HAMILTONIAN DYNAMICS ON THE MODULI SPACE OF HALF WEIGHTED BOHR - SOMMERFELD LAGRANIGIAN SUBCYCLES OF A FIXED VOLUME

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Abstract. Connecting ideas of [2], [8] and [1] in the present paper a new approach to geometrical quantization procedure is introduced. As a first result we verify that the correspondence between "classical" Poisson bracket on based symplectic manifold and "quantum" Poisson bracket on the moduli space takes place.

§0. Introduction

In recent papers [2] and [8] it was represented construction of moduli space of Bohr - Sommerfeld Lagrangian subcycles of fixed topological type over a symplectic manifold. Starting with a simple description of the tangent bundle to the moduli space authors propose an additional superstructure — fibered over the first moduli space the second one of half - weighted Bohr - Sommerfeld Lagrangian subcycles of a fixed volume. This moduli space is endowed with structure of an infinite dimensional kahler manifold. It's remarkable that this infinite dimensional manifold is constructed almost canonically over a fixed compact symplectic manifold — one has to add just a fixed metaplectic structure (see [2], [8]).

On the other hand one has an interesting approach to the description of the quantum mechanics (see [1]). Instead of a Hilbert space usual in the known descriptions one can use the projectivization of this space as the phase space — and this projectivization is an infinite dimensional kahler manifold, endowed as a kahler manifold with an integrable complex structure, a symplectic 2- form and the corresponding Riemannian metric. Then (see [1]) the Schroedinger equation describing the dynamics is replaced by "classical" Hamilton equation using the symplectic structure. The Riemannian metric on the same projective space reflects the probabilistic aspects of the model.

So dealing with these two constructions it is a natural idea in the geometric quantization programme. Namely for a classical mechanical system represented by the corresponding compact symplectic manifold (M, ω) with symplectic form ω one can consider an infinite dimensional kahler manifold defined as the moduli space of half - weighted Bohr - Sommerfeld Lagrangian subcycles of a fixed volume. This infinite dimensional kahler manifold can be regarded as an analogy of the projectivized Hilbert space corresponds to an appropriate quantization of the system. In this setup there is a natural correspondence between the space of smooth functions on (M, ω) and a special subspace in the space of smooth functions over the moduli space. For a compact symplectic manifold (M, ω) and the corresponding infinite dimensional kahler manifold $\mathcal{B}^{hw,1}_{BS}$ one has an inclusion

$$F : C^\infty(M \to \mathbb{R}) \to C^\infty(\mathcal{B}^{hw,1}_{BS}), \tag{0.1}$$

explicitly described below in the formula (3.3). So the natural question arises: what is the relationship between the Poisson brackets on the original finite dimensional symplectic manifold and on the induced infinite dimensional kahler manifold which admits canonical symplectic structure?

The main result of the present paper is the following (Proposition 3.2): inclusion F preserves the structure of Lee algebra on $C^\infty(M \to \mathbb{R})$, and the Poisson bracket induced by ω maps to the Poisson bracket induced by the canonical symplectic form Ω up to a constant multiple (at the end of section 4 we discuss what is the constant).

In other words if the infinite dimensional kahler manifold consists of half - weighted Bohr- Sommerfeld Lagrangian subcycles of fixed volume is a good candidate on the role of quantum phase space of the quantized system then the correspondence between the Poisson brackets ensures us that the Dirac condition holds in this quantization procedure (about the Dirac condition, prequantization, quantization etc. see [3], [4], [7]).

1. Moduli spaces of half- weighted Bohr - Sommerfeld Lagrangian subcycles

Let us recall briefly the main constructions and formulae from [2] and [8] which we use in what follows.

Let (M, ω) be a compact symplectic manifold of real dimension $2n$ endowed with symplectic form ω. For an appropriate real smooth n- dimensional manifold S one considers the space of all smooth inclusions

$$\phi : S \to M \tag{1.1}$$

such that $\phi^*\omega$ is identically zero on S. It's easy to see that this space just corresponds to the space of all smooth Lagrangian cycles on M, homologically equivalent to S. Let moreover the pair (M, ω) satisfies the integer Dirac condition

$$[\omega] \in H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R}).$$

Then one has on (M, ω) a prequantization quadruple (see [7]) consists of the manifold M, the symplectic form ω and additionally the complex line bundle $L \to M$ uniquely defined by the topological condition

$$c_1(L) = [\omega]$$

and a hermitian connection $a \in \mathcal{A}_h(L)$ which is a solution of the natural equation

$$F_a = 2\pi \imath \omega. \tag{1.2}$$

A Lagrangian inclusion ϕ (1.1) is Bohr - Sommerfeld if the following condition is satisfied: for the pair $(\phi^* L, \phi^* a)$ over S where $\phi^* L$ is topologically trivial over S and $\phi^* a$ is a flat connection (see (1.2)) the last one is gauge equivalent to the ordinary d.

It's easy to see that if S is simply connected then every Lagrangian inclusion is Bohr - Sommerfeld.

After factorization of the space of such inclusion by the $Diff S$- action one gets a space which was called the moduli space of Bohr - Sommerfeld Lagrangian subcycles of fixed topological type. In the pioneer papers [2] and [8] the tangent space to the moduli space in a smooth point (S, ϕ) is represented by the space of exact 1-forms on S. For simplicity let us fix an appropriate S and denote, following the original papers, as $\mathcal{B}_{BS}$ the moduli space. But the space $\mathcal{B}_{BS}$ doesn't admit any natural symplectic structure.

So the next step in the construction to consider the space of Planckian cycles $\mathcal{P}_S$ fibered over $\mathcal{B}_{BS}$ with $U(1)$- fibers. Take now the space of all half- weighted Planckian cycles $\mathcal{P}^{hw}_{BS}$ consists of pairs

$$\mathcal{P}^{hw}_{BS} = \{(\rho, \theta)\},$$

where ρ is a Planckian cycle over M and θ is a half - form over ρ (see [2], [8]). The point is that the last space admits a natural symplectic structure and a natural compatible integrable complex structure so it is a kahler manifold ([2], [8]). Further, on the kahler manifold $\mathcal{P}^{hw}_{BS}$ one has the natural $U(1)$- action, preserving both the structures. And we have as a result of kahler reduction (in dependence with choosing fixed volume) the moduli space $\mathcal{B}^{hw,r}_{BS}$ of half- weighted Bohr - Sommerfeld Lagrangian cycles with fixed volume $r \in \mathbb{R}$. In what follows we work with cycles of volume 1 just for simplicity but evidently all the results hold for any r.

For the computations below we need to recall the local description of $\mathcal{B}^{hw,1}_{BS}$ and the explicit formula for Ω. Points of $\mathcal{B}^{hw,1}_{BS}$ are represented by pairs (ϕ, θ) where ϕ is a Bohr - Sommerfeld Lagrangian inclusion of S into M and θ is a half - form on $\phi S \subset M$ such that

$$\int_{\phi S} \theta^2 = 1.$$

The tangent space $T_{(\phi_0, \theta_0)} \mathcal{B}^{hw,1}_{BS}$ is represented by pairs (f, θ) where f - is a smooth function on $\phi_0 S$ and θ is a half- form on the same sub manifold such that

$$\int_{\phi_0 S} f \theta_0^2 = 0 \tag{1.4}$$

and

$$\int_{\phi_0 S} \theta_0 \theta = 0. \tag{1.4'}$$

The symplectic form Ω at the point (ϕ_0, θ_0) has the form

$$\Omega(v_1, v_2) = \int_{\phi_0 S} (f_1 \theta_2 - f_2 \theta_1) \theta_0 \tag{1.5}$$

for every two tangent vectors

$$v_1 = (f_1, \theta_1), \qquad v_2 = (f_2, \theta_2).$$

It's not hard to see that this 2- form is nondegenereted everywhere and is closed. All details can be found in [2] and [8].

2. The background: geometrical formulation of quantum mechanics

So we have seen that for every compact symplectic manifold (M, ω) there exists a set of infinite dimensional kahler manifolds in dependence of the homological classes. It looks very interesting if we turn to the framework of "geometrical formulation of quantum mechanics". Since the author learned this subject from the article [1] it is recommended as a reference together with [6].

Let $\mathcal{H}$ is a Hilbert space (here and in what follows in the present section we will use the notations of [1], [6]) corresponding to a quantum mechanical system. Consider $\mathcal{H}$ as a real vector space endowed with complex structure J. Then the hermitian inner product can be decomposed into the sum of real and imaginary parts

$$< \Phi, \Psi >= \frac{1}{2\hbar} G(\Phi, \Psi) + \frac{\imath}{2\hbar} \Omega(\Phi, \Psi). \tag{2.1}$$

Thus on the real vector space $\mathcal{H}$ we have the corresponding triple (G, J, Ω), consists of positive defined real inner product, complex structure and the corresponding symplectic form.

On the Hilbert vector space every observable (being represented by a self adjoint operator) can be regarded as a vector field on $\mathcal{H}$: really in each point of $\mathcal{H}$ (more precisely at each vector) we have another vector - the volume of this operator. The Schroedinger equation reads as

$$\Psi' = -\frac{1}{\hbar} J \hat{H} \Psi, \tag{2.2}$$

so it's convenient to use the following notation

$$Y_{\hat{F}}(\Psi) = -\frac{1}{\hbar} J \hat{F} \Psi \tag{2.3}$$

for the vector field corresponding to an observable $\hat{F}$. For the same observable $\hat{F}$ let us consider the expectation value function

$$F : \mathcal{H} \to \mathbb{R}, \qquad F(\Psi) =< \Psi;, \hat{F}\Psi >= \frac{1}{2\hbar} G(\Psi; \hat{F}\Psi).$$

It's easy to see ((2.5) in [1]) that

$$i_{Y_{\hat{F}}} \Omega = dF, \tag{2.4}$$

so the Schroedinger equation reads as the Hamilton equation. Moreover, for two observables $\hat{F}, \hat{K}$ the Poisson bracket of the functions F and K is represented as the expectation value function of the self - adjoint operator equals to a multiple of their commutator (see (2.6), [1]).

The relationships of these types use to be applicable after we turn from the real kahler space to the projectivization of $\mathcal{H}$, which was denoted as $\mathcal{P}$ in [1]. Section B of the paper [1] is dedicated to a construction of such projective space using the language of constraints. From the mathematical point of view it is equivalent to construct the projective space using the kahler reduction under the natural $U(1)$-action with moment map

$$S : \mathcal{H} \to \mathbb{R}, \qquad S(\Psi) =< \Psi; \Psi > \tag{2.5}$$

for any level, for example for level $S(\Psi) = 1$. Under the projective space $\mathcal{P}$, realized by this procedure, one has the corresponding kahler structure consists of Riemannian metric g, integrable complex structure I and symplectic 2-form Ω. Instead of observables $\hat{F}$ one could consider the induced expectation value functions F which are invariant under the ($U(1)$- action being restricted on the hypersurface $S(\Psi) = 1$ thus these function are correctly defined on $\mathcal{P}$. In section C of the paper [1] one shows that the correspondence between usual observables on $\mathcal{H}$ and special functions on $\mathcal{P}$ is one - to - one, so one can reconstruct the original self - adjoint operator from the corresponding expectation function on $\mathcal{P}$. On the other hand one has to specify the subspace of observable functions over $\mathcal{P}$, induced by self - adjoint operators. The answer is: smooth function

$$f : \mathcal{P} \to \mathbb{R}$$

is induced by a self - adjoint operator if and only if its Hamiltonian vector field H_f over $\mathcal{P}$ is a Killing vector field for the Riemannian metric g (Corollary 1, Theorem 2.1 in [1]0.

Before forgetting about the original Hilbert space it's necessary to translate the notions "eigenvector" and "eigenvalue" to the projective language. It's not hard to see (subsection 3, section C of [1]) that

— for observable $\hat{F}$ with induced function $f : \mathcal{P} \to \mathbb{R}$ every eigenvector after the projectivization uses to be a critical point of the function f;

— and the corresponding critical value equals to the original eigenvalue.

Now we are ready to formulate the postulates of the quantum mechanics in these projective terms following the authors of [1] (see section D [1]). Quantum phase space is represented by an appropriate projective space $\mathcal{P}$ with is a kahler manifold (finite dimensional or infinite dimensional); the space of observable is the set of real smooth functions on $\mathcal{P}$ such that their Hamiltonian vector fields are Killing vector fields with respect to the Riemannian metric; the dynamics is described by the Hamilton equation; all probabilistic aspects, state reductions and so on are described in terms of geodesic distances (so based on the Riemannian structure on $\mathcal{P}$). One has the state reductions in both cases (of discrete spectrum and of non isolated critical points).

Roughly speaking the authors show that the difference between the classical mechanics and the quantum one is in the presence of an appropriate Riemannian metric. In both cases one has symplectic structures, reflecting the dynamics, but additionally in the quantum case one has a Riemannian metric, reflects the probabilistic aspects. At the end of the paper the author propose the following question: is there exist a quantization procedure of classical mechanical systems which gives directly from a given classical system an appropriate infinite dimensional kahler manifold $\mathcal{P}$ with Riemannian metric g and symplectic 2- form Ω and doesn't use known approaches, deriving form the system Hilbert vector spaces and doesn't refer to such Hilbert spaces?

3. Preferred functions on $\mathcal{B}_S^{hw,1}$

In this section we construct the inclusion F (0.1) and consider some geometrical objects on $\mathcal{B}_S^{hw,1}$ induced by a smooth real function f defined on the based symplectic manifold (M,ω).

Let $f \in C^\infty(M \to \mathbb{R})$ is a smooth function. Then the differential of this function df being restricted on a Bohr - Sommerfeld cycle $\phi S \subset M$ gives us a tangent vector to $\mathcal{B}_S^{hw,1}$ in point (ϕ,θ) for any θ. This gives a vector field on $\mathcal{B}_S^{hw,1}$ denoted as A_f. This vector field doesn't depend on the second "coordinate" on $\mathcal{B}_S^{hw,1}$, in other words it is constant along fibers of

$$\mathcal{B}_S^{hw,1} \to \mathcal{B}_S.$$

We have the following simple

Proposition 3.1. *The set of singular points of A_f consists of such Bohr - Sommerfeld cycles ϕS that the function f is constant being restricted on ϕS.*

Together with the vector field for a function f one has a natural induced 1- form on $\mathcal{B}_S^{hw,1}$. At a point (ϕ_0,θ_0) this form denoted as B_f reads as

$$B^f_{(\phi_0,\theta_0)}(f_1,\theta_1) = \int_{\phi S} f\theta_0\theta_1. \tag{3.1}$$

Since the symplectic form Ω is described by the formula (1.5) one gets by direct computation that the vector field A^f and the 1- form B_f are related as follows

$$B_f = \Omega^{-1}(A^f). \tag{3.2}$$

Now we want to define the inclusion F. For each smooth function f on M let us define the following function F_f, naturally induced on $\mathcal{B}_S^{hw,1}$:

$$F_f(\phi,\theta) = \int_{\phi S} f|_{\phi S}\theta^2. \tag{3.3}$$

First of all let us remark that the images of constant functions are constant functions on $\mathcal{B}_S^{hw,1}$ (compare (3.3) and (1.3)).

Consider the image of the smooth function space on M as a subspace of the smooth function space on $\mathcal{B}_S^{hw,1}$:

$$\mathcal{N} = Im_F(C^\infty(M \to \mathbb{R})), \quad \mathcal{N} \subset C^\infty(\mathcal{B}_S^{hw,1} \to \mathbb{R}).$$

It's easy to see that $\mathcal{N}$ is a linear subspace. But if we consider the space $C^\infty(\mathcal{B}_S^{hw,1} \to \mathbb{R})$ as the algebra with pointwise multiplication then the subspace $\mathcal{N}$ were not a subalgebra. Really if F_f and F_g are two induced functions form $\mathcal{N}$ then the product $F_f \cdot F_g$ isn't a priori induced by a real function and doesn't lie in $\mathcal{N}$. But this shortage is compensated by the fact that this $\mathcal{N}$ is a Lie subalgebra with respect to the Poisson bracket induced by the symplectic structure Ω. We have the following

Proposition 3.2. *The identity*

$$\{F_f; F_g\}_\Omega = 2F_{\{f;g\}_\omega}$$

holds.

Proposition 3.2 together with definition (3.3) represent an answer (or just a part of an answer) on the question arises in Introduction. Before we will prove the statement let us input few remarks.

The definition (3.3) of the correspondence $f \mapsto F_f$ can be easily modified scaling by an appropriate constant

$$F_f(\phi_0, \theta_0) = \int_{\phi_0 S} \tau f|_{\phi_0 S}\theta_0^2.$$

This modification changes the identity (3.5) as follows

$$\{F_f; F_g\}_\Omega = 2\tau^2 F_{\{f;g\}_\omega}.$$

So one can rearrange the identity (3.5) such that any physical constant (Planck constant etc.) will be consistent. Thus the Dirac condition is realizable in the framework.

On the other hand, it has been remarked that the pointwise multiplication doesn't preserve the subspace $\mathcal{N}$. This uninvariance gives us an interesting effect. Let us suppose that the given classical mechanical system, represented by symplectic manifold (M, ω), is completely integrable. Thus we have a number of the integrals $f_1, ..., f_n$, which commute each with others. This set generates a subalgebra $\mathcal{V} \subset C^\infty(M \to \mathbb{R})$ with usual pointwise multiplication, and this $\mathcal{V}$ is the maximal commutative Lie subalgebra in $C^\infty(M \to \mathbb{R})$, viewing as a Lie algebra. In other words the maximal commutative Lie subalgebra is finitely generated as usual algebra: for any element h from this maximal Lie subalgebra there are exist a n- tuple $(r_1, ..., r_n), r_i \in \mathbb{N}$ such that

$$h = f_1^{r_1} \cdot ... \cdot f_n^{r_n}.$$

Let us turn now to the infinite dimensional moduli space $\mathcal{B}_S^{hw,1}$. For the completely integrable situation one has the following infinite set of commuting function

$$\{F_{f_1^{r_1} \cdot ... \cdot f_n^{r_n}}\},$$

but for this maximal commutative Lie subalgebra the same isn't true — we have at least $\mathbb{N}^n$ generators because $F_{f_1 \cdot f_2} \neq F_{f_1} \cdot F_{f_2}$. So one could define the dimension of symplectic manifold $\mathcal{B}_S^{hw,1}$ using this arguments.

4. Computations

Here we prove Proposition 3.2. Let f is a smooth function on M. If we take the corresponding $F_f \in C^\infty(\mathcal{B}_S^{hw,1})$ then it has the differential equals to

$$dF_f(\phi_0, \theta_0)(f_1, \theta_1) = \int_{\phi_0 S} 2f\theta_1\theta_0 + \int_{\phi_0 S} df_1(\omega^{-1}(df)|_{\phi_0 S})\theta_0^2, \qquad (4.1)$$

so the first summand in (4.1) is constant under the varying of the first "variable" and the second summand is constant under the varying of the second one. One could recognize the form B_f defined by (3.1) as the first summand in (4.1). As we have seen above (see (3.2)) the vector field A^f is symplectically dual to the form. Thus the Hamiltonian vector field of the function F_f has the form

$$H_{F_f} = \Omega^{-1}(dF_f) = 2A^f + C^f,$$

where C^f is the vector field, symplectically dual to the 1- form

$$\int_{\phi_0 S} df_1(\omega^{-1}(df)|_{\phi_0 S})\theta_0^2. \tag{4.2}$$

we need not to compute this C^f explicitly because of the following argument. Vector field A^f is constant on the half form coordinates while the symplectically dual 1-form B_f has as the kernel all vector field of the shape $(f_1(\phi,\theta),0)$. Vice versa for the vector field C^f and for the symplectic dual 1- form one has the same properties if one changes coordinates "functions — half forms". From the formula (1.5) we see that the symplectic form Ω separates the coordinates f and θ. So one can see that

$$\begin{aligned} \{F_f, F_g\}_\Omega = \Omega(H_{F_f}; H_{F_g}) = \\ \Omega(2A^f + C^f; 2A^g + C^g) = 2\Omega(A^f; C^g) - 2\Omega(A^g; C^f) \end{aligned} \tag{4.3}$$

since $\Omega(A^f; A^g) = \Omega(C^f; C^g) = 0$ for every pair f, g. Further, as a continuation of (4.3) one gets

$$\{F_f; F_g\}_\Omega = 2B_f(C^g) - 2B_g(C^f) = 2C_f^*(A^g) - 2C_g^*(A^f), \tag{4.4}$$

where we denote the form (4.2) as C_f^*. Thus we need not to compute the explicit expression for C^f. Really substituting the explicit expressions for the vector fields and 1- forms we get

$$\{F_f; F_g\}_\Omega = 2\int_{\phi_0 S} dg|_{\phi_0 S}(\omega^{-1}(df)|_{\phi_0 S})\theta_0^2 - 2\int_{\phi_0 S} df|_{\phi_0 S}(\omega^{-1}(dg)|_{\phi_0 S})\theta_0^2. \tag{4.5}$$

Let us take the integrand from (4.5)

$$2(dg|_{\phi_0 S}(\omega^{-1}(df)|_{\phi_0 S}) - df|_{\phi_0 S}(\omega^{-1}(df)|_{\phi_0 S})). \tag{4.6}$$

It is a function on $\phi_0 S$. Let us show that this function coincides with the restriction on $\phi_0 S$ of the Poisson bracket $\{f; g\}_\omega$, multiplied by 2. The rest of this section is dedicated to the proof of this coincidence.

First of all it's easy to see that the following identity takes place

$$2\{f; g\}_\omega = 2df(\omega^{-1}(dg)) = -2dg(\omega^{-1}(df)) = (df(\omega^{-1}(dg)) - dg(\omega^{-1}(df)). \tag{4.7}$$

But we work now near a Lagrangian submanifold — we are interesting in the expression (4.7) only in a neighborhood of our $\phi_0 S$. For simplicity let us choose any compatible almost complex structure J on M, getting the corresponding hermitian triple (g, J, ω) on the based symplectic manifold. The corresponding Riemannian metric g splits near our Lagrangian submanifold $\phi_0 S$ which means that

$$v \in T_m \phi_0 S \subset T_m M \implies G(v, Jv) = 0. \tag{4.8}$$

In the expression (4.6) the restrictions on $\phi_0 S$ of 1- forms df, dg and vector fields $\omega^{-1}(dg), \omega^{-1}(df)$ take place. So let us decompose every ingredient in formula (4.7) into horizontal and vertical parts with respect to the tangents subspaces to $\phi_0 S$ and their orthogonal complements with respect to the Riemannian metric g. For the first summand in (4.7) one gets

$$\begin{gathered}(df_{vert}((Jg^{-1}(dg))_{vert}) + df_{vert}((Jg^{-1}(dg))_{hor}) + \\ df_{hor}((Jg^{-1}(dg))_{vert}) + df_{hor}((Jg^{-1}(dg))_{hor})).\end{gathered} \tag{4.9}$$

In the last expression (4.9) one has only two nontrivial summands due to the orthogonality (4.8) — namely the first and the forth ("vert - vert" and "hor - hor"). Analogously for the second summand in (4.7) one has

$$-(dg_{vert}((Jg^{-1}(df))_{vert}) + dg_{hor}((Jg^{-1}(dg))_{hor})), \tag{4.10}$$

so we have again only two nontrivial terms. But from the compatibility condition for the Riemannian metric and the almost complex structure one has

$$df_{vert}((Jg^{-1}(dg))_{vert}) = -dg_{hor}((Jg^{-1}(df))_{hor}) \tag{4.11}$$

and as well

$$df_{hor}((Jg^{-1}(dg))_{hor}) = -dg_{vert}((Jg^{-1}(df))_{vert}). \tag{4.12}$$

Thus we can rewrite the expression for the Poisson bracket $\{f; g\}_\omega$ restricted on the Lagrangian submanifold $\phi_0 S$ using only horizontal components. This gives us

$$\{f; g\}_\omega|_{\phi_0 S} = df_{hor}((\omega^{-1}(dg))_{hor}) - dg_{hor}((\omega^{-1}(df))_{hor}). \tag{4.13}$$

Now it remains to recall that the *hor* - components just correspond to the restrictions of vector fields and 1- forms to the Lagrangian submanifold $\phi_0 S$ thus we have the following identity

$$\{f; g\}_\omega|_{\phi_0 S} = (df|_{\phi_0 S}(\omega^{-1}(dg)|_{\phi_0 S}) - dg|_{\phi_0 S}(\omega^{-1}(df)|_{\phi_0 S}). \tag{4.14}$$

Comparing (4.6) and (4.14) one gets the statement of Proposition 3.2.

Final remarks

This paper contains first results about the moduli spaces of half - weighted Lagrangian subcycles of fixed volume. We got an interesting result about the relationship between the Poisson structures on the based manifold and on the moduli space, nothing else. But in the framework on the geometrical quantization programme this result looks like a hint that the approach mentioned in the introduction to this paper could be exploited. Really, as usual one understands the geometric quantization of a classical mechanical system as a procedure gives an appropriate Hilbert space together with a correspondence "classical observables — quantum observables" such that the irreducibility and the Dirac conditions would be satisfied (see [3], [4], [7] etc.). Leaving the question of the possibility to construct such correspondence (we mean the celebrated van Hove theorem, see f.e. section 5.2 in [3])let us turn to the geometrical formulation of quantum mechanics. As we have seen in section 2 (following A. Ashtekar and T. Schilling) one could try to find an appropriate kahler manifold instead of a Hilbert space directly. It has to have some very special properties (maximal symmetries and so on) but as well it has to be defined directly from the based symplectic manifold. At the present time we have a candidate — the moduli space of half- weighted Bohr - Sommerfeld Lagrangian subcycles constructed by A.Tyurin and A.Gorodentsev. Their moduli space has to be studied in the following directions:

1. irreducibility of the moduli space (for the irreducibility condition);
2. maximal symmetries (to be really a geometrical model of quantum mechanics);
3. the functions F_f from $\mathcal{N}$ (to be really quantum observables).

Preliminary results, which we got during the work on this paper (but were not included here because of the negativity), show that at least for the third question the answer has to be negative. The point is that for any F_f the critical points form a continuous set, and there are exist such functions from $\mathcal{N}$ which have only one critical value. But the situation is very rich to avoid a lot of difficulties

— we can study some special submanifolds of $\mathcal{B}_S^{hw,1}$ to get a more appropriate kahler manifold; this way can give us right constructions to move in the 1st and the 2nd directions;

— we can modify the definition of the correspondence F (0.1) to move in the 3d direction;

— at the end, we have seen that our result (Proposition 3.2) depends on the symplectic structure on $\mathcal{B}_S^{hw,1}$ and doesn't depend on Riemannian metric or integrable complex structure. So we can perturb the original complex structure together with the Riemannian metric (constructed in [2], [8]) getting an almost complex symplectic manifold. May be this almost complex manifold gives the right construction of geometric quantization. If this approach will be succesfull we would relax the celebrated Penrose slogan: "The Nature is complex". We will add:"or almost complex".

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