# Local Structure Of Quasiperiodic Tilings Having 8-Fold Symmetry


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# LOCAL STRUCTURE OF QUASIPERIODIC TILINGS HAVING 8-FOLD SYMMETRY

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**Abstract**

Local rules, enforcing quasiperiodicity, are introduced for the class of planar tilings having 8-fold symmetry. The local structure, when decorations are not involved, is studied.

## Introduction

In the first section we present a local rule, or matching rule, for the class of planar tilings having 8-fold symmetry. Without decorations this class of tilings does not admit even weak local rule in the sense of Levitov (cf.[B], [dB2], [Le2]. But it admits another type of local rule described below. This local rule is similar to that of Penrose-de Bruijn [dB]. In fact the local rule presented here is equivalent to the matching rule suggested by Ammann [ABS]. Here we show how to get it. This paper is a concrete realization of the general case presented in [LPS1]. However, this case has many specific properties and it is worthy to treat separately. We will try to keep the paper self-contained and make the method simpler and more understandable.

In §1.3 we define the class of quasiperiodic tilings having 8-fold symmetry. Then we try to find local rule such that every tiling satisfying this local rule must belong to this class. Without decorations on the tiles one can never succeed. In section 2 we characterize the set of all tilings having the same local structure *up to a fixed radius* as all the quasiperiodic tilings with 8-fold symmetry have. We prove that in some sense they are close to quasiperiodic tilings having 8-fold symmetry.

# 1 Local rules with decorations

## 1.1 Description of local rule

On the Euclidean space $\mathbb{R}^2$ consider a fixed regular octagon with vertices marked by $0, 1, \ldots, 7$; two rhombs and a square colored by numbers as indicated in fig. 1a. Here a *colored polygon* is just a pair $(P, j)$ where $P$ is a polygon and $j$ is a number from $\{0, 1, 2, 3, 4, 5, 6, 7\}$, called the color of this polygon. The colored square will be denoted by $(S, 1)$ and two colored rhombs by $(R, 1)$ and $(R, 4)$. Two colored polygons $(P, j)$ and $(P', j')$ are congruent if $P'$ is a translate of $P$ and $j = j'$. In fig.1a all the sides of the square and two rhombs have the same length.

Consider the group $D_8$ of all symmetries of the octagon. Each $\varphi \in D_8$ is a permutation of numbers $0, 1, \ldots, 7$. In addition, if $P$ is a polygon in the plane then $\varphi(P)$ is a polygon. For a colored polygon $(P, j)$ let $\varphi(P, j)$ be the colored polygon $(\varphi(P), \varphi(j))$. Note that here

$\varphi$ also acts on colors. For example, from $(R,1),(R,4),(S,1)$ by actions of $D_8$ we can get 24 non-congruent colored polygons. (If we don't take into account the colors then there are only 6 non-congruent polygons).

*A star* is any collection of colored polygons having a common vertex. Two stars are *congruent* if the second is a translate of the first and the corresponding colors are coincident. If $\mathcal{S}$ is a star consisting of $(P_1,j_1),(P_2,j_2),\dots,(P_k,j_k)$ then let $\varphi(\mathcal{S})$ be the star consisting of $\varphi(P_1,j_1),\varphi(P_2,j_2),\dots,\varphi(P_k,j_k)$ where $\varphi\in D_8$.

Let $\mathcal{A}$ be the set of all stars congruent to one of $\varphi(\mathcal{S})$ where $\mathcal{S}$ is one of 13 stars indicated in fig. 1b and $\varphi\in D_8$. There are 152 non-congruent stars in $\mathcal{A}$.

*A colored tiling* is a tiling whose tiles are colored polygon. If $T$ is a colored tiling of the plane $\mathbb{R}^2$ then the *star-configuration* of a vertex $v$ of $T$ is the collection of all tiles incident to $v$.

**Definition:***A colored tiling $T$ of $\mathbb{R}^2$ satisfies local rule $\mathcal{A}$ if the star-configuration of every its vertex is congruent to one of $\mathcal{A}$.*

We will prove that every tiling satisfying this local rule must belong to a special class of tilings, called the class of quasiperiodic tilings having 8-fold symmetry.

### 1.2 Quasiperiodic tilings having 8-fold symmetry

1.The cut method. We introduce here the cut method to obtain planar quasiperiodic tilings having 8-fold symmetry. Let's consider the Euclidean space $\mathbb{R}^4$ equipped with a normal base $\varepsilon_0,\varepsilon_1,\varepsilon_2,\varepsilon_3$, and actions of the cyclic group $\mathbb{Z}_8=\{g\mid g^8=1\}$ on $\mathbb{R}^4$ as follow: $g(\varepsilon_i)=\varepsilon_{i+1}$ for $i=0,1,2; g(\varepsilon_3)=-\varepsilon_1$. The space $\mathbb{R}^4$ falls into two 2-dimensional invariant subspaces $\mathbf{E}$ and $\bar{\mathbf{E}}=\mathbf{E}^{\perp}$. On $\mathbf{E}$ $g$ acts as rotation by $\pi/4$ while on $\bar{\mathbf{E}}$ $g$ acts as rotation by $3\pi/4$. The 2-plane $\mathbf{E}$ is spanned by vectors $(2,\sqrt{2},0,-\sqrt{2})$ and $(-\sqrt{2},2,\sqrt{2},0)$. The 2-plane $\bar{\mathbf{E}}$ is its orthogonal complement. We have $\mathbb{R}^4=\mathbf{E}\oplus\bar{\mathbf{E}}$, let $\mathbf{p}$ and $\bar{\mathbf{p}}$ be the projectors corresponding to this decomposition, $\mathbf{p}:\mathbb{R}^4\to\mathbf{E},\bar{\mathbf{p}}:\mathbb{R}^4\to\bar{\mathbf{E}}$. Put $e_i=\mathbf{p}(\varepsilon_i)$ and $\bar{e}_i=\bar{\mathbf{p}}(\varepsilon_i)$, for $i=0,1,2,3$.

A set $A$ in $\mathbb{R}^4$ is called an $\mathbf{E}$-prism (or simply a prism in this section) if $A=\mathbf{p}(A)+\bar{\mathbf{p}}(A)$. For a prism $A$ we define its parallel boundary $\partial^{\|}A$ as $\mathbf{p}(A)+\partial(\bar{\mathbf{p}}(A))$, where $\partial X$ is the boundary of the set $X$, in what follows $X$ is a polygon.

Let $\Lambda$ be the lattice generated by the base $\varepsilon_0,\varepsilon_1,\varepsilon_2,\varepsilon_3:\Lambda=\mathbb{Z}^4$. Then it is easy to check that both projectors $\mathbf{p},\bar{\mathbf{p}}$, when restricted to $\Lambda$, are one-to-one. Let $\gamma$ be the unit cube: $\gamma=\{\sum_{i=0}^{3}\lambda_i\varepsilon_i\mid\lambda_i\in[0,1]\}$ and $M$ be the set of pairs $(i,j)$ with $0\le i<j\le 3$. There are six elements in $M$. The cube $\gamma$ is a fundamental domain of group $\Lambda$ but it is not a prism. We now introduce another fundamental domain consisting of six prisms. For $I=(i,j)\in M$ let $P_I$ be the parallelogram generated by $e_i$ and $e_j, P_I=\{\lambda_1e_i+\lambda_2e_j\mid\lambda_1,\lambda_2\in[0,1]\}$. If $I=(i,j)$ let

$I^c = (k, l)$ where $(i, j, k, l)$ is a permutation of (0, 1, 2, 3). Put $P_I^\perp = \{-\lambda_1 \bar{e}_k - \lambda_2 \bar{e}_l \mid \lambda_1, \lambda_2 \in [0,1]\}$, where $I^c = (k, l)$. Each $P_I^\perp$ is a parallelogram lying in $\bar{\mathbf{E}}$. The set $C_I = P_I + P_I^\perp$ is a prism. A theorem of [ODK] asserts that $\bigcup_{I\in M} C_I$ is a fundamental domain of $\Lambda$, i. e. the family $\mathcal{O}$ of prisms $\{C_{I,\xi}, I \in M, \xi \in \Lambda\}$, where $C_{I,\xi} = C_I + \xi$, is a tiling of $\mathbb{R}^4$. Two fundamental properties of $\mathcal{O}$ are:

*i*)it is periodic, i. e. invariant under translations by vectors from $\Lambda$.

*ii*)every its tile is a prism.

Every family of polyhedra which covers the whole $\mathbb{R}^4$ without holes and overlaps and satisfies these two conditions is called an *oblique periodic tilings of* $\mathbb{R}^4$ (the terminology is due to [ODK]). For an oblique periodic tiling we define its *parallel boundary* as the union of all the parallel boundaries of all the prisms contained in this family. Let $\mathbf{B}$ be the parallel boundary of $\mathcal{O}$: $\mathbf{B} = \cup\partial^{||}(C_{I,\xi}), I \in M, \xi \in \Lambda$. The intersection of a plane $\mathbf{E} + \alpha$, where $\alpha \in \bar{\mathbf{E}}$, with a prism $C_{I,\xi}$ is congruent to $P_I$ if it is not empty. When $\mathbf{E} + \alpha$ dose not intersect $\mathbf{B}$ the intersections of $\mathbf{E} + \alpha$ and members of the family $\mathcal{O}$ form a tiling of $\mathbf{E} + \alpha$, and hence a tiling $T_\alpha$ of $\mathbf{E}$ by projecting. A point $\alpha \in \bar{\mathbf{E}}$ is called *regular* with respect to an oblique periodic tiling if $\mathbf{E} + \alpha$ dose not meet the parallel boundary of this oblique periodic tiling.

The set of points irregular with respect to $\mathcal{O}$ can be described as follow. Let $\bar{f}_0$, $\bar{f}_1$, $\bar{f}_2$, $\bar{f}_3$ are lines in $\bar{\mathbf{E}}$ generated by $\bar{e}_0$, $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$. Lemma IV.2 of [ODK] states that $\alpha$ is not regular iff $\alpha$ belongs to $\Phi = (\bar{f}_0 \cup \bar{f}_1 \cup \bar{f}_2 \cup \bar{f}_3) + \bar{\mathrm{p}}(\Lambda)$. Note that the set $\bar{\mathrm{p}}(\Lambda)$ is a dense set in $\bar{\mathbf{E}}$. The set $\Phi$ is the union of 4 families of parallel lines, each family is dense in $\bar{\mathbf{E}}$, but the set $\Phi$ has measure zero.

If $\alpha$ is not regular then all the intersections of $\mathbf{E} + \alpha$ and $\mathcal{O}$ do not form a tiling, they have overlaps. In this case one can delete some intersections such that the remainings form a proper tiling. There are many ways, sometime even infinitely, to do this. We can say that when $\alpha$ is irregular, it defines not one but a whole family of tilings, among them some will be considered to be quasiperiodic.

Let $\mathcal{T}$ denote the set of all tilings of the type $T_\alpha$ or its translates for regular $\alpha \in \bar{\mathbf{E}}$. A sequence of tilings $T_i, i = 1, 2, 3, \ldots$ converges to a tiling $T$ means that for every real number $r > 0$ there exists a natural number $N$ such that $T$ coincides with $T_i$ for $i \geq N$ inside the circle with center at the origin of $\mathbf{E}$ and radius $r$. Let $\overline{\mathcal{T}}$ be the closure of $\mathcal{T}$, i.e. $\overline{\mathcal{T}}$ is the set of all the limits of sequence of tilings $T_1, T_, \ldots$ with $T_i \in \mathcal{T}$.

**Definition:***Every tiling $T$ from $\overline{\mathcal{T}}$ is called a quasiperiodic tiling having 8-fold symmetry.*

2.Theorem about local rule. Now we can state the main result of section 1.

**Theorem 1.1:***a)Every quasiperiodic tiling having 8-fold symmetry can be colored such that the resulted tiling satisfying local rule $\mathcal{A}$.*

*b) Every tiling satisfying local rule $\mathcal{A}$ is a quasiperiodic tiling having 8-fold symmetry.*

The remaining of section 1 is devoted to a proof of this theorem.

**Remark:**

1)There are two non-congruent tilings satisfying local rule $\mathcal{A}$ but they are the same if we ignore color.

2)It's easy to see that if all the tilings $T_i$ satisfy our local rule $\mathcal{A}$ (or any local rule in Levitov's sense [L]) then their limit also satisfies this local rules. So, if a local rule always enforces quasiperiodicity, the set of all quasiperiodic tilings must be closed under the operation "limit". This is the reason why we consider not the set $\mathcal{T}$ but its closure $\overline{\mathcal{T}}$. By this definition there will be two different quasiperiodic tilings coinciding on a half-plane of $\mathbf{E}$.

3)If $\alpha$ and $\beta$ are regular then the tilings $T_\alpha, T_\beta$ have *the same local structure*, this means that every finite part of $T_\alpha$ is congruent to a finite part of $T_\beta$ and vice-versa (cf. [L], [LPS2]).

4)Another ways to get quasiperiodic tilings having 8-fold symmetry are the strip projection method, the dual multigrid method (cf. for example [dB1], [GR]). But for us the geometry of the cut method is more convenient.

We briefly explain why the class of tiling $\overline{\mathcal{T}}$ is called the class of tilings having 8-fold symmetry. If $T$ is a tiling from $\overline{\mathcal{T}}$ then generally $T$ is not congruent to the tiling $gT$ obtained from $T$ by rotation by $\pi/4$. But every tiling $T \in \overline{\mathcal{T}}$ has the same local structure as its rotation $gT$. Some authors say that they are locally isomorphic. To see that $T$ has the same local structure as $gT$ let us consider actions of group $\mathbb{Z}_8$ in $\mathbb{R}^4$. This group does not leave the family $\mathcal{O}$ invariant but the group $g_1\mathbb{Z}_8g_1^{-1}$ leaves $\mathcal{O}$ invariant, where $g_1$ is the translation in $\mathbb{R}^4$ by vector $\delta = \frac{1}{2}(e_1 + e_2 + e_3 + e_0)$. The element $g_1gg_1^{-1}$ acts on $\mathbf{E}$ as the rotation by $\pi/4$ around point $\delta$ and on $\bar{\mathbf{E}} + \delta$ as the rotation by $3\pi/4$ around point $\delta$, it transforms $E + \alpha$ into $E + g(\alpha)$. Hence the rotation on $\mathbf{E}$ by $\pi/4$ around $\delta$ transforms the tiling $T_\alpha$ into the tiling $T_{g(\alpha)}$. Because both have the same local structure, it follows that $gT$ and $T$ have the same local structure.

3.Lifting a tiling Suppose $P$ is a polygon congruent to one of six $P_I$ (without any color), that is, $P = v + P_I$ for some vector $v$ in $\mathbf{E}$. If in addition $P$ has one vertex lying in $\mathbf{p}(\Lambda)$ then all the vertices of $P$ lie in $\mathbf{p}(\Lambda)$ and there is a unique prism $C$ from $\mathcal{O}$ such that $\mathbf{p}(C) = P$. Hence if $T$ is a tiling whose tiles congruent to $P_I$ and one vertex of $T$ is lying in $\mathbf{p}(\Lambda)$ then there is a map $l$ : {tiles of $T$} $\rightarrow$ {prisms of $\mathcal{O}$} such that $\mathbf{p}(l(P)) = P$ for every tile $P$ of $T$. This map $l$ is called the lift of $T$ into $\mathcal{O}$.

### 1.3 Refined oblique periodic tilings

Let $f_i$ for $i = 0, 1, 2, 3$ be the line spanned by $e_i$. From now on we identify the space $\mathbf{E}$ with the 2-dimensional Euclidean space $\mathbb{R}^2$ introduced in §1.1 such that the origin of $\mathbf{E}$ coincides with the center of the octagon, vertex 0 lies on $f_0$, vertex 1 lies on $f_1$ and the rhomb $R$ is congruent to $P_{(0,1)}$, then the square $S$ is congruent to $P_{(0,2)}$.

Each prism $C_I$ is the sum of $P_I$ and $P_I^{\perp}$. There are 6 $P_I^{\perp}$ up to congruence, which are listed in fig. 2.

In order to point out the orientation in fig. 2 we also draw 4 vectors $-\bar{e}_1, -\bar{e}_2, -\bar{e}_3, -\bar{e}_0$. $P_I^{\perp}$ is a rhomb or a square. We divide each $P_I^{\perp}$ into 4 parts by its diagonals (punctured segments in fig. 2) and color these parts by numbers as there shown: $P_I^{\perp} = \bigcup_j P_I^{\perp;j}$, here index $j$ denotes the color , for each $I$ the color $j$ takes 4 different values.

The prism $C_I$ is divided into 4 prisms, $C_I = \bigcup_j C_I^j$, where $C_I^j = P_I + P_I^{\perp;j}$. We color $C_I^j$ by $j$. Instead of 6 prisms $C_I$ we get 24 colored prisms $C_I^j$. They also form a fundamental domain of $\Lambda$. The family of colored prisms $\{C_I^j + \xi, I \in M, \xi \in \Lambda\}$ is called a refinement of $\mathcal{O}$, and denoted by $\mathcal{O}^{\mathcal{A}}$. This family is also an oblique periodic tiling of $\mathbb{R}^4$. This family has relation with local rule $\mathcal{A}$ which will be explained later. Here $C_I^j + \xi$ has color $j$.

Let $\mathbf{B}^{\mathcal{A}}$ be the parallel boundary of the refined family $\mathcal{O}^{\mathcal{A}}$, this is the union of the parallel boundaries of all prisms from the refined $\mathcal{O}^{\mathcal{A}}$. Of course $\mathbf{B}^{\mathcal{A}} \supset \mathbf{B}$.

Let $\bar{f}_4$ be the line in $\bar{\mathbf{E}}$ generated by vectors $\bar{e}_1 + \bar{e}_2$ and $\bar{f}_5 = g\bar{f}_4$, that is, $\bar{f}_5$ is obtained from $\bar{f}_4$ by rotation by $3\pi/4$. Similarly let $\bar{f}_6 = g\bar{f}_5$, $\bar{f}_7 = g\bar{f}_6$. Eight lines $\bar{f}_0, \ldots, \bar{f}_7$ form a regular star as in fig.3.

Let $\tilde{\Phi} = (\bigcup_{i \in \{0,1,\ldots,7\}} \bar{f}_i) + \bar{\mathrm{p}}(\Lambda)$. This is the union of 8 families of lines in $\bar{\mathbf{E}}$. Each family is dense in $\bar{\mathbf{E}}$ and contains lines parallel to one of $\bar{f}_i, i = 0, 1, \ldots, 7$. Note that if $C$ is a prism from $\mathcal{O}^{\mathcal{A}}$ then the projection of $C$ on $\bar{\mathbf{E}}$ is a triangle having sides lying in $\tilde{\Phi}$, hence we get

**Proposition 1.2:** *The projection* $\mathrm{p}(\mathbf{B}^{\mathcal{A}})$ *on* $\bar{\mathbf{E}}$ *is* $\tilde{\Phi}$, $\bar{\mathrm{p}}(\mathbf{B}^{\mathcal{A}}) = \tilde{\Phi}$.

If $\alpha \notin \tilde{\Phi}$ then the intersection of $\mathbf{E} + \alpha$ with members of the refined family $\mathcal{O}^{\mathcal{A}}$ form a colored tiling on $E + \alpha$, and on $\mathbf{E}$ by projecting. If we ignore colors then this tiling coincides with $T_\alpha$, we denote the colored tiling by $T_\alpha^c$ to emphasize the color. Every tile of $T_\alpha^c$ is congruent to one of $\mathcal{P}^{\mathcal{A}}$.

**Proposition 1.3:** *The set $\mathcal{A}$, up to congruence, is the set of all star-configurations of $T_\alpha^c$ for a point $\alpha \in \bar{\mathbf{E}}$ not lying in $\tilde{\Phi}$.*

In fact we begin with $\mathcal{O}^{\mathcal{A}}$ and then find that the set of all star-configurations of tilings of type $T_\alpha^c$ is $\mathcal{A}$. Note that since $\bar{\mathrm{p}}(\Lambda)$ is dense in $\bar{\mathbf{E}}$, if $\alpha$ and $\beta$ are two points of $\bar{\mathbf{E}}$ not lying in $\tilde{\Phi}$ then the set of all star-configurations of $T_\alpha^c$ and that of $T_\beta^c$ are the same, up to translations.

This proposition can be proved by analyzing the tiling $T_\alpha^c$. We present a proof in the §1. 9.

If $P$ is a colored polygon congruent to one of $\mathcal{P}^{\mathcal{A}}$ and if one vertex of $P$ is lying in $\mathrm{p}(\Lambda)$ then there is a unique prism $l(P)$ from $\mathcal{O}^{\mathcal{A}}$ such that $\mathrm{p}(l(P)) = P$ and the colors of $P$ and $l(P)$ are the same. Hence if $T$ is a tiling whose tiles are congruent to elements of $\mathcal{P}^{\mathcal{A}}$ and one vertex of $T$ is in $\mathrm{p}(\Lambda)$ then there is a unique map $l$ : {tiles of $T$} $\rightarrow$ {prisms of $\mathcal{O}^{\mathcal{A}}$} such that $\mathrm{p}(l(P)) = P$ and colors of $P$ and $l(P)$ are the same for every tile of $T$. This map is called the lift of $T$ into $\mathcal{O}^{\mathcal{A}}$.

### 1.4 Non-planar sections and idea of proof of theorem 1.1

1.Section.*A section* $\Omega$ is a 2-dimensional surface in $\mathbb{R}^4$ such that $\mathrm{p}_{|\Omega} : \Omega \rightarrow E$ is a homeomorphism. If a section $\Omega$ dose not meet $\mathbf{B}^{\mathcal{A}}$ then it defines a colored tiling $T_\Omega$ by projecting on $\mathbf{E}$ all the colored prisms meeting $\Omega$. It is easy to see that all the star-configurations of this tiling are the same as that of tiling of the type $T^c_\alpha$(cf. also [ODK]). Hence if $\Omega$ does not meet $\mathbf{B}^{\mathcal{A}}$ then the colored tiling $T_\Omega$ satisfies local rule $\mathcal{A}$. Conversely, if $T$ satisfies $\mathcal{A}$, then after a shift we may suppose that $T$ has a vertex in $\mathrm{p}(\Lambda)$ and hence $T$ has a lift $l$ into $\mathcal{O}^{\mathcal{A}}$. Because the star of every vertex of $T$ is a translate of a star of a vertex of $T^c_\alpha$, the lift of a star of $T$ is a translate of the lift of a star of $T^c_\alpha$. It is easy to prove the following

**Proposition 1.4:***Suppose $T$ is a colored tiling satisfying local rule $\mathcal{A}$ then $T$ is congruent to a tiling $T_\Omega$ for some section $\Omega$ not meeting* $\mathbf{B}^{\mathcal{A}}$.

For a rigorous proof we refer to [LPS1],[LPS2].

In order to prove theorem 1.1 we have to prove that every section not meeting $\mathbf{B}^{\mathcal{A}}$ defines a quasiperiodic tiling having 8-fold symmetry. If $\alpha \in \bar{\mathbf{E}}$ and not lies in $\tilde{\Phi}$ then the colored tiling defined by section $\mathbf{E} + \alpha$ is exactly $T^c_\alpha$.

For a section $\Omega$ not meeting $\mathbf{B}^{\mathcal{A}}$ let $\mathcal{P}(\Omega)$ be the set of all prisms from $\mathcal{O}^{\mathcal{A}}$ meeting $\Omega$. This is just the set of all the prisms $l(P)$ where $P$'s are tiles the colored tiling defined by $\Omega$. For a point $x$ in $\mathbf{E}$ let $\Omega$ let $\Omega(x)$ be the point of $\Omega$ lying upon $x$, i. e. $\Omega(x) = \Omega \cap \mathrm{p}^{-1}(x)$. In the case when $\Omega = \mathbf{E} + \alpha$ with $\alpha \notin \tilde{\Phi}$ all the projections $\bar{\mathrm{p}}(C)$ where $C$'s are prisms from $\mathcal{P}(\Omega)$ have a common point, it is $\alpha$. Moreover each projection $\bar{\mathrm{p}}(C)$ is a triangle and $\alpha$ lies *inside* this triangle because it is regular and all the sides of this triangle are in $\tilde{\Phi}$. This means $\alpha$ is an interior point of this triangle.

If $\Omega$ is a section not meeting $\tilde{\Phi}$ such that all the projections $\bar{\mathrm{p}}(C)$ where $C$'s are prisms from $\mathcal{P}(\Omega)$ have a common point $\alpha$ and this common point is regular then it is easy to see that $\Omega$ defines the same colored tiling as $\alpha$, hence $T_\Omega$ belongs to $\overline{\mathcal{T}}$. We shall prove first that for a section $\Omega$ not meeting $\mathbf{B}^{\mathcal{A}}$ all the projections $\bar{\mathrm{p}}(C)$ where $C$'s are prisms from $\mathcal{P}(\Omega)$ have a common, but in general not a *regular* point, the common point maybe irregular. One makes use of the following

**Proposition 1.5:***Suppose $\Omega$ is a section not meeting $\mathbf{B}^{\mathcal{A}}$ and all the projections $\bar{\mathrm{p}}(C)$ where $C$'s are prisms from $\mathcal{P}(\Omega)$ have a common point which is interior with respect to every projection $\bar{\mathrm{p}}(C)$. Then the tiling $T_\Omega$ is a quasiperiodic tiling having 8-fold symmetry.*

**Proof:** We number the tiles of $T_\Omega$, $T_\Omega = \{P_1, P_2, \dots\}$ such that for every $r > 0$ the disk $U_r$ in $\mathbf{E}$ with center at $\mathbf{0}$ and radius $r$ is covered by $N$ first tiles, here $N$ depends on $r$. Let $l(P)$ for a tile $P$ of $T$ be the prism from $T_\Omega$ lying upon $P$, $\mathrm{p}(l(P)) = P$. Since the polygons $\bar{\mathrm{p}}(l(P_1)), \dots, \bar{\mathrm{p}}(l(P_N))$ have non-empty interior intersection, the intersection of these polygons is a polygon (with non-empty interior). There is a regular point $\alpha_r$ belonging to this polygon. Then the colored tiling $T^c_{\alpha_r}$ is the same as $T$ inside the disk $U_r$. Hence the sequence of colored tiling $T_{\alpha_i}, i = 1, 2, \dots$ converges to $T$. □

2.Idea of proof of theorem 2.1 For proving theorem 2.1 we need to prove that if $\Omega$ is a section not meeting $\mathbf{B}^{\mathcal{A}}$ then for every finite number of prisms $C_1, \dots, C_m$ of $T_\Omega$ the projections on $\bar{\mathbf{E}}$ of $C_1, \dots, C_m$ have non-empty interior intersection. Note that sides of a triangle $\bar{\mathrm{p}}(C)$ where $C$ is a prism of $\mathcal{P}(\Omega)$ are lying in the set $\tilde{\Phi}$. This set is a family of lines. We will introduce *orientation* on lines from $\tilde{\Phi}$, i.e. for each line we associate a normal vector. Each line divides $\bar{\mathbf{E}}$ into two half-planes, the half to which the normal vector directs is called the positive half-plane of the oriented line. Here we consider that the half-plane does not contain the line itself, i.e. the half-plane is an open subset of $\bar{\mathbf{E}}$. We will find the way to orient all the lines in $\tilde{\Phi}$ such that

(*) *if the triangle $\bar{\mathrm{p}}(C)$, where $C$ is a prism of $\mathcal{P}(\Omega)$, has a side lying on a line $h$ then the interior of this polygon is lying in the positive half-plane of $h$.*

Moreover if in addition the following condition is fulfilled

(**) *for every finite number of lines $h_1, \dots, h_n$ from $\tilde{\Phi}$ their positive half-planes have a common point.*

then it is easy to prove that for every finite number of prisms $C_1, \dots, C_m$ of $\mathcal{P}(\Omega)$ the projections $\bar{\mathrm{p}}(C_1), \dots, \bar{\mathrm{p}}(C_m)$ have non-empty *interior* intersection. Hence we can apply proposition 1.5 to conclude that $T_\Omega$ is a quasiperiodic tiling having 8-fold symmetry.

### 1.5 Structure of the parallel boundary

Let $f_i$ be the lines on $\mathbf{E}$ generated by $e_i, i = 0, 1, 2, 3$; $f_4$ by $e_1 + e_2$, $f_5 = g(f_5), f_6 = g(f_5), f_7 = g(f_6)$. For $i = 0, \dots, 7$ let $F_i = \bar{f}_i + f_i$ and $F'_i = F_i + \delta$ (recall that $\delta = (e_0 + e_1 + e_2 + e_3)/2$).

Each $F'_i$ is a prism, and $\bar{\mathrm{p}}(F'_i) = \bar{f}_i, \mathrm{p}(F'_i) = f_i + \delta$. In the coordinate system $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ the planes $F_0$ and $F_4$ are given by $F_0 = \{\lambda_1 + \lambda_3 = 0, \lambda_2 = 0\}, F_4 = \{\lambda_1 + \lambda_2 = 0, \lambda_0 - \lambda_3 = 0\}$. All the others are obtained by actions of the group $\mathbb{Z}_8$. All the 2-planes $F_i$ are rational.

If $C$ is a colored prism from $\mathcal{P}(\Omega)$ then the parallel boundary of $C$ is $\partial(\bar{\mathrm{p}}(C)) + \mathrm{p}(C)$. The first term is three sides of a triangle, hence the parallel boundary of $C$ consists of 3 parts, each is the sum of a side and $\mathrm{p}(C)$. We call each part a *small wall* of $C$. Obviously $\mathbf{B}^{\mathcal{A}}$ is the union of all the small walls. Up to translations from $\Lambda$ there are 24 colored prisms and $24 \times 3 = 72$ small walls.

**Proposition 1.6:** *All 8 planes $F_i', i = 0, \ldots, 7$ are contained in the parallel boundary* $\mathbf{B}^{\mathcal{A}}$ .

Remark: In fact we begin with the system of 2-planes $\bigcup_{i=0}^{7} F_i'$ and then find a refinement of the oblique periodic tiling such that the parallel boundary of the refiment contains this system of 2-planes.

**Proof:** We prove, for example, that $F_0'$ is contained in $\mathbf{B}^{\mathcal{A}}$. Let $\mathcal{F}_0' = F_0' + \Lambda$, it is a family of parallel planes in $\mathbb{R}^4$. This family is locally finite in the sense that every compact meets only a finite numbers of planes from it because $F_0$ is a rational plane. We shall investigate the intersection of the original family $\mathcal{O}$ with $F_0'$. Both are invariant under translations by vectors from $\Lambda$. For each $I \in M$ the prism $C_I$ meets only a finite numbers of planes from $\mathcal{F}_0'$, more precisely, it is easy to check that each $C_I$ meets no more than two 2-planes from $\mathcal{F}_0'$, if we do not take into account those which intersect $C_I$ by a set of positive codimension, that is, a segment. The projections of the intersections of $C_I$ and these planes on $\bar{\mathbf{E}}$ are segments lying on the boundary of $\bar{\mathrm{p}}(C_I)$ or on its diagonals. This can be checked easily. From this (and the closedness of the intersection of $\mathcal{F}_0'$ with $\mathbf{B}^{\mathcal{A}}$) one sees at once that $\mathcal{F}_0'$ is contained in $\mathbf{B}^{\mathcal{A}}$ . □

Denote $\mathcal{F}_i' = F_i' + \Lambda$ and $\mathcal{F}' = \bigcup_{i=0,\ldots,7} \mathcal{F}_i'$. Then $\bar{\mathrm{p}}(\mathcal{F}') = \tilde{\Phi}$. We have $\mathcal{F}' \subset \mathbf{B}^{\mathcal{A}}$. Every two 2-planes from $\mathcal{F}'$ either are parallel or have a unique intersection point.

**Proposition 1.7:** *If the projection of a small wall $w$ on $\bar{\mathbf{E}}$ is contained in the projection of a 2-plane $F$ from $\mathcal{F}'$ then $w$ meets $F$.*

**Proof:** Due to the actions of groups $\Lambda$ and $G$ we may assume that $F = F_0'$. It's easy to check that there are exactly 24 small walls having projections lying on the line $\bar{f}_0$, and others with this property can be obtained from these 24 by translations by integer vectors from $F_0$. Small walls and $F$ are prisms, to investigate their intersection is very simple, it is sufficient to consider the intersection of their projections on $\mathbf{E}$ and on $\bar{\mathbf{E}}$. One can check that these 24 small walls intersect $F_0'$. Translations by vectors from $F_0$ do not change $F_0'$ , hence all the above mentioned small walls intersect $F_0'$. □

## 1.6 Orientations for lines from $\tilde{\Phi}$ and property (*)

From now on we fix a section $\Omega$ not meeting $\mathbf{B}^{\mathcal{A}}$. Then $\Omega$ can be regarded as the graph of a continuous map $\rho : \mathbf{E} \to \bar{\mathbf{E}}$, $\Omega = \{x + \rho(x) | x \in \mathbf{E}\}$.

A line $h$ in $\bar{\mathbf{E}}$ is called oriented if one half-plane separated by $h$ is marked, this marked half-plane is called the positive half-plane of the oriented line. Note that a half-plane here is an open subset of $\bar{\mathbf{E}}$, i.e. it dose not contain the line $h$. A point $x$ is *greater than an oriented line (we write $x > h$)* if $x$ belongs to the positive half-plane. The notion $x \geq h$ means that $x > h$ or $x \in h$, for a set $X \subset \bar{\mathbf{E}}$ the notion $X > h$ (resp. $X \geq h$) means $x > l$ (resp. $x \geq h$) for every $x \in X$. A set of oriented lines is *compatible* if there exists a point greater than all of them. Two parallel oriented lines have *the same orientation* if the intersection of their positive half-planes is a half-plane.

We introduce orientations for lines from $\tilde{\Phi}$ as follow.

The section $\Omega$ does not meet $F$ for every 2-plane $F$ from $\mathcal{F}$because $\mathcal{F} \subset \mathbf{B}^{\mathcal{A}}$. Consider two lines $\bar{f} = \bar{\mathrm{p}}(F)$ and $f = \mathrm{p}(F)$. The first is in $\bar{\mathbf{E}}$ and the second is in $\mathbf{E}$. We define $\Omega(f)$ as the set of point from $\Omega$ lying upon $f$, $\Omega(f) = \Omega \cap \mathbf{p}^{-1}(f)$. It is a connected set in $\mathbb{R}^4$. The set $\mathbf{p}^{-1}(f)$ is a 3-plane. In this 3-plane lie two sets: the 2-plane $F = f + \bar{f}$ and $\Omega(f)$. They do not have intersection hence $\Omega(f)$ lies in one half-space separated by $F$. By projecting on $\bar{\mathbf{E}}$ we see that $\rho(f) = \bar{\mathrm{p}}(\Omega(f))$ does not meet $\bar{f}$. This can be proved more rigorously as follow. If $x \in (\bar{f} \cap \bar{\mathrm{p}}(\Omega(f))$ then $x = \rho(z)$ with $z \in f$. Then $y = (x + z) \in (f + \bar{f})$ and at the same time $y \in \Omega(f)$. This is a contradiction because $F \cap \Omega(f) = \emptyset$.

We have just seen that for every 2-plane $F$ from $\mathcal{F}$ the set $\rho(\mathbf{p}(F))$ does not meet the line $\bar{\mathrm{p}}(F)$. We orient the line $\bar{\mathrm{p}}(F)$ such that $\rho(\mathbf{p}(F)) > \bar{\mathrm{p}}(F)$. For the correctness we have to prove that two different 2-planes from $\mathcal{F}'$ have different projections on $\bar{\mathbf{E}}$.

**Proposition 1.8:***a) Suppose* $\mathrm{p}(F) = \mathrm{p}(F')$ *where* $F, F'$ *are 2-planes from* $\mathcal{F}'$ *then* $F = F'$.

*b) Suppose* $\bar{\mathrm{p}}(F), \bar{\mathrm{p}}(F')$ *and* $\bar{\mathrm{p}}(F'')$ *where* $F, F'$ *and* $F''$ *are 2-planes from* $\mathcal{F}'$ *have a common point then* $F, F', F''$ *also have a common point.*

This proposition is easily proved and we omit the proof. It follows from the total irrationality of $\mathbf{E}$,$\bar{\mathbf{E}}$ and rationality of $F_i$ (cf.[LPS] for a rigorous proof).

Now we can establish property (*). Suppose $C = l(P)$ is a lift of a tile of $T$. Then $\bar{\mathrm{p}}(C)$ is a polygon having sides lying on lines from $\tilde{\Phi}$.

**Proposition 1.9:***Suppose* $Q = \bar{\mathrm{p}}(C)$ *has a side* $s$ *lying on the line* $\bar{f}$ *from* $\tilde{\Phi}$. *Then* $Q \geq \bar{f}$.

**Proof:** There is a 2-plane $F$ from $\mathcal{F}'$ such that $f = \bar{\mathrm{p}}(F)$. The set $\mathbf{p}(C) + s$ is a small wall. By proposition 1. 7 this small wall $\mathbf{p}(C) + s$ and the 2-plane $F$ have a common point $y$. Let $x = p(y)$. Then $\rho(x)$ is an interior point of $\bar{\mathbf{p}}(C)$ and by definition of the orientation $\rho(x) > f$, hence $\bar{\mathrm{p}}(C) \geq \bar{f}$. □

## 1.7 Boundedness of sections

Four planes $F_0, F_1, F_2$ and $F_4$(not $F_3$!) are *complete* in the following sense: there is no linear transformation $\psi : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\psi(F_i) = F_i, i = 0, 1, 2, 4$ and $\psi(E) = \bar{\mathbf{E}}$. This can be checked easily. Another definition of completeness is the following. Every 2-dimensional plane going through the origin in $\mathbb{R}^3$ can be given by two linear equations:

$$a_0\lambda_0 + a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 = 0$$

$$b_0\lambda_0 + b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 = 0$$

$$\text{Put } A_{ij} = \det \begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}$$

Six numbers $A_{ij}$ for $0 \leq i < j \leq 3$ are called the projective coordinates of the 2-dimensional plane. Four planes are complete if their projective coordinates are linear independent. This definition is equivalent to the previous one (cf. [Le2] for a comprehensive consideration of completeness)

It can be checked directly that the four 2-planes $F_0, F_1, F_2, F_4$ are also complete in this definition . From the completeness one can prove the following

**Proposition 1.10**: *the map $\rho$ is bounded, that is $\rho(\mathrm{E})$ is a bounded set in $\bar{\mathbf{E}}$.*

This is proved in [LPS1]. It is a generalization of a proof of Levitov.

As a consequence we see that for a fixed $i \in \{0, 1, \ldots, 7\}$ all the lines from $\tilde{\Phi}_i = \bar{\mathrm{p}}(\mathcal{F}_i)$ can not have the same orientation.

## 1.8 Property (**)

**Proposition 1.11:** *If several lines $h_1, \ldots, h_m$ from $\tilde{\Phi}$ have a common point then they are compatible.*

**Proof**: Suppose $H_1, \ldots, H_m$ are 2-planes from $\mathcal{F}'$ such that $\bar{\mathrm{p}}(H_i) = h_i, i = 1, \ldots, m$. By proposition 1. 8b all the 2-planes $H_1, \ldots, H_m$ have a common point $y$. Let $x = \mathbf{p}(y)$. Then by definition of orientations the point $\rho(x)$ is greater than all the lines $h_1, \ldots, h_m$.□

A set $X$ is called *bootstrapped* by $Y$ and $Z$ if $X \cap Y = X \cap Z$. For example, it's easy to check that $\tilde{\Phi}_0 = \bar{\mathrm{p}}(\mathcal{F}_0)$ is bootstrapped by $\tilde{\Phi}_1 = \bar{\mathrm{p}}(\mathcal{F}_1)$, $\tilde{\Phi}_3 = \bar{\mathrm{p}}(\mathcal{F}_3)$. The set $\tilde{\Phi}_2 = \bar{\mathrm{p}}(\mathcal{F}_2)$ is also bootstrapped by $\tilde{\Phi}_1$, $\tilde{\Phi}_3$. From the bootstrapped property one can prove

**Proposition 1.12:** *Every two lines $h_1, h_2$ from $\tilde{\Phi}_0$ or $\tilde{\Phi}_2$ are compatible.*

**Proof**: If they are parallel and are not compatible then by the bootstrapped property there are lines from $\tilde{\Phi}_1$ and $\tilde{\Phi}_3$ which look like in fig. 4.

Here the two shadowed lines are $h_1, h_2$, the shadowing shows the orientations (i.e. the positive half-planes are shadowed), all the other lines are from $\tilde{\Phi}_1$ and $\tilde{\Phi}_3$, all the intersection points are triple. With the help of the previous proposition we can easily find the orientation of all the lines in fig. 4 if we know the orientation of *one* line, say the line going through points $A, B$. There are two possibilities for orientation of the line going through $A, B$. In both cases it is easy to see that all the lines in fig. 4 from $\tilde{\Phi}_1$ have the *same direction.* This is contradict to the fact that $\rho(\mathrm{E})$ is bounded. (For a more rigorous proof see [LPS1]). □

From this proposition and the boundedness of $\rho(\mathbf{E})$ one sees easily that there is a unique line $l_0$ such that $l_0 \geq h$ for every line $h$ from $\tilde{\Phi}_0$, similarly there is a unique line $l_2$ such that

$l_2 \geq h$ for every line $h$ from $\tilde{\Phi}_2$. Let $\alpha$ be the intersection point of $l_0$ and $l_2$. Then proposition 8 of [LPS1] asserts that $\alpha \geq h$ for every line $h$ from $\tilde{\Phi}$. Now we can prove

**Proposition 1.13:** *every finite number of lines $h_1, \ldots, h_m$ from $\tilde{\Phi}$ are compatible.*

**Proof:** If the point $\alpha$ defined above does not belong to any of $h_i, i = 1, ..., m$ then $\alpha$ is a point greater than all these lines. Suppose some of $h_i$ contains $\alpha$, say $h_1, h_2, ..., h_n$ where $n \leq m$. Since $\alpha$ is greater than $h_i$ for $i = n+1, ..., m$ there is a neighborhood $V$ of $\alpha$ such that $V > h_i$ for $i = n+1, ..., m$. Since all the lines $h_i$ for $i = 1, ...n$ go through $\alpha$, they must be compatible by proposition 1. 11. The set of all points greater than all $h_i$ for $i = 1, ..., n$ is a corner (or an angle) with vertex at $\alpha$ and this set have non-empty intersection with $V$. Hence we can choose a point which is greater than all $h_i$, for $i = 1, ..., m$.□.

Together with this proposition we have proved theorem 1.1.

### 1.9 Star-configurations, other local rules

Star-configurations of vertices of colored tiling $T^c_\alpha$ can be classified as follow. Let $x$ be a point of $\mathbf{p}(\Lambda), x = \mathbf{p}(\xi), \xi \in \Lambda$. For a colored polygon $(P, j)$ there is at most one colored prism $C^j_{I,\eta}$ which projects on $(P, j)$, we call it the lift (if it exists) of the colored polygon. Let $K = -\bar{\mathbf{p}}(\gamma)$, it is an octagon lying in $\bar{\mathbf{E}}$. The tiling $T_\alpha$ has $x$ as a vertex if and only if $\xi + K$ intersects $E + \alpha$(cf. [ODK] ). Let $(P, j)$ be a colored polygon having vertex $x$ then $T_\alpha$ has $(P, j)$ as a colored tile iff $E + \alpha$ meets $C$ where $C$ is the lift of $(P, j)$. In this case $C$ must intersect $\xi + K$. Let $Q(P, j)$ be the intersection of $C$ and the octagon $\xi + K$. It is easy to see that $Q(P, j)$ is congruent to $\bar{\mathbf{p}}(C)$ and lying on the boundary of $C$. All the triangles $Q(P, j)$, when $(P, j)$ runs through the set of all colored polygons having $x$ as vertex, partition the octagon $\xi + K$ into many small convex polygons. Each small polygon defines a type of star-configuration, i. e. if $\alpha$ and $\beta$ s. t. the intersection points of $\alpha + E$ and $\beta + E$ with $\xi + K$ lie in the same small polygon then both tilings $T_\alpha$ and $T_\beta$ have the same star-configuration at vertex $x$. Because everything in invariant under $\Lambda$ the partition of $\xi + K$ just looks like in the case $\xi = 0$. In this case the partition of $K$ is given in fig. 5.

There are 152 small polygons, they define 152 stars - exactly 152 stars defined in §1.1. This set of stars has the symmetry described in §1.1 due to the following fact. Consider the group $G$ generated by two linear transformations of $\mathbb{R}^4, g$ and $\tilde{g}$, where $g$ is given in §1, $\tilde{g}$ acts on $\mathbf{E}$ as the reflection relative to the line $f_0 + \delta$ and on $\bar{\mathbf{E}}$ as the reflection relative to the line $\bar{f}_0$. It is easy to check that $G$ consists of 16 elements and leaves invariant $\mathbf{E}$ and $\bar{\mathbf{E}}$. The restriction of $G$ on $\mathbf{E}$ is $g_1 D_8 g_1^{-1}$. If $C$ is a prism of the refined $\mathcal{O}$ with color $j$ then $\varphi(C)$ is another prism, with color $\varphi(j)$. In fact we have chosen colors for $C^j_I$ such that this property holds. In figure 5 we note 13 regions of $K$ by numbers, the corresponding stars are in figure 1b.

**Remark:**

1.Consider the following local rule. *An edge-configuration* of a tiling $T$ at an edge of this tiling is the collection of two (colored) tiles incident to this edge. Two edge-configurations are congruent if the second is a translate of the first and the corresponding colors are the same. A tiling satisfies the local rule $\mathcal{B}$ if every edge-configuration of this tiling is congruent to an edge-configuration of an edge of a colored tiling $T_\alpha^c$ for some $\alpha \notin \tilde{\Phi}$. Then every tiling satisfies this local rule has a lift $l$ which is connected in the sense that if $P_1, P_2$ are two tiles sharing a common edge then the projections $\bar{\mathrm{p}}(l(P_1)), \bar{\mathrm{p}}(l(P_2))$ have non-empty interior intersection. By using the similar method as used here it can be proved that every connected lift defines a tiling belonging to $\overline{\mathcal{T}}$ (cf. a similar proof in [Le1]). Hence the the local rule $\mathcal{B}$ is as good as $\mathcal{A}$. From the local rule $\mathcal{B}$ one can easily construct a local rule involving only decorations on edges. This means, there are decorations by colors on edges of $P_I$ such that every tiling by these polygon such that colors of a common edge coming from different edge must be the same is a quasiperiodic tiling having 8-fold symmetry.

2.If we divide 6 parallelograms $P_I^\perp$ as in fig. 6 (more simply than what we do in fig. 2) then we get a new oblique periodic tiling which is simpler than the above.

The parallel boundary of this new oblique priodic tiling contains $F_0, F_1, F_2, F_3, F_4$, (but not $F_5, F_6, F_7$), and this oblique priodic tiling also defines a local rule, enforcing quasiperiodicity, with 66 stars. The number of stars is less than that of $\mathcal{S}$ but the symmetry is broken. The group of symmetries consists of 4 elements. With the help of this group the number of stars is reduced to 19.

3.The local rule $\mathcal{A}$ is equivalent to the matching rule A5 suggested by R. Ammann (cf. [AGS], [GS]). This means every tiling satisfies $\mathcal{A}$ can be decorated such that the resulted tiling satisfies matching rule A5, and vice-versa. The fact that every tiling satisfying A5 is aperiodic was proved in [AGS] by the composition method. DeBruijn proved, also using the composition method, that every tiling satisfying A5 must be quasiperiodic tiling having 8-fold symmetry. But the method used here point out how *to get* such matching rule, and it can be easily generalized to other cases, both for 2-dimensional and higher dimensional tilings (cf. [LPS1], [LPS2], [LP], [Le1]).

## 2 Local rules without decorations

### 2.1 Definitions

Note that we begin with the class $\overline{\mathcal{T}}$ of quasiperiodic tilings having 8-fold symmetry. There are no colors in any of tilings from $\overline{\mathcal{T}}$. Now we consider the question about local rule without any decorations. First we introduce the exact definition of local rules.

*An r-map of a tiling T at vertex v* is the collection of all the tiles lying inside the ball with center at $v$ and radius $r$. *An r-map with center at v* is any $r$-map of any tiling at $v$. Two $r$-maps are congruent if the second is a translate of the first .

**Definition:** *A set $\mathcal{A}$ of r-maps is called a local rule of radius r (or briefly a local rule). A tiling T satisfies a local rule $\mathcal{A}$ of radius r if every r-map of T is a congruent to one from $\mathcal{A}$. A set of tilings admit a local rule if this is the set of all tilings satisfying some local rule of some radius.*

What we have proved in the previous part of this paper can be stated as follow. There is a class of colored tilings which admits local rules, and this class of tilings, when ignoring the colors is exactly the class of quasiperiodic tiling having 8-fold symmetry. Hence we can say that the class of quasiperiodic tilings having 8-fold symmetry, after coloring , admits local rules.

The question whether the class $\overline{\mathcal{T}}$ admits local rule or not can be formulated more explicitly as follows. Let $\mathcal{B}(r)$ be the set of all $r$-maps of all tilings from $\mathcal{T}$. Let $\mathcal{T}(r)$ be the set of all tilings satisfying this local rules, then $\mathcal{T}(r)$ is the sets of all tilings whose $r$-maps are translates of $r$-maps of tilings from $\mathcal{T}$. The set $\overline{\mathcal{T}}$ admits local rule if there is $r$ such that $\mathcal{T}(r)$ is the set of all quasiperiodic tilings having 8-fold symmetry, $\overline{\mathcal{T}} = \mathcal{T}(r)$.

A negative answer to this question is given by Burkov in [B]. This result is also proved by DeBruijn in [dB2]. In [Le2] we prove a more general result on the absence of local rules, and the case of quasiperiodic tilings having 8-fold symmetry can be obtained as a special case of a theorem there.

Here we prove that although $\mathcal{T}(r)$ is different from $\overline{\mathcal{T}}$, but the set $\mathcal{T}(r)$ contains only "pseudo-periodic" tilings very close to quasiperiodic tilings. Some tilings from $\mathcal{T}(r)$ may be even *periodic.*

At first consider $r$-maps of tiling $T_\alpha$. Suppose $\mathcal{M}$ is an $r$-map with center at $\mathbf{0}$ such that every polygon in $\mathcal{M}$ is congruent to one of $P_I$. Let $v_1, v_2, \dots, v_m$ be vertices of this $r$-map. Then all $v_i$ are projections of integer points, $v_i = \mathbf{p}(\xi_i), \xi_i \in \Lambda$. The intersection of $K$ and several translate of itself, each of the form $K - \sum_{i=1}^{i=m} a_i \xi_i$ where $a_i = 0$ or 1, is called the existence domain of the $r$-map $\mathcal{M}$. It is a polygon lying in $K$. The easy fact to check is that $T_\alpha$ has $\mathcal{M}$ as the $r$-map at $\mathbf{0}$ if and only if $\mathbf{E} + \alpha$ meets the existence domain of $\mathcal{M}$. Similarly a tiling $T_\alpha$ has the $r$-map at a vertex $x$ congruent to $\mathcal{M}$ if and only if $\mathbf{E} + \alpha$ meet the polygon ($\xi+$ the existence domain of $\mathcal{M}$). Here $\xi$ is the only point from $\Lambda$ projecting into $x$.

The set of the existence domains of all possible $r$-maps with center ot $\mathbf{0}$ forms a partition of $K$, for a fixed $r$. If $\alpha$ and $\beta$ are two regular points such that $\mathbf{E}+\alpha$ and $\mathbf{E}+\beta$ meets $K$ at two different domains then the $r$-maps of $T_\alpha$ and $T_\beta$ at $\mathbf{0}$ are different. If $Q$ is a polygon lying in the octagon $K$ and $Q$ has sides lying on lines from $\Phi = \bar{\mathrm{p}}(\mathcal{F})$ then for sufficiently large $r$ every existence domain of $r$-maps with vertex at $\mathbf{0}$ is either lying in $Q$ or has no interior intersection with $Q$. This follows from the construction of the existence domains.

## 2.2 lifting a tiling

Let $U_r$ be the disk with center at $\mathbf{0}$ and radius $r$. Recall that $\mathcal{F}$ is the union of 4 families of parallel 2-planes, $\mathcal{F} = \bigcup_{i=0}^{3}(F_i + \Lambda)$.

**Proposition 2.1:** *a)There is $r$ such that the set $\mathcal{F}$ is contained in $\mathbf{B} + U_r$.*

*b)For every $r$ there is $d$ such that $\mathbf{B} + U_r$ is a subset of $\mathcal{F} + U_d$.*

**Proof**: a) We have seen that $\mathbf{B}^{\mathcal{A}}$ contains $\mathcal{F}$. Hence it is sufficient to prove that there is $r$ such that $\mathbf{B}^{\mathcal{A}}$ is contained in $\mathcal{F} + U_r$. If $w$ is a small wall of a prism $C_I^{\perp;ij}$ then there is a prism $C$ from the original family $\mathcal{O}$ such that $\bar{\mathrm{p}}(w)$ is contained in $\bar{\mathrm{p}}(C)$. This can be checked directly. Since $w = \bar{\mathrm{p}}(w) + \mathrm{p}(w)$ we can choose $r$ such that $w$ is contained in $\bar{\mathrm{p}}(C) + \mathrm{p}(C) + U_r$ for every small wall of one of $C_I^{\perp;ij}$. Then evidently $\mathbf{B}^{\mathcal{A}}$ is a subset of $\mathbf{B} + U_r$.

b) It suffices to prove that $\mathbf{B}$ is a subset of $\mathcal{F} + U_d$ for some $d$. Because up to translations from $\Lambda$ there are a finite number of small walls of prisms from $\mathcal{O}$ and the projection on $\bar{\mathbf{E}}$ of every small wall is contained in the projection of $\mathcal{F}$, this can be proved just as in the previous case.□

Choose an $r_0$ such that $\mathcal{F} \subset (\mathbf{B} + U_{r_0})$ and $\mathbf{B} \subset \mathcal{F} + U_{r_0}$. From now on fix such an $r_0$. Note that if $w$ is a small wall of a prism from $\mathcal{O}$ and $\bar{\mathrm{p}}(w)$ lies on the line $\bar{\mathrm{p}}(F)$ for a 2-plane from $\mathcal{F}$ then $w$ has non-empty intersection with $F$. Hence the sets $\mathrm{p}(w)$ and $\mathrm{p}(F)$ have non-empty intersection. It follows that two sets $\mathrm{p}(w)$ and $\mathrm{p}(F) + U_{r_0}$, being 2-dimensional in $\mathbf{E}$, have non-empty interior intersection.

Recall that a section $\Omega$ not meeting the parallel boundary $\mathbf{B}$ defines a tiling $T_\Omega$.

**Proposition 2.2:***For a fixed $r$ there is $r'$ such that if $T$ is a tiling satisfying $\mathcal{B}(r')$, that is $T \in \mathcal{T}(r')$ then after a shift $T$ is a tiling $T_\Omega$ for a section $\Omega$ not meeting $\mathbf{B} + U_r$.*

**Proof**: There is $d_1$ such that $\mathbf{B} + U_r$ is a subset of $\mathcal{F} + U_{d_1}$. We define a big wall as a set of type $F + U_{d_1}$ where $F$ is a 2-plane from $\mathcal{F}$. Every big wall is contained in a unique 3-plane. Each prism $C_I$ intersects with only a finite number of big walls. The 3-planes going through theses big walls divide $C_I$ into smaller prisms, each has the same projection on $\mathbf{E}$ as the prism $C_I$ itself has, only the projection on $\bar{\mathbf{E}}$ of each smaller prism is a polygon which is a part of the polygon $P_I^{\perp}$. We spread this division to every prism from $\mathcal{O}$ by translation. By this way we get a new family $\hat{\mathcal{O}}$. This is also an oblique periodic tiling of $\mathbb{R}^4$. The parallel boundary of this family is denoted by $\hat{\mathbf{B}}$. By the construction all the big walls are contained in the parallel boundary $\hat{\mathbf{B}}$, $(\mathcal{F} + U_{d_1}) \subset \hat{\mathbf{B}}$. But we have $\hat{\mathbf{B}} \subset (\mathcal{F} + U_{d_1+1})$ where we consider that 1 is the maximal diameter of polygons $P_I$. The set $\mathbf{B} + U_r$ is a subset of $\hat{\mathbf{B}}$.

For a point $x = \mathbf{p}(\xi)$ let $K(x)$ be the octagon $K + \xi$. The intersections of $K(x)$ with prisms from the refined family $\hat{\mathcal{O}}$ divide $K(x)$ into many regions, each is a convex polygon having sides lying in $\Phi = \bar{\mathrm{p}}(\mathcal{F})$.

**Lemma 2.3:** *There exists $d_2$ such that for regular $\alpha, \beta$ where $E + \alpha$ and $E + \beta$ intersect $K(x)$ at different regions, the $d_2$-maps of $T_\alpha$ and $T_\beta$ at $x$ are different.*

**Proof:** We choose $d_2$ sufficiently large so that the existence domain of every $d_2$-map is lying in some region.□

We claim that we can take $r' = d_2 + 1$. Suppose $T$ is a tiling from $\mathcal{T}(r')$. First we will try to find a lift of $T$ into the refined family $\hat{\mathcal{O}}$. After a shift we may assume that vertices of $T$ are in $\mathbf{p}(\Lambda)$.

The intersections of $K(x)$ with prisms from the refined family $\hat{\mathcal{O}}$ divide $K(x)$ into many regions. For a every region $Z$ and a prism $C$ from the original family $\mathcal{O}$ there are at most one prism $D$ from the refined family $\hat{\mathcal{O}}$ such that $D$ is contained in $C$ and at the same time the region $Z$ is lying in $D$.

Let $P$ be a tile of $T$ and $x$ be a vertex of $P$. There is a prism $C$ from the original family $\mathcal{O}$ projecting into $P$. By definition of the local rule there is a regular $\alpha$ such that $\mathbf{E} + \alpha$ intersects $K(x)$ and the $d_2$-map of $T_\alpha$ at $x$ coincides with the $d_2$-map of $T$ at $x$. Let $Z$ be the region containing the intersection point of $\mathbf{E} + \alpha$ with $K(x)$. Since $\alpha$ is regular this intersection point is an interior point of $Z$. This region $Z$ is unique in the sense that if $\beta$ is regular such that $\mathbf{E} + \beta$ intersects $K(x)$ at a point not in the region $Z$ then the $d_2$-map of $T_\beta$ at $x$ is different from that of $T$ by lemma 2.3. By the above observation there is a unique prism $D$ of the refined family $\hat{\mathcal{O}}$ which projects into $P$ and contains the region $Z$. We see that $D$ contains the set $\alpha + P$. Let this prism $D$ be the lift $l(P)$ of $P$.

We have to check the correctness: the lift of the tile $P$ must not depend on vertex $x$. Suppose $y$ is another vertex of $P$. Then the distance between $x$ and $y$ is less than or equal to the maximal diameter of $P_I$. Hence both $r$-maps of $T$ at $x$ and at $y$ contain $d_2$-map at $x$ and $d_2$-map at $y$. Suppose $Z'$ is the corresponding region of $y$ contained in $K(y)$. There is a regular $\alpha$ such that $\mathbf{E} + \alpha$ intersects the region $Z$ and $r$-map of $T_\alpha$ at $x$ coincides with $r$-map of $T$ at $x$. Since $r$-map at $x$ contains $d_2$ map at $y$ we see that $\mathbf{E} + \alpha$ must intersect $Z'$. We conclude that the lift is uniquely defined.

So far we have defined $l(P)$ for every tile $P$ of $T$. For each vertex $x$ we choose a point $\tilde{x}$ lying inside the region $Z$ defined above. If $y$ is a neighboring vertex of $T$, that is, the segment $[x, y]$ is a side of a tile $P$ of $T$, then by the construction the segment $[\tilde{x}, \tilde{y}]$ is contained in $l(P)$. Consider $T$ as a polygonal structure of $\mathbf{E}$. We transfer this polygonal structure into a simplicial structure by putting a diagonal in every tile of $T$, then linearly lift the simplicial complex to anther (2-dimensional) simplicial complex with the help of $x \to \tilde{x}$ and linearity. The new simplicial complex is a surface lying inside the union of all the prisms $l(P)$. This surface defines a tiling which is exactly $T$ and it does dot meet the parallel boundary $\hat{\mathbf{B}}$ of $\hat{\mathcal{O}}$ and hence does not meet $\mathbf{B} + U_r$. Proposition 2.2 is proved.□

When $r \to \infty$ the above $d_1$ also tends to infinity, hence the maximal diameter of $\bar{\mathrm{p}}(C)$ for prism $C$ from the refined family $\hat{\mathcal{O}}$ tends to zero. We get the following

**Corollary 2.4:** *For every $t_1 > 0$ there is $r, r'$ such that if $T \in \mathcal{T}(r')$ then $T = T_\Omega$ for a section $\Omega$ not meeting $\mathbf{B} + U_r$ and the slop of $\Omega$ is less than $t_1$, that is, if $\Omega$ is the graph of the continuous map $\rho : \mathbf{E} \to \bar{\mathbf{E}}$ then there is a constant $c$ such that*

$$|\rho(x) - \rho(y)| < t_1|x - y| + c$$

*for every $x, y \in \mathbf{E}$.*

### 2.3 Approximation of E

Recall that the intersections $f_i = F_i \cap \mathbf{E}$ and $\bar{f}_i = F_i \cap \bar{\mathbf{E}}$ are lines, for $i = 0, 1, 2, 3$. Choose the coordinate systems $(a_0, a_1)$ in $\mathbf{E}$ and $(b_0, b_1)$ in $\bar{\mathbf{E}}$ such that $f_0$ is given by $\{a_0 = 0\}, f_1$ by $\{a_1 = 0\}, f_2$ by $\{a_0 + a_1 = 0\}$, $\bar{f}_0$ is given by $\{b_0 = 0\}$, $\bar{f}_1$ by $\{b_1 = 0\}$, $\bar{f}_2$ by $\{b_0 + b_1 = 0\}$.

Together $(a_0, a_1, b_0, b_1)$ form a coordinate system of $\mathbb{R}^4$. In this coordinate system the 2-plane $\mathbf{E}$ is given by $b_0 = b_1 = 0$. Let $\mathbf{E}^{(t)}$ be the 2-plane given by $b_0 = ta_0, b_1 = ta_1$ for real $t \in [-1, 1]$. The most important property of these 2-planes is that they intersects all four 2-planes $F_0, F_1, F_2, F_3$ by lines: $\dim(F_i \cap \mathbf{E}^{(t)}) = 1, i = 0, 1, 2, 3$. Note also that the 2-plane $\mathbf{E}^{(t)}$ is either totally irrational or rational. The family of 2-planes $\mathbf{E}^{(t)}$ has been used in [B], [Le2] to prove the absence of local rule for the class of quasiperiodic tilings having 8-fold symmetry.

We have $\mathbb{R}^4 = \mathbf{E} \oplus \mathbf{E}^{(t)}$ for every $t$. Denote $\mathbf{p}^{(t)}$,$\bar{\mathbf{p}}^{(t)}$the corresponding projectors, $\mathbf{p}^{(t)}$ is the projector on $\mathbf{E}^{(t)}$ along $\bar{\mathbf{E}}$, $\bar{\mathbf{p}}^{(t)}$ is the projector on $\bar{\mathbf{E}}$ along $\mathbf{E}^{(t)}$. A set $A$ in $\mathbb{R}^4$ is called a $\mathbf{E}^{(t)}$-prism if $A = \mathbf{p}^{(t)}(A) + \bar{\mathbf{p}}^{(t)}(A)$. For example the 2-planes $F_i, i = 0, 1, 2, 3$ are $\mathbf{E}^{(t)}$-prisms for every $t$.

If we use the pair $\mathbf{E}^{(t)}$,$\bar{\mathbf{E}}$ to construct the oblique periodic tiling as in §1.2 then we get a new family of $\mathbf{E}^{(t)}$-prisms $\mathcal{O}^{(t)}$. That is, consider six $\mathbf{E}^{(t)}$-prisms $C_I^{(t)}$ where $I \in M$ and $C_I^{(t)} = \mathbf{p}^{(t)}(\gamma_I) - \bar{\mathbf{p}}^{(t)}(\gamma_I)$. These six $\mathbf{E}^{(t)}$-prisms and their translates by vectors from $\Lambda$ form a cover of $\mathbb{R}^4$ without holes and overlaps. This family of $\mathbf{E}^{(t)}$-prisms is invariant under translations by vectors from $\Lambda$. The parallel boundary is denoted by $\mathbf{B}^{(t)}$. It is easy to see (as in the case $t = 0$) that the projection of $\mathbf{B}^{(t)}$ on $\bar{\mathbf{E}}$ (by $\bar{\mathbf{p}}^{(t)}$) is the projection of $\mathcal{F}$ on $\bar{\mathbf{E}}$.

If $\alpha$ is a point in $\bar{\mathbf{E}}$ not lying in $\bar{\mathrm{p}}^{(t)}(\mathcal{F})$ then the 2-plane $\mathbf{E}^{(t)}+\alpha$ does not meet the parallel boundary $\mathbf{B}^{(t)}$ and by projecting on $\mathbf{E}$ the intersections of $\mathbf{E}^{(t)}+\alpha$ with members of the family $\mathcal{O}^{(t)}$ we get a tiling, called the tiling defined by $\alpha$ and $\mathcal{O}^{(t)}$. The set of all such tilings (when $\alpha$ varies in the set $\bar{\mathbf{E}}$ and not belongs to $\bar{\mathrm{p}}^{(t)}(\mathcal{F})$), their translates and their limits is denoted by $\overline{\mathcal{T}}^{(t)}$ and a tiling of this set is called a quasiperiodic tiling associate with $\mathbf{E}^{(t)}$. Note that all they are tilings of $\mathbf{E}$, and every tile is conruent to one of the six $P_I$.

If $\mathbf{E}^{(t)}$ is *rational* then every tiling from $\overline{\mathcal{T}}^{(t)}$ is periodic, and up to translations there are only a finite number of tilings from $\overline{\mathcal{T}}^{(t)}$. We consider a larger class of tilings $\tilde{\mathcal{T}}^{(t)}$ as follow. When $\mathbf{E}^{(t)}$ is rational the set $\bar{\mathrm{p}}^{(t)}(\Lambda)$ is a discrete 2-dimensional lattice in $\bar{\mathbf{E}}$, and $\bar{\mathrm{p}}^{(t)}(\mathcal{F})$ is the union of four families of parallel lines, each family is locally finite. If $\alpha \in \bar{\mathrm{p}}^{(t)}(\mathcal{F})$ then all the possible intersections of $\mathbf{E}^{(t)}+\alpha$ with members of the family $\mathcal{O}^{(t)}$ cover the whole $\mathbf{E}^{(t)}+\alpha$, but with overlaps. We can delete some of them such that the remaining form a proper tiling of $\mathbf{E}^{(t)}+\alpha$, and hence of $\mathbf{E}$ by projecting. Let $\tilde{\mathcal{T}}^{(t)}$ be the set of all such tilings and their translates. Other words, $T \in \tilde{\mathcal{T}}^{(t)}$ if and only if after a shift $T$ has a lift into $\mathcal{O}^{(t)}$, $l$ : {tiles of $T$} $\rightarrow$ {prisms from $\mathcal{O}^{(t)}$}, such that all the projections $\bar{\mathrm{p}}^{(t)}(l(P))$ have a common point where $P$'s are all the tiles of $T$. Perhaps a tiling from $\tilde{\mathcal{T}}^{(t)}$ should be called "pseudo-periodic".

For $t$ such that $\mathbf{E}^{(t)}$ is totally irrational let $\tilde{\mathcal{T}}^{(t)}$ be simply the set $\overline{\mathcal{T}}^{(t)}$. We will prove that if $r$ is large and $T \in \mathcal{T}(r)$ then $T$ is a tiling from $\tilde{\mathcal{T}}^{(t)}$ for some $t$ very close to 0.

For a prism $C$ from $\mathcal{O}^{(t)}$ its parallel boundary is the sum of $\partial(\bar{\mathrm{p}}^{(t)}(C))$ and $\mathrm{p}^{(t)}(C)$. The first term is the boundary of a polygon. We call the sum of a side of this polygon and $\mathrm{p}^{(t)}(C)$ *a small wall of* $C$. The parallel boundary $\mathbf{B}^{(t)}$ is the union of all the small walls of all prisms from $\mathcal{O}^{(t)}$.

If $\Omega$ is a section not meeting $\mathbf{B}^{(t)}$ then by projecting on $\mathbf{E}$ the intersections of $\Omega$ with members of the family $\mathcal{O}^{(t)}$ we get a tiling, called the tiling defined by $\Omega$ and $\mathcal{O}^{(t)}$.

Recal that $U_r$ is the disk in $\mathbf{E}$ with center at $\mathbf{0}$ and radius $r$. Let $U_r^{(t)}$ be the projection of $U_r$ on $\mathbf{E}^{(t)}$, $U_r^{(t)} = \mathrm{p}^{(t)}(U_r)$. We have seen that if $w$ is a small wall of $\mathcal{O}$ and $\bar{\mathrm{p}}(w)$ lies on the line $\bar{f} = \bar{\mathrm{p}}(F)$ for some 2-plane $F$ from $\mathcal{F}$ then $\mathrm{p}(w)$ and $\mathrm{p}(F)+U_{r_0}$ have non-empty interior intersection. This means the small wall and the set $F+U_{r_0}$ have non-empty intersection. It is easy to see that when $t$ is small enough then this property still hold: if $w$ is a small wall of a prism from $\mathcal{O}^{(t)}$ and $\bar{\mathrm{p}}^{(t)}(w)$ is lying on the line $\bar{\mathrm{p}}^{(t)}(F)$ for some 2-plane $F$ from $\mathcal{F}$ then the small wall $w$ and the set $F+U_{r_0}^{(t)}$ have non-empty intersection. Suppose $t_0$ is a number such that for $|t| < t_0$ this property holds. Fix such $t_0$.

**Proposition 2.5**:*Suppose $\Omega$ is a section not meeting $\mathcal{F}+U_{r_0}^{(t)}$ and not meeting $\mathbf{B}^{(t)}$. If $\bar{\mathrm{p}}^{(t)}(\Omega)$ is bounded then the tiling defined by $\Omega$ and $\mathcal{O}^{(t)}$ is a quasiperiodic tiling associate with $\mathbf{E}^{(t)}$, that* *is , it belongs to* $\tilde{\mathcal{T}}^{(t)}$.

**Proof**: First note that the set $\mathcal{F}_0$ is bootstrapped by $\mathcal{F}_1$ and $\mathcal{F}_3$, that means if $F, F'$ are two 2-planes from $\mathcal{F}_0$ then there are 2-planes $H_i, i \in \mathbb{Z}$, $H_{2j+1}$ lies in $\mathcal{F}_1$, $H_{2j}$ lies in $\mathcal{F}_3$ such that $F, H_{2j}, H_{2j+1}$ have a common point, $F', H_{2j+1}, H_{2j+2}$ have a common point for every $j \in \mathbb{Z}$. This can be checked easily, for example, by considering the projections (by $\bar{\mathbf{p}}$) on $\bar{\mathbf{E}}$ of these family of 2-planes. The set $\mathcal{F}_1$ is bootstrapped by $\mathcal{F}_0$ and $\mathcal{F}_2$, the set $\mathcal{F}_3$ is also bootstrapped by $\mathcal{F}_0$ and $\mathcal{F}_2$.

We can regard $\Omega$ as the graph of a continuous map $\rho : \mathbf{E}^{(t)} \to \bar{\mathbf{E}}$, that is, $\Omega = \{x + \rho(x) | x \in \mathbf{E}^{(t)}\}$. Since $F + U^{(t)}_{r_0}$ does not meet $\Omega$ where $F$ is a 2-plane from $\mathcal{F}$ and since $F + U^{(t)}_{r_0}$ is a $\mathbf{E}^{(t)}$-prism, it is easy to check that the set $\rho(F + U^{(t)}_{r_0})$ does not meet the line $h = \bar{\mathbf{p}}^{(t)}(F) = \bar{\mathbf{p}}^{(t)}(F + U^{(t)}_{r_0})$. A set of type $F + U^{(t)}_{r_0}$ is called here a big wall.

Suppose now $\mathbf{E}^{(t)}$ is totally irational. Then analogue of proposition 1.8 holds true, that is, if $F, F'$ are two different 2-planes from $\mathcal{F}$ then their projections $\bar{\mathbf{p}}^{(t)}(F), \bar{\mathbf{p}}^{(t)}(F')$ are different and if three 2-planes have projections on $\bar{\mathbf{E}}$ having a common point then they themselves have a common point. In adition, if $w$ is a small wall of a prism $C$ from $\mathcal{O}^{(t)}$ and $\bar{\mathbf{p}}^{(t)}(w)$ lies on the lines $\bar{\mathbf{p}}^{(t)}(F)$ then the small wall $w$ and the set $F + U^{(t)}_{r_0}$ have non-empty intersection. Hence the proof of theorem 1.1 can be applied and we see that the tiling defined by $\Omega$ and $\mathcal{O}^{(t)}$ belongs to $\overline{\mathcal{T}}^{(t)} = \tilde{\mathcal{T}}^{(t)}$.

Now if $\mathbf{E}^{(t)}$ is not totally irrational (then it can be checked that $\mathbf{E}^{(t)}$ is rational). The trouble is that in this case there may be different 2-planes $F, F'$ from $\mathcal{F}$ such that $\bar{\mathbf{p}}^{(t)}(F), \bar{\mathbf{p}}^{(t)}(F')$ are the same. For the correctness we will say about "orientation" of 2-planes from $\mathcal{F}$. By orientation of a 2-plane $F$ from $\mathcal{F}$ (or the orientation of the big wall $F + U^{(t)}_{r_0}$) we mean the orientation of the line $\bar{\mathbf{p}}^{(t)}(F) = \bar{\mathbf{p}}^{(t)}(F + U^{(t)}_{r_0})$ such that $\rho(\mathbf{p}(F + U^{(t)}_{r_0})) > \bar{\mathbf{p}}^{(t)}(F)$. If $h$ is a line from $\bar{\mathbf{p}}^{(t)}(\mathcal{F})$ and $h = \bar{\mathbf{p}}^{(t)}(H)$ then *the orientation of $h$ induced from $H$*, by definition, is just the orientation of $H$. Several 2-planes from $\mathcal{F}$ (or several big walls) are *compatible* if their projections on $\bar{\mathbf{E}}$ (by $\bar{\mathbf{p}}^{(t)}$) are compatible. We call a line from $\bar{\mathbf{p}}^{(t)}(\mathcal{F})$ *special* if there are two 2-planes from $\mathcal{F}$ projecting into this line and induce different orientaions.

First note that if three big walls have a common point then they are compatible. Hence if $F, F'$ are two parallel 2-planes from $\mathcal{F}$ such that $\bar{\mathbf{p}}^{(t)}(F) \neq \bar{\mathbf{p}}^{(t)}(F')$ then they are compatible. This follows from the bootstrapped property as in the case $t = 0$. Hence in the system of parallel lines $\bar{\mathbf{p}}^{(t)}(\mathcal{F}_i)$ there is at most one special lines, for a fixed $i \in \{0, 1, 2, 3\}$. Excluding the special lines and then applying the proof of theorem 1.1 we see that all the lines from $\bar{\mathbf{p}}^{(t)}(\mathcal{F})$ are compatible, except special lines. It follows that there is a point $\alpha$ greater than or equal to every lines from $\bar{\mathbf{p}}^{(t)}(\mathcal{F})$ and every special line goes through $\alpha$. This means the projection on $\bar{\mathbf{E}}$ (by $\bar{\mathbf{p}}^{(t)}$) of every prism from $\mathcal{O}^{(t)}$ meeting $\Omega$ contains $\alpha$. Hence the tiling defined by $\Omega$ and

$\mathcal{O}^{(t)}$ belongs to $\tilde{\mathcal{T}}^{(t)}$.□

## 2.4 Local structure of $\mathcal{T}$

**Lemma 2.6:** *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying $|f(x+y) - f(x) - f(y)| < 1$ for every $x, y \in \mathbb{R}$ and $|f(x) - f(y)| < 1$ when $|x - y| < 1$ then there is $a \in \mathbb{R}$ such that $|f(x) - ax| < 11$.*
**Proof:** Let $a$ be the upper limit of $\frac{f(x)}{x}$ when $x \to \infty$, this exists due to the second condition. Set $h(x) = f(x) - ax$. First we prove that for $x > 0$ the value of $h(x)$ is less than 2, $h(x) < 2$. In fact, if $h(x) \geq 2$ then $h(2x) \geq (2h(x) - 1) = 3, \ldots, h(2^n x) \geq (2^{n-1} + 1)$. Hence the upper limit of $\frac{h(x)}{x}$ is not less than 1/2x, a contradiction.

Now we prove that there is a sequence $x_1, x_2, \ldots$, tending to infinity such that $h(x_i) > -2$. If not such the case, then there is $r$ such that $h(x) < -2$ for every $x > r$. Then $h(x) < -3$ for $r < x < 2r$, $h(x) < -5$ for $2r < x < 4r, \ldots, h(x) < -(2^{n-1} + 1)$ for $2^n r < x < 2^{n+1} r$. Hence the upper limit of $\frac{h(x)}{x}$ when $x \to \infty$ is less than $-r/4$ and can not be 0.

Now if $x > 0$ then we can choose $x_n > x$ such that $h(x_n) > -2$. Let $y = x_n - x$ then $h(x_n) - h(x) - h(y) < 1$ and $h(y) < 2$ hence $h(x) > -5$. We see that for $x > 0$ $|h(x)| < 5$.

If $x < 0$, choose $y, z > 0$ such that $y = x + z$, using the fact that $|f(y) - f(x) - f(z)| < 1$ one easily see that $|f(x)| < 11$. □

A corollary of theorem 3.2.1 and proposition 3.3.2 of [Le2] is the following
**Proposition 2.7:** *There is $t_1, 0 < t_1 < t_0$ such that if $|t| < t_1$ then there is a continuous map $\varphi^{(t)} : \mathbb{R}^4 \to \mathbb{R}^4$ such that*

*a)* $\varphi^{(t)}(x + \bar{\mathbf{E}}) = x + \bar{\mathbf{E}}$ *for every* $x \in \mathbb{R}^4$.

*b)* $\varphi^{(t)}(\mathcal{F} + U_{r_0}) = \mathcal{F} + U_{r_0}$, $(\varphi^{(t)})^{-1}(\mathcal{F} + U_{r_0}) = \mathcal{F} + U_{r_0}$, $\varphi^{(t)}(\mathcal{F}) = \mathcal{F}$, $(\varphi^{(t)})^{-1}(\mathcal{F}) = \mathcal{F}$.

*c) If $C$ is a prism of $\mathcal{O}$ then $\varphi^{(t)}(C)$ is a prism of $\mathcal{O}^{(t)}$.*

*d)* $|\varphi^{(t)}(x) - x| < const$ *for every* $x \in \mathbb{R}^4$.

It follows that if $\Omega$ is a section not meeting $\mathcal{F} + U_{r_0}$ (hence not meeting $\mathbf{B}$) then the tiling defined by $\Omega$ and $\mathcal{O}$ is the same as the tiling defined by $\varphi^{(t)}(\Omega)$ and $\mathcal{O}^{(t)}$.
**Proposition 2.8:** *There is $r_1$, $r_1 > r_0$ such that if $T \in \mathcal{T}(r_1)$ then there exists $t, |t| < t_1$ such that:*

*a) After a shift $T$ is defined by a section $\Omega$ and $\mathcal{O}$ where $\Omega$ does not meet $\mathcal{F} + U_{r_0}$.*

*b) The set $\varphi^{(t)}(\Omega)$ is bounded in $\bar{\mathbf{E}}$.*

**Proof:** By corollary 2.4 for $r = r_0$ there is $r_1$ such that if $T \in \mathcal{T}(r_1)$ then after a shift $T$ is the tiling defined by a section $\Omega$ and $\mathcal{O}$ where $\Omega$ does not meet $\mathbf{B} + U_{r_0}$. Recall that we have a coordinate system $(a_0, a_1, b_0, b_1$ in $\mathbf{E}$. The section $\Omega$ can be regarded as the graph of a map $\rho : \mathbf{E} \to \bar{\mathbf{E}}$, $\rho$ is defined by two functions $b_0(a_0, a_1), b_1(a_0, a_1)$. The notation $f \equiv g$ for two functions $f, g$ on $\mathbf{E}$ will mean that $max|f - g| < const$. A corollary of the proof of proposition

7 of [LPS1] is there is a function $f$ such that

$$b_0(a_0, a_1) \equiv f(a_0)$$

$$b_1(a_0, a_1) \equiv f(a_1)$$

in addition $|f(x+y) - f(x) - f(y)| < const$ and $|f(x) - f(y)| < const$ if $|x - y| < 1$. By lemma 2.6 there is a real number $t$ such that $|f(x) - tx| < const$. This means $\bar{\mathbf{p}}^{(t)}(\Omega)$ is a bounded set because $\bar{\mathbf{p}}^{(t)}$ is the projector along $\mathbf{E}^{(t)}$ and $\mathbf{E}^{(t)}$ is given by $\{b_0 = ta_0, b_1 = ta_1\}$.

Since the slop of $\Omega$ is less than $t_1$ by corollary 2.4, we have $|t| < t_1$.□

Now we can prove the main result of this section.

**Theorem 2.9:** *There exist $r > 0$ such that if $T$ is a tiling satisfying $\mathcal{B}(r)$, that is, $T \in \mathcal{T}(r)$ then $T$ belongs to $\tilde{\mathcal{T}}^{(t)}$ for some $t$.*

**Proof:** Choose $r = r_1$ of the previous proposition. If $T \in \mathcal{T}(r)$ then after a shift $T = T_\Omega$ for some section $\Omega$ not meeting $\mathcal{F} + U_{r_0}$ and there is $t$ such that $\bar{\mathbf{p}}^{(t)}(\Omega)$ is bounded. Consider $\varphi^{(t)}(\Omega)$. It is also a section. By proposition 2.7 it does not meet $\mathbf{B}^{(t)}$ and $\mathcal{F}$. The projection $\bar{\mathbf{p}}^{(t)}(\varphi^{(t)}(\Omega))$ is bounded by boundedness of $\bar{\mathbf{p}}^{(t)}(\Omega)$ and proposition 2.7d. Hence by proposition 2.5 the tiling defined by $\varphi^{(t)}(\Omega)$ and $\mathcal{O}^{(t)}$ belongs to $\tilde{\mathcal{T}}^{(t)}$. But the tiling defined by $\varphi^{(t)}(\Omega)$ and $\mathcal{O}^{(t)}$ is the same as the tiling defined by $\Omega$ and $\mathcal{O}$, it is $T$. Hence $T$ belongs to $\tilde{\mathcal{T}}^{(t)}$.□

**Remark:** One might hope that for large $r$ and $T \in \mathcal{T}(r)$ the tiling $T$ must belong not only to $\tilde{\mathcal{T}}^{(t)}$ but also to $\overline{\mathcal{T}}^{(t)}$ for some $t$, that is, $T$ must be quasiperiodic in our definition. But it can be proved that for every $r$ there is a tiling $T \in \mathcal{T}(r)$ which does not belong to $\overline{\mathcal{T}}^{(t)}$ for every $t$.

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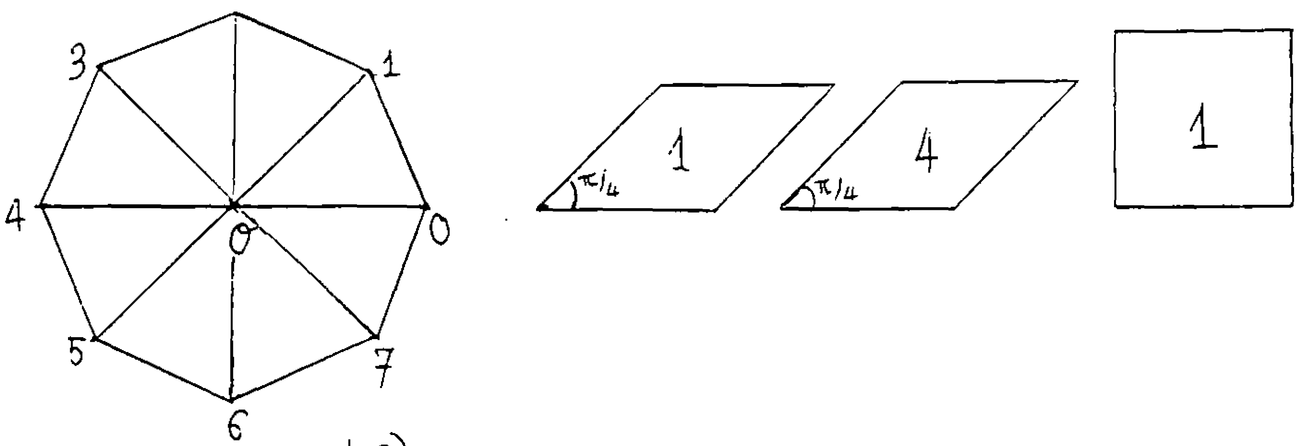
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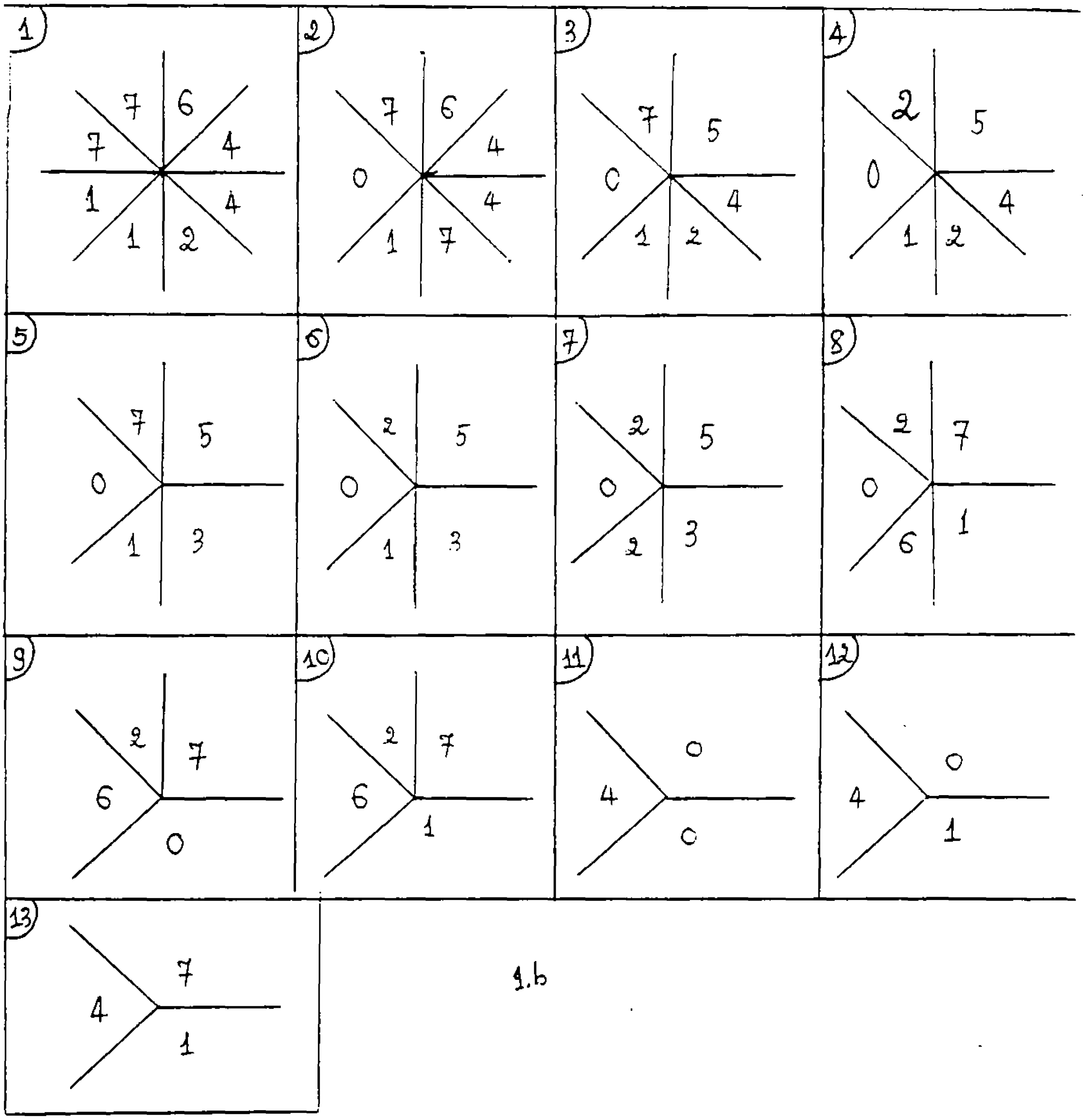
1.a)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1.b

fig. 1

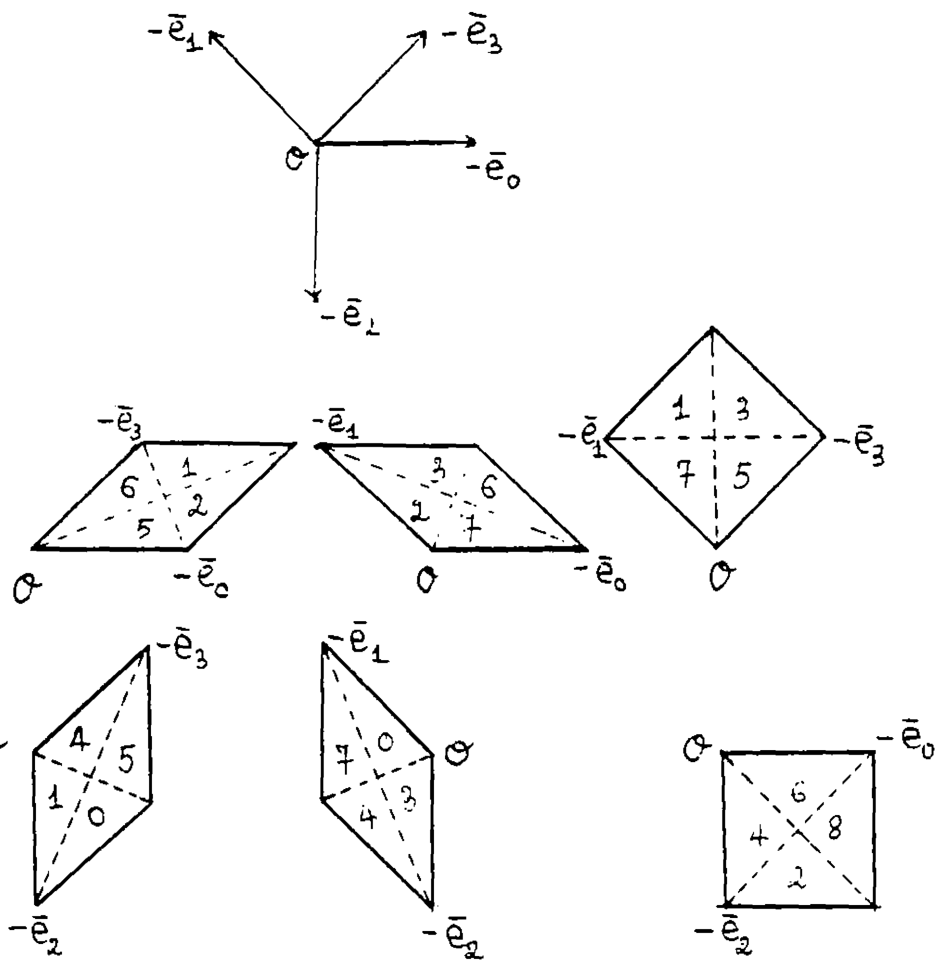
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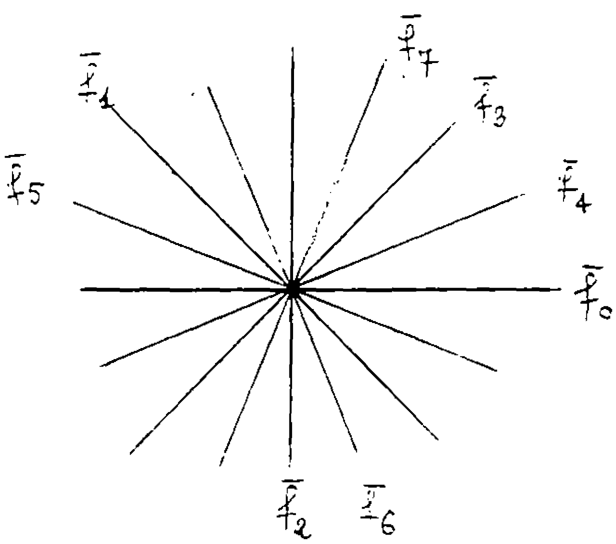
fig. 2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$\bar{f}_2$
$\bar{f}_6$

fig. 3

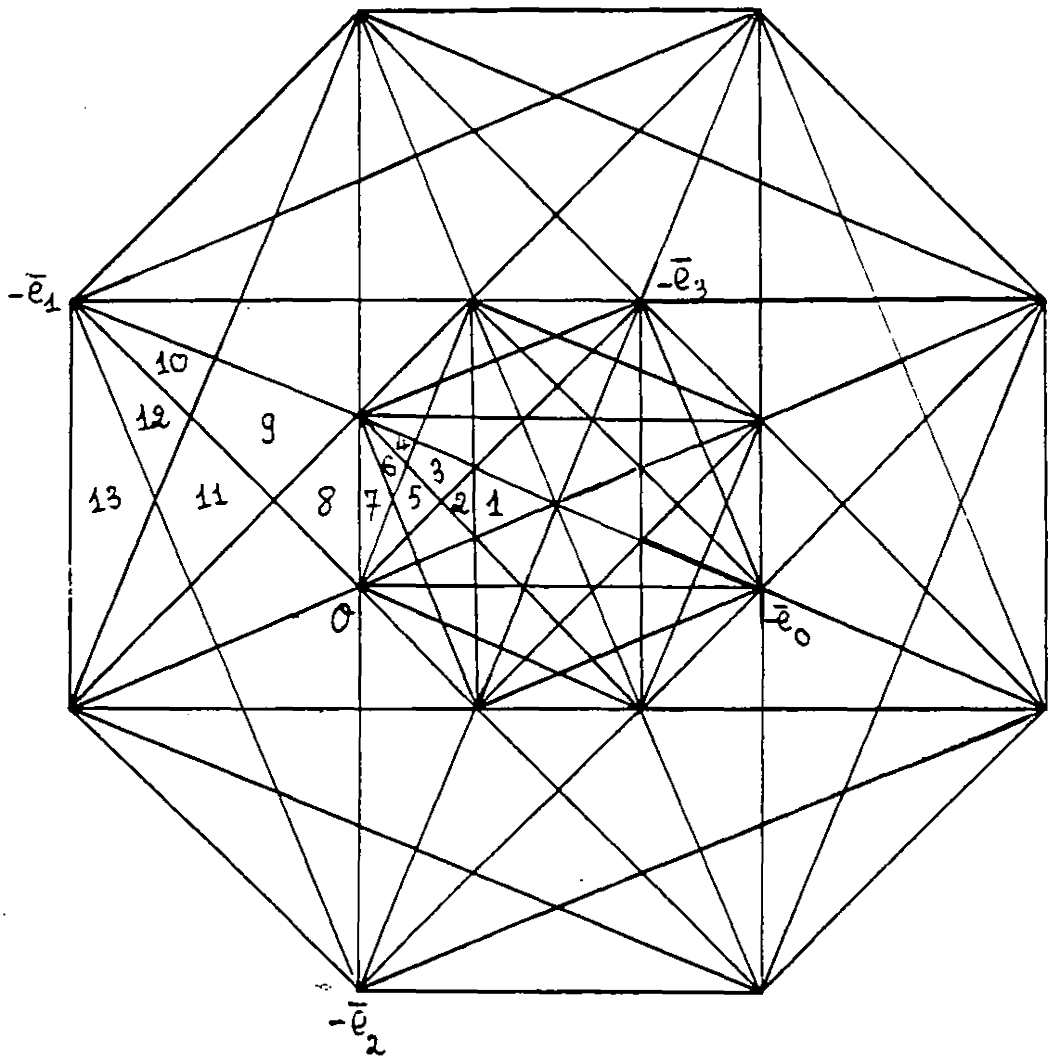


fig. 5

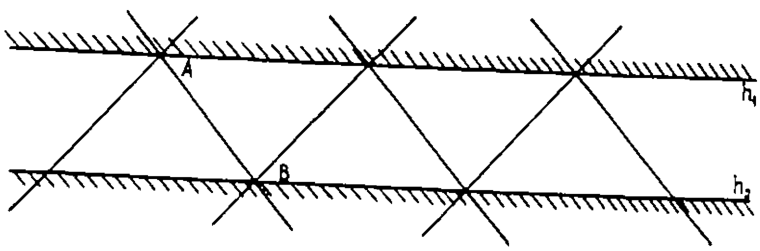


fig. 4

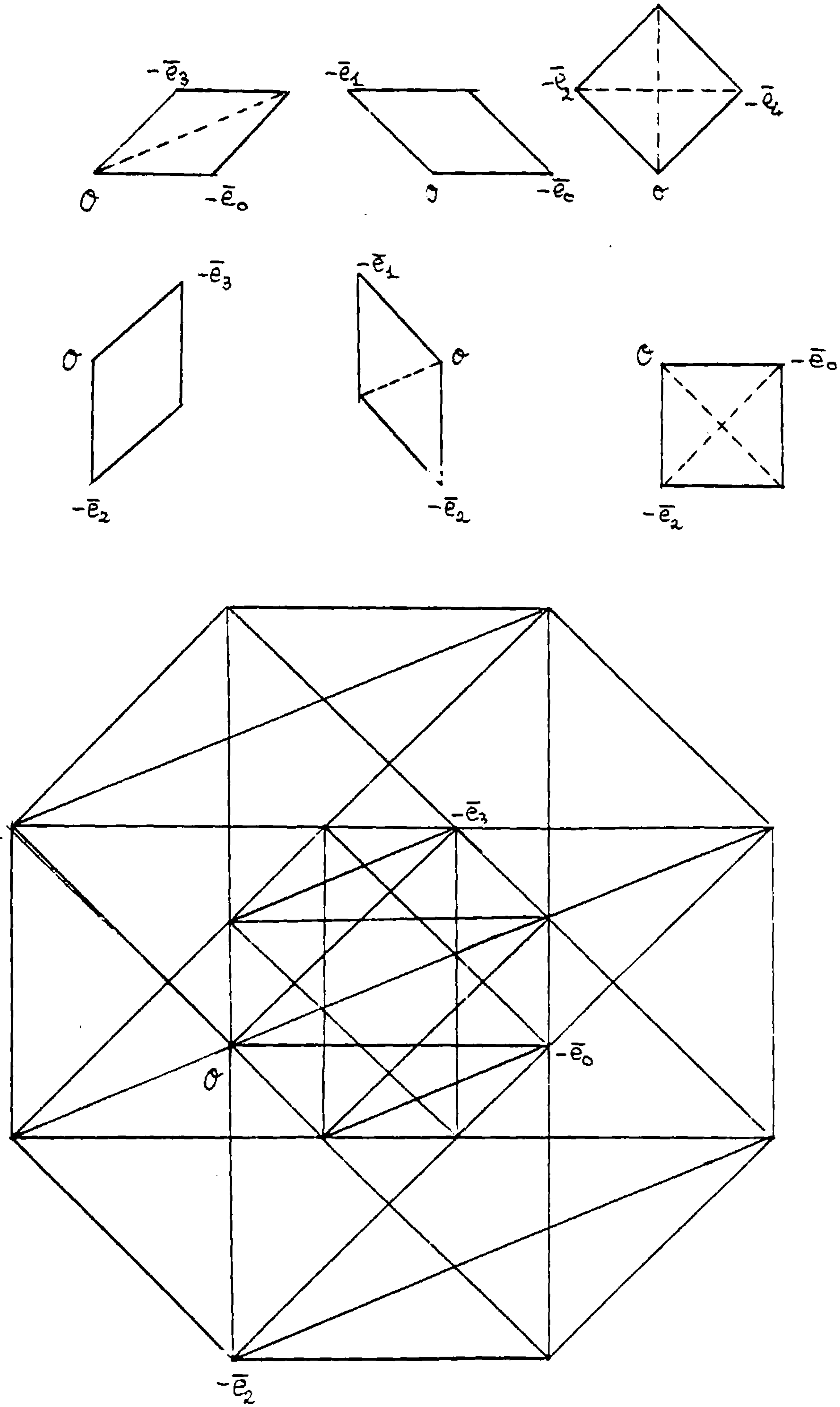


fig. 6