Codazzi Tensor Fields, Curvature
and Pontryagin forms

by

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AND PONTRYAGIN FORMS

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§ 1 . Introduction

A symmetric (0,2) tensor field b on a Riemannian manifold (M,g) is said to be a Codazzi tensor if it satisfies the Codazzi equation

$$(\nabla_X b)(Y,Z) = (\nabla_Y b)(X,Z)$$

for arbitrary vector fields X, Y, Z . In this case, the self-adjoint section B of End TM , character-ized by $g(BX,Y) = b(X,Y)$, will also be called a Codazzi tensor. The Codazzi tensor b will be call-ed non-trivial if it is not a constant multiple of the metric.

The aim of the present paper is to study some geometric and topological consequences of the exist-ence of a non-trivial Codazzi tensor on a given Riemannian manifold. Results of this type were obtain-ed by J. P. Bourguignon [3] , who proved that the existence of such a tensor imposes strong restrictions on the curvature operator ([3] , Théorème 5.1. and Corollaire 5.3.) and, as a consequence, obtained the following theorem ([3] , Corollaire 7.3.) : A compact orientable Riemannian four-manifold admitting a non-trivial Codazzi tensor with constant trace must have signature zero. Our main results consist in generaliz-ing these theorems, in particular in seeing what can be said when the assumption on the trace is dropped.

In §2 of this paper we observe that, in the C^∞ category, every manifold admits a Riemannian metric with a non-trivial Codazzi tensor (Example 7), so

that topological consequences may be expected only if some sort of analytic behaviour is assumed. §3 is devoted to the particular consequences of the existence of a non-trivial Codazzi tensor B for the structure of the curvature operator (Theorem 1) :

For any point x of the manifold M and arbitrary eigenspaces V_λ , V_μ of B_x , the span $V_\lambda \wedge V_\mu \subset \Lambda^2 T_x M$ of all exterior products of elements of V_λ and V_μ is invariant under the curvature operator R_x acting on 2-forms. As a consequence, we obtain in §4 a relation between the eigenspaces of any Codazzi tensor and the Pontryagin forms (Propositions 3 and 4), which, together with an extra argument for the case of a Codazzi tensor having only two distinct eigenvalues (Lemma 1) implies that a compact orientable Riemannian four-manifold (M,g) admitting a non-trivial Codazzi tensor b must have signature zero unless the restriction of b to some non-empty open subset of M is a constant multiple of g (Theorem 2). Another consequence of Proposition 4 is that for any n-dimensional Riemannian manifold with a Codazzi tensor having n distinct eigenvalues almost everywhere, all the real Pontryagin classes are zero (Corollary 3).

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§ 2 . Examples of Codazzi tensors

Codazzi tensors appear in a natural way in many geometric situations :

EXAMPLE 0 . The simplest Codazzi tensors are parallel ones ; non-trivial (i.e. not proportional to the metric) tensors of this type exist only in locally reducible manifolds.

EXAMPLE 1 . For a space (M,g) of constant sectional curvature κ and any function f on M , the formula b = ∇df + κfg defines a Codazzi tensor. As shown by D. Ferus [8], every Codazzi tensor in a space of constant curvature is, locally, of this type.

EXAMPLE 2 . The second fundamental form of any hypersurface (M,g) in a space of constant curvature is a Codazzi tensor (non-trivial, unless M is totally umbilic).

EXAMPLE 3 . Let (M,g) be a conformally flat manifold, n = dim M ≥ 3 . Then b = Ric - $(2n-2)^{-1}$ scal.g is a Codazzi tensor (non-trivial, unless (M,g) is of constant curvature). In fact, for n = 3 , the Codazzi equation for b is equivalent to the conformal flatness of g , while, for n ≥ 4 , the Weyl conformal tensor W of any Riemannian n-manifold satisfies the well-known divergence formula

$$(n-2)\nabla^r W_{rkij} = (n-3)(\nabla_i b_{kj} - \nabla_j b_{ki}) .$$

XAMPLE 4 . A Riemannian manifold is said to have <u>harmonic cur-</u>
<u>vature</u> if $\delta R = 0$ (in local coordinates, $\nabla^r R_{rkij} = 0$). This
happens if and only if the Ricci tensor Ric satisfies the Co-
dazzi equation. There exist various examples of compact manifolds
with this property and with $\nabla \text{Ric} \neq 0$ ([7], [5], [13]). In par-
ticular, such a metric always exists on the product $S^1 \times N$, N
being any compact Einstein manifold of positive scalar curvature.

EXAMPLE 5 . Consider a Riemannian manifold (M,g) admitting a
function f (not identically zero) such that

$$(1). \qquad \nabla df = f[\text{Ric} - (n-1)^{-1}\text{Scal}.g] , \quad n = \dim M .$$

It is well-known ([4], [9], cf. also [14]) that such f exists
if and only if the mapping assigning to metrics on M their sca-
lar curvature functions is not submersive at g . Moreover, (1)
is necessary and sufficient in order that the metric $g + f^2.dt^2$
on $(M - f^{-1}(0)) \times S^1$ be Einsteinian. There are some obvious ex-
amples for (1) (the standard sphere $M = S^n \subset \mathbb{R}^{n+1}$, f linear ;
a Riemannian product $M = S^1 \times N$, N an Einstein space of positive
scalar curvature, f being the composite of a linear function on
$S^1 \subset \mathbb{R}^2$ with the projection $S^1 \times N \to S^1$). Moreover, as shown by
J. Lafontaine [16], and, independently, by O. Kobayashi, for any
Einstein manifold N of positive scalar curvature (dim $N \geq 2$), the
product $S^1 \times N$ admits a metric g (essentially different from the
product $S^1 \times N$) such that (1) has a non-trivial solution f . If
$n = 3$ and f satisfies (1) on (M,g) , then $b = f^2\text{Ric} +$
$+ (\frac{2}{3}|\nabla f|^2 - \frac{1}{3}\text{Scal}.f^2)g$ is a Codazzi tensor on (M,g) (in fact,
the Codazzi equation for b is just the integrability condition
for (1) with $n = 3$).

EXAMPLE 6 . Fix a basis X,Y,Z of left-invariant vector fields
on the Lie group $S^3 = SU(2)$, satisfying the bracket relations

$[X,Y] = Z$, $[Y,Z] = X$, $[Z,X] = Y$. Given a number
$\gamma > 0$ and mutually distinct real numbers λ,μ,ν ,
we define a left-invariant metric g and a $(1,1)$ ten-
sor field B on S^3 by $g(X,X) = \gamma(\mu-\nu)^2$, $R(X,Y) =$
$= \gamma(\lambda-\nu)^2$, $R(X,Y) = g(X,Z) =$
$= 0$ and $BX = \lambda X$, $BY = \mu Y$, $BZ = \nu Z$. Then B is a
Codazzi tensor on (S^3,g) . Conversely, it is easy to
show that if (M,g) is a three-dimensional, complete,
locally irreducible Riemannian manifold with a non-tri-
vial Codazzi tensor B having constant eigenvalues,
then the universal covering of the triple (M,g,B) is
isometric to some (S^3,g,B) of the type just described.

EXAMPLE 7 . Every manifold M carries a C^∞ metric g
such that (M,g) admits a non-trivial C^∞ Codazzi ten-
sor b . In fact, let E be a subset of M , diffeo-
morphic to a closed ball in $\mathbb{R}^n$, $n = \dim M$. Using a
suitable embedding of E in $\mathbb{R}^{n+1}$ (cf. Example 2), we
can find a C^∞ metric g_1 on E and a non-trivial Co-
dazzi tensor b on (E,g_1) , vanishing near ∂E . Set-
ting $b = 0$ in $M \setminus E$, we may choose our g to be
any C^∞ metric on M such that $g = g_1$ wherever $b \neq 0$.

EXAMPLE 8 . For Riemannian manifolds (M_i,g_i) , $i = 1,2$,
and a function $f > 0$ on M_1 , one defines the <u>warped</u>
<u>product</u> $(M,g) = (M_1,g_1) \times_f (M_2,g_2)$ by $M = M_1 \times M_2$ and
$g = \pi^*_1 g_1 + (f^2 \circ \pi_1)^2 \pi^*_2 g_2$, $\pi_i : M \to M_i$ being the natural
projection (cf. [2],[12]). If b' is a Codazzi tensor
of type $(0,2)$ on (M_2,g_2) , it is easy to verify that
$b = (f \circ \pi_1)^2 \pi^*_2 b'$ is a Codazzi tensor on (M,g) (for the
Riemannian connection of g , see [2], p. 24).

EXAMPLE 9 . Let (M,g,J) be a Kähler manifold. If B is a Codazzi tensor of type $(1,1)$ on (M,g) which is Hermitian with respect to J, i.e., $BoJ = JoB$, then B is parallel. In fact, the $(1,1)$ tensor field $\zeta = JoB$ is skew-adjoint and hence so is $\nabla_X \zeta$ for any vector X; on the other hand, $\nabla J = 0$ yields $(\nabla_X \zeta)Y = J((\nabla_X B)Y)$ for arbitrary vectors X, Y. Thus, the expression $\langle(\nabla_X \zeta)Y,Z\rangle$ is symmetric in X,Y and skew-symmetric in Y,Z and therefore it must vanish identically (cf. [18]).

EXAMPLE 10. For a smooth manifold M endowed with a linear connection ∇, the $(1,1)$ tensor fields B on M, satisfying the Codazzi equation $(\nabla_X B)Y = (\nabla_Y B)X$ for arbitrary vectors X,Y can be called the Codazzi tensors on (M,∇). They can be interpreted as follows :

(a) Denoting by d^∇ the exterior differentiation operator on TM-valued forms in M, determined by ∇, we see that a TM-valued 1-form is a Codazzi tensor on (M,∇) if and only if it is d^∇-closed (cf. [3]).

(b) Considering a Codazzi tensor B on (M,∇) in the case where M is compact, we may always assume that B is non-degenerate everywhere, replacing it, if necessary, by $B + t.\mathrm{Id}$ for a sufficiently large t . Any non-degenerate $(1,1)$ tensor field B on M is a vector bundle automorphism of TM, transforming ∇ into the connection $\bar{\nabla} = B*\nabla$ characterized by $B(\nabla_X Y) = \bar{\nabla}_X(BY)$. The torsion tensors $T, \bar{T}$ of ∇ and $\bar{\nabla}$ satisfy then the relation ([8])

$$B[T(X,Y) - \bar{T}(X,Y)] = (\nabla_X B)Y - (\nabla_Y B)X .$$

Consequently, a section B of $\mathrm{Aut}_M(TM)$ satisfies the Codazzi equation with respect to ∇ if and only if ∇ and $\bar{\nabla} = B*\nabla$ have equal torsion tensors ; since, in this case, $\nabla = (B^{-1})*\bar{\nabla}$, it follows that B^{-1} satisfies the Codazzi equation with respect to $\bar{\nabla}$. From the above equality it also follows that a symmetric connection ∇ on a compact manifold M admits a Codazzi tensor which is not a constant multiple of Id if and only if ∇ can be transformed

into some symmetric connection by an M-automorphism of TM not proportional to the identity.

It is clear that every Riemannian product carries non-trivial Codazzi tensors, which are parallel. However, by suitably deforming any product metric, one can obtain locally irreducible metrics on product manifolds, which still admit non-trivial Codazzi tensors:

PROPOSITION 1. For arbitrary C^∞ (resp., analytic) manifolds $M_1, \ldots, M_k$, $k \geq 1$, the product $M = M_1 \times \ldots \times M_k$ carries a C^∞ (resp., analytic) locally irreducible Riemannian metric g such that (M,g) admits a C^∞ (resp., analytic) Codazzi tensor b which has precisely k distinct eigenvalues at every point, all the eigenvalues being bounded on M, and whose eigenspace bundles coincide with the natural foliations of M coming from the product structure. Moreover, the metric induced by such a g on any integral manifold of an eigenspace foliation of b can be prescribed arbitrarily (unless k = 1, in which case it must be locally irreducible).

PROOF: We proceed by induction on k. For k = 1, our assertion is obvious. Suppose now that we have already found a metric g' with a Codazzi tensor b' on M' = $M_1 \times \ldots \times M_{k-1}$, having the required properties. Adding to b' a suitable constant multiple

of g', we may assume that 0 is not an eigenvalue of b' at any point. For an arbitrary metric g_k on M_k, we can clearly find a positive function f on M_k, bounded away from zero and such that the warped product $(M,g) = (M_k, g_k) \times_f (M',g')$ is locally irreducible. The Codazzi tensor b on (M,g), defined in terms of b' and f as in Example 8, will then have k distinct and uniformly bounded eigenvalues at each point, which completes the proof.

§ 3 . Consequences for the curvature

The following proposition, due to N. Hicks [10] (cf. also [19]) is the main step for proving our Theorem 1. It reduces all the arguments involving differentiations of a given Codazzi tensor to the mere fact that the curvature tensor to some new Riemannian metric has the generally valid symmetries, which makes the proof of Theorem 1 purely algebraic.

PROPOSITION 2 (N. Hicks). Let (M,g) be a Riemannian manifold, B a section of End TM , non-degenerate at each point. Define a new Riemannian metric G and a linear connection $\overline{\nabla}$ on M by $G(X,Y) = g(BX,BY)$ and $\overline{\nabla}_X Y = \nabla_X Y + B^{-1}((\nabla_X B)Y)$, i.e., $B(\overline{\nabla}_X Y) = \nabla_X(BY)$ for arbitrary vector fields X,Y , ∇ being the Riemannian connection of g . Then

(i) $\overline{\nabla}G = 0$ and the curvature tensor $\overline{R}$ of $\overline{\nabla}$ is characterized by $G(\overline{R}(X,Y)Z,U) = R(X,Y,BZ,BU)$.

(ii) If B satisfies the Codazzi equation $(\nabla_X B)Y = (\nabla_Y B)X$, then $\overline{\nabla}$ is the Riemannian connection of G and the curvature tensor R^G (of type (0,4)) for (M,G) is given by

(2) $R^G(X,Y,Z,U) = R(X,Y,BZ,BU)$.

(iii) If B is a (self-adjoint) Codazzi tensor on (M,g) , then the symmetric tensor field b , defined by $b(X,Y) = g(BX,Y)$, is a Codazzi tensor on (M,g) as well as on (M,G) .

PROOF : B is an automorphism (a "gauge transformation") of the vector bundle TM and our definitions of G and $\overline{\nabla}$ can be written as $G = B^*g$ and $\overline{\nabla} = B^*\nabla$. Consequently, $\overline{\nabla}G = 0$ and $\overline{R} = B^*R$, which implies (i). If, moreover, B is a Codazzi tensor on (M,∇) , then $\overline{\nabla}$ is torsion-free and B^{-1} is a Codazzi tensor on $(M,\overline{\nabla})$ (cf. Example 10), which, together with $G(B^{-1}X,Y) = g(X,BY)$, implies (ii) and (iii). This completes the proof.

Using Proposition 2, we can now prove our basic result on the structure of the curvature tensor R of a Riemannian manifold (M,g) admitting a non-trivial Codazzi tensor B . For any point $x \in M$ and eigenvalues λ, μ of B_x , we shall denote by V_λ , $V_\mu \subset T_xM$ the corresponding eigenspaces and by $V_\lambda \wedge V_\mu \subset \Lambda^2 T_xM$ the subspace spanned by all exterior products $X \wedge Y$ with $X \in V_\lambda$, $Y \in V_\mu$. The curvature tensor R will be viewed, in the obvious way, as an operator acting on 2-forms.

THEOREM 1 . Let B be a Codazzi tensor on a Riemannian manifold (M,g) . For any $x \in M$ and arbitrary eigenvalues λ, μ of B_x , the subspace $V_\lambda \wedge V_\mu$ of $\Lambda^2 T_xM$ is invariant under the curvature operator $R_x \in End \Lambda^2 T_xM$. In other words, given eigenvalues λ, μ, ν of B_x and vectors $X \in V_\lambda$, $Y \in V_\mu$, $Z \in V_\nu$, we have

$$R(X,Y)Z = 0$$

provided that λ, μ, ν are mutually distinct or
$\lambda = \mu \neq \nu$.

PROOF : Replacing B by $B + t.Id$ for a suitable
constant t, we may assume that B is non-degene-
rate in a neighbourhood of x . Consider eigenvalues
λ, μ, ν, ξ of B_x and vectors $X \in V_\lambda$, $Y \in V_\mu$,
$Z \in V_\nu$, $U \in V_\xi$. Using (2), we obtain $\nu\xi R(X,Y,Z,U) =$
$= R^G(X,Y,Z,U) = R^G(Z,U,X,Y) = \lambda\mu R(X,Y,Z,U)$ and, simi-
larly, $(\mu\xi-\lambda\nu)R(X,Z,U,Y) = (\mu\nu-\lambda\xi)R(X,U,Y,Z) = 0$.
In view of (2), we also have $0 = R^G(X,Y,Z,U) +$
$+ R^G(X,Z,U,Y) + R^G(X,U,Y,Z) = \nu\xi R(X,Y,Z,U) +$
$+ \mu\xi R(X,Z,U,Y) + \mu\nu R(X,U,Y,Z)$, so that the preceding
equalities, together with $\mu\nu\xi \neq 0$, yield the matrix
equation

$$\begin{bmatrix} \lambda & \xi & \nu \\ \xi & \lambda & \mu \\ \nu & \mu & \lambda \end{bmatrix} \cdot \begin{bmatrix} R(X,Y,Z,U) \\ R(X,Z,U,Y) \\ R(X,U,Y,Z) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Suppose that $R(X,Y,Z,U) \neq 0$. Then, the coefficient
matrix above is of rank at most two, which implies
the cofactor relations $(\lambda-\xi)(\lambda+\xi-\mu-\nu) = (\nu-\lambda)(\nu+\lambda-\mu-\xi) =$
$= (\lambda-\mu)(\lambda+\mu-\nu-\xi) = 0$. If we now had $\lambda \neq \mu$, $\lambda \neq \nu$
and $\lambda \neq \xi$, these relations would give a contradic-
tion $(\lambda = \mu = \nu = \xi)$. Consequently, $R(X,Y,Z,U)$
can be non-zero only if λ is equal to one of μ,ν,ξ .

The symmetries of R imply now that, for arbitrary eigenvectors
$X_1,...,X_4$ of B_x , $R(X_1,X_2,X_3,X_4)$ can be non-zero only if
$X_1,...,X_4$ belong to at most two distinct eigenspaces. On the
other hand, if $X,Y \in V_\lambda$, $Z \in V_\nu$ and $\lambda \neq \nu$, (2) yields
$0 = B [R^G(X,Y)Z + R^G(Y,Z)X + R^G(Z,X)Y] = R(X,Y)(BZ) + R(Y,Z)(BX)$
$+ R(Z,X)(BY) = (\nu-\lambda)R(X,Y)Z$. This completes the proof.

Any two self-adjoint (1,1) tensor fields A, B on a
Riemannian manifold (M,g) give rise to a section $A \otimes B$ of
End Λ^2TM defined by $(A \otimes B)(X \wedge Y) = \frac{1}{2}(AX \wedge BY + BX \wedge AY)$. Thus,
(2) can be rewritten as $R^G = (B \otimes B) \circ R$, where R^G is viewed
as a section of End Λ^2TM with the aid of g . The Weyl confor-
mal tensor W of (M,g) , acting on 2-forms, is given by $W =$
$= R - 2(n-2)^{-1}Id \otimes Ric + (n-1)^{-1}(n-2)^{-1}Scal.Id \otimes Id$, $n = \dim M$.
If $x \in M$ and V_λ , $V_\mu \subset T_x M$ are the eigenspaces corresponding
to the eigenvalues λ , μ of B_x , then the restriction of
$(Id \otimes B)_x$ to $V_\lambda \wedge V_\mu$ equals $\frac{1}{2}(\lambda+\mu)$ times the identity. Sup-
pose now that B is a Codazzi tensor. If $X \in V_\lambda$, $Y \in V_\mu$,
where λ, μ are distinct eigenvalues of B_x , then Theorem 1
yields $R(X,Z,Y,Z) = 0$ for any eigenvector Z of B_x . There-
fore $Ric(X,Y) = 0$. Hence the Ricci tensor commutes with B
and, since $\Lambda^2T_x M$ is spanned by all the $V_\lambda \wedge V_\mu$, Theorem 1
implies that the endomorphisms R , $Id \otimes Ric$ and W of Λ^2TM
commute with $Id \otimes B$. Consequently, we obtain the following com-
mutation theorem, due to J. P. Bourguignon ([3], Théorème 5.1.) ;
note that this result implies our Theorem 1 except for the
case where B_x has two distinct pairs (λ,μ) , (ν,ξ) of
eigenvalues with $\lambda+\mu = \nu+\xi$: in this case, $V_\lambda \wedge V_\mu$ and $V_\nu \wedge V_\xi$
are proper subspaces of some eigenspace of $(Id \otimes B)_x$ and so.

their R_x-invariance is not an algebraic consequence of the fact that the endomorphisms commute.

COROLLARY 1 (J. P. Bourguignon). Let B be a Codazzi tensor on a Riemannian manifold (M,g). Then

(i) B commutes with the Ricci tensor Ric,

(ii) The endomorphisms R, Id Ⓢ Ric and W of $\Lambda^2 TM$ commute with Id Ⓢ B.

Let us now consider a non-trivial Codazzi tensor B on a Riemannian manifold (M,g). Given a point x ∈ M, it is natural to study the components of the curvature tensor R_x with respect to an orthonormal basis of eigenvectors of B_x. According to Theorem 1, there are, essentially, only two types of these components, which may be non-zero; they have the form R(X,Y,Z,U) with X,Z ∈ V_λ, Y,U ∈ V_μ, λ, μ being eigenvalues of B_x such that either

(I)   λ = μ,

or

(II)   λ ≠ μ.

The obvious question that arises is whether the Codazzi equation for B imposes any algebraic restriction on these components, which are not consequences of the general curvature symmetries. For the components of type (I), the answer is negative, the simplest example being the parallel tensors on Riemannian product manifolds; moreover, the same situation occurs when the underlying manifold is assumed locally irreducible.

In fact, restricting our consideration to the connected components of some open dense subset of M, we may view the eigenvalues of the given self-adjoint (1,1) tensor field B as smooth functions, while the eigenspaces of B form differentiable distributions.

If B is a Codazzi tensor, these distributions must be integrable and their leaves are totally umbilic in M ([7], Lemma 2; more general results can be found in[11]). The curvature components of type (I) are, thus, closely related to the intrinsic curvatures of the leaves, which, by Proposition 1, may be completely arbitrary also in the locally irreducible case.

On the other hand, for a Riemannian manifold with a non-trivial Codazzi tensor, the curvature components of type (II) must have some special algebraic properties, at least in the case of a Codazzi tensor with only two distinct eigenvalues. Namely, we have

LEMMA 1. Let (M,g) be a Riemannian manifold with a Codazzi tensor B which has, at each point x ∈ M, exactly two distinct eigenvalues λ(x), μ(x), λ(x) < μ(x). This gives rise to smooth eigenvalue functions λ, μ and eigenspace distributions V_λ, V_μ on M. For any point x ∈ M and arbitrary eigenvectors X,Z ∈ $V_\lambda(x)$, Y,U ∈ $V_\mu(x)$, we have

$$(\lambda-\mu)^2 R(X,Y,Z,U) = -g(\nabla\lambda,\nabla_\mu)g(X,Z)g(Y,U) +$$
$$+ g(X,Z)A_\mu(Y,U) + A_\lambda(X,Z)g(Y,U),$$ (3)

A_λ (resp., A_μ) being the symmetric bilinear form

on V_λ (resp. on V_μ), obtained by restricting the bilinear form $(\mu - \lambda)\nabla d\nu - 2d\nu \otimes d\nu$ to V_λ (resp. by restricting $(\lambda - \mu)\nabla d\lambda - 2d\lambda \otimes d\lambda$ to V_μ). Consequently, there exist orthonormal bases $e_1, \ldots, e_s$ of $V_\lambda(x)$, $e_{s+1}, \ldots, e_m$ of $V_\mu(x)$ ($m = \dim V_\lambda$, $s = \dim M$) such that, for certain real numbers $c_1, \ldots, c_n$:

(4) $\qquad R(e_a, e_i, e_j, e_b) = (c_i + c_a)\delta_{ij}\delta_{ab}$

whenever $1 \le i, j \le m < a, b \le n$.

PROOF : The vector bundle decomposition $TM = V_\lambda \oplus V_\mu$ together with the Riemannian connection ∇ in TM gives rise to the connections ∇^λ, ∇^μ in V_λ, V_μ, respectively, defined in the obvious way. On the other hand, for arbitrary local sections X, Z of V_λ and Y, U of V_μ, we have $(\lambda - \mu)g(\nabla^\lambda_X Z, Y) = g(X, Z)\nabla_Y \lambda$ (by Lemma 2 of [7], V_λ is integrable, so that both sides of this equality are symmetric in X, Z, while the equality holds for $X = Z$ by (4†) of Lemma 1 of [7]), and, similarly, $(\mu - \lambda)g(\nabla^\mu_Y U, X) = g(Y, U)\nabla_X \mu$. Consequently, for X, Y, Z, U as above, $\nabla_X Y - \nabla^\mu_X Y + \nabla^\lambda_X X$, $\nabla_Y U - \nabla^\mu_Y U + \nabla^\lambda_Y U$,

$+ (\lambda - \mu)^{-1}g(X, Z)(\nabla\lambda)_\mu$, $\nabla_X Y - (\mu - \lambda)^{-1}g(Y, U)_\lambda X + \nabla^\mu_Y Y$,

$\nabla_X Y - (\lambda - \mu)^{-1}g(X, U)^{-1}g(X, U)(\nabla\lambda)_\lambda$,

$(\nabla\lambda)_\mu$ being the V_μ component of $\nabla\lambda$. This implies

$g(\nabla^\lambda_X \nabla^\lambda_Y Z) - (\lambda - \mu)^{-1}g(\nabla\mu, \nabla^\lambda_X Z)\nabla_Y \mu$,

$g(\nabla^\lambda_X Z) - (\lambda - \mu)^{-1}g(\nabla\mu, \nabla^\lambda_X Z) +$

$+ (\lambda - \mu)^{-2}\nabla_Y(\mu - \lambda)g(X, Z)\nabla_\mu$.

$+ g(X, \nabla^\lambda Z))\nabla_\mu + (\lambda - \mu)^{-1}g(\nabla_X(\nabla\lambda)_\mu, U)g(X, Z)$ and

$-g(U, \nabla_X \nabla_Y Z) - (\lambda - \mu)^{-2}g(X, \nabla^\lambda_X(\lambda - \mu)) \cdot g(Y, U)\nabla_X \mu -$

$-(\lambda - \mu)^{-1}g(X, \nabla_X \nabla_\mu U) - (\lambda - \mu)^{-2}g(X, U)\nabla^\lambda_X \mu -$

$-(\lambda - \mu)^{-1}g(X, \nabla^\lambda_X Z)\nabla_Y \mu$, and $g(U, \nabla[X, Y]Z) -$

$-(\lambda - \mu)^{-1}g(\nabla^\mu_Y X, U)\nabla_\mu - (\lambda - \mu)^{-1}g(\nabla^\lambda_X Z)\nabla_\mu -$

$-(\lambda - \mu)^{-2}g(X, Z)\nabla^\lambda_X \lambda - (\lambda - \mu)^{-2}\nabla_X \lambda \cdot \nabla_\mu \lambda \cdot g(Y, U)$.

Adding up the last relations and using the obvious equalities $g(\nabla_X(\nabla\lambda)_\mu, U) = (\nabla d\lambda)(U, Y) + g(\nabla_Y(\nabla\lambda)_\mu, U)$

$\nabla_X \nabla_\mu - g(\nabla_X(\nabla\mu), Z) + g(\nabla^\lambda_X Z, \nabla_\mu) = (\nabla d\mu)(X, Z) +$

$- g(\nabla_Y(\nabla\lambda)_\mu, U) = (\nabla d\lambda)(U, Y) + (\mu - \lambda)^{-1}g((\nabla\lambda)_\mu, \nabla_\mu)g(X, Z)$

$\nabla_X \nabla_\mu - g(\nabla_X(\nabla\mu), Z) + g(\nabla^\lambda_X Z, \nabla\mu) = (\nabla d\mu)(X, Z) +$

$+ g(\nabla^\lambda_X Z, \nabla_\mu) + (\lambda - \mu)^{-1}g((\nabla\lambda)_\mu, \nabla\mu)$ and $\nabla\lambda =$

$(\nabla\lambda)_\mu + (\nabla\lambda)_\mu$, we obtain (3). The symmetry of A_λ and A_μ, immediate from the Bianchi identity for R together with Theorem 1, can also be verified as follows : the only term of A_λ which is not obvious-ly symmetric is the restriction of $d\lambda \otimes d\mu$ to V_λ, however, by Lemma 2 of [7], λ is constant along V_λ unless $\dim V_\lambda = 1$. Finally, (4) can be obtained by diagonalizing A_λ and A_μ, which completes the proof.

REMARK 1 : Assuming in Lemma 1 that $\dim V_\lambda \ge 2$ and $\dim V_\mu \ge 2$ we can reduce formula (3) to the simpler form

$\nabla R(Z, Y, Z, U) = -g(X, Z)\nabla d\mu(Z, U) - \nabla d\mu(X, Z)g(Y, U) +$

$+ \frac{1}{6-\phi}|\nabla\phi|^2 g(X, Z)g(Y, U)$

for $X, Y \in V_\lambda$, $Y, U \in V_\mu$; where $\phi = (\lambda - \mu)^{-1}$. In fact, the constancy of λ and μ along V_λ and V_μ, respectively ([7]), yields $g(\nabla\lambda, \nabla\mu) = 0$ and $A_\lambda(X, Z)_\mu = -\phi^{-3}d\phi(X, Z) +$

$+ (\mu - \lambda)\nabla d\lambda(X, Z)$, while $\nabla d\lambda(X, Z) = -d\lambda(\nabla^\lambda_X Z) =$

$(\lambda - \mu)(\nabla^\lambda_X Z)_\mu = g(X, Z)\nabla\lambda$ (cf. the first equality in the proof of Lemma 1), we obtain $A_\lambda(X, Z) = -\phi^{-3}d\phi(X, Z) +$

similarly, $A_\mu(Y, U) = -\phi^{-3}d\phi(Y, U) + |\nabla\lambda|^2 g(X, Z)$ as restriction is immediate from (1).

§ 4. Codazzi tensors and Pontryagin forms

Given a Riemannian manifold (M,g) and a positive integer k, one defines the k-th Pontryagin form $P_k = P_k(M,g)$ of (M,g), which is a closed $4k$-form on M, obtained, locally, by applying some standard homogeneous polynomials of degree $2k$ to the local curvature components ([17], p. 308) and represents, in the de Rham cohomology of M, the real Pontryagin class $P_k \in H^{4k}(M,\mathbb{R})$. By the Pontryagin algebra of (M,g) we shall mean the graded subalgebra of $\Gamma(\Lambda M)$ (sections of $\Lambda M = \Lambda^0 T^*M + \ldots + \Lambda^n T^*M$, $n = \dim M$, with pointwise exterior multiplication), generated by all the P_k. It is sometimes more convenient to consider, instead of P_k, another system Ω_k $(k = 1,2,\ldots)$ of generators for the Pontryagin algebra of (M,g), described as follows: Ω_k is the $4k$-form on M, obtained by alternating (taking the skew-symmetric part of) the tensor field ω_k of type $(0,4k)$ on M, given by

$$\omega_k(X_1,\ldots,X_{4k}) =$$

$$(5) \qquad = \mathrm{Trace}\,[R(X_1 \wedge X_2) \circ R(X_3 \wedge X_4) \circ \ldots \circ R(X_{4k-1} \wedge X_{4k})]\,,$$

the curvature tensor R of (M,g) being viewed as a homomorphism $\Lambda^2 TM \to \mathrm{End}\ TM$. Every Pontryagin form can be obtained by applying some universal polynomial to the Ω_k.

Consider now a finite dimensional real vector space T. A fixed inner product $\langle,\rangle$ allows to identify T with its dual T^*, $T \otimes T$ with $T^* = \mathrm{End}\ T$ and $\Lambda^2 T$ with a subspace of $\mathrm{End}\ T$. Thus, for vectors $X,Y,Z,U \in T$, we have

$$(6) \qquad \mathrm{Trace}\,(X \otimes Y) = \langle X,Y\rangle$$

and

$$(7) \qquad (X \otimes Y) \circ (Z \otimes U) = \langle Y,Z\rangle . X \otimes U\,.$$

$$X \wedge Y = X \otimes Y - Y \otimes X\,.$$

For subspaces V, V' of T, we define $V \otimes V' \subset T \otimes T$ to be the span of all tensor products $X \otimes X'$ with $X \in V$, $X' \in V'$. In an analogous way, we define $V \wedge V' \subset \Lambda^2 T$ and, for subspaces P, P' of $\mathrm{End}\ T$, the subspaces $P \circ P'$ and $P + P'$.

LEMMA 2. Suppose we are given a finite dimensional real vector space T with an inner product, a subspace V of T and an endomorphism R of $\Lambda^2 T$ leaving the subspaces $V \wedge V$, $V \wedge V^\perp$ and $V^\perp \wedge V^\perp$ of $\Lambda^2 T$ invariant. For any sequence $X_1,\ldots,X_{2r}$ $(r \geq 1)$ of vectors, containing exactly m elements of V and $2r-m$ elements of $V^\perp$, we have

$$(1) \qquad R(X_1 \wedge X_2) \circ R(X_3 \wedge X_4) \circ \ldots \circ R(X_{2r-1} \wedge X_{2r}) \in$$

$$\in \begin{cases} (V \otimes V) + (V^\perp \otimes V^\perp)\,, & \text{if } m \text{ is even,} \\ (V \otimes V^\perp) + (V^\perp \otimes V)\,, & \text{if } m \text{ is odd.} \end{cases}$$

(ii) Trace $[R(X_j \wedge X_2) \circ \dots \circ R(X_{2r-1} \wedge X_{2r})] = 0$

if m is odd.

PROOF : Set $V_j = V$, $V_{-j} = V^\perp$ and $P(1) =$
$= (V \otimes V) + (V^\perp \otimes V^\perp)$, $P(-1) = (V \otimes V^\perp) + (V^\perp \otimes V)$.
By (7) and our hypothesis on R , $P(\delta) \circ P(\varepsilon) \subset P(\delta\varepsilon)$
and $R(V_\delta \wedge V_\varepsilon) \subset V_\delta \wedge V_\varepsilon \subset P(\delta\varepsilon)$ whenever $\delta, \varepsilon = \pm 1$.
Therefore, if $X_i \in V_{\delta_i}$, $i = 1, \dots, 2r$, then
$R(X_1 \wedge X_2) \circ \dots \circ R(X_{2r-1} \wedge X_{2r}) \in P(\delta_1 \delta_2 \dots \delta_{2r})$,
which proves (i). Assertion (ii) is now immediate
from (6), which completes the proof.

As an immediate consequence of Theorem 1 and
Lemma 2, we obtain

PROPOSITION 3 . Let (M,g) be a Riemannian manifold
with a Codazzi tensor B , x a point of M . $V_\lambda \subset$
$\subset T_x M$ the eigenspace corresponding to an eigenvalue
λ of B . Given a positive integer k , an odd in-
teger m $(0 \le m \le 4k)$ and arbitrary vectors $X_1, \dots,$
$\dots, X_{4k} \in T_x M$ such that $X_1, \dots, X_m \in V_\lambda$ and
$X_{m+1}, \dots, X_{4k} \in V_\lambda^\perp$, every element Ω of degree 4k
in the Pontryagin algebra of (M,g) satisfies the
relation

$$\Omega(X_1, \dots, X_{4k}) = 0 .$$

In particular, if λ is a simple eigenvalue of B_x
and $X \in V_\lambda$, then

$$\iota_X P_k = P_k(X, \dots, \cdot) = 0$$
for each Pontryagin form P_k , $k \ge 1$.

PROOF : In view of Theorem 1, we can apply Lemma 2
to $T = T_x M$ and $V = V_\lambda$. Consequently, the tensor
ω_k given by (5) has the property that $\omega_k(X_1, \dots, X_{4k}) =$
$= 0$ whenever, for some odd integer m , the se-
quence $X_1, \dots, X_{4k}$ contains m elements of V_λ
and 4k-m elements of $V_\lambda^\perp$. The same must hold
for the alternation Ω_k of ω_k and hence for all
forms in the Pontryagin algebra of (M,g) , which
completes the proof.

From Proposition 3 we obtain now the follow-
ing sufficient conditions for the vanishing of cer-
tain Pontryagin forms :

PROPOSITION 4 . Let (M,g) be a Riemannian mani-
fold with a Codazzi tensor B , x a point of M .
If B_x has exactly p distinct eigenvalues $\lambda_1, \dots$
$\dots, \lambda_p$ of multiplicities $m_1, \dots, m_p$, respective-
ly, then every homogeneous element of degree great-
er than

$$2 \sum_{i=1}^{p} [m_i/2] ,$$

in the Pontryagin algebra of (M,g) vanishes at
x , $[m_i/2]$ being the integer part of $m_i/2$.

PROOF : Let Ω be a homogeneous element of degree

$4r$ in the Pontryagin algebra of (M,g) such that $\Omega_x \neq 0$. Thus, $\Omega(X_1,\ldots,X_{4r}) \neq 0$ for some sequence $X_1,\ldots,X_{4r}$ of linearly independent eigenvectors of B_x. By Proposition 3, for any eigenvalue λ_i of B_x, the number of times that eigenvectors corresponding to λ_i occur among the $X_1,\ldots,X_{4r}$ is even and not greater than the multiplicity m_i, i.e., it does not exceed $2[m_i/2]$. Consequently, $\Omega_x \neq 0$ implies that $\deg \Omega = 4r \leq 2 \Sigma_{i=1}^p [m_i/2]$, which completes the proof.

In particular, we have

COROLLARY 2. Let (M,g) be an n-dimensional Riemannian manifold with a Codazzi tensor B, x a point of M. If B_x has exactly k simple eigenvalues, $0 \leq k \leq n$, then every homogeneous element of degree greater than n-k in the Pontryagin algebra of (M,g) vanishes at x.

PROOF: Let $m_1 \leq \ldots \leq m_p$ be the multiplicities of the distinct eigenvalues of B_x, so that $m_1 = \ldots = m_k = 1$. Our assertion follows now from Proposition 4 together with $n-k = \Sigma_{i>k}m_i \geq 2\Sigma_{i>k}[m_i/2] = 2\Sigma_{i=1}^p[m_i/2]$.

COROLLARY 3. Let B be a Codazzi tensor on an n-dimensional Riemannian manifold (M,g). If, for some $x \in M$, B_x has n distinct eigenvalues, then all the Pontryagin forms of (M,g) vanish at x.

REMARK 2. It seems useful to consider the following concept, introduced by H. Maillot ([15],[16]): A Riemannian manifold (M,g) is said to have pure curvature operator at a point $x \in M$ if, for some orthonormal basis

$X_1,\ldots,X_n$ of T_xM, each exterior product $X_i \wedge X_j$, $(1 \leq i < j \leq n = \dim M)$ is an eigenvector of $R_x \in \text{End } \Lambda^2 T_xM$. This happens, e.g., if $R_x = \Sigma_\alpha A_\alpha \otimes B_\alpha$ with mutually commuting self-adjoint (1,1) tensors A_α, B_α, so that the simplest examples where the curvature operator is pure at each point are provided by conformally flat manifolds and by submanifolds of space forms having flat normal connection. As observed by Maillot, pure curvature operator at a point x implies that all the Pontryagin forms vanish at x (this is also immediate from Lemma 2 applied to $V = \text{span}(X_i)$, $i=1,\ldots,n$). By Theorem 1, if B is a Codazzi tensor on an n-dimensional Riemannian manifold (M,g) and, for some point x, B_x has n distinct eigenvalues, then (M,g) has pure curvature operator at x (cf. Corollary 3).

For Codazzi tensors on four-dimensional manifolds, the preceding results imply the following

THEOREM 2. Let B be a Codazzi tensor on a four-dimensional Riemannian manifold (M,g). The Pontryagin form P_1 of (M,g) satisfies then the relation

$$P_1 \otimes (B - \tfrac{1}{4}\text{Trace}B.\text{Id}) = 0.$$

PROOF: We shall prove that $P_1(x) = 0$ at all points x at which the number of eigenvalues of B is locally constant and greater than one; the set of these points is dense in the subset of M defined by $B \neq \tfrac{1}{4}\text{Trace}B.\text{Id}$. For such an x, the number of eigenvalues of B_x may equal two (multiplicities: (1,3) or (2,2)), three (multiplicities: (1,1,2)), or four (multiplicities: (1,1,1,1)). By Corollary 2, $P_1(x) = 0$ unless the distribution of multiplicities is

(2,2) . However, in the latter case formula (4) to-
gether with Theorem 1 imply that (M,g) has pure
curvature operator, which again yields $P_1(x) = 0$
(Remark 2). This completes the proof.

In the case of compact oriented four-manifolds,
Theorem 2 together with Hirzebruch's signature for-
mula $3\tau(M) = \int_M P_1$ yields

COROLLARY 4 . Let (M,g) be a compact, orientable,
analytic four-dimensional Riemannian manifold admit-
ting a non-trivial analytic Codazzi tensor. Then,
the signature of M is zero.

An oriented Riemannian four-manifold (M,g) is
called self-dual [1] if $W \circ * = W$, the Weyl tensor
W and the Hodge star $*$ being viewed as endomor-
phisms of $\Lambda^2 TM$. For any compact self-dual manifold
(M,g), the signature $\tau(M) \geq 0$, the inequality be-
ing strict unless $W = 0$ identically (see [1]). On
the other hand, one can define the divergence δW
of W by the local coordinate formula $(\delta W)_{ijk}$ =
$-\nabla^r W_{rijk}$. Corollary 4 gives now a new proof of
the following result ([6], Proposition 7 and [3],
Proposition 9.1.).

COROLLARY 5 . Let (M,g) be a compact, analytic,
oriented Riemannian four-manifold. If (M,g) is
self-dual and $\delta W = 0$, then (M,g) is conformally
flat or Einsteinian.

PROOF : Condition $\delta W = 0$ means that $b = Ric - \frac{1}{6} Scal.g$ is a Codazzi tensor (Example 3). If $\tau(M)$
= 0 , then $W = 0$ in view of self-duality. On the
other hand, if $\tau(M) \neq 0$, Corollary 4 implies that
b is a multiple of g , i.e., g is an Einstein
metric, which completes the proof.

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