Perturbation theory for quasiperiodic solutions of infinite–dimensional Hamiltonian systems

2. Statement of the main theorem and its consequences

by

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Ball model for Hilbert's twelvth problem

by

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Contents

In this paper we consider Hamiltonian perturbations of an infinite–dimensional linear system with pure imaginary spectrum $\{\pm i\,\tau_j \mid j = 1,2,...\}$. We study perturbations of a quasiperiodic solution of linear system with finitely–many frequencies $\tau_{j_1},...,\tau_{j_n}$ being exited. It is proved that for most values of frequency vector $(\tau_{j_1},...\tau_{j_n})$ this solution is preserved under small Hamiltonian perturbations, if roughly speaking the perturbed system is quasilinear and frequencies τ_j grows linearly or super–linearly when $j \longrightarrow \infty$.

Our theorems have well–known finite–dimensional analogues. The preservation of most quasiperiodic motions with n fundamental frequencies in the integrable 2n–dimensional Hamiltonian system under small Hamiltonian perturbations was proved in the classical works of Kolmogorow, Arnold and Moser (see [AA], [Mo] and their bibliography). Theorems on the preservation of quasiperiodic motions with $k \leq n$ fundamental frequencies were formulated by V. Melnikov. For their proves and discussions see [E,P1]. As an infinite–dimensional analog of our results we want to mention the paper [W] devoted to the perturbed wave equation with random potential.

In this paper we formulate our main theorem and give its applications to some nonlinear equations of mathematical physics. The theorem generalize results of works [K1 – K3]. Its proof will be given in the next paper (part 3 of our text).

The following notations are used: for Hilbert spaces Y and Z the norms are denoted by $|\cdot|_Y$, $|\cdot|_Z$ and inner products by $<\cdot,\cdot>_Y$, $<\cdot,\cdot>_Z$; dist_Z – distance in the space Z.

The usual norms in $\mathbb{R}^n$ and $\mathbb{C}^n$ $(n \geq 1)$ are denoted $|\cdot|$. For metric spaces B_1, B_2, for a subset $Q_1 \subset B_1$ and a mapping $h : Q_1 \longrightarrow B_2$ we denote

$$\text{Lip } h = \text{Lip}(h : Q_1 \longrightarrow B_2) = \sup_{b_1 \neq b_2} \frac{\text{dist}_{B_2}(h(b_1);h(b_2))}{\text{dist}_{B_1}(b_1;b_2)}$$

If the space B_2 is a Banach one with a norm $|\cdot|_{B_2}$, we denote

$$|h|_{B_2}^{Q_1,\text{Lip}} = \max\{\sup_{b \in Q_1} |h(b)|_{B_2}, \text{Lip } h\} . \qquad (0.1)$$

Let B_1, B_2 be Banach spaces with norms $|\cdot|_{B_1}$, $|\cdot|_{B_2}$, let $B_1{}^c$, $B_2{}^c$ be their complexifications, let $V_j{}^c$ be an (open) domain in $B_j{}^c$ $j = 1,2$. We denote by $\mathscr{A}^R(V_1{}^c ; V_2{}^c)$ the set of Fréchet complex–analytical mappings from $V_1{}^c$ to $V_2{}^c$ which map $V_1{}^c \cap B_1$ into $V_2{}^c \cap B_2$. Let M be some metric space. We denote by $\mathscr{A}^R{}_M(V_1{}^c ; V_2{}^c)$ a class of mappings $G : V_1{}^c \times M \longrightarrow V_2{}^c$ with the following properties:

i) $G(\cdot\,; m) \in \mathscr{A}^R(V_1{}^c ; V_2{}^c)$ $\forall m \in M$,

ii) the map $G(b\,; \cdot) : M \longrightarrow V_2{}^c$ is Lipschitz $\forall b \in V_1{}^c$ and

$$|G|_{B_2}^{V_1{}^c ; M} \equiv \sup_{b \in V_1{}^c} |G(b\,;\cdot)|_{B_2}^{M,\text{Lip}} < \infty \qquad (0.2)$$

(the norm in $B_2{}^c$ is denoted by $|\cdot|_{B_2}$).

For domains $V_Y \subset Y$, $V_Z \subset Z$ we use standard notations $C^k(V_Y ; V_Z)$ $(k \in \mathbb{Z}, k \geq 0)$ for the spaces of Fréchet–differentiable mappings $\varphi : V_Y \longrightarrow V_Z$ and notation $\varphi_*(\varphi^*)$ for tangent (cotangent) map.

For abstract sets $\mathfrak{A}$, $\mathcal{J}$, for a subset Θ of their product $\mathfrak{A} \times \mathcal{J}$ and for $I \in \mathcal{J}$ we denote by $\Theta[I]$ a subset of $\mathfrak{A}$ of the form

$$\Theta[I] = \{a \in \mathfrak{A} \mid (a,I) \in \Theta\} \qquad (0.3)$$

In the notations of functions and mappings we sometimes omit a part of arguments; we denote by C, C_1, C_2 etc. different positive constants which arrive at estimates and denote by K, K_1 etc. constants at the assumptions of theorems.

1. Statement of the main theorem

Let $\{Z,\{Z_s \mid s \in \mathbb{R}\}\ ,\ \alpha = < J^Z dz\ ,\ dz >_Z\}$ be a symplectic Hilbert scale as it was defined in [1]. It means that Z is a Hilbert space, $\{Z_s\}$ is a scale of Hilbert spaces with norms $\|\cdot\|_s$ and inner products $<\cdot,\cdot>_s$, $Z_{s_1} \subset Z_{s_2}$ if $s_1 \geq s_2$, Z_s^* is adjoint to Z_s with respect to scalar product $<\cdot,\cdot>_0$ and $Z_0 = Z$. J^Z is an isomorphism of scale $\{Z_s\}$ of order $-d_J \leq 0$, i.e. $J^Z : Z_s \xrightarrow[\sim]{} Z_{s+d_J}$ $\forall s \in \mathbb{R}$. Operator $J^Z : Z \longrightarrow Z_{d_J} \subset Z$ is supposed to be antisymmetric in Z. Operator $\bar{J}^Z = -(J^Z)^{-1}$ is an isomorphism of the scale $\{Z_s\}$ of order d_J, its restriction on Z is aniselfadjoint (and possibly unbounded). The 2–form $\alpha = < J^Z dz\ ,\ dz >_Z$,

$$< J^Z dz\ ,\ dz >_Z [z_1,z_2] \equiv < J^Z z_1, z_2 >_Z,$$

is continous, antisymmetric and nondegenerate in any space Z_s, $s \geq 0$. Now every $Z_s (s \geq 0)$ is a linear symplectic space. See [1] for more details.

Let us suppose that operator J^Z depends on a vector–parameter $a \in \mathfrak{A}$, $\mathfrak{A}$ is a bounded open domain in $\mathbb{R}^n$. So the symplectic form α depends on the parameter a, too. Let $A^Z(a)$ be a self–adjoint in Z operator depending on $a \in \mathfrak{A}$ and let $\forall a \in \mathfrak{A}$ $A^Z(a)$ defines an isomorphism of the scale $\{Z_s\}$ of order $d_A \geq 0$,

$$A^Z(a) : Z_s \xrightarrow[\sim]{} Z_{s-d_A} \qquad \forall s \in \mathbb{R} \tag{1.1}$$

Let us suppose that there exists a basis $\{\varphi_j^{\pm} \mid j \geq 1\}$ of the space Z with the following properties:

i) there exist positive numbers $\lambda_j^{(s)}$, $s \in \mathbb{R}$, $j \in \mathbf{N}$, such that $\lambda_j^{(-s)} = (\lambda_j^{(s)})^{-1}$ $\forall j,s$,

$$K^{-1} j^s \leq \lambda_j^{(s)} \leq K j^s \quad \forall j \geq 1, \forall s \in \mathbf{R}, \tag{1.2}$$

and $\{\varphi_j^{\pm} \lambda_j^{(-s)} \mid j \geq 1\}$ is a Hilbert basis of the space Z_s $\forall s \in \mathbf{R}$, i.e.
$< \varphi_j^{\sigma_1} \lambda_j^{(-s)}, \varphi_k^{\sigma_2} \lambda_k^{(-s)} >_{Z_s} = \delta_{j,k}\, \delta_{\sigma_1,\sigma_2}$ $\forall\, j,k \in \mathbf{N}$, $\forall\, \sigma_1,\sigma_2 = \pm$;

ii) $$J^Z(a)\, \varphi_j^{\pm} = \mp \lambda_j^J(a)\, \varphi_j^{\mp} \quad \forall j \geq 1, \forall a \tag{1.3}$$

$$A^Z(a)\, \varphi_j^{\pm} = \lambda_j^A(a)\, \varphi_j^{\pm} \; \forall j, \forall a, \tag{1.3'}$$

Here real numbers λ_j^J, λ_j^A are positive for j large enough:

$$\lambda_j^A(a) > 0,\; \lambda_j^J(a) > 0 \;\forall a,\;\; \forall j \geq j_0 \tag{1.4}$$

Let us consider a hamiltonian

$$\mathscr{H}(z;a,\varepsilon) = \frac{1}{2} < A^Z(a)\, z, z >_Z + \varepsilon\, H(z;a,\varepsilon)$$

depending on a parameter $a \in \mathfrak{A}$ and a small parameter $\varepsilon \in [0,1]$. Corresponding Hamiltonian equation (with respect to 2–form $\alpha(a)$) has the form

$$\dot{z} = J^Z(a)\, (A^Z(a) z + \varepsilon\, \nabla H(z;a,\varepsilon)) . \tag{1.5}$$

Here and in what follows, ∇ is the gradient in $z \in Z$ with respect to the scalar product

$<\cdot,\cdot>_Z$. Equation (1.5) is a perturbation of linear Hamiltonian equation

$$\dot{z} = J^Z(a)A^Z(a)z \tag{1.6}$$

In view of conditions (1.3), (1.3′) the spectrum of operator $J^Z(a)\,A^Z(a)$ is purely imaginary,

$$\sigma(J^Z(a)A^Z(a)) = \{\pm\, i\, \lambda_j(a) \mid j \geq 1\}\ ,\ \lambda_j(a) = \lambda_j^{\,J}(a)\ \lambda_j^{\,A}(a)\ .$$

It is supposed that the functions

$$a \longmapsto \lambda_j^J(a)\ ,\ \ a \longmapsto \lambda_j^A(a)\ ,\ \ j \leq n\ ,$$

are C^2–smooth and for $j \leq n$, $\alpha \in \mathbb{Z}^n$, $|\alpha| \leq 2$

$$|\,\partial_a^\alpha\, \lambda_j^J(a)\,| + |\,\partial_a^\alpha\, \lambda_j^A(a)\,| \leq K_1 \tag{1.7}$$

and the mapping $a \longmapsto \omega = (\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n$ is nondegenerate at some point $a_0 \in \mathfrak{A}$,

$$|\det(\partial\omega_j/\partial a_k)(a_0)\,| \geq K_0 > 0\ (j = 1,\dots,n)\ . \tag{1.8}$$

Let us denote

$$\lambda_{j0}^A = \lambda_j^A(a_0)\ ,\ \lambda_{j0}^J = \lambda_j^J(a_0)\ ,\ \lambda_{j0} = \lambda_j(a_0),\ \omega_0 = \omega(a_0) \tag{1.9}$$

Let us set $Z^0 \subset Z$ be a 2n–dimensional linear span of the vectors $\{\varphi_j^\pm \mid j \leq n\}$. The space Z^0 is foliated into tori $T(I)$ which are invariant for linear equation (1.6),

$$T(I) = \{ \sum_{j=1}^{n} \alpha_j^+ \varphi_j^+ + \alpha_j^- \varphi_j^- \mid {\alpha_j^+}^2 + {\alpha_j^-}^2 = 2\ I_j \geq 0\ ,\ 1 \leq j \leq n \}\ . \tag{1.9'}$$

A torus $T(I)$ with $I_j > 0$ $\forall j$ is n–dimensional, $T(I) \simeq \mathbf{T}^n$ and it is filled with quasiperiodic solutions of the form

$$\dot{q} = \omega(a)\ . \tag{1.10}$$

Here q is a coordinate on $T(I)$,

$$q_j = \mathrm{Arg}(\alpha_j^+ + i\ \alpha_j^-)\ ,\ j = 1,\dots,n\ .$$

Let us set

$$\sum\nolimits_I^0 : \mathbf{T}^n \longrightarrow Z^0 \subset Z$$

be an imbedding identifying a point of $\mathbf{T}^n$ with a point of $T(I)$ having the same coordinates.

Let us consider a family of tori $\{T(I) \mid I \in \mathcal{J}\}$ where

$$\mathcal{J} \subset \{I \in \mathbb{R}^n \mid K_1^{-1} \leq I_j \leq K_1\ ,\ j = 1,\dots,n\} \tag{1.11}$$

is some Borel set (possibly $\mathcal{J}$ consists of the only point, $\mathcal{J} = \{I_0\}$). Let us denote

$$\mathcal{T} = \cup\{T(I) \mid I \in \mathcal{J}\}\ .$$

Let us fix some number d ,

$$d_A/2 \leq d \ , \tag{1.12}$$

and let choose a domain O_d^c in the complexification of the space Z_d, $O_d^c \subset Z_d^c = Z_d \underset{\mathbb{R}}{\otimes} \mathbb{C}$, such that $\mathcal{T} \subset O_d^c$ and

$$\mathrm{dist}_{Z_d}(\mathcal{T}; Z_d^c \backslash O_d^c) \geq K_1^{-1} \ . \tag{1.13}$$

We suppose that the function H may be extended to a function $H : O^c \times \mathfrak{A} \times [0,1] \longrightarrow \mathbb{C}$ which is complex–analytical on $z \in O_d^c$ and Lipschitz on $a \in \mathfrak{A}$, i.e. $H \in \mathcal{A}_{\mathfrak{A}}^{R}(O_d^c;\mathbb{C}) \ \forall \epsilon$.

<u>Theorem 1.1</u>. Let the conditions mentioned above hold together with

1) (analyticity and quasilinearity): for some $d_H \in \mathbb{R}$ such that

$$d_H \leq d_A - 1 \ , \ d_J + d_H \leq 0 \ , \tag{1.14}$$

and for all $\epsilon \in [0,1]$

$$|H(\cdot;\cdot,\epsilon)|^{O_d^c;\mathfrak{A}} \leq K_1, \ \ |\nabla H(\cdot;\cdot,\epsilon)|^{O_d^c;\mathfrak{A}}_{Z_{d-d_H}} \leq K_1 \tag{1.15}$$

(see (0.2));

2) (spectral asymptotics):

$$d_1 \equiv d_A + d_J \geq 1$$

and there exists an asymptotic expansion for the frequencies λ_{j0} , $j \longrightarrow \infty$:

$$|\lambda_{j0} - K_2 j^{d_1} - K_2^1 j^{d_{1,1}} - \ldots - K_2^{r-1} j^{d_{1,r-1}}| \leq K_1 j^{d_{1,r}} , \tag{1.16}$$

here $K_2 > 0$, $r \geq 1$ and $d_1 > d_{1,1} > \ldots > d_{1,r}$, $d_1 - 1 > d_{1,r}$;

$$K_1^{-1} j^{d_A} \leq |\lambda_j^A(a)| \leq K_1 j^{d_A} \quad \forall j \geq 1 , \tag{1.17}$$

$$K_1^{-1} j^{d_J} \leq |\lambda_j^J(a)| \leq K_1 j^{d_J} \quad \forall j \geq 1 ; \tag{1.18}$$

a Lipschitz constants of functions λ_j^A , λ_j^J , λ_j are bounded above:

$$\text{Lip}\ \lambda_j^A \leq K_1 j^{d_A} , \text{Lip}\ \lambda_j^J \leq K_1 j^{d_J} , \text{Lip}\ \lambda_j \leq K_1 j^{d_{1,r}} ; \tag{1.19}$$

Then there exist integers j_1, M_1 such that if condition

3) (nonresonance):

$$\begin{gathered} |\ell_1 \lambda_{10} + \ell_2 \lambda_{20} + \ldots + \ell_{j_1} \lambda_{j_1 0}| \geq K_3 > 0 \\ \forall \ell \in \mathbb{Z}^{j_1} \quad |\ell| \leq M_1 , 1 \leq |\ell_{n+1}| + \ldots + |\ell_{j_1}| \leq 2 \end{gathered} \tag{1.20}$$

is satisfied, then for sufficiently small $\epsilon > 0$ there exist $\delta_* > 0$ (sufficiently small and independent from ϵ), Borel set $\Theta_\epsilon^{a_0}$ of vectors (a,I) ,

$$\Theta_\epsilon^{a_0} \subset \Theta^{a_0} \equiv \mathfrak{A}(a_0,\delta_*) \times \mathcal{J}, \quad \mathfrak{A}(a_0,\delta_*) \equiv \{a \in \mathfrak{A} \mid |a-a_0| < \delta_*\} , \tag{1.21}$$

and analytical embeddings

$$\sum\nolimits_{(a,I)}^{\epsilon} : \mathbf{T}^n \longrightarrow Z_{d_c}, \quad (a,I) \in \Theta_\epsilon^{a_0}, \quad d_c = d+d_A-d_H-1 \tag{1.22}$$

with the following properties a)–d):

$$\text{a) } \operatorname{mes} \Theta_\epsilon^{a_0}[I] \underset{\epsilon \longrightarrow 0}{\longrightarrow} \operatorname{mes} \mathfrak{A}(a_0,\delta_*) \tag{1.23}$$

uniformly with respect to $I \in \mathcal{J}$ (see (0.3));

b) the mapping

$$\sum\nolimits^{\epsilon} : \mathbf{T}^n \times \Theta_\epsilon^{a_0} \longrightarrow Z_{d_c}, \quad (q,a,I) \longmapsto \sum\nolimits_{(a,I)}^{\epsilon}(q)$$

is Lipschitz and is close to the mapping $\sum^0 : (q;a,I) \longmapsto \sum_I^0(q)$

$$|\sum^{\epsilon} - \sum^{0}|_{Z_{d_c}}^{T^n \times \Theta_{\epsilon}^{a_0}, Lip} \leq C\,\epsilon\,;$$

c) every torus $\sum^{\epsilon}_{(a,I)}(T^n)$, $(a,I) \in \Theta_{\epsilon}^{a_0}$, is invariant for the equation (1.5) and is filled with weak in Z_d solutions of (1.5) of the form $z^{\epsilon}(t) = \sum^{\epsilon}_{(a,I)}(q+\omega' t)$, here $q \in \mathbb{T}^n$, $\omega' = \omega'(a,I,\epsilon) \in \mathbb{R}^n$ and $|\omega - \omega'| \leq C\epsilon$;

d) all Liapunov exponents of solutions $z^{\epsilon}(t)$ are equal to zero.

The theorem will be proved in a part 3 of the text. Indeed we shall formulate and prove more general result applicable to some systems with $d_H + d_J > 0$ which are of physical interest.

An immediate consequence of the stated result is a strong averaging principle for nonresonant systems of form (1.5):

Corollary 1.2. Under the assumptions of Theorem 1.1 for every $(a,I) \in \Theta_{\epsilon}^{a_0}$, $q \in \mathbb{T}^n$, and for all t a curve $t \longmapsto \sum^{0}_{I}(q+\omega' t)$ for ϵ small enough is $C\epsilon$–close to some weak solution of (1.5). Here ω' is an averaged frequency vector, $|\omega' - \omega| \leq C\epsilon$.

Remarks. 1) From the second estimate in (1.15) one can see that the order of nonlinear operator in equation (1.5) is equal to $d_J + d_H$. The order of linear one is equal to $d_J + d_A$. So the condition (1.14) of theorem 1.1 indeed means the quasilinearity of equation (1.5) because the order of linear term exceeds the order of nonlinear one at least by one.

2) If $d_a \le d_c - d_1 = d - d_J - d_H - 1$ then the r.h.s. in (1.5) with $z(t) = z^\epsilon(t)$ belongs to $C([0,T];Z_{d_a})$. So $z^\epsilon \in C^1([0,T];Z_{d_a})$ is a strong in Z_{d_a} solution of (1.5).

3) The numbers j_1, M_1 in the assumption 3) of Theorem 1.1 depends on K, K_0–K_2, K_2^j, d_1, $d_{1,j}$, d_A, d_J, d_H, d, n and j_0 only. The maximal possible values of ϵ, δ_* and the rate of convergence in (1.23) depends on the same quantities and on K_3.

4) All the tori $\sum^\epsilon_{(a,I)}(\mathbf{T}^n)$ are isotropic, i.e. $\left[\sum^\epsilon_{(a,I)}\right]^* \alpha = 0 \ \forall (a,I) \in \Theta^{a_0}_\epsilon$.

5) The frequencies $\{\lambda_{j0}\}$ are ordered asymptotically only (see (1.16)). So for a space Z^0 one can choose any 2n–dimensional invariant subspace of operator $J(a)A(a)$.

6) If instead of the condition $d_1 \ge 1$ a weaker condition $d_1 > 0$ takes place then the statements of Theorem 1.1 seems to be wrong in a general case. But the statement of Corollary 1.2 remains true for $0 \le t \le \epsilon^{-\chi}$ for some $\delta > 0$, $\chi > 1$ (see [K4]).

7) The form (1.16) of a spectral condition is not the most general one we need for our proof. For example for $d_1 > 1$ it is sufficient to demand that

$$C^{-1} j^{d_1} \le |\lambda_j| \le C j^{d_1}, \quad |\lambda_{j+1} - \lambda_j| \ge C_1 j^{d_1 - 1} \ \forall j. \tag{1.24}$$

See [K1] for (1.24) and [K2] for a possible form of a spectral condition with $d_1 = 1$. For the profound investigation of this problem see [DPRV].

8) The necessity of the quasilinearity condition $d_H \le d_A - 1$ results from (1.16) (or (1.24)). Indeed for arbitrary $d_H^1 > d_A - 1$ one can easily find perturbation H of the form

$$H = \frac{1}{2} < A^p(a,\epsilon)z,z >_Z , \quad A^p A^Z = A^Z A^p ,$$

such that condition (1.15) is satisfied with $d_H = d_H^1$ and for the operator $A^Z(a) + A^p(a,\epsilon)$ condition $|\lambda_{j+1} - \lambda_j| \ge C_1 j^{d_1 - 1}$ is broken for some j large enough.

9) The analyticity of tori $\sum_{(a,I)}^{\epsilon}(T^n)$ was observed by J. Pöschel [P1]. In the author's works [K1–K3] only smoothness of the tori was stated.

10) If all the numbers d, d_H, d_A, d_J are the integers then Theorem 1.1 may be stated in the framework of discrete symplectic Hilbert scales $\{Z,\{Z_s \mid s \in \mathbb{Z}\},\alpha\}$ (see [1], part 5).

2. Reformulation of Theorem 1.1

Let us suppose that the boundary $\partial\mathfrak{A}$ is smooth, domain $\mathfrak{A}$ is connected, all eigen–values λ_j^J, λ_j^A are analytical functions of $a \in \mathfrak{A}$ and

$$\det\{\partial\omega_j/\partial a_k(a) \mid 1 \leq j,k \leq n\} \not\equiv 0 \ . \tag{2.1}$$

For some fixed point $a_0 \in \mathfrak{A}$ we define numbers λ_{j0}^A, λ_{j0}^J, λ_{j0} and a vector ω_0 as in (1.9).

Let us consider some resonance relation of a form

$$s \cdot \omega(a) + \Lambda(a) \equiv 0 \ , \ \ \Lambda = \ell_1 \lambda_{n+1}(a) + \ldots + \ell_p \lambda_{n+p}(a) \ , \tag{2.2}$$

$$s \in \mathbb{Z}^n \ , \ \ 1 \leq |\ell|_1 \equiv |\ell_1| + |\ell_2| + \ldots + |\ell_p| \leq 2 \ . \tag{2.3}$$

<u>Lemma 2.1</u>. Let all the functions λ_j^J, λ_j^A be analytical in $\mathfrak{A}$, $d_1 \geq 1$ and asymptotic (1.16) together with assumptions (1.19) and (2.1) take place. Then there exist numbers M_2, j_2 with the following property: if some relation of form (2.2), (2.3) holds and in (2.2) $\ell_p \neq 0$, then $|s| \leq M_2$, $p \leq j_2$.

<u>Proof</u>. For the assumption (2.1) there exist a point $a' \in \mathfrak{A}$ such that

$$C^{-1} \leq |\omega_*(a')|_{\mathbb{R}^n,\mathbb{R}^n} \leq C \tag{2.4}$$

for some C. Let ∇_a be a gradient with respect to the usual scalar product in $\mathbb{R}^n$. Then

$\nabla_a(s \cdot \omega(a)) = \omega^*(a)s$ and for (2.4)

$$|\nabla_a(s \cdot \omega(a'))| \geq C^{-1}|s| \tag{2.5}$$

We consider two possibilities:

a) $\underline{d_1 > 1}$. As $|\ell|_1 \leq 2$, then for the assumption (1.16)

$$|\Lambda(a_0)| \geq |\lambda_p(a_0)| - |\lambda_{p-1}(a_0)| - C_0 \geq C_1 p^{d_1 - 1} - C_2 \tag{2.6}$$

So assumption (2.2) imply inequality

$$|s \cdot \omega(a_0)| \geq C_1 p^{d_1 - 1} - C_2$$

and

$$|s| \geq C_1^1 p^{d_1 - 1} - C_2^1 . \tag{2.7}$$

We may suppose that in (1.19) $d_{1,r} > 0$. Then $|\nabla_a \Lambda(a')| \leq 2K_1 p^{d_{1,r}}$ and for (2.5), (2.2) $|s| \leq C_3 p^{d_{1,r}}$. From this estimate and (2.7) the statements of the lemma results immediately.

b) $\underline{d_1 = 1}$. By the assumption (1.19) $|\nabla_a \Lambda(a')| \leq 2K_1$. From this estimate and (2.2), (2.5), (1.19) results an estimate on s :

$$|s| \leq C|\nabla_a(s \cdot \omega(a'))| = C|\nabla_a \Lambda(a')| \leq 2CK_1 . \tag{2.8}$$

If $|\ell|_1 = 1$ or $|\ell|_1 = 2$ and $|\ell_p| = 2$ then $|\Lambda(a_0)) \geq K_2p - C_5$ for (1.16). So by (2.2)

$$K_2p \leq C_5 + |s \cdot \omega(a_0)| \leq C_5 + |s|\,|\omega(a_0)| \quad . \tag{2.9}$$

The estimates (2.8), (2.9) imply the lemma's statements if $|\ell|_1 = 1$ or $|\ell_p| = 2$.

Now let $|\ell|_1 = 2$ and $|\ell_p| = 1$. Then the set $\{j|\ell_j \neq 0\}$ consists of two elements and contains p. Let us denote the second element as $p-\Delta$, $\Delta > 0$. Then for (1.16) $|\Lambda(a_0)| \geq K_2\Delta - C'$ and for estimates (2.8) and relation (2.2) $K_2\Delta - C' \leq |s \cdot \omega(a_0)| \leq C_6$. So $\Delta \leq C_7$ and for (1.19)

$$|\nabla_a\Lambda(a'))| \leq 2K_1(p-\Delta)^{d_{1,r}}, \; d_{1,r} < d_1 - 1 = 0 \; .$$

So, $|\nabla_a(s \cdot \omega(a'))| \leq 2K_1(p-\Delta)^{d_{1,r}}$, too. If $s \neq 0$ then for (2.5)

$$C' \leq |\nabla_a(s \cdot \omega(a'))| \leq 2K_1(p-\Delta)^{d_{1,r}}$$

and the estimate for p is obtained. If $s = 0$ then

$$0 = |\Lambda(a_*)| \geq K_2\Delta - 2K_1(p-\Delta)^{d_{1,r}}$$

and the estimate on p is obtained again. Now the lemma is proved. ∎

<u>Theorem 2.2</u>. Let all eigen–values λ_j^A, λ_j^J be analytical functions of a parameter $a \in \mathfrak{A}$

and conditions (1.2), (1.3), (1.3′), (1.7), (1.11)–(1.13), (2.1) hold together with assumptions 1), 2) of Theorem 1.1. Then there exist integers j_1, M_1 such that if an assumption

$$s_1\lambda_1(a) + s_2\lambda_2(a)+\dots+s_{j_1}\lambda_{j_1}(a) \not\equiv 0 \tag{2.10}$$
$$\forall s \in \mathbb{Z}^{j_1}, \ |s| \le M_1, \ 1 \le |s_{n+1}|+\dots+|s_{j_1}| \le 2$$

is satisfied then for every $\delta > 0$ and for sufficiently small $\epsilon > 0$ there exist a Borel subset $\mathfrak{A}_\epsilon^\delta \subset \mathfrak{A}$ and analytical embeddings

$$\sum\nolimits_a^\epsilon : \mathbf{T}^n \longrightarrow \mathbb{Z}_{d_c}, \ a \in \mathfrak{A}_\epsilon^\delta, \ d_c = d+d_A-d_H-1 \ ,$$

with the following properties a)–c):

a) mes $\mathfrak{A}\setminus\mathfrak{A}_\epsilon^\delta < \delta$,

b) the mapping

$$\sum\nolimits^\epsilon : \mathbf{T}^n \times \mathfrak{A}_\epsilon^\delta \longrightarrow Z_{d_c}, \ (q,a) \longmapsto \sum\nolimits_a^\epsilon(q)$$

is Lipschitz and

$$|\sum\nolimits^\epsilon - \sum\nolimits^0| {}^{\mathbf{T}^n\times\mathfrak{A}_\epsilon^\delta,\mathrm{Lip}}_{Z_{d_c}} \le C_\delta \epsilon \ , \tag{2.11}$$

c) every torus $\sum_a^{\epsilon}(\mathrm{T}^n)$, $a \in \mathfrak{A}_{\epsilon}^{\delta}$, is invariant for the equation (1.5) and is filled with weak in Z_d quasiperiodidc solutions of the form $\sum_a^{\epsilon}(q+\omega' t)$ and

$$|\omega-\omega'| \le C_{\delta}^{1}\epsilon \ . \tag{2.12}$$

All Liapunov exponents of these solutions are equal to zero.

<u>Proof</u>. By the analyticity of functions λ_j^J, λ_j^A and by the assumption (2.1) the set $\{a \in \mathfrak{A} \,|\, |\det \partial\omega_j/\partial a_k| > 0\}$ is open and of full Lebesque measure in $\mathfrak{A}$ (i.e. a measure of its complement is equal to zero). Let set

$$\mathfrak{A}_t = \{a \in \mathfrak{A} \,|\, |\det \partial\omega_j/\partial a_k| \ge t \ , \ \operatorname{dist}(a,\partial\mathfrak{A}) \ge t\} \ .$$

Then $\mathfrak{A}_t$, $t \longrightarrow 0$, is an increasing sequence of compact sets and $\cup\{\mathfrak{A}_t \,|\, t > 0\}$ is of full measure. So there exists $K_0 = K_0(\delta) > 0$ such that

$$\operatorname{mes} \mathfrak{A}\backslash\mathfrak{A}_{K_0} < \gamma = \delta/4 \ . \tag{2.13}$$

Let us choose $j_1 \ge j_2+n$, $M_1 \ge M_2+2$ with j_2, M_2 as in Lemma 2.1. Then for the assumption (2.10) and Lemma 2.1 there is no identical resonance relation of the form (2.2), (2.3). So every set

$$\{a \in \mathfrak{A} \,|\, s_1\lambda_1(a)+\ldots+s_p\lambda_p(a) \ne 0\} \tag{2.14}$$

with $1 \le |s_{n+1}|+\ldots+|s_p| \le 2$, is of full measure in $\mathfrak{A}$.

Let us take a point $a_0 \in \mathfrak{A}_{K_0}$. For the remark 3, Theorem 1.1 is applicable with this choice of a_0 if condition (1.20) is fulfilled with some $j_1 = j_{1,0}$, $M_1 = M_{1,0}$ which does not depend on a_0 . Let us consider a set

$$\overline{\mathfrak{A}}_t = \{a \in \overline{\mathfrak{A}} \mid |s_1\lambda_1(a) + \ldots + s_{j_{1,0}} \lambda_{j_{1,0}}(a)| \geq t$$

$$\forall \ |s| \leq M_{1,0} , \ 1 \leq |s_{n+1}| + \ldots + |s_{j_{1,0}}| \leq 2\} \ .$$

As the sets (2.14) are of full measure, then for some $t_0 > 0$

$$\operatorname{mes} \mathfrak{A} \backslash \overline{\mathfrak{A}}_{t_0} < \gamma \ . \tag{2.15}$$

Theorem 1.1 is applicable with arbitrary $a_0 \in \mathfrak{A}_{K_0} \cap \mathfrak{A}_{t_0}$, $\mathcal{J} = \{I_0\}$ and a constant K_0 in the assumption (1.8) as in (2.13). In this situation for remark 3 δ_* does not depend on a_0 and set $\Theta_\epsilon^{a_0}$ is of the form

$$\Theta_\epsilon^{a_0} = \mathfrak{A}_\epsilon^{a_0} \times \{I_0\} \ , \ \mathfrak{A}_\epsilon^{a_0} \subset \mathfrak{A}(a_0, \delta_*) \tag{2.16}$$

The open balls $\mathfrak{A}(a_0,\delta_*)$, $a_0 \in \mathfrak{A}_{K_0} \cap \overline{\mathfrak{A}}_{t_0}$, form a covering of compact $\mathfrak{A}_{K_0} \cap \overline{\mathfrak{A}}_{t_0}$. Let us fix some finite subcovering, $\mathfrak{A}_{K_0} \cap \overline{\mathfrak{A}}_{t_0} \subset \bigcup_{j=1}^{M} D_j$, $D_j = \mathfrak{A}(a_{0j}, \delta_*)$. For the statement a) of Theorem 1.1

$$\operatorname{mes} D_j \backslash \mathfrak{A}_\epsilon^{a_{0j}} < \gamma/M \quad \forall \ j=1,\ldots,M \ \text{ if } \ \epsilon < \epsilon(\delta) \tag{2.17}$$

For every $j = 1,\dots,M$ let us choose a closed subset $D_j^0 \subset D_j$ such that

$$\operatorname{dist}(D_j^0, D_k^0) \geq \delta' > 0 \quad \forall j \neq k \tag{2.18}$$

and

$$\operatorname{mes}(\cup D_j \backslash \cup D_j^0) < \gamma . \tag{2.19}$$

Let us set

$$\mathfrak{A}_\epsilon^\delta = \bigcup_{j=1}^{M} (\mathfrak{A}_\epsilon^{a_{0j}} \cap D_j^0)$$

and define a map $\sum_a^\epsilon$, $a \in \mathfrak{A}_\epsilon^{a_{0j}} \cap D_j^0$, being equal to the map $\sum_{(a,I_0)}^\epsilon$ constructed by means of Theorem 1.1 for $a \in \mathfrak{A}_\epsilon^{a_{0j}}$. This definition is correct because every point $a \in \mathfrak{A}_\epsilon^\delta$ belongs to the only set $\mathfrak{A}_\epsilon^{a_{0j}} \cap D_j^0$.

The statements a)–c) of the theorem are true with this choice of $\mathfrak{A}_\epsilon^\delta$ and $\sum_a^\epsilon$. Indeed, the assertion a) results from the estimates (2.13), (2.15), (2.17), (2.19). The assertion c) is local with respect to the parameter a and it results from Theorem 1.1.

For to prove the assertion b) let us mention that by Theorem 1.1 for $\Delta\Sigma = \Sigma^\epsilon - \Sigma^0$ we have

$$|\Delta\,\Sigma|_{Z_{d_c}}^{\mathbb{T}^n\times\mathfrak{A}_\epsilon^{a_{0j}},Lip} \le C'_\delta\epsilon \tag{2.20}$$

If $b_j \in \mathfrak{A}_\epsilon^{a_{0j}} \cap D_j^0$ then for $j_1 \neq j_2$ by (2.18) $|b_{j_1}-b_{j_2}| \ge \delta'$. So by (2.20)

$$\|\Delta\,\Sigma(q;b_1) - \Delta\,\Sigma(q;b_2)\|_{d_c} \le 2\,C'_\delta\,\delta'^{-1}\,|b_1-b_2|\,\epsilon \quad \forall b_1,b_2 \in \mathfrak{A}_\epsilon^\delta\,. \tag{2.21}$$

By (2.20), (2.21) we get an estimate (2.11) with $C_\delta = C'_\delta(1+2\delta'^{-1})$. ∎

<u>Corollary 2.3</u>. If under the assumptions of Theorem 2.2 condition (2.10) is satisfied then for arbitrary $\rho \in (0,1)$ and for $0 < \epsilon << 1$ there exist a Borel subset $\mathfrak{A}_\epsilon \subset \mathfrak{A}$ and analytical embeddings $\sum_a^\epsilon : \mathbb{T}^n \longrightarrow Z_{d_c}$, $a \in \mathfrak{A}_\epsilon$, $d_c = d + d_A - d_H - 1$, with the following properties:

a) $\operatorname{mes} \mathfrak{A}\backslash\mathfrak{A}_\epsilon \longrightarrow 0 \quad (\epsilon \longrightarrow 0)$,

b) $\left|\sum^\epsilon - \sum^0\right|_{Z_{d_c}}^{\mathbb{T}_n\times\mathfrak{A}_\epsilon,Lip} \le \epsilon^\rho$,

c) every torus $\sum_a^\epsilon(\mathbb{T}^n)$, $a \in \mathfrak{A}_\epsilon$, is invariant for the equation (1.5) and is filled with weak in Z_d solutions of the form $\sum_a^\epsilon(q+\omega' t)$, $|\omega'-\omega| \le \epsilon^\rho$. All Liapunov exponents of these solutions are equal to zero.

<u>Proof</u>. By Theorem 2.2 with $\delta = 1/n$, $n = 1,2,\ldots$ for $\epsilon \le \epsilon_n$, $\epsilon_n > 0$, we have the

sets $\mathfrak{A}_\epsilon^{1/n}$ and maps $\sum_a^\epsilon$ satisfying the assertions a)–c) of the theorem. If $\epsilon_n << 1$ then

$$C_\delta \epsilon \leq \epsilon^\rho \,,\ C_\delta^1 \epsilon \leq \epsilon^\rho \quad \forall\, \epsilon < \epsilon_n \ . \tag{2.22}$$

We may assume that $\epsilon_n \searrow 0$ $(n \longrightarrow \infty)$ and set $\delta(\epsilon) = 1/n$ if $\epsilon \in (\epsilon(n+1), \epsilon(n)]$. Now the assertions of the corollary result from Theorem 2.2 and (2.22). ∎

3. On systems with random spectrum

Theorems 1.1 and 2.2 may be applied to the Hamiltonian perturbations of random linear system for proving that quasiperiodic solutions of the unperturbed linear system survive in perturbed system with probability 1 (w.pr.1). Here we prove a simple theorem of this sort which deals with perturbations of a linear system equivalent to a countable set of free harmonic oscillators with random frequencies $\omega_1, \omega_2, \dots$.

The perturbations of a countable system of random oscillators by means of a short range interected hamiltonians have been studied in a number of works (see [FSW], [P2] and bibliography of these papers). For applications of our theorems we don't need short range interaction assumption. Instead of the last we use assumption of linear or super–linear growth of frequencies $(\omega_j \sim Cj^d \,,\ d \geq 1)$.

In the work [W] non–linear perturbations of the string equation with a random potential were studied. The theorems of [W] are similar to our results of this section.

Let Z be a Hilbert space with an orthobasis $\{\varphi_j^{\pm} \mid j \geq 1\}$; Z_s , $s \in \mathbb{R}$, be a Hilbert spaces with the orthobasis $\{j^{-s}\varphi_j^{\pm} \mid j \geq 1\}$ and

$$J : Z \longrightarrow Z \,,\ J(\varphi_j^{\pm}) = \mp \varphi_j^{\mp} \quad \forall j \,. \tag{3.1}$$

Then $\bar{J} = (-J)^{-1} = J$ and the triple $\{Z, \{Z_s\} \,,\ \langle \bar{J}dz, dz \rangle_Z\}$ is a symplectic Hilbert scale with properties (1.3) being fulfilled with $\lambda_j^J \equiv 1$.

Let $(\mathcal{U}, \mathcal{F}, \mathcal{P})$ be some probability space and $A = A(\mu)$, $\mu \in \mathcal{U}$, be a random selfadjoint operator in Z such that $\forall j \in \mathbb{N}$

$$A(\varphi_j^{\pm}) = \lambda_j^A(\mu)\varphi_j^{\pm},\ \lambda_j^A(\mu) = Kj^{d_A} + \Lambda_j(\mu)\ . \tag{3.2}$$

Here $K > 0$ and $\{\Lambda_j \mid j \geq 1\}$ are independent random variables (r.v.) such that every Λ_j is uniformly distributed on a segment

$$\Delta_j = [-\tfrac{1}{2} j^p, \tfrac{1}{2} j^p]\ . \tag{3.3}$$

Let O_d^c be a neighborhood of Z_d in $Z_d^c = Z_d \underset{\mathbb{R}}{\otimes} \mathbb{C}$ and $H \in \mathscr{A}^R(O_d^c;\mathbb{C})$. Let us consider a Hamiltonian equation with a hamiltonian $\mathscr{H} = \frac{1}{2} < Az,z >_Z + \epsilon\, H(z)$, i.e. the equation

$$\dot{z} = J(Az + \epsilon\, \nabla H(z))\ . \tag{3.4}$$

<u>Theorem 3.1</u>. Let $d_A \geq 1$, $d \geq \frac{1}{2} d_A$, $H \in \mathscr{A}^R(O_d^c;\mathbb{C})$, $\nabla H \in \mathscr{A}^R(O_d^c;Z_{d-d_H})$ with some $d_H \leq \min(0,d_A-1)$ and $H, \nabla H$ are bounded on bounded subsets of O_d^c. Let in (3.3) $p < d_A-1$. Let Q_d be an arbitrary open domain in Z_d. Then $\forall\, \epsilon > 0$ there exists a set $\mathscr{U}_\epsilon \in \mathscr{F}$ such that

a) $\mathscr{P}(\ \mathscr{U}_\epsilon) \longrightarrow 0\ (\epsilon \longrightarrow 0)$,

b) if $\mu \notin \mathscr{U}_\epsilon$ then the equation (3.4) has a quasiperiodic solution passing through

Q_d. All Liapunov exponents of this solution are equal to zero.

Remark. For a "not so small ϵ" one has $\mathcal{U}_\epsilon = \mathcal{U}$ and the statement of the theorem is empty.

Corollary 3.2. Let $\{\epsilon_j\}$ be a sequence such that $\epsilon_j \searrow 0$ for $j \longrightarrow \infty$. Then under the assumptions of Theorem 3.1 w.pr.1 equation (3.4) has a quasiperiodic solution through Q_d for ϵ equal to some ϵ_j.

Proof. Let us set $\mathcal{U}_0 = \cap\, \mathcal{U}_{\epsilon_j}$. For Theorem 3.1 $\mathcal{P}(\mathcal{U}_0) = 0$. If $\mu \notin \mathcal{U}_0$ then μ lies out of some $\mathcal{U}_{\epsilon_j}$ and equation (3.4) with $\epsilon = \epsilon_j$ has a quasiperiodic solution though Q_d. ■

Corollary 3.3. Let $\epsilon_j \searrow 0$ $(j \longrightarrow \infty)$ and $QP(\epsilon_j)$ be the union of all quasiperiodic trajectories of equation (3.4) with $\epsilon = \epsilon_j$. Then w.pr.1 $\bigcup_j QP(\epsilon_j)$ is dense at Z_d.

Proof. As the Hilbert space Z_d is separable there exists a countable system $\{B_j \mid j \in \mathbb{N}\}$ of balls $B_j \subset Z_d$ such that any open set B_* contains some ball B_{j_*}. Now the statement results from Corollary 3.2 being applied to the balls B_j $(j = 1,2,\ldots)$, because the intersection of a countable system of sets of full measure is of full measure, again. ■

Proof of the theorem. Let us take some point $z_0 \in Q_d$ of the form

$$z_0 = \sum_{j=1}^{n} z_{0j}^{\pm} \varphi_j^{\pm}, \quad n = n(z_0) < \infty ,$$

and denote

$$\mathrm{dist}_{Z_d}(z_0, Z_d \setminus Q_d) = \delta_0, \ \ \delta_0 > 0 \ . \tag{3.5}$$

After the rearrangement of n first pairs of basis vectors $\{\varphi_j^{\pm} \mid j = 1,\dots,n\}$ and decreasing the number n (if there is need in it) one may suppose that $z_{0j}^{+2} + z_{0j}^{-2} > 0 \ \ \forall\, j = 1,\dots,n$. So the point z_0 belongs to some torus $T(I_0) \simeq \mathbb{T}^n$, $I_0 \in \mathbb{R}_+^n$.

Let us denote $\omega_j = \lambda_j^A(\mu)$, $j \in \mathbb{N}$. By Corollary 2.3 for every fixed $\omega_\infty = (\omega_{n+1}, \omega_{n+2}, \dots)$ there exists a set $\Omega_\epsilon = \Omega_\epsilon(\omega_\infty)$ of vectors $\omega = (\omega_1,\dots,\omega_n)$, $\Omega_\epsilon \subset \Delta_1 \times \Delta_2 \times \dots \times \Delta_n$, such that

$$\mathrm{mes}\, \Omega_\epsilon \le m(\epsilon) \ , \ \ m(\epsilon) \xrightarrow[\epsilon \to 0]{} 0 \ , \tag{3.6}$$

(here mes is the normalized Lebesque measure) and for $\omega \notin \Omega_\epsilon$ the equation (3.4) has an invariant torus $T_\epsilon \simeq \mathbb{T}^n$ at a distance $< \epsilon^{1/2}$ from the torus $T(I)$. The torus T_ϵ is filled with the quasiperiodic solutions. So if $\epsilon < \delta_0^2$ then equation (3.4) has a quasiperiodic solution passing through Q_d provided ω lies out of Ω_ϵ .

In the present situation a n–dimensional parameter of the problem (3.4) is the frequency vector ω itself. So condition (1.8) is fulfilled with $K_0 = 1$. All the constants mentioned in the remark 3 (see part 1) are uniform with respect to ω_∞. So the remark and an analysis of the proof of Theorem 2.2 (we omit the routine) show that in (3.6) the function $m(\epsilon)$ does not depend on ω_∞. Let us set $\mathcal{U}_\epsilon = \{\mu \in \mathcal{U} \mid \omega \in \Omega_\epsilon(\omega_\infty)\}$. As the r.v. ω and ω_∞ are independent, then $\mathcal{P}(\mathcal{U}_\epsilon) \leq m(\epsilon)$. So the theorem is proved because for $\mu \notin \mathcal{U}_\epsilon$ equation (3.4) has a quasiperiodic solution through Q_d. ■

4. Nonlinear Schrödinger equation

A nonlinear Schrödinger equation

$$\dot{u} = i(-u_{xx} + V(x)u + \epsilon \frac{\partial}{\partial |u|^2} \chi(x, |u|^2)u)$$

will be considered under the Dirichlet boundary condition

$$0 \leq x \leq \pi \, , \quad u(t,0) \equiv u(t,\pi) \equiv 0 \quad .$$

Let $Z = L_2(0,\pi;\mathbb{C})$ which is regarded as a real Hilbert space with inner product

$$< u,v >_Z = \mathrm{Re} \int u(x) \, \overline{v(x)} \, dx \quad .$$

A differential operator $-\partial^2/\partial x^2$ with the Dirichlet boundary conditions defines a positive selfadjoint operator $\mathscr{A}_0$ in Z with the domain of definition $D(\mathscr{A}_0) = (\overset{\circ}{H}{}^1 \cap H^2)(0,\pi;\mathbb{C})$. For $s \geq 0$ let Z_s be the domain of definition of the operator $\mathscr{A}_0^{s/2}$. Every space Z_s is a closed subspace of $H^s(0,\pi;\mathbb{C})$ and the norm in Z_s is equivalent to the norm induced from $H^s(0,\pi;\mathbb{C})$. In particular

$$Z_1 = \overset{\circ}{H}{}^1(0,\pi;\mathbb{C}) \, , \quad Z_2 = (\overset{\circ}{H}{}^1 \cap H^2)(0,\pi;\mathbb{C}) \quad . \tag{4.1}$$

Let Z_{-s} be the space adjoint to Z_s with respect to the scalar product in Z.

Let us consider antiselfadjoint operator J,

$$J : Z \longrightarrow Z\ ,\ \ u(x) \longmapsto i\, u(x)\ .$$

Then $J^2 = -E$, so $\bar{J} \equiv -(J^{-1}) = J$ and the triple $\{Z,\{Z_s\}\ ,\ < Jdz,dz >_Z\}$ is a symplectic Hilbert scale [1].

Let $\mathfrak{A}$ be a bounded domain in $\mathbb{R}^n$ and $V : [0,\pi] \times \bar{\mathfrak{A}} \longrightarrow \mathbb{R}$ be a C^2–function. The differential operator $-\partial^2/\partial x^2 + V(x;a)$ defines a selfadjoint operator $\mathscr{A}(a)$ in Z with the domain of definition Z_2. $\mathscr{A}(a)$ depends on a parameter $a \in \mathfrak{A}$. For a full system of eigen–vectors of $\mathscr{A}(a)$ let us take $\{\varphi_j^{\pm}(a)\}$. Here $\varphi_j^{+}(a) = \varphi_j(x;a)$, $\varphi_j^{-}(a) = i\, \varphi_j(x;a)$ and $\{\varphi_j(x;a)\}$ is the full in $L_2(0,\pi;\mathbb{R})$ system of real eigen–functions of the operator $-\partial^2/\partial x^2 + V(x;a)$ under Dirichlet boundary conditions. So

$$\mathscr{A}(a)\, \varphi_j^{\pm}(a) = \lambda_j^{A}(a)\, \varphi_j^{\pm}(a) \quad \forall\, j \geq 1\ .$$

Let us suppose that the numbers $\{\lambda_j^A(a)\}$ are asymptotically ordered, i.e. $\lambda_j^A(a) > \lambda_k^A(a)$ if $j > k$ and k is large enough.

Let $O^c \subset \mathbb{C}$ be a complex neighborhood of $\mathbb{R}$ and $\chi : O^c \times [0,\pi] \times \bar{\mathfrak{A}} \longrightarrow \mathbb{C}$ be a function such that

$$\begin{gathered} \chi(\cdot,\cdot;a) \in C^2(O^c \times [0,\pi];\mathbb{C}) \ \ \forall a \in \mathfrak{A}\ , \\ \frac{\partial^s}{\partial x^s}\, \chi(\cdot,x;\cdot) \in \mathscr{A}_{\mathfrak{A}}^{R}(O^c;\mathbb{C}) \ \ \forall s \leq 2,\ \forall x \in [0,\pi]\ . \end{gathered} \tag{4.2}$$

Let us set

$$H_0(u;a) = \frac{1}{2}\int_0^{\pi} \chi(|u(x)|^2,x;a)dx \ . \tag{4.3}$$

Lemma 4.1. For any $R > 0$ there exists a complex δ–neighbourhood $B_R^c \subset Z_2^c$ of a ball $\{u \in Z_2 | \ \|u\|_2 \le R\}$ such that $\delta = \delta(R) > 0$ and $H_0 \in \mathcal{A}_{\mathfrak{A}}^R(B_R^c;\mathbb{C})$,

$$\nabla H_0(u;a) = \frac{\partial}{\partial |u|^2}\chi(x,|u|^2;a)u \tag{4.4}$$

and $\nabla H_0 \in \mathcal{A}_{\mathfrak{A}}^R(B_R^c;Z_2^c)$.

Proof. The existence of the set B_R^c and analyticity of H_0 result from Corollary A2 from Appendix. Relation (4.4) results from the identities

$$< v(x),\nabla H_0(u(x);a) >_Z = d\, H_0(u;a)(v) =$$

$$= \frac{1}{2}\int_0^{\pi} \frac{\partial}{\partial |u|^2}\chi(x,|u|^2;a)(u\,\bar{v} + \bar{u}\,v)dx =$$

$$= < v(x), \frac{\partial}{\partial |u|^2}\chi(x,|u(x)|^2;a)u >_Z \ .$$

The last statement results from Corollary A2 again. ∎

So the Hamiltonian equation with a hamiltonian $\frac{1}{2} < \mathcal{A}(a)u,u >_Z + H_0(u;a)$ has the form

$$\dot{u} = i(-u_{xx} + V(x;a)u + \epsilon \frac{\partial}{\partial |u|^2}\chi(x,|u|^2;a)u) \tag{4.5}$$

This equation is of the form (1.5) but operators $\mathcal{A}(a)$ don't commute one with another and the condition (1.3′) is not satisfied. For applying the theorem we at first must do linear transformations U_a of the phase space depending on a parameter a ,

$$U_a : Z \longrightarrow Z\,,\ \ z\varphi_j(x) \longmapsto z\varphi_j(x;a)\quad \forall z \in \mathbb{C}\quad \forall j\ .$$

Here $\varphi_j(x) = (2/\pi)^{1/2} \sin jx$.

Lemma 4.2. For every $a \in \mathfrak{A}$ the transformation U_a is canonical and orthogonal with respect to scalar product $< \cdot,\cdot >_Z$. For every $a, a_1, a_2 \in \mathfrak{A}$ and every $s \in [0,2]$

$$|\lambda_j(a_1) - \lambda_j(a_2)| \le C|a_1 - a_2|\ , \tag{4.6}$$

$$\|U_{a_1} - U_{a_2}\|_{s,s} \le C_s|a_1 - a_2|\ , \tag{4.7}$$

$$\|U_a\|_{s,s} \le C'_s\ . \tag{4.8}$$

Here $\|\cdot\|_{s,s} = |\cdot|_{Z_s,Z_s}$.

Proof. The orthogonality of U_a results from the fact that it maps one Hilbert basis of the space Z into another. The canonicity results from identities

$$< i\, U_a u, U_a v >_Z = < U_a iu, U_a v >_Z = < iu, v >_Z$$

(we use the orthogonality of U_a).

The estimate (4.6) for the spectrum of Sturm–Liouville problem is well–known [PT,Ma].

For to prove (4.7) let us mention that for the eigen–functions $\varphi_j(x;a)$ one has the estimate

$$\|\varphi_j(a_1)-\varphi_j(a_2)\|_0 \le C \sup_x |V(x;a_1)-V(x;a_2)|/j \le C_1 |a_1-a_2|/j \tag{4.9}$$

(see [PT,Ma]). As

$$\frac{\partial^2}{\partial x^2} \varphi_j(x;a) = (V(x;a) - \lambda_j(a))\varphi_j(x;a)$$

then we get from estimates (4.6), (4.9) that

$$\|\varphi_j(a_1) - \varphi_j(a_2)\|_2 \le C_2 |a_1-a_2| j \ . \tag{4.10}$$

From (4.9), (4.10) and interpolation inequality [RS2] it follows that estimate (4.10) holds for all $s \in [0,2]$.

Let $u' \in Z_s$ and

$$u = \sum (u_k^+ + iu_k^-)\varphi_k(x) \ , \ \|u\|_s^2 = \sum |u_k^\pm|^2 k^{2s} < \infty$$

(one has to mention that $\|\varphi_j\|_s = j^s$) . Then

$$\|U_{a_1}u - U_{a_2}u\|_s = \|\sum_k (u_k^+ + iu_k^-)(U_{a_1} - U_{a_2})\varphi_k(x)\|_s \le$$

$$\leq \sum_k |u_k^+ + iu_k^-| \; \|\varphi_k(x;a_1) - \varphi_k(x;a_2)\|_s \leq$$

$$\leq C_s'(\sum_k |u_k^+ + iu_k^-|^2 k^{2s})^{1/2} |a_1 - a_2| (\sum k^{-2})^{1/2} \leq$$

$$\leq C_s'' \; |a_1 - a_2| \; \|u\|_s$$

and we get the estimate (4.7). The estimate (4.8) results from the inequality $\|\varphi_j(x;a) - \varphi_j(x)\|_j \leq C_j^1 j^{s-1}$ in the same way as (4.7) results from (4.10). ■

For Lemma 4.2 and Theorem 2.2 from [1] the substitution

$$u = U_a \, v \tag{4.11}$$

transforms solutions of equation (4.5) to solutions of equation

$$\dot{v} = J(A(a)v + \epsilon \, \nabla H(v;a)) \tag{4.12}$$

with

$$A(a) = U_a^* \, \mathscr{A}_a U_a \, , \; H = H_0(U_a v;a) \; .$$

So

$$\nabla \, H(v;a) = U_a^* \nabla H_0(U_a v;a) \; ,$$

$$A(a)\,\varphi_j^{\pm}(a) = \lambda_j^A(a)\,\varphi_j^{\pm} \quad \forall j \ .$$

Equation (4.5) with $\epsilon = 0$ is a linear Schrödinger equation

$$\dot{u} = i(-u_{xx}+V(x;a)u)\ , \quad u(t) \in (\overset{\circ}{H}{}^1 \cap H^2)(0,\pi;\mathbb{C})\ \forall t$$

and it has invariant n–tori

$$T_a^n(I) = \left\{ \sum_{j=1}^{n} (\alpha_j^+ + i\alpha_j^-)\varphi_j(x;a) \,|\, \alpha_j^{+2} + \alpha_j^{-2} = 2\,I_j > 0 \right\}$$

Let a Borel set $\mathcal{J} \subset \mathbb{R}_+^n$ be as in (1.11) and $\mathcal{T}_a = \cup\,\{T_a^n(I)\,|\,I \in \mathcal{J}\}$. For every $a \in \mathfrak{A}$ $U_a^{-1}(T_a^n(I))$ is an invariant torus $T(I)$ of equation (4.12) with $\epsilon = 0$. It is of the form (1.9′), does not depend on a and

$$U_a^{-1}\,\mathcal{T}_a = \mathcal{T} = \cup\,\{T(I)\,|\,I \in \mathcal{J}\}\ .$$

Moreover, if R is large enough then one can choose a domain O_2^c

$$O_2^c \subset \bigcap_{a\in\mathfrak{A}} U_a^{-1}\,B_R^c \tag{4.13}$$

which satisfies relation (1.13) with $s = 2$.

Let us check that Theorem 1.1 with

$$\lambda_j^J \equiv 1\ ,\ d_J = 0\ ,\ d_A = 2\ ,\ d_H = 0\ ;\ \dot{d} = 2\ ,\ d_c = 3$$

may be applied to the equation (4.12). Indeed, the validity of assumption 1) with $O_d^c = O_2^c$ (see (4.13)) results from (4.3), (4.7), (4.8); assumption 2) with $d_1 = d_A = 2$ results from (4.6) and from the well–known asymptotic $\lambda_j = j^2 + O(1)$ (see [PT], [Ma]). So we get the following statement.

Theorem 4.3. Let a_0 be a point in $\mathfrak{A}$ such that

$$|\det(\partial \lambda_j^A(a_0)/\partial a_k \,|\, 1 \le j, k \le n)| \ge K_0 > 0\ . \tag{4.14}$$

Then there exist integer j_1, M_1 such that if

$$\begin{gathered} \lambda_1^A(a_0)s_1 + \lambda_2^A(a_0)s_2 + \ldots + \lambda_{j_1}^A(a_0)s_{j_1} \neq 0 \\ \forall s \in \mathbb{Z}^{j_1}\ ,\ |s| \le M_1,\ 1 \le |s_{n+1}| + \ldots + |s_{j_1}| \le 2\ , \end{gathered} \tag{4.15}$$

then for sufficiently small $\epsilon > 0$ there exists $\delta_* > 0$ (sufficiently small and independent on ϵ), Borel subset

$$\Theta_\epsilon^{a_0} \subset \Theta^{a_0} = \mathfrak{A}(a_0, \delta_*) \times \mathcal{J}$$

and analytic embeddings

$$\Sigma_{(a,I)}^\epsilon : \mathbf{T}^n \longrightarrow (\overset{\circ}{H}{}^1 \cap H^3)(0,\pi;\mathbf{C}),\ (a,I) \in \Theta_\epsilon^{a_0}\ ,$$

with the following properties:

a) mes $\Theta_\epsilon^{a_0}[I] \longrightarrow$ mes $\mathfrak{A}(a_0,\delta_*)$ $(\epsilon \longrightarrow 0)$ uniformly with respect to I ;

b) every torus $\sum_{(a,I)}^{\epsilon}(\mathbb{T}^n)$ is invariant for the equation (4.5) and is filled with weak in $(\overset{\circ}{H}{}^1 \cap H^2)$ solutions of (4.5) of the form $\sum_{(a,I)}^{\epsilon}(q_0 + \omega' t)$ (q_0 is an arbitrary point from $\mathbb{T}^n$, $\omega' = \omega'(a,I,\epsilon) \in \mathbb{R}^n$);

c) $\operatorname{dist}_{H^2}(\sum_{(a,I)}^{\epsilon}(\mathbb{T}^n), T_a^n(I)) \leq C\epsilon$ and $|\omega - \omega'| \leq C\,\epsilon$;

d) the numbers j_1, M_1 depend on K_0, n and C^2–norm of V(x;a) only.

Let us discuss assumptions (4.14), (4.15) of the theorem. For this purpose let us consider a mapping $\mathcal{U}$ from the set $\mathfrak{A}$ into the space $C[0,\pi]$ of potentials V(x) ,

$$\mathcal{U} : \mathfrak{A} \longrightarrow C[0,\pi] \ , \ a \longmapsto V(\cdot;a) \ .$$

Every λ_j^A is an analytical function of potential V(x) . So condition (4.1) means that the point $\mathcal{U}(a_0)$ lies in the space $C[0,\pi]$ out of the zero set of some nontrivial analytical function. For to discuss assumption (4.14) let us mention that

$$\frac{\partial \lambda_j}{\partial a_k}(a_0) = \int_0^\pi \varphi_j^2(x;a_0)\,\frac{\partial V(x;a_0)}{\partial a_k}\,dx$$

(see [PT, Ma]). It is proved in [PT] that the system of the functions $\{\varphi_1^2(\cdot;a),\ldots,\varphi_n^2(\cdot;a)\}$ is linearly independent for all a . So the function

$$(\xi_1(x),\dots,\xi_n(x)) \longmapsto \det\left(\int \varphi_j^2(x;a)\,\xi_\ell(x)dx \mid 1\le j,\ \ell\le n\right)$$

is non–trivial n–form on the space $C[0,\pi]$ and the condition (4.14) means that the restriction of this n–form on the image of the tangent mapping

$$\mathcal{U}_*(a_0) : \mathbb{R}^n \longrightarrow T_{\mathcal{U}(a_0)}C[0,\pi] \simeq C[0,\pi]$$

is nondegenerate, too.

So the assumption (4.14)+(4.15) is an non–degeneracy condition on the 1–jet of the map $\mathcal{U}$ at the point a_0.

Remark. Theorem 1.1 is applicable to study equation (4.5) under Neumann boundary conditions, or in the space of even periodic with respect to x functions,

$$x\in\mathbb{R},\ \ u(t,x) \equiv u(t,x+2\pi),\ \ u(t,x) \equiv u(t,-x)\ , \tag{4.16}$$

if the functions V and χ are even periodic and smooth on x. In the last situation one has to take for spaces $\{Z_s\}$ the spaces of even periodic Sobolev functions. In such a case relation (4.4) defines an analytical mapping from the space Z_s into itself for every $s \ge 1$. So Theorem 1.1 is applicable with arbitrary $d \ge 1$ and in the case of the problem (4.5), (4.16) one may prove the existence of arbitrary smooth invariant tori (i.e. being in the space $H^k(0,\pi;\mathbb{C})$ with k arbitrary large) at a distance of order ϵ from $\mathcal{T}_a$.

5. Nonlinear string equation

The next application of our theorem will be to the equation of oscillation of a string with the fixed ends in a nonlinear–elastic medium:

$$\frac{\partial^2}{\partial t^2} w = (\partial^2/\partial x^2 - V(x))w - \epsilon \frac{\partial}{\partial w} \chi(x,w) \ ; \tag{5.1}$$

$$w = w(t,x), \ \ 0 \le t \le \pi; \ \ w(t,0) \equiv w(t,\pi) \equiv 0 \tag{5.2}$$

For writing down this non–linear boundary value problem in a form (1.5) we need some preliminary work. Let $V : [0,\pi] \times \mathfrak{A} \longrightarrow \mathbb{R}_+$ be a smooth function. The differential operator $-\partial^2/\partial x^2 + V(x;a)$ defines a positive selfadjoint operator in the space $L_2(0,\pi;\mathbb{R})$ with the domain of definition $(\overset{\circ}{H}{}^1 \cap H^2)(0,\pi;\mathbb{R})$. The space $\mathcal{Z} = D(\sqrt{\mathcal{A}_a})$ is the Sobolev space $\overset{\circ}{H}{}^1(0,\pi;\mathbb{R})$ with the scalar product

$$< u,v >^{(a)} = \int_0^\pi (u_x v_x + V(x;a)uv)dx$$

For $t \ge 0$ let $\mathcal{Z}_t$ be the space $\mathcal{Z}_t = D(\mathcal{A}_a^{(t+1)/2})$ with the norm $\|u\|_t^{(a)} = \| \mathcal{A}_a^{t/2} u \|_0^{(a)}$. In particular $\|u\|_0^{(a)} = (< u,u >^{(a)})^{1/2}$. For $-t \le 0$ let $\mathcal{Z}_{-t}$ be a space dual to $\mathcal{Z}_t$ with respect to scalar product $< \cdot ; \cdot >^{(a)}$. Let us set $Z_t^{(a)} = \mathcal{Z}_t \times \mathcal{Z}_t$ with the natural norm and scalar product which will be denoted as $< \cdot , \cdot >^{(a)}$, too. In the scale $\{Z_t^{(a)}\}$ let us consider an operator J_a of order $d_J = 1$,

$$J_a : Z_t^{(a)} \longrightarrow Z_{t-1}^{(a)},\ w = (w_1, w_2) \longmapsto (\mathcal{A}_a^{1/2} w_2, - \mathcal{A}_a^{1/2} w_1)$$

This operator is anti–selfadjoint in $Z^{(a)} = Z_0^{(a)}$ with the domain of definition $D(J_a) = Z_1^{(a)}$. The triple

$$\{Z^{(a)}, \{Z_s^{(a)} \mid s \in \mathbb{R}\}, < \bar{J}_a dw, dw >^{(a)}\}, \bar{J}_a = -(J_a)^{-1}$$

is a symplectic Hilbert scale [1] depending on a parameter a.

Let $\{\varphi_j^{(a)} \mid j \geq 1\}$ be a full in $L_2(0,\pi;\mathbb{R})$ system of eigen–functions of operator $-\partial^2/\partial x^2 + V(x;a)$,

$$(-\partial^2/\partial x^2 + V(x;a))\varphi_j^{(a)} = \lambda_j^{(a)}\varphi_j^{(a)},\ |\varphi_j^{(a)}|_{L_2} = 1 ,$$

and $\lambda_j^{(a)} > \lambda_k^{(a)}$ for $j > k$ and k large enough. Let us set

$$\varphi_j^{+(a)} = (\varphi_j^{(a)}(x),0)(\lambda_j^{(a)})^{-1/2},\ \varphi_j^{-(a)} = (0,\varphi_j^{(a)}(x))(\lambda_j^{(a)})^{-1/2} .$$

Then the set of functions $\{(\lambda_j^{(a)})^{-s/2}\varphi_j^{\pm(a)} \mid j \geq 1\}$ is a Hilbert basis of $Z_s^{(a)}\ \forall\ s \in \mathbb{R}$ and

$$J_a\varphi_j^{\pm(a)} = \mp(\lambda_j^{(a)})^{1/2}\varphi_j^{\mp(a)}\ \forall\ j \tag{5.3}$$

Let the function $\chi(x,w;a)$ and domain $O^c \subset \mathbb{C}$ be the same as in § 4 and

$$H^0(w_1,w_2) = \int_0^\pi \chi(x,w_1(x);a)dx$$

Lemma 5.1. For any $R > 0$ there exists a complex δ–neighborhood $B_R^c \subset Z_1^{(a)c} = Z_1^{(a)} \otimes \mathbb{C}$ of a ball $\{u \in Z_1^{(a)} \mid \|u\|_1^{(a)} \le R\}$ such that $\delta = \delta(R) > 0$ and $H^0 \in \mathcal{A}_{\mathfrak{A}}^R(B_R^c;\mathbb{C})$,

$$\nabla^a H^0(u;a) = (\mathcal{A}_a^{-1} \frac{\partial}{\partial w_1} \chi(x,w_1(x);a),0) \ , \qquad (5.4)$$

$\nabla^a H^0 \in \mathcal{A}_{\mathfrak{A}}^R (B_R^c;Z_3^{(a)c})$ (here ∇^a is the gradient with respect to scalar product $<\cdot;\cdot>^{(a)}$).

Proof of analyticity of H^0 and $\nabla^a H^0$ is the same as in Lemma 4.1. The formula for $\nabla^a H^0$ results from identities

$$< (v_1,v_2), \nabla^a H^0(w) >^{(a)} = dH^0(w)(v_1,v_2) =$$

$$= \int (\frac{\partial}{\partial w_1} \chi(x,w_1(x);a)v_1(x))dx =$$

$$= \int (\mathcal{A}_a^{-1} \frac{\partial}{\partial w_1} \chi(x,w_1(x);a) \ \mathcal{A}_a v_1(x))dx =$$

$$= < (v_1,v_2), (\mathcal{A}_a^{-1} \frac{\partial}{\partial w_1} \chi(x,w_1(x);a), 0) >^{(a)} . \qquad \blacksquare$$

The Hamiltonian equation corresponding to a hamiltonian $\mathcal{H}_a(w) = \frac{1}{2} \|w\|_0^2 + \epsilon H^0(w)$ in a symplectic structure with the 2–form $< J_a dw,dw >^{(a)}$ is the following:

$$(\dot{w}_1,\dot{w}_2) = \dot{w} = J_a \nabla \mathscr{H}_a =$$

$$= (\mathscr{A}_a^{1/2}\, w_2, - \mathscr{A}_a^{1/2}(w_1 + \mathscr{A}_a^{-1}\, \epsilon \frac{\partial}{\partial w_1} \chi(x,w_1(x);a)))$$

or

$$\begin{aligned} \dot{w}_1 &= \mathscr{A}_a^{1/2} w_2 \\ \dot{w}_2 &= - \mathscr{A}_a^{1/2}(w_1 + \mathscr{A}_a^{-1}\, \epsilon \frac{\partial}{\partial w_1} \chi(x,w_1(x);a)) \ . \end{aligned} \tag{5.5}$$

After elimination w_2 from this equation one gets an equation on w_1,

$$\frac{\partial^2}{\partial t^2} w_1 = (\frac{\partial^2}{\partial x^2} - V(x;a))w_1 - \epsilon \frac{\partial}{\partial w_1} \chi(x,w_1(x);a) \ . \tag{5.6}$$

So equation (5.5) is equivalent to equation (5.1). In what follows we shall discuss equation (5.1) in the form (5.5).

As in § 4 we have to do some linear transformation before we apply our theorem. So let $\{Z_s\}$ be the scale of spaces of form $\{Z_s^{(a)}\}$ with $V(x;a) \equiv 0$, i.e. defined by operator $-\partial^2/\partial x^2$ instead of $-\partial^2/\partial x^2 + V(x;a)$. Let us set

$$\varphi_j^+(x) = (\sin jx,0)(2/\pi j)^{1/2}, \ \ \varphi_j^- = (0,\sin jx)(2/\pi j)^{1/2}$$

and denote an antiselfadjoint operator $J(a)$ of order 1 in the scale $\{Z_s\}$,

$$J(a)\, \varphi_j^{\pm} = \mp (\lambda_j^{(a)})^{1/2}\, \varphi_j^{\mp} \tag{5.7}$$

The triple $\{Z = Z_0, \{Z_s\}, \ < \bar{J}(a)dw, dw >_0\}$ is a symplectic Hilbert scale depending on a parameter a of the same sort as in § 1, i.e. with condition (1.3) being fulfilled.

For the relations (5.3), (5.7) the mapping

$$U_a : Z \longrightarrow Z^{(a)}, \ \varphi_j^{\pm} \longmapsto \varphi_j^{\pm(a)}$$

defines a canonical transformation from Z_s to $Z_s^{(a)}$ for every $s \geq 0$. So U_a transforms solutions of equation

$$\dot{v} = J(a)(v + \epsilon \nabla H(v;a)), \ H(v;a) = H^0(U_a(v);a) \tag{5.8}$$

into solutions of (5.5). As in § 4 one can prove the following statement.

Lemma 5.2. For any $R > 0$ there exists a complex δ–neighborhood $O_1^c \subset Z_1^c$ of a ball $\{u \in Z_1 \mid \|u\|_1 \leq R\}$ such that $\delta = \delta(R) > 0$ and $H \in \mathscr{A}_{\mathfrak{A}}^R(O_1^c;\mathbb{C})$, $\nabla H \in \mathscr{A}_{\mathfrak{A}}^R(O_1^c;Z_3^c)$. Let us check that Theorem 1.1 with

$$\lambda_j^A = 1, \ \lambda_j^J = (\lambda_j^{(a)})^{1/2}, \ d_J = 1, \ d_A = 0, \ d_H = -2, \ d = 1, \ d_c = 2$$

is applicable to equation (5.8). Indeed, assumption 1) results from Lemma 5.2, assumption 2) with $r = 2$, $K_2 = 1$, $d^{1,1} = 0$ and some $K_2^1 \in \mathbb{R}$ is satisfied because $\lambda_j(a) = \lambda_j^J(a) = (\lambda_j^{(a)})^{1/2}$, where $\{\lambda_j^{(a)} = j^2 + C(a) + O(1)\}$ is a spectrum of the Sturm–Liouville problem. So we get the following statement on equation (5.5) (or (5.6)).

Theorem 5.2. Let a Borel set $\mathcal{J}$ be as in (1.11) and a_0 be a point in $\mathfrak{A}$ such that

$$\det(\partial\lambda_j^{(a_0)}/\partial a_k \,|\, 1\leq j,\ k\leq n) \neq 0 \ . \tag{5.9}$$

Then there exist integers j_1, M_1 such that if

$$\begin{gathered}(\lambda_1^{(a_0)})^{1/2} s_1 + (\lambda_2^{(a_0)})^{1/2} s_2 + \ldots + (\lambda_{j_1}^{(a_0)})^{1/2} s_{j_1} \neq 0 \\ \forall\, s \in \mathbb{Z}^{j_1},\ \ |s| \leq M_1\,,\ 1 \leq |s_{n+1}| + \ldots + |s_{j_1}| \leq 2\end{gathered} \tag{5.10}$$

then for sufficiently small $\epsilon > 0$ there exists $\delta_* > 0$, Borel subset $\Theta_\epsilon^{a_0} \subset \Theta^{a_0} = \mathfrak{A}(a_0,\delta_*) \times \mathcal{J}$ and smooth embeddings $\sum_{(a,I)}^{\epsilon} : \mathbf{T}^n \longrightarrow Z_2$, $(a,I) \in \Theta_\epsilon^{a_0}$, with the following properties:

a) mes $\Theta_\epsilon^{a_0}[I] \longrightarrow$ mes $\mathfrak{A}(a_0,\delta_*)$ $(\epsilon \longrightarrow 0)$ uniformly with respect to I;

b) every torus $\sum_{(a,I)}^{\epsilon}(\mathbf{T}^n)$ is invariant for the equation (5.5) and is filled with weak in Z_1 solutions.

The conditions (5.9), (5.10) are ones of non–degeneracy in the same hence as in § 4.

6. On real points in the spectrum

The proof of the statements a)–c) of Theorem 1.1 is valid if some finite number of eigenvalues of the operator $J^Z(a)\,A^Z(a)$ is real, i.e. if for some finite number of indexes j instead of the conditions (1.3′), (1.3) one has

$$A^Z(a)\varphi_j^{\pm} = \mp\,\lambda_j^A(a)\varphi_j^{\pm},\ J^Z(a)\varphi_j^{\pm} = \mp\,\lambda_j^J(a)\varphi_j^{\mp},\ j = n+1,\dots,n+p \tag{6.1}$$

(see [K3]). In such a case

$$\begin{gathered}\sigma(J^Z(a)A^Z(a)) = \{\pm i\lambda_j(a)\,|\,j=1,\dots,n,\ \ n+p+1,\ \ n+p+2,\dots\}\ \cup \\ \cup\ \{\pm\lambda_j(a)\,|\,j=n+1,\dots,n+p\},\ \ \lambda_j(a) = \lambda_j^J(a)\lambda_j^A(a)\ .\end{gathered} \tag{6.2}$$

So the spectrum contains p pairs of real eigenvalues.

<u>Example</u>. Let us consider the problem (5.6), (5.2) without the limitation $V(x;a) \geq 0$ (and, so, with the possibility of negative points in the spectrum $\sigma(\mathcal{A}_a)$ of the operator $\mathcal{A}_a = -\partial^2/\partial x^2 + V(x;a))$. Let us suppose that $0 \notin \sigma(\mathcal{A}_a)$ and denote by $\mathcal{Z}_t$, $t \geq 0$, a space $\mathcal{Z}_t = D(|\mathcal{A}_a|^{(t+1)/2})$. Let us define spaces $\{Z_s^{(a)}\}$, $\{Z_s\}$ and operators J_a, $J(a)$ and function H^0 in the same way as in § 5 but with the operator $|\mathcal{A}_a|$ instead of $\mathcal{A}_a$ and $|\lambda_j^{(a)}|$ instead of $\lambda_j^{(a)}$, $j = 1,2,\dots$ (by definition, $|\mathcal{A}_a|\varphi_j^{(a)} = |\lambda_j^{(a)}|\varphi_j^{(a)}\ \forall j)$.

Let us consider a hamiltonian

$$\mathcal{H}(w;a) = \frac{1}{2}\,\|w\|^2_{Z_0^{(a)}} + \frac{1}{2} < \operatorname{sgn} \mathcal{A}_a w^1, w^1 >_{Z_0^{(a)}} + \epsilon\, H^0(w;a); \tag{6.3}$$

$$\operatorname{sgn} \mathcal{A}_a\, \varphi_j^{(a)} = \operatorname{sgn} \lambda_j^{(a)}\, \varphi_j^{(a)}\ \forall\, j\ .$$

Corresponding Hamiltonian equations have the form

$$\begin{aligned} \dot{w}^1 &= |\, \mathcal{A}_a|^{1/2} w^2\ , \\ \dot{w}^2 &= -|\, \mathcal{A}_a|^{1/2}((\operatorname{sgn} \mathcal{A}_a)w^1 + \epsilon\, |\, \mathcal{A}_a|^{-1} \frac{\partial}{\partial w^1}\, \chi) \end{aligned} \tag{6.4}$$

and we get the equation (5.6) for the function $w^1(t,x)$, again. One can repeat the proofs of § 5 and to write down the equations (6.4) in a form (5.8) which satisfies the conditions of Theorem 1.1 with the condition (6.1) instead of (1.3′), (1.3). So the statements of Theorem 5.2 are true without the assumption $V(x;a) \geq 0$.

Appendix. On superposition operator in Sobolev spaces.

Let $O^c \subset \mathbb{C}$ be a complex neighborhood of the real line and $\chi : O^c \times [0,\pi] \longrightarrow \mathbb{C}^p$ be a C^k–function which is real for real arguments. Let $H^k(0,\pi;\mathbb{C}^p)$ $(H^k(0,\pi;\mathbb{R}^p))$ be the usual Sobolev space of $\mathbb{C}^p(\mathbb{R}^p)$–valued functions on $[0,\pi]$; B_R be a ball in $H^k(0,\pi;\mathbb{R}^p)$ of radius R centered at zero and $B_R^c(\delta)$ be a δ–neighborhood of B_R in $H^k(0,\pi;\mathbb{C}^p)$. As $H^k(0,\pi;\mathbb{R}^p) \subset C(0,\pi;\mathbb{R}^p)$ for $k \geq 1$ then for such a k $B_R^c(\delta) \subset C(0,\pi;O^c)$ if $\delta = \delta(R) << 1$. So the superposition operator

$$\phi : B_R^c(\delta) \longrightarrow C(0,\pi;\mathbb{C}^p), \ u(x) \longmapsto \chi(u(x),x)$$

is well–defined.

Theorem A1. Let $k \in \mathbb{N}$, $\chi \in C^k(O^c \times [0,\pi])$ and $\forall s \leq k$

$$\frac{\partial^s}{\partial x^s}\chi(\cdot,x) \in \mathcal{A}^R(O^c;\mathbb{C}) \ \forall x \in [0,\pi], \ \left|\frac{\partial^s}{\partial x^s}\chi(u,x)\right| \leq K_* \quad \forall u \in O^c, \ x \in [0,\pi] .$$

Then $\phi \in \mathcal{A}^R(B_R^c(\delta); H^k(0,\pi;\mathbb{C}^p))$ and

$$|\phi(u)|_{H^k(0,\pi;\mathbb{C}^p)} \leq C(R)K_* \quad \forall u \in B_R^c(\delta) \tag{A1}$$

Proof. By taking a derivative of order $\ell \leq k$ from the function $\chi(u(x),x)$, $u \in B_R^c(\delta)$, one gets the estimate (A1). If $u \in B_R^c(\delta)$ and $v,w \in H^k(0,\pi;\mathbb{C}^p)$ then the function

$$\lambda \longmapsto \langle \phi(u+\lambda v),w \rangle_{H^k(0,\pi;\mathbb{C}^p)}$$

is complex–analytic in some neighborhood of the origin in $\mathbb{C}$; so the map ϕ is weakly analytic on $B_R^c(\delta)$.

As ϕ is bounded and weakly analytic then it is Fréchet–analytic (see [PT], Appendix A). ■

Let the function $\chi = \chi(u,x;a)$ depends on a parameter $a \in \mathfrak{A}$ in a Lipschitzian way, i.e. $\chi(\cdot,\cdot;a) \in C^k(O^c \times [0,\pi])$ $\forall a \in \mathfrak{A}$ and

$$\frac{\partial^s}{\partial x^s}\chi(\cdot,x;\cdot) \in \mathcal{A}^R(O^c;\mathbb{C}^p) \ \forall s \leq k, \ \forall x \in [0,\pi] . \tag{A2}$$

Then by applying Theorem A1 to functions $\chi(u(x),x,a)$ and $\chi(u(x),x,a_1) - \chi(u(x),x,a_2)$ $(a,a_1,a_2 \in \mathfrak{A})$ we get

<u>Corollary A2</u>. If assumption (A2) takes place for some $k \in \mathbb{N}$, then $\phi \in \mathcal{A}_{\mathfrak{A}}^R(B_R^c(\delta);\ H^k(0,\pi;\mathbb{C}^p))$. In particular, a function $u(x) \longmapsto \int \phi(u)(x)dx$ belongs to $\mathcal{A}_{\mathfrak{A}}^R(B_R^c(\delta);\mathbb{C}^p)$.

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