ will be denoted by $\overline{}$. We also set

$$\overline{\mathcal{F}} = \{\overline{U} \mid U \in \mathcal{F} = \mathcal{F}(G)\} \quad .$$

Our main interest will be in the case when N is torsion-free and has finite index in G .

Since $F[N]$ has finite global dimension (equal to the Hirsch number $h(N)$ of N), any $F[G]$-module V satisfies

$$\operatorname{Tor}_n^{F[G]}(\overline{F[G]},V) = \operatorname{Tor}_n^{F[G]}(F[G] \otimes_{F[N]} F,V) \cong \operatorname{Tor}_n^{F[N]}(F,V) = H_n(N,V) = 0$$

for all sufficiently large n . This allows us to define a homomorphism

$$\gamma = \gamma_{F[G],N} : G_0(F[G]) \longrightarrow G_0(F[\overline{G}]), [V] \longmapsto \sum_{i \geq 0} (-1)^i [H_i(N,V)]$$

(cf. [Ba1, p.454]). For example, if $V|_{F[N]}$ is projective, then $H_i(N,V) = 0$ for all $i > 0$ and so $\gamma([V]) = [V/(\omega N)V]$. In particular, if $F[G]$ itself has finite global dimension (i.e. $\operatorname{char} F \nmid f(G)$) , then $G_0(F[G]) \cong K_0(F[G])$ and γ reduces to the canonical map $K_0(F[G]) \longrightarrow K_0(F[\overline{G}]) \xrightarrow{\text{Cartan}} G_0(F[\overline{G}])$. In this section, we describe some properties of the map γ . The connection of γ with (normalized) Goldie ranks is explained in the following lemma.

LEMMA 4.1. Assume that $F[G]$ is prime and that N is torsion-free and has finite index in G . Then, for any finitely generated $F[G]$-module V ,

$$\chi_{F[G]}(V) = [G:N]^{-1} \cdot \dim_F \circ \gamma_{F[G],N}([V]) \quad .$$

PROOF. This is part of [Lo1, Corollary 9].

$\blacksquare$

In the next lemma, we collect some elementary properties of the map γ .

LEMMA 4.2. i. If N is finite, then γ is onto.

ii. Let $H \leq G$ be a subgroup of G . Then the following diagram commutes:

$$\begin{array}{ccc} G_0(F[H]) & \xrightarrow{\gamma_{F[H],N\cap H}} & G_0(F[\overline{H}]) \\ \Big\downarrow \mathrm{Ind}_H^G & & \Big\downarrow \mathrm{Ind}_{\overline{H}}^{\overline{G}} \\ G_0(F[G]) & \xrightarrow[\gamma]{} & G_0(F[\overline{G}]) \end{array} \quad .$$

In particular, $\gamma(G_0(F[G])_{\mathcal{F}}) = G_0(F[\overline{G}])_{\overline{\mathcal{F}}}$.

iii. Let (K,A,k) be as in Section 2, notations, and let $d : G_0(K[G]) \longrightarrow G_0(k[G])$ and $\overline{d} : G_0(K[\overline{G}]) \longrightarrow G_0(k[\overline{G}])$ denote the decomposition maps. Then the following diagram commutes:

$$\begin{array}{ccc} G_0(K[G]) & \xrightarrow{\gamma_{K[G],N}} & G_0(K[\overline{G}]) \\ \Big\downarrow d & & \Big\downarrow \overline{d} \\ G_0(k[G]) & \xrightarrow[\gamma_{k[G],N}]{} & G_0(k[\overline{G}]) \end{array} \quad .$$

iv. Assume that N has finite index in G . Then $\gamma(G_0(F[\overline{G}]))$ is an ideal of $G_0(F[\overline{G}])$. Indeed, viewing $G_0(F[G])$ as a module over $G_0(F[\overline{G}])$ via inflation and $\otimes_F$, we have

$$\gamma(\alpha\cdot\beta) = \gamma(\alpha)\cdot\beta \quad (\alpha \in G_0(F[G]), \beta \in G_0(F[\overline{G}])) \quad .$$

<u>PROOF</u>. (i). Let V be a finitely generated $F[G]$-module and let $N \triangleleft G$ be finite, with $\operatorname{char} F \nmid |N|$. Then $V|_{F[N]}$ is

projective and so $\gamma([V]) = [V/(\omega N)V]$. Therefore, $\gamma \circ \mathrm{Inf}_{\bar{G}}^{G}$ is the identity on $G_0(F[\bar{G}])$.

(ii). Let V be a finitely generated $F[H]$-module. Since $F[N \cap H]$ has finite global dimension, we may assume that $V\big|_{F[N \cap H]}$ is projective so that $\gamma_{F[H],N\cap H}([V]) = [V/\omega(N\cap H)\cdot V]$. Moreover,

$$\mathrm{Res}_N^G\ \mathrm{Ind}_H^G(V) = \bigoplus_x \mathrm{Ind}^N_{x(N\cap H)x^{-1}}\ \mathrm{Res}^{xHx^{-1}}_{x(N\cap H)x^{-1}}\ (x\otimes V)$$

(x runs over a set of right coset representatives of NH in G) is projective over $F[N]$. Therefore,

$$\gamma_{F[G],N} \circ \mathrm{Ind}_H^G([V]) = [\mathrm{Ind}_H^G V\ /\omega N \cdot \mathrm{Ind}_H^G V]$$

$$= [\mathrm{Ind}_{\bar{H}}^{\bar{G}}(V/\omega(N\cap H)\cdot V)] = \mathrm{Ind}_{\bar{H}}^{\bar{G}} \circ \gamma_{F[H],N\cap H}([V]) \ .$$

This proves the commutativity of the diagram. The equality $\gamma(G_0(F[G])_{\mathcal{F}}) = G_0(F[\bar{G}])_{\bar{\mathcal{F}}}$ now follows from the fact that $\gamma_{F[U],N\cap U}$ is onto for all $U \in \mathcal{F}$, by (i).

The proofs of (iii) and (iv) are similarly straightforward and are omitted.

■

In the proof of Proposition 4.4 below, we will need a version of the Artin induction theorem for finite groups which we now explain. For this, let H denote a finite group and set

$$H_{reg} = \{x \in H \mid \mathrm{char}\, F \nmid \mathrm{order}(x)\} \qquad (= H \text{ if } \mathrm{char}\, F = 0).$$

Furthermore, let $\mathfrak{U}$ denote a family of subgroups of H and set

$$G_0(F[H];\mathfrak{U}) = \{\alpha \in G_0(F[H]) \mid \varphi_\alpha \quad \text{vanishes on} \quad H_{reg} \setminus \bigcup_{U\in\mathfrak{U}} U\}.$$

Here, φ_α denotes the virtual character or, in case $\mathrm{char}\, F = p > 0$, the virtual modular character of $\alpha \in G_0(F[H])$ (cf. [Se 1, p. 161]). Recall that $G_0(F[H])_{\mathfrak{U}}$ is the image of $\mathrm{Ind}_{\mathfrak{U}}^{H}: \bigoplus_{U\in\mathfrak{U}} G_0(F[U]) \to G_0(F[H])$. $G_0(F[H])_{\mathfrak{U}}$ and $G_0(F[H];\mathfrak{U})$ are ideals of $G_0(F[H])$, with $G_0(F[H])_{\mathfrak{U}} \subseteq G_0(F[H];\mathfrak{U})$.

Fix an integer m which is a multiple of the exponent of H. Then $(\mathbf{Z}/m\mathbf{Z})^*$ acts as a permutation group on H via $x \mapsto x^t$ $(x \in H,\ t \in \mathbf{Z}/m\mathbf{Z})$. This action commutes with the conjugation action of H, so $H \times (\mathbf{Z}/m\mathbf{Z})^*$ acts, and H_{reg} is stable under this action. Let F_1 be the field obtained from F by adjoining to F all m-th roots of unity or, equivalently, all m'-th roots of unity, where m' is the part of m which is prime to $\mathrm{char}\, F$ $(= m$ if $\mathrm{char}\, F = 0)$. Then the extension F_1/F is Galois, and

$$\Gamma_F := \mathrm{Gal}(F_1/F) \subseteq (\mathbf{Z}/m'\mathbf{Z})^* \underset{\sim}{\subseteq} (\mathbf{Z}/m\mathbf{Z})^* .$$

Thus $H \times \Gamma_F$ acts on H.

LEMMA 4.3. (notation as above) The groups $G_0(F[H];\mathfrak{U})$ and $G_0(F[H])_{\mathfrak{U}}$ both have rank equal to the number of $(H \times \Gamma_F)$-conjugacy classes in $\bigcap_{h \in H} \bigcup_{U \in \mathfrak{U}} U^h_{reg}$. In particular, $G_0(F[H];\mathfrak{U})/G_0(F[H])_{\mathfrak{U}}$ is finite.

PROOF. First assume that $\operatorname{char} F = 0$. Using [Se 1, Cor. 1 on p. 110], we identify each $G(U) := F \otimes_{\mathbf{Z}} G_0(F[U])$, $U \in \mathfrak{U}$ or $U = H$, with the algebra of $(U \times \Gamma_F)$-invariant F-valued functions on U. We will show that

$$X := F \otimes_{\mathbf{Z}} G_0(F[H]) = Y := \{\varphi \in G(H) \mid \varphi \text{ vanishes on } H \setminus \bigcup_{U \in \mathfrak{U}} U\}$$

$$= \{\varphi \in G(H) \mid \varphi \text{ vanishes on } H \setminus \bigcap_{h \in H} \bigcup_{U \in \mathfrak{U}} U^h\}.$$

Since, clearly, $X \subseteq F \otimes_{\mathbf{Z}} G_0(F[H];\mathfrak{U}) \subseteq Y$, this will prove that $G_0(F[H])_{\mathfrak{U}}$ and $G_0(F[H];\mathfrak{U})$ both have rank equal to the number of $(H \times \Gamma_F)$-conjugacy classes in $\bigcap_{h \in H} \bigcup_{U \in \mathfrak{U}} U^h$, as required.
The usual scalar product $<.,.>_U$ of central functions on U satisfies Frobenius reciprocity, and the different characters of irreducible representations of U over F form an orthogonal basis of G(U) [Se 1, Théorème 13 on p. 73, Prop. 32 on p. 105, and Cor. 2 on p. 111]. Using this, we obtain

$$\begin{aligned} Y \cap X^{\perp} &= \{\varphi \in Y \mid <\varphi, \operatorname{Ind}_U^H \psi>_H = 0 \text{ for all } \psi \in G_0(F[U]),\ U \in \mathfrak{U}\} \\ &= \{\varphi \in Y \mid <\operatorname{Res}_U^H \varphi, \psi>_U = 0 \text{ for all } \psi \in G_0(F[U]),\ U \in \mathfrak{U}\} \\ &= \{\varphi \in Y \mid \operatorname{Res}_U^H \varphi = 0 \text{ for all } U \in \mathfrak{U}\} \\ &= (0) . \end{aligned}$$

Since $Y \supseteq X$, we conclude that $X \cap X^{\perp} = (0)$ and $G(H) = X + X^{\perp}$, whence $Y = X$, as we have claimed.

In the case when $\operatorname{char} F = p > 0$, let (K,A,F) be a p-modular system (i.e., A is a discrete valuation ring with residue field $F = A/\mathfrak{m}$ and field of fractions K) such that A is $\mathfrak{m}$-adically complete and $\operatorname{char} K = 0$. Consider the decomposition map $d : G_0(K[H]) \longrightarrow G_0(F[H])$. By [Se1, Remarque and Cor. 2 on p. 161], we have

$$G_0(F[H];\mathfrak{U}) = d(Z) \quad \text{with}$$

$$Z := \{\alpha \in G_0(K[H]) \mid \varphi_\alpha \text{ vanishes on } H_{reg} \setminus \bigcup_{U \in \mathfrak{U}} U\}$$

and

$$\operatorname{Ker} d = \{\alpha \in G_0(K[H]) \mid \varphi_\alpha \text{ vanishes on } H_{reg}\} .$$

Therefore, $G_0(K[H];\mathfrak{U}) + \operatorname{Ker} d \subseteq Z$. Again identifying $K \otimes_{\mathbf{Z}} G_0(K[H])$ with the algebra $G(H)$ of $(H \times \Gamma_K)$-invariant K-valued functions on H, we will show that

$$K \otimes_{\mathbf{Z}} (G_0(K[H])_{\mathfrak{U}} + \operatorname{Ker} d) = K \otimes_{\mathbf{Z}} Z$$

$$= I := \{\varphi \in G(H) \mid \varphi \text{ vanishes on } H_{reg} \setminus \bigcap_{h \in H} \bigcup_{U \in \mathfrak{U}} \dot{U}^h\} .$$

First note that, under the above identification of $K \otimes_{\mathbf{Z}} G_0(K[H])$ with $G(H)$, the map

$$\begin{array}{ccccc} K \otimes_{\mathbf{Z}} G_0(K[H]) & \xrightarrow{\mathrm{id}_K \otimes d} & K \otimes_{\mathbf{Z}} G_0(F[H]) & \longrightarrow & \left\{ \begin{array}{l} \overline{K}\text{-valued } H\text{-invariant} \\ \text{functions on } H_{reg} \end{array} \right\} \\ & & \psi & & \psi \\ & & \alpha & \longmapsto & \varphi_\alpha \end{array}$$

($\bar{K}$ = algebraic closure of K) becomes restriction of functions from H to H_{reg} (cf. [Se 1, p. 163]). Therefore, $K \underset{\mathbf{Z}}{\otimes} \operatorname{Ker} d$, the kernel of this map, corresponds to the ideal $J :=$ $\{\varphi \in G(H) \mid \varphi$ vanishes on $H_{reg}\}$. Moreover, as we have shown above, $K \underset{\mathbf{Z}}{\otimes} G_0(K[H])_{\mathfrak{U}}$ corresponds to the ideal $Y = \{\varphi \in G(H) \mid \varphi$ vanishes outside of $\bigcap_{h \in H} \bigcup_{U \in \mathfrak{U}} U^h\}$. Now, for any $\varphi \in I$, define φ' to be identical with φ outside of H_{reg} but $\varphi' = 0$ on H_{reg}. Then $\varphi' \in K \underset{\mathbf{Z}}{\otimes} \operatorname{Ker} d$ and $\varphi - \varphi' \in K \otimes G_0(K[H])_{\mathfrak{U}}$, because $\varphi - \varphi'$ vanishes outside of H_{reg} and on $H_{reg} \setminus \bigcap_{h \in H} \bigcup_{U \in \mathfrak{U}} U^h$, hence on $H \setminus \bigcap_{h \in H} \bigcup_{U \in \mathfrak{U}} U^h$. As the inclusions $K \underset{\mathbf{Z}}{\otimes} (G_0(K[H])_{\mathfrak{U}} + \operatorname{Ker} d)$ $\subseteq K \underset{\mathbf{Z}}{\otimes} Z \subseteq I$ are clear, this shows that equality holds throughout. In particular, $d(G_0(K[H])_{\mathfrak{U}}) = G_0(F[H])_{\mathfrak{U}}$ has finite index in $d(Z) = G_0(F[H];\mathfrak{U})$. Moreover,

$$\begin{aligned} \operatorname{rank} G_0(F[H];\mathfrak{U}) &= \dim_K I - \dim_K J \\ &= \#(H \setminus H_{reg} \,\dot{\cup}\, \bigcap_{h \in H} \bigcup_{U \in \mathfrak{U}} U^h_{reg})/H \times \Gamma_K - \#(H \setminus H_{reg})/H \times \Gamma_K \\ &= \#(\bigcap_{h \in H} \bigcup_{U \in \mathfrak{U}} U^h_{reg})/H \times \Gamma_K \,. \end{aligned}$$

It remains to replace Γ_K by Γ_F in this formula. For this, let m be a multiple of the exponent of H and let m' denote the p'-part of m, as in the paragraph preceding the statement of the lemma. Note that the action of $(\mathbf{Z}/m\mathbf{Z})^*$ on H_{reg} factors through $(\mathbf{Z}/m'\mathbf{Z})^*$. Let K_1, resp. K_1', denote the field obtained from K by adjoining the m-th, resp. m'-th, roots of unity, and similarly for F (so $F_1 = F_1'$). Then $\Gamma_K = \operatorname{Gal}(K_1/K) \subseteq (\mathbf{Z}/m\mathbf{Z})^*$ maps onto $\Gamma_K' := \operatorname{Gal}(K_1'/K) \subseteq (\mathbf{Z}/m'\mathbf{Z})^*$, and the action of Γ_K on

H_{reg} factors through Γ'_K. Finally, as subgroups of $(\mathbf{Z}/m'\mathbf{Z})^*$, Γ'_K and Γ_F coincide (cf. [Se 2, Prop. 16, p. 77]). This finishes the proof of the lemma.

∎

We now apply the foregoing to study the image of the map $\gamma = \gamma_{F[G],N}$ for N normal of finite index in G. The essence of the following proof is extracted from [Ga-Ro].

PROPOSITION 4.4. Let N be normal of finite index in G, with $\mathrm{char}\, F \nmid f(N)$. Then $G_0(F[G])_{\mathcal{F}} + \mathrm{Ker}\,\gamma$ has finite index in $G_0(F[G])$.

PROOF. We have to show that $\gamma(G_0(F[G])_{\mathcal{F}}) = G_0(F[\bar{G}])_{\bar{\mathcal{F}}}$ (Lemma 4.2ii) has finite index in $\gamma(G_0(F[G]))$. In view of Proposition 2.2i, this amounts to showing that the image of $\gamma(c\, K_0(F[G]))$ modulo $G_0(F[\bar{G}])_{\bar{\mathcal{F}}}$ is finite. By Lemma 4.3, it suffices to show that for all finitely generated projective F[G]-modules P, we have $\gamma([P]) \in G_0(F[\bar{G}];\bar{\mathcal{F}})$. For this, we recall some facts about Hattori-Stallings ranks (cf. [Ba 2]).

Let R be any commutative ring and let P denote a finitely generated projective R[G]-module. Then the Hattori-Stallings rank r_P of P is an R-valued function on G which is central (i.e., constant on G-conjugacy classes) and vanishes on all but finitely many conjugacy classes of G. More precisely, an important result due to Formanek, Farkas and Snider, and Cliff (cf. [Ga-Ro, Theorem 2.2]) asserts that $r_P(x) = 0$ for all $x \in G$ of infinite order. The rank $r_{\bar{P}}$ of the finitely generated projective $R[\bar{G}]$-module $\bar{P} = P/(\omega N)\cdot P$ is given by

$$r_{\bar{P}}(\bar{x}) = \sum_{\substack{c(y) \\ c(\bar{y})=c(\bar{x})}} r_P(y) \quad ,$$

where $c(.)$ denotes G-conjugacy classes [Ba 2, 5.4]. Now suppose that $x \in G$ satisfies $\bar{x} \notin \bar{U}$ for all $U \in \mathcal{F} = \mathcal{F}(G)$. Then each $y \in G$ with $c(\bar{y}) = c(\bar{x})$ has infinite order. Therefore, $r_P(y) = 0$ and so $r_{\bar{P}}(\bar{x}) = 0$. The rank $r_{\bar{P}}$ and the character $\chi_{\bar{P}}$ of $\bar{P}$ (i.e., the traces of the operators given by the action of G on the finitely generated projective R-module $\bar{P}$, cf. [Bou, p. 78]) are related by the formula

$$\chi_{\bar{P}}(\bar{x}) = |\mathbf{C}_{\bar{G}}(\bar{x})| \cdot r_{\bar{P}}(\bar{x}^{-1}) \qquad (\bar{x} \in \bar{G})$$

[Ba 2, 5.8]. Thus, if $x \in G$ satisfies $\bar{x} \notin \bar{U}$ for all $U \in \mathcal{F}$, then $\chi_{\bar{P}}(\bar{x}) = 0$.

In the case when $\operatorname{char} F = 0$ simply take $R = F$ in the above to conclude that $\gamma([P]) = [\bar{P}] \in G_0(F[G];\bar{\mathcal{F}})$ holds for all finitely generated projective $F[G]$-modules P, as required. Thus assume that $\operatorname{char} F = p > 0$ and let (K,A,F) be a p-modular system, with K complete of characteristic 0. Let $\mathfrak{m}$ denote the maximal ideal of A and, for each $n \geq 1$, set $A_n = A/\mathfrak{m}^n$. Since the kernel of $A_n[G] \to F[G]$ is nilpotent, there exists a unique (up to isomorphism) finitely generated projective $A_n[G]$-module P_n whose reduction modulo $\mathfrak{m}^n$ is P [Ba 1, p. 90]. By the result of the preceding paragraph of the proof, applied to $R = A_n$ and P_n, we know that the rank of $\bar{P}_n = P_n/(\omega N)\cdot P_n$ satisfies $r_{\bar{P}_n}(\bar{x}) = 0$ for all $\bar{x} \in \bar{G} \setminus \bigcup_{U \in \mathcal{F}} \bar{U}$. Let Q denote the unique (up to isomorphism) finitely generated projective $A[\bar{G}]$-module whose reduction modulo $\mathfrak{m}^n$ equals $\bar{P}_n$ for all $n \geq 1$ (cf. [Se, p. 133]). Then the rank $r_{\bar{P}_n}$ of $\bar{P}_n$ is the reduction of r_Q modulo $\mathfrak{m}^n$ ([Ba2, 2.9]). From the foregoing we conclude that, for all $\bar{x} \in \bar{G} \setminus \bigcup_{U \in \mathcal{F}} \bar{U}$, $r_Q(\bar{x}) \in \bigcap_n \mathfrak{m}^n = (0)$. Therefore, the character of Q also vanishes on $\bar{G} \setminus \bigcup_{U \in \mathcal{F}} \bar{U}$. As the restriction of this character to $\bar{G}_{reg}$ is the Brauer character of $\bar{P}$, we have again shown that $\gamma([P]) = [P] \in G_0(F[\bar{G}];\bar{\mathcal{F}})$. This completes the proof of the proposition.

■

For finitely generated abelian-by-finite groups G, Proposition 4.4 above is an immediate consequence of the main result of [Br-H-Lo] which asserts that $G_0(F[G])/G_0(F[G])_{\mathcal{F}}$ is periodic. Indeed, it is not hard to show directly, using the Artin induction theorem, that if G has a finitely generated free abelian normal subgroup A of finite index a, then

$$a \cdot G_0(F[G]) \subseteq G_0(F[G])_{\mathcal{F}} + \operatorname{Ker} \gamma_{F[G],A} .$$

Proposition 4.4 can be used to derive an upper bound for the Goldie rank $\rho(F[G])$ entirely in terms of finite images of G. For this, let H denote any finite group and

$\mathfrak{U}$ a family of subgroups of H . We put

$$|\mathfrak{U}| := \ell.\text{c.m.}\left\{ |U| \;\middle|\; U \in \mathfrak{U} \right\} \quad ,$$

$$T(\mathfrak{U}) := \left\{ \alpha \in G_0(F[H]) \;\middle|\; n\,\alpha \in G_0(F[H])_{\mathfrak{U}} \quad \text{for some} \quad n \right\} ,$$

the isolator of $G_0(F[H])_{\mathfrak{U}}$ in $G_0(F[H])$, and

$$t(\mathfrak{U}) := \#\ \operatorname{Dim} T(\mathfrak{U}) \,/\, \operatorname{Dim} G_0(F[H])_{\mathfrak{U}} \quad ,$$

where $\operatorname{Dim} : G_0(F[H]) \longrightarrow \mathbf{Z}$ sends $[V]$ to $\dim_F V$ for any $F[H]$-module V . Note that $\operatorname{Dim} G_0(F[H])_{\mathfrak{U}} = |H| \cdot |\mathfrak{U}|^{-1} \cdot \mathbf{Z}$.

<u>COROLLARY 4.5</u> (notation as above). Assume that $F[G]$ is prime and that N is torsion-free and has finite index in G . Let $\mathfrak{U}$ be any family of subgroups of $\bar{G} = G/N$ such that for all $U \in \mathcal{F}(G)$ there exists $U_1 \in \mathfrak{U}$ with $\bar{U} \subseteq U_1$. Then

$$\rho(F[G]) \;\Big|\; |\mathfrak{U}| \cdot t(\mathfrak{U}) \quad .$$

<u>PROOF</u>. Setting $\gamma = \gamma_{F[G],N}$ we have, by Proposition 4.4 and our assumption on $\mathfrak{U}$, $\gamma(G_0(F[G])) \subseteq T(\bar{\mathcal{F}}) \subseteq T(\mathfrak{U})$. Thus, by definition of $t(\mathfrak{U})$,

$$t(\mathfrak{U}) \cdot \operatorname{Dim} \circ \gamma(G_0(F[G])) \subseteq \operatorname{Dim} G_0(F[\bar{G}])_{\mathfrak{U}} = |\bar{G}| \cdot |\mathfrak{U}|^{-1} \cdot \mathbf{Z} \quad .$$

Finally, Lemma 4.1 implies that, for any $\alpha \in G_0(F[G])$, $\mathrm{Dim}(\gamma(\alpha)) = |\bar{G}| \cdot \chi_{F[G]}(\alpha)$. So we get

$$|\mathfrak{U}| \cdot t(\mathfrak{U}) \cdot \chi_{F[G]}(\alpha) \in \mathbb{Z} ,$$

which proves the corollary.

■

Note that, in the situation of the corollary, we always have $f(G) = |\bar{\mathcal{F}}| \,\Big|\, |\mathfrak{U}|$. The troublesome part of the above formula is the factor $t(\mathfrak{U})$ whose explicit determination appears far from trivial in general. Clearly, $t(\mathfrak{U})$ divides the exponent of $T(\mathfrak{U})/G_0(F[\bar{G}])_{\mathfrak{U}}$, and our next lemma gives a bound for the latter in the very special case where $\mathfrak{U}$ consists of a single normal subgroup. So let H be a finite group and let B be a normal subgroup of H . For any simple $F[B]$-module V , let $I_H(V)$ denote its inertia group in H and put

$$s(V) := \min\left\{\frac{\dim W}{\dim V} \,\middle|\, W \text{ a simple } F[I_H(V)]\text{-module with } W|_B \supseteq V\right\}$$

(cf. Remark 4.7 below).

<u>LEMMA 4.6</u> (notation as above). Assume that F is a splitting field for B with $\mathrm{char}\, F \nmid H$. Then the exponent of $T(B)/G_0(F[H])_B$ divides $\ell.c.m.\{s(V) \mid V \text{ a simple } F[B]\text{-module}\}$.

PROOF. Fix a complete set $\{V_j \mid j \in J\}$ of pairwise non-isomorphic and not H-conjugate simple F[B]-modules and let $\alpha \in G_0(F[H])$ be torsion modulo $G_0(F[H])_B$. Then $G_0(F[H])_B = \sum_{j\in J} \mathbf{Z} \cdot \mathrm{Ind}_B^H[V_j]$ and so $r\alpha = \sum_{j\in J} z_j \cdot \mathrm{Ind}_B^H[V_j]$ for suitable $r, z_j \in \mathbf{Z}$. For each j, choose a simple $F[I_H(V_j)]$-module W_j with $\dim W_j = s(V_j) \cdot \dim V_j$ and $W_j|_B \supseteq V_j$, and put $S_j = \mathrm{Ind}_{I_H(V_j)}^H W_j$. Then $\mathrm{Ind}_B^H V_j$ maps onto S_j, and

$$S_j|_B \cong \bigotimes_{x\in X_j} \left(V_j^{s(V_j)} \right)^{(x)} ,$$

where X_j is a transversal for $I_H(V_j)$ in H and $(.)^{(x)}$ denotes conjugate modules. Therefore, writing $<.,.>_H = \dim_F \mathrm{Hom}_{F[H]}(.,.)$ as usual, we obtain

$$\begin{aligned} r \cdot <\alpha, S_j>_H &= \sum_{c\in J} z_c < \mathrm{Ind}_B^H V_c, S_j >_H \\ &= \sum_{c\in J} z_c < V_c, S_j|_B >_B \\ &= \sum_{x\in X_j} \sum_{c\in J} z_c \cdot s(V_j) \cdot <V_c, V_j^{(x)} >_B \\ &= z_j \cdot s(V_j) \quad . \end{aligned}$$

So r divides $z_j \cdot s$, where $s = \ell.c.m. \{s(V_j) \mid j \in J\}$, and we conclude that $g.c.d.(r,s) \cdot \alpha \in G_0(F[H])_B$, as it was to be shown.

■

<u>REMARKS 4.7.</u> We now comment on the numbers $s(V)$. Let $H, B,$ and F be as in Lemma 4.6 and let V be a simple $F[B]$-module, with corresponding representation $\varphi : B \longrightarrow GL(V_F)$. Put

$$H(V) = I_H(V) / B \quad .$$

Then V yields a 2-cocycle ω_V of $H(V)$ with values in F^* whose class in $H^2(H(V), F^*)$ is uniquely determined by V . Explicitely, let $X = X(V) = \{x_h \mid h \in H(V)\}$ be a fixed transversal for B in $I_H(V)$ and let $M_x (x \in X)$ be fixed matrices in $GL(V_F)$ (unique up to scalars $\in F$) such that, for all $b \in B$,

$$\varphi(xbx^{-1}) = M_x \varphi(b) M_x^{-1} \quad .$$

Then ω_V is given by the scalar matrices

$$\omega_V(h_1, h_2) = \varphi\left(x_{h_1h_2} x_{h_2}^{-1} x_{h_1}^{-1}\right) \cdot M_{x_{h_1}} M_{x_{h_2}} M_{x_{h_1h_2}}^{-1}$$

for $h_1, h_2 \in H(V)$. The number $s(V)$ is the smallest dimension of a (nonzero) module for the twisted group algebra $R_V = F^{\omega_V}[H(V)]$ (cf. [C-R, Theorem 11.20]). In particular, <u>the order of the class of</u> ω_V <u>in</u> $H^2(H(V), F^*)$ <u>divides</u> $s(V)$.

In the special case where <u>$H(V)$ is abelian and F is algebraically closed</u>, $s(V)$ can be determinded as follows. Put

$$H_V = \left\{ h \in H(V) \mid \omega_V(h,k) = \omega_V(k,h) \quad \text{for all} \quad k \in H(V) \right\} .$$

It is easily checked that H_V is a subgroup of $H(V)$ which only depends on the class of ω_V in $H^2(H(V),F^*)$. Moreover, the center Z_V of R_V is the subalgebra generated by H_V , Z_V is isomorphic to the group algebra $F[H_V]$, and R_V is free of rank $|H(V)/H_V|$ over Z_V . Finally, it is not hard to show that all two-sided ideals of R_V are generated by their intersections with Z_V . Consequently, the dimension of all simple R_V-modules equals $|H(V)/H_V|^{\frac{1}{2}}$, whence

$$s(V) = |H(V)/H_V|^{\frac{1}{2}} .$$

In particular, if $H(V)$ is cyclic, then $s(V) = 1$ ([Sri]).

Of course, if B is a direct factor of H , or if H is abelian and F is a splitting field for H , then $s(V) = 1$ holds for all simple $F[B]$-modules V . R. Knörr has pointed out to me that if $s(V) = 1$ holds for all one-dimensional $F[B]$-modules V , then all subgroups D of H with $B \subseteq D \subseteq H$ satisfy $[D,B] = [D,D] \cap B$.

THEOREM 4.8. Assume that $F[G]$ is prime and that $\operatorname{char} F \nmid f(G)$. Let N be a torsion-free normal subgroup of finite index in G. For each prime ℓ, let G_ℓ denote the preimage in G of a Sylow ℓ-subgroup of $\bar{G} = G/N$, and let B_ℓ be a normal subgroup of $\bar{G}_\ell = G_\ell/N$ containing the images of all finite subgroups of G_ℓ. Then:

i. $\rho(F[G])_\ell \,\Big|\, |B_\ell| \cdot \ell.\mathrm{c.m.}\{s(V) \mid V$ a simple $E[B_\ell]$-module$\}$, where E is any splitting field for B_ℓ with $E \supseteq F$.

ii. If $\bar{G}_\ell$ is abelian, or $\bar{G}_\ell/B_\ell$ is cyclic, or B_ℓ is a direct factor of $\bar{G}_\ell$, then $\rho(F[G])_\ell \,\Big|\, |B_\ell|$.

iii. (Gabber-Rosset [Ga-Ro]) If $\bar{G}_\ell$ is cyclic, then $\rho(F[G])_\ell = f(G)_\ell$.

PROOF. Since $\rho(F[G])$ divides $\rho(E[G])$ for any field extension E/F (by [W, Theorem 3], for example), we may assume that F is large enough. Moreover, using Corollary 1.3 ii, we reduce to the case where $G = G_\ell$ for some ℓ and $\operatorname{char} F \nmid |\bar{G}|$. Part (i) is now immediate from Corollary 4.5 and Lemma 4.6. In view of our preceding remarks, (ii) is a special case of (i). Finally, if $\bar{G}_\ell$ is cyclic, then its subgroups are linearly ordered by inclusion, and we can take B_ℓ to be the largest image of a finite subgroup of G_ℓ. Then $|B_\ell| = f(G_\ell) = f(G)_\ell$, and so (iii) follows from (ii).

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