CHERN CLASSES OF BIRATIONAL VARIETIES

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Abstract. Let $\varphi : V \dashrightarrow W$ be a birational map between smooth algebraic varieties which does not change the canonical class (in the sense of Batyrev, [Bat99]). We prove that the total homology Chern classes of V and W are push-forwards of the same class from a resolution of indeterminacies of φ.

For example, it follows that the push-forward of the total Chern class of a crepant resolution of a singular variety is independent of the resolution.

1. Introduction and statement of the result

There is a strong motivic feel to the theory of Chern-Schwartz-MacPherson classes, although this does not seem to have yet congealed into a precise statement in the literature. In this note we give an instance of a result exploiting this heuristic observation. Our statement is an analog of Victor Batyrev's well-known theorem showing that the Betti numbers of birational manifolds 'in the same K-class' are equal, see [Bat99]. Batyrev's theorem has been generalized to many other numerical invariants, such as the Hodge numbers or certain Chern numbers. Our statement is not numerical; it refers to the total Chern class of the tangent bundle, a class in the Chow group of the variety.

Theorem 1.1. *Let $\varphi : V \dashrightarrow W$ be a birational morphism of nonsingular algebraic varieties over an algebraically closed field of characteristic 0. Assume that there is a resolution of indeterminacies of φ, Z:*

$$\begin{tikzcd} & Z \arrow[dl, "v"'] \arrow[dr, "w"] & \\ V \arrow[rr, dashed] & & W \end{tikzcd}$$

such that v, w are proper and birational, and the Jacobian ideals of v and w coincide.

Then there exists a class $C \in (A_*Z)_{\mathbb{Q}}$ such that

$$c(TV) \cap [V] = v_*(C) \quad \textit{and} \quad c(TW) \cap [W] = w_*(C)$$

in $(A_*V)_{\mathbb{Q}}$, $(A_*W)_{\mathbb{Q}}$ respectively.

The condition on the Jacobians implies that the pull-backs v^*K_V and w^*K_W of the respective canonical classes coincide. In fact, as Lawrence Ein kindly pointed out to me, the condition *is equivalent* to requiring that $v^*K_V \equiv_{\mathrm{Num}} w^*K_W$, an easy consequence of the 'Kodaira lemma' (see for example 2.19 in [Kol92]).

Theorem 1.1 has a number of immediate consequences: for instance the equality of Euler characteristics of V and W, or more generally the equality of Chern numbers

$$c_1(V)^i \cdot c_{n-i}(V) = c_1(W)^i \cdot c_{n-i}(W)$$

for all $i \geq 0$ (where $n = \dim V = \dim W$) in the stated hypotheses, when V and W are complete. Identities of this type are not new; for example, Anatoly Libgober and John Wood have proved ([LW90]) that $c_1 \cdot c_{n-1}$ is determined by the Hodge numbers, and these agree for V and W, as mentioned above. Our argument is very direct, and yields the equality for all i; suprisingly, this simple equality does not seem to be explicitly in the literature for $i \geq 2$, although it is certainly implied by recent and powerful results on elliptic genera ([BL03], [Wan03]). Also:

Corollary 1.2. *Let $\alpha : Y \to X$ be a crepant resolution. Then the class*

$$\alpha_*(c(TY) \cap [Y])$$

in $(A_*X)_{\mathbb{Q}}$ is independent of Y.

Again, this is immediate from Theorem 1.1; it also follows from the more sophisticated technology in Lev Borisov and Libgober's work. It would be interesting to extend this notion of 'Chern class' to arbitrary singular varieties, and to compare it to other known intrinsic definitions such as the Chern-Fulton class ([Ful84], Example 4.2.6), or the Chern-Schwartz-MacPherson class ([Mac74]).

Similarly to analogous numerical statements, Theorem 1.1 is an easy consequence of a basic formula in motivic integration, to wit Proposition 6.3.2 in [DL99]. This formula is usually obtained as a corollary of Kontsevich's change of variable formula; we refer the reader to loc. cit. or to the excellent surveys [DL01], [Lo02] for a discussion in the context of motivic integration.

In order to provide a self-contained treatment of the material, we take the liberty of providing an alternative proof of this 'motivic' statement, using the factorization theorem of [AKMW02]; our argument has the (very small) advantage of proving the needed formula without requiring completions. This is done in §2. The idea of using the factorization theorem as an alternative to motivic integration is of course not new; for example, Willem Veys uses it to prove a similar (in fact, stronger) statement, in §2 of [Wey03]. Also, the factorization theorem is a key tool in the work of Borisov and Libgober on elliptic genera. In fact, Borisov has pointed out to me that Theorem 1.1 can also be derived from Theorem 3.5 in [BL], and that this work may be used to extend the class given in Corollary 1.2 to all varieties with log-terminal singularities.

Regardless of how the basic formula is established, however, our main observation is that this formula alone—and in fact even just the Euler characteristic version of the basic formula, which goes back to [DL92]—implies an analog at the level of Chern-Schwartz-MacPherson classes, and that this immediately implies Theorem 1.1. The upgrade of the basic formula to Chern-Schwartz-MacPherson classes is Theorem 3.1; Theorem 1.1 follows easily from this result, as shown in §3.

Theorem 3.1 must thus have been known to the experts very early (and François Loeser confirms this); the fact that it has nice applications, such as the invariance of $c_1^i \cdot c_{n-i}$ under K-equivalence, seems to have been inexplicably overlooked thus far. It may be argued that Theorem 1.1, and hence these straightforward consequences, should have been immediately noticed after the diffusion of [DL92]; in particular, long before the discovery of the factorization theorem of [AKMW02], which has fueled the more recent and sophisticated work in this direction.

In any case, we believe that Theorem 3.1 is of independent interest in the context of Chern-Schwartz-MacPherson classes, and that it deserves to be out in the open.

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2. The basic formula

We work over an algebraically closed field k. We will assume k has characteristic 0, as this is at present needed for resolution of singularities (needed in the factorization theorem of [AKMW02]) and the theory of Chern-Schwartz-MacPherson classes used in §3.

If X is a variety, $[X]$ will denote its class in the Grothendieck ring of algebraic varieties over k; that is, the abelian group generated by isomorphism classes of varieties modulo the relation $[X-Y]=[X]-[Y]$ whenever Y is a closed subvariety of X, with multiplication defined by product over k. The unit element is $1=[\operatorname{Spec} k]$. It is costumary to denote by $\mathbb{L}$ the 'Tate motive' $[\mathbb{A}^1]$.

The basic formula is a relation in this ring, depending on the datum of a 'modification' of nonsingular varieties. It is in fact convenient to localize the ring at elements $[\mathbb{P}^\mu]=\frac{\mathbb{L}^{\mu+1}-1}{\mathbb{L}-1}$, and this will be done without further mention in what follows.

Notation: if $\{D_j\}_{j\in J}$ is a set of irreducible divisors in a variety and $I\subset J$, then D_I° will denote the complement of $\cup_{j\in J\setminus I}D_j$ in $\cap_{i\in I}D_i$.

Let $v: Z\to V$ be a proper birational morphism of nonsingular varieties, such that the exceptional divisor of v has normal crossings, with nonsingular irreducible components E_j, $j\in J$.

Assume that the Jacobian ideal of v is principal, with divisor $\sum_j \mu_j E_j$, so that $v^*K_V=K_Z+\sum_j\mu_jE_j$. The following formula is (essentially) Proposition 6.3.2 in [DL99].

Theorem 2.1. *Let U be a closed subvariety of V. Then*

$$[U]=\sum_{I\subset J}\frac{[E_I^\circ\cap v^{-1}(U)]}{\prod_{i\in I}[\mathbb{P}^{\mu_i}]}\quad .$$

As mentioned in §1, we provide a motivic-integration-free proof of Theorem 2.1 in this section, which the hurried reader may safely skip.

The factorization theorem of [AKMW02] will reduce the proof of Theorem 2.1 to the case in which v is a blow-up at a smooth center; this case will be dealt with by analyzing the situation in the exceptional divisor, and this in turn will be reduced to the analysis of a single fiber, a projective space. The real reason behind Theorem 2.1 can then be traced to the following simple lemma, whose statement requires no localization.

Lemma 2.2. *Let $d\geq k>0$ and $\mu_i\geq 0$, $i=1,\dots,k$, be integers, and let $K=\{1,\dots,k\}$. Let $H_1,\dots,H_k$ be linearly independent hyperplanes in $\mathbb{P}^{d-1}$. Then*

$$\sum_{I\subset K}\left([H_I^\circ]\prod_{i\in K\setminus I}[\mathbb{P}^{\mu_i}]\right)=[\mathbb{P}^{\sum_{j\in K}\mu_j+d-1}]\quad .$$

Proof. With evident coordinates, the subset H_I° consists of homogeneous d-tuples in which $|I|$ fixed components are 0 and $k-|I|$ fixed components are nonzero. It follows that

$$[H_I^\circ] = \begin{cases} \dfrac{(\mathbb{L}-1)^{k-|I|}\mathbb{L}^{d-k}}{\mathbb{L}-1} & |I| \neq k \\ \dfrac{\mathbb{L}^{d-k}-1}{\mathbb{L}-1} & |I| = k \end{cases} \quad ,$$

giving

$$[H_I^\circ] \prod_{i\in K\setminus I} [\mathbb{P}^{\mu_i}] = \begin{cases} \dfrac{\mathbb{L}^{d-k}}{\mathbb{L}-1} \displaystyle\prod_{i\in K\setminus I} (\mathbb{L}^{\mu_i+1}-1) & I \neq K \\ \dfrac{\mathbb{L}^{d-k}-1}{\mathbb{L}-1} - \dfrac{1}{\mathbb{L}-1} & I = K \end{cases} \quad .$$

Hence the left-hand-side in the stated equality is

$$\frac{\mathbb{L}^{d-k}}{\mathbb{L}-1} \sum_{I\subset K} \prod_{i\in K\setminus I} (\mathbb{L}^{(\mu_i+1)}-1) - \frac{1}{\mathbb{L}-1} = \frac{\mathbb{L}^{d-k}}{\mathbb{L}-1} \prod_{j\in K} (1 + (\mathbb{L}^{\mu_j+1}-1)) - \frac{1}{\mathbb{L}-1}$$

which immediately yields the right-hand-side. □

The form in which Lemma 2.2 will be used is the following.

Corollary 2.3. *With the same notation as in Lemma 2.2, set $\mu_0 = \sum_{j\in K} \mu_j + d - 1$; then*

$$\sum_{I\subset K} \frac{[H_I^\circ]}{\prod_{i\in(\{0\}\cup I)} [\mathbb{P}^{\mu_i}]} = \frac{1}{\prod_{j\in K} [\mathbb{P}^{\mu_j}]} \quad .$$

This formula will be used to control the situation across blow-ups. Let X be a nonsingular algebraic variety, and assume given a set $\{E_j\}_{j\in J}$ of irreducible nonsingular divisors in X. Let $S \subset X$ be a nonsingular subvariety of codimension $d \geq 1$, intersecting $\cup E_j$ with normal crossings: that is, at each point $s \in S$ there is an analytic system of parameters $x_1, \dots, x_n$ for X at s such that S is given by $x_1 = \cdots = x_d = 0$, and $\cup E_j$ is given by a monomial in the x_i's.

Let $\pi : Y \to X$ be the blow-up of X along S; denote by $\widetilde{E}_0$ the exceptional divisor $\pi^{-1}(S)$, and let $\widetilde{E}_j$ be the proper transform of E_j. The following lemma is a straightforward computation, which we leave to the reader.

Lemma 2.4. *—The divisor $\widetilde{E}_0 \cup (\cup_{j\in J} \widetilde{E}_j)$ has normal crossings at all points of $\widetilde{E}_0$;*

—If E_j does not contain S and $s \in S \cap E_j$, then $\widetilde{E}_j$ contains the fiber $\pi^{-1}(s)$;

—Let K denotes the set of indices $j \in J$ such that E_j contains S. Then for all $s \in S$ and $j \in K$, the intersections of $\widetilde{E}_j$ with $\pi^{-1}(s) \cong \mathbb{P}^{d-1}$ consist of linearly independent hyperplanes in $\mathbb{P}^{d-1}$.

Now assign to each E_j and $\widetilde{E}_j$, $j \in J$ a nonnegative integer μ_j, and assign to the exceptional divisor $\widetilde{E}_0$ the combination

$$\mu_0 := \sum_{E_j \supset Z} \mu_j + d - 1 \quad .$$

Note that, with this choice,

$$(\sum_{j\in(\{0\}\cup J)} \mu_j \widetilde{E}_j) - \pi^*(\sum_{j\in J} \mu_j E_j)$$

is the divisor of the Jacobian ideal of π.

For any algebraic subset $T \subset X$, let

$$\chi_X(T) := \sum_{I \subset J} \frac{[E_I^\circ \cap T]}{\prod_{i\in I}[\mathbb{P}^{\mu_i}]}$$

and

$$\chi_Y(\pi^{-1}(T)) := \sum_{I \subset (\{0\}\cup J)} \frac{[\widetilde{E}_I^\circ \cap \pi^{-1}(T)]}{\prod_{i\in I}[\mathbb{P}^{\mu_i}]} \quad .$$

Proposition 2.5. *With notation as above, $\chi_Y(\pi^{-1}(T)) = \chi_X(T)$.*

Proof. The contributions to $\chi_X(T)$ coming from the complement of S equal the contributions to $\chi_Y(\pi^{-1}(T))$ coming from the complement of $\widetilde{E}_0$. Thus it suffices to prove that

$$\sum_{I \subset J} \frac{[E_I^\circ \cap T \cap S]}{\prod_{i\in I}[\mathbb{P}_i^\mu]}$$

equals the contribution to $\chi_Y(\pi^{-1}(T))$ supported within $\widetilde{E}_0$, which consists of

$$\sum_{I \subset J} \frac{[\widetilde{E}_{\{0\}\cup I}^\circ \cap \pi^{-1}(T)]}{[\mathbb{P}^{\mu_0}]\prod_{i\in I}[\mathbb{P}_i^\mu]} \quad .$$

Now every point of $T\cap S$ belongs to exactly one E_K°. For all $K \subset J$, $\widetilde{E}_{\{0\}\cup K}^\circ \cap \pi^{-1}(T)$ fibers onto $E_K^\circ \cap T \cap S$ with fibers equal to the union over all $I \subset K$ of

$$\widetilde{E}_I^\circ \cap \pi^{-1}(x) \cong \widetilde{E}_I^\circ \cap \mathbb{P}^{d-1}$$

for $x \in E_K^\circ \cap T \cap S$ (identifying the fiber of $\widetilde{E}_0$ with $\mathbb{P}^{d-1}$). By the multiplicativity of products in the Grothendieck ring, it suffices to show that for all $K \subset J$

$$\sum_{I \subset K} \frac{[\widetilde{E}_I^\circ \cap \pi^{-1}(x)]}{[\mathbb{P}^{\mu_0}]\prod_{i\in I}[\mathbb{P}^{\mu_i}]} = \frac{1}{\prod_{j\in K}[\mathbb{P}^{\mu_j}]} \quad ,$$

for all $x \in T \cap S \cap E_K^\circ$. By Lemma 2.4, indices in K corresponding to divisor which do not contain S only contribute common multiplicative factors to the denominators in both sides of this equality; so we may assume that, for all j in K, $S \subset E_j$.

Again by Lemma 2.4, the intersections $\pi^{-1}(x) \cap \widetilde{E}_j$ consist of linearly independent hyperplanes in $\pi^{-1}(x) \cong \mathbb{P}^{d-1}$; with the given μ_0, the sought formula is then precisely the statement of Corollary 2.3. $\square$

We are now ready to prove Theorem 2.1.

Proof. By the factorization theorem of [AKMW02], v may be decomposed as a sequence:

$$Z = V_0 \dashrightarrow V_1 \dashrightarrow V_2 \dashrightarrow \cdots \dashrightarrow V_M = V$$

where each map is either a blow-up or a blow-down at a smooth center. More precisely, for each i one has one of the possibilities

$$\begin{tikzcd} V_i \arrow[rr] \arrow[dr] & & V_{i+1} \arrow[dl] \\ & V & \end{tikzcd} \qquad \begin{tikzcd} V_{i+1} \arrow[rr] \arrow[dr] & & V_i \arrow[dl] \\ & V & \end{tikzcd}$$

where the diagonal (regular!) maps are obtained by composing the rational maps to V; and the horizontal map is a blow-up of a nonsingular variety at a nonsingular center. This center may be chosen so that it meets with normal crossings the exceptional divisor of the map to V (cf. [AKMW02], expecially the 'Furthermore' section of the main statement, and part 6 of Theorem 0.3.1).

Therefore, at each stage of the factorization we are in the hypotheses of Proposition 2.5, with the normal crossing divisor on the base of the blow-up equal to the Jacobian of the map to V; this is compatible with the prescribed multiplicities μ_j, since the composition of the differentials is the differential of the composition. By pasting together the equalities thereby obtained at each stage, we get that $\chi_Z(\pi^{-1}(U)) = \chi_V(U) = [U]$; that is, the stated equality. □

3. Proof of the theorem

A *constructible function* on a variety Z is a finite $\mathbb{Z}$-linear combination of characteristic functions of subvarieties: $\sum_{S \subset Z} m_S 1\!\!1_S$ with $m_S \in \mathbb{Z}$, and $1\!\!1_S(s) = 1$ for $s \in S$, $1\!\!1_S(s) = 0$ otherwise.

Denote by $F(Z)$ the group of constructible functions on the variety Z. This assignment is in fact a covariant functor under proper morphisms, as follows: if $v : Z \to V$ is a proper morphism, and S is a subvariety (or more generally a constructible set) in Z, define the push-forward of the function $1\!\!1_S$ by

$$\forall p \in V \quad , \quad v_*(1\!\!1_S)(p) = \chi(v^{-1}(p) \cap S) \quad .$$

Here χ denotes the topological Euler characteristic if the ground field is $\mathbb{C}$; this definition may be extended to arbitrary algebraically closed fields of characteristic 0, see [Ken90].

In [Mac74], Robert MacPherson proved (over $\mathbb{C}$) that there exists a natural transformation c from F to homology, such that if V is a *nonsingular* variety, then $c(1\!\!1_V)$ equals the total homology Chern class of the tangent bundle of V, $c(TV) \cap [V]$. MacPherson's natural transformation can in fact be lifted to the Chow group and over any algebraically closed field of characteristic 0, see Example 19.1.7 in [Ful84] and [Ken90].

MacPherson's natural transformation may be used to define a notion extending to possibly singular varieties the total Chern class of the tangent bundle. This notion is commonly named 'Chern-Schwartz-MacPherson class', since it can be shown that the class so obtained agrees with a notion previously defined by Marie-Hélène Schwartz, cf. [BS81].

Analogously, we may define a 'Chern class' of any constructible set S in a variety Z, *as an element of* A_*Z, by applying MacPherson's transformation to its characteristic function: that is, we set

$$c_{\mathrm{SM}}(S) := c(1\!\!1_S) \in A_*Z \quad .$$

Now consider the situation of §2: $v : Z \to V$ is a proper birational morphism of nonsingular varieties; the exceptional divisor has normal crossings, with nonsingular components E_j, and the Jacobian ideal of v is principal, with divisor $\sum_j \mu_j E_j$.

Theorem 3.1. *Let U be a constructible subset of V. Then*

$$c_{\mathrm{SM}}(U) = v_* \sum_{I \subset J} \frac{c_{\mathrm{SM}}(E_I^\circ \cap v^{-1}(U))}{\prod_{i\in I}(\mu_i + 1)}$$

in $(A_*V)_{\mathbb{Q}}$.

Proof. For all $p \in V$,

$$\sum_{I\subset J} \frac{[E_I^\circ \cap v^{-1}(p)]}{\prod_{i\in I}[\mathbb{P}^{\mu_i}]} = [p]$$

by Theorem 2.1. Taking Euler characteristics:

$$\sum_{I\subset J} \frac{\chi(E_I^\circ \cap v^{-1}(p))}{\prod_{i\in I}(\mu_i+1)} = 1 \quad .$$

Thus,

$$\sum_{I\subset J} \frac{\chi(E_I^\circ \cap v^{-1}(U) \cap v^{-1}(p))}{\prod_{i\in I}(\mu_i+1)} = \begin{cases} 0 & p \notin U \\ 1 & p \in U \end{cases} \quad .$$

by definition of push-forward of constructible functions, this tells us that

$$v_* \sum_{I\subset J} \frac{1\!\!1_{E_I^\circ \cap v^{-1}(U)}}{\prod_{i\in I}(\mu_i+1)} = 1\!\!1_U$$

in $F(V)\otimes\mathbb{Q}$. Applying MacPherson's natural transformation gives the statement. □

Theorem 1.1 follows immediately. Indeed, if V and W are both dominated birationally by Z, and $v : Z \to V$, $w : Z \to W$ have the same Jacobian, we may assume (after resolution of singularities) that Z is nonsingular, with normal crossing exceptional divisor $\cup E_j$, and both v and w have Jacobian with divisor $\sum \mu_j E_j$. Let

$$C = \sum_{I\subset J} \frac{c_{\mathrm{SM}}(E_I^\circ)}{\prod_{i\in I}(\mu_i+1)} \in (A_*Z)_{\mathbb{Q}} \quad ;$$

then $c_{\mathrm{SM}}(V) = v_*(C)$, $c_{\mathrm{SM}}(W) = w_*(C)$ by Theorem 3.1, and these classes agree with $c(TV)\cap[V]$, $c(TW)\cap[W]$ respectively, by the normalization property of Chern-Schwartz-MacPherson's classes.

Theorem 3.1 appears to be of independent interest. For example, let U be a singular variety embedded in a nonsingular variety V, and let Z be the variety obtained from V as U undergoes a resolution of singularities à la Hironaka. Then Theorem 3.1 gives an explicit expression for the Chern-Schwartz-MacPherson class of U, with rational coefficients, in terms of classes of loci in its inverse image in Z (note that in this case $E_I^\circ \subset v^{-1}(U)$ for $I \neq \emptyset$). To our knowledge, this expression is new.

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