Symplectic fixed points, Calabi invariant and Novikov homology

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CONTENTS

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§1. Introduction.

A symplectic structure ω on a manifold M provides the one-to-one correspondence between closed 1-forms and infinitesimal automorphisms of (M, ω), i.e. vector fields X satisfying $\mathcal{L}_X \omega = 0$. An infinitesimal automorphism X is called a Hamiltonian vector field, if it corresponds to a exact 1-form. A symplectomorphism φ on M is called exact, if it is the time 1 map of a time-depending Hamiltonian vector field. In fact, one can find a periodic Hamiltonian function such that φ is the time 1 map of the Hamiltonian system. For each symplectomorphism φ isotopic to the identity through symplectomorphisms, one can assign a cohomology class $Cal(\varphi)$, which is called the Calabi invariant of φ. Banyaga [B] showed that φ is exact if and only if $Cal(\varphi) = 0$. The Arnold conjecture states that the number of fixed points of an exact symplectomorphism on a compact symplectic manifold can be estimated below by the sum of the Betti numbers of M provided that all the fixed points are non-degenerate. Arnold came to this conjecture by analysing the case that φ is close to the identity (see [A]). If φ is the time 1 map of a time-independent Hamiltonian vector field corresponding to a Morse function f which is C^2-small, the fixed points coincides with the critical points of f and the conjecture is verified in this case, which is nothing but the Morse theory.

There are many partial results in the Arnold conjecture. A great progress was done by Floer, who combined the variational approach (see [C-Z]) and theory of pseudo-holomorphic curves due to Gromov and proved the Arnold conjecture for monotone symplectic manifolds [F1]. He developed an analogue of the Morse theory for the action functional on the loop space and led to the notion of Floer homology. The Arnold conjecture is derived from the fact that the Floer homology group is isomorphic to the ordinary homology group of M. Recently, Hofer and Salamon [H-S] define the Floer homology group for a wider class of symplectic manifolds (which are called weakly monotone symplectic manifolds). An almost complex structure J on M is calibrated by ω, if

$$\langle \xi, \eta \rangle = \omega(\xi, J\eta) \tag{1.1}$$

defines a Riemannian metric on M. J is defined unique up to homotopy and we denote $c_1 = c_1(M)$ the first Chern class of the almost complex manifold (M, J). (M, ω) is called monotone, if the evaluation of c_1 on $\pi_2(M)$ is positively proportional to the one of ω. The condition of weak monotonicity [H-S] implies the non-existence of J-holomorphic spheres with negative Chern number for a generic J. Hofer and Salamon computed the Floer homology in the case that (M, ω) is monotone, or $c_1(\pi_2(M)) = 0$, or the minimal Chern number is at least half of the dimension of M, and verified the Arnold conjecture in these cases. Later, the second author defined the "modified Floer homology group" and verified the conjecture for weakly monotone symplectic manifolds. However we do not know whether the Floer homology group defined by Hofer and Salamon and the modified Floer homology group coincide or not.

In this note, we consider an analogue of the Arnold conjecture for non-exact symplectomorphisms isotopic to the identity through symplectomorphisms. In the case of non-exact symplectomorphisms, the fixed point set may be empty. For example, an irrational rotation on an even dimensional torus with the standard symplectic structure preserves the symplectic form, however the fixed point set is empty. That is the reason why we have to consider the Novikov homology instead of the ordinary homology. The aim of this note is to show the following:

Main Theorem. *Let (M,ω) be a closed symplectic manifold which satisfies the following condition*

$$c_1\,_{|\pi_2(M)} = \lambda\omega_{|\pi_2(M)},\ \lambda \neq 0,$$

and if $\lambda < 0$, the minimal Chern number $N > n-3$. Suppose φ is a symplectomorphism on (M,ω) which is isotopic to the identity through symplectomorphisms. If all the fixed points of φ are non-degenerate, the number of fixed points of φ is at least the sum of the Betti numbers of the Novikov homology over $\mathbf{Z}_2$ associated to the Calabi invariant of φ.

It is well-known that Novikov homology groups are isomorphic for almost all cohomology class in $H^1(M;\mathbf{R})$ and the rank of this group is minimal in the class of Novikov homology groups associated to all the cohomology class in $H^1(M;\mathbf{R})$ (see Appendix 3). Hence we get the estimate of the number of fixed points in terms of Novikov homology of generic 1-forms. We reduce the problem to the one concerning the 1-periodic solutions of a periodic Hamiltonian system and define the Floer homology in this setting. The argument in [F1],[H-S] yields that Floer homology groups are isomorphic under the deformation keeping the Calabi invariant constant. However we have to consider deformations changing the Calabi invariant in order to compute it. To apply the weak compactness argument, necessary is the estimate of the energy functional for solutions of "chain homomorphism" equation. This is done for specific deformations with the help of a variant of the Palais-Smale condition (§5).

This note also contains Appendices. The first one is concerning the classification of loops in a symplectic manifold under symplectomorphisms. We prove that two embedded contractible loops are congruent under a time 1-map of a time-dependent Hamiltonian flow if and only if the Poincare integral invariant of them coincide. The second one contains a proof of a fact, which is needed in the computation of the Floer homology groups (see also [H-S]). The third one is a note on Novikov homology theory. Since it seems difficult to find a reference containing proofs, we give proofs for the sake of completeness.

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§2. Calabi invariant and a variational approach.

Given a symplectic form ω on M there is an isomorphism Φ from the space of vector fields to the space of differential forms on M:

$$\Phi(V)(W) = -\omega(V, W). \tag{2.1}$$

Let g_1 be an element of the identity component $\mathit{Diff}^0_\omega M$ of the symplectomorphism group and $\{g_t\}$ a path connecting the identity element and g_1. We define the vector field $D_t(g_t)$ by

$$D_t(g_t)(x) = (\frac{\partial}{\partial t} g_t)(g_t^{-1} x).$$

Clearly, $\Phi(D_t(g_t))$ are closed differential forms. Recall that g_1 is said to be an *exact symplectomorphism* if all $\Phi(D_t(g_t))$ are exact 1-forms. The *flux homomorphism* Φ from the universal cover $\widetilde{\mathit{Diff}^0_\omega M}$ to $H^1(M, \mathbf{R})$ is defined as follows (see [B]):

$$\Phi(\tilde{g_1}) = [\int \Phi(D_t(g_t))\, dt].$$

This homomorphism was first considered by Calabi and we call the image $\Phi(\tilde{g})$ the *Calabi invariant* of an element $\tilde{g} \in \widetilde{\mathit{Diff}^0_\omega M}$. Passing to the group $\mathit{Diff}^0_\omega M$ the Calabi invariant of a symplectomorphism on M is an element of the quotient $H^1(M, \mathbf{R})/\Gamma$, where Γ is the image of the subgroup $\pi_1(\mathit{Diff}^0_\omega M)$ (which is identified with the kernel of the projection from $\widetilde{\mathit{Diff}^0_\omega M}$ to $\mathit{Diff}^0_\omega M$) under the homomorphism Φ. It is known ([B], see also Lemma 2.1) that Γ is a discrete subgroup in $H^1(M, \mathbf{R})$ if and only if the subgroup of exact symplectomorphisms is closed in $\mathit{Diff}^0_\omega M$. Kählerian manifolds, or more generally, any symplectic manifold M such that the multiplication by ω^{n-1} induces an isomorphism from $H^1(M, \mathbf{R})$ to $H^{2n-1}(M, \mathbf{R})$, are such examples [B].

Deformation Lemma 2.1. *Let $[\theta] \in H^1(M, \mathbf{R})$ be the Calabi invariant of an element $\tilde{g}_1$. Then there exists a smooth path $\{g_t\}$ in $Diff^0_\omega M$, joining the identity element Id and g_1, and a periodic Hamiltonian H_t on M such that $\Phi(D_t(g_t)) = \theta + dH_t$ for all t.*

Proof. First, we show that we can choose a path g_t connecting the identity element and g_1 such that all the cohomology class of $\theta_t = \Phi(D_t(g_t))$ is $[\theta]$ and $\theta_0 = \theta_1$. Put

$$V(x) = \int_0^1 D_t(g_t)(x)\, dt.$$

Let $G_t(x)$ be the one-parameter subgroup of symplectomorphisms generated by the vector field $V(x)$: $G_t = \exp tV$. Then the Calabi invariant for the path $g_t \cdot G_{-t}$ is zero. It is known that there exists a smooth path p_t in the subgroup of exact symplectomorphisms, $Diff_{ex}M$, such that $p_0 = Id$, $p_1 = g_1 \cdot G_{-1}$ (see [B]). By reparametrizing the parameter t, we may assume that p_t is constant around 0 and 1. Now let us consider the path $g'(t) = p_t \cdot G_t$, so we have $g'_0 = Id, g'_1 = g_1$. We obtain

$$\Phi(D_t(p_t \cdot G_t)) = \Phi(dG_t(D_t(p_t)) + \Phi(V). \tag{2.2}$$

From (2.2) we obtain that the form $\Phi(D_t(g'(t))) = \theta_t$ is of the same cohomology class $[\theta]$ for all t. Let p_t be as above. Then $\Phi(D_t(p_t))$ is the differential dH_t for a smooth function H_t on M. Now, the periodicity condition $\theta_0 = \theta_1$ is equivalent to the following:

$$dH_0 + \Phi(V) = dG_1^*(dH_1) + \Phi(V). \tag{2.3}$$

Since p_t is constant around 0 and 1, $dH_0 = dH_1 = 0$, i.e. H_0 and H_1 are constant functions. Hence (2.3) holds. Consequently, we have that $H_1 = H_0 + a$, where a is some constant. Now put $H'_t = H_t + \psi(t)$, where $\xi(0) = 0$; $\psi(1) = -a$ and $\psi'(0) = \psi'(1) = 0$. Then the path g_t associated with $\theta_t = \theta + dH'_t$ satisfies the condition of Deformation Lemma. □

With the help of our Deformation Lemma we reduce the problem of finding fixed points of a symplectomorphism g to a variational problem on the loop space $\mathcal{L}(M)$. Suppose that g is given by the following equation:

$$g_0 = Id;\ \Phi(D_t(g_t)) = \theta_t,$$

where θ_t satisfy the condition in Deformation Lemma. Clearly, the fixed points of g are in 1-1 correspondence with the 1-periodic solutions of the following differential equation

$$\dot{x}(t) = X_\theta(t, x(t)), \tag{2.4}$$

where $X_\theta = \Phi^{-1}(\theta)$. (It is easy to see that $\Phi^{-1}(\theta) = J\nabla P_\theta$ where P_θ is a local primitive of the closed 1-form θ.) The set $\mathcal{P}(\theta_t)$ of 1-periodic solutions of (2.4) coincide with the critical points set (i.e. the zero set) of the following closed 1-form $d\mathcal{A}_\theta$ on $\mathcal{L}(M)$:

$$d\mathcal{A}_\theta(z,\xi) = \int \omega(\dot{z},\xi) + \theta_t(z(t))(\xi). \tag{2.4'}$$

The (minus) gradient flow for the multi-valued functional $\mathcal{A}_\theta$ is defined by the following equation.

$$\partial_{J,\theta}(u) = \frac{\partial u}{\partial s} + J(u)(\frac{\partial u}{\partial t} - X_\theta(u)) = 0, \tag{2.5}$$

where u is a mapping $\mathbf{R} \times S^1 \to M$. For a solution u of (2.5) we define its energy as follows.

$$E(u) = \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1 (|\frac{\partial u}{\partial s}|^2 + |\frac{\partial u}{\partial t} - X_\theta(t,u)|^2)\, dtds. \tag{2.6}$$

As in the case of exact symplectomorphisms, the space of the solution (2.5) with the bounded energy is the space of connecting orbits, that is, $\lim_{s\to\pm\infty} u(s,t) = x^\pm(t)$ where $x^\pm(t)$ are periodic solutions of (2.4).

We will restrict ourselves in the component of contractible loops on M. For the sake of simplicity, henceforth, we also denote the latter space by $\mathcal{L}(M)$. We construct an associated covering space $\tilde{\mathcal{L}}\widetilde{M}$ such that the action functional $\mathcal{A}_\theta$ on this cover is well-defined function. Consider the following commutative diagram.

$$\begin{array}{ccccc}
\tilde{\mathcal{L}}\widetilde{M} & \xrightarrow{\tilde{j}} & \mathcal{L}\widetilde{M} & \xrightarrow{\tilde{e}} & \widetilde{M} \\
\downarrow{\tilde{\Pi}} & & \downarrow{\Pi} & & \downarrow{\pi} \\
\tilde{\mathcal{L}}M & \xrightarrow{j} & \mathcal{L}M & \xrightarrow{e} & M
\end{array}$$

Here $\widetilde{M}$ denotes the covering space of M associated to the period homomorphism of θ, $I_\theta : \pi_1(M) \to \mathbf{R}$. This means that the covering transformation group is isomorphic to the quotient group

$$\Gamma_1 = \pi_1(M)/\ker I_\theta.$$

Furthermore, e denotes the evaluation map: $x(t) \mapsto x(0)$, and j denotes the projection from the covering space $\tilde{\mathcal{L}}M$ of $\mathcal{L}M$ associated to the action of homomorphims ϕ_{c_1}, ϕ_ω:

$\pi_2(M) \to \mathbf{R}$. This means that the covering transformation group is isomorphic to the quotient group

$$\Gamma_2 = \frac{\pi_2(M)}{\ker \phi_{c_1} \cap \ker \phi_\omega}.$$

An element of $\widetilde{\mathcal{L}M}$ is represented by an equivalence class of pairs $(\tilde{x}, \tilde{u})$, where $\tilde{x}$ is a loop in $\widetilde{M}$ and $\tilde{u}$ is a disk in $\widetilde{M}$ bounding $\tilde{x}$. A pair $(\tilde{x}, \tilde{u})$ is equivalent to $(\tilde{y}, \tilde{v})$ if and only if $\tilde{x} = \tilde{y}$ and the values of ϕ_{c_1} and ϕ_ω are zero for $u\sharp(-v)$, where $u = \pi(\tilde{u}), v = \pi(\tilde{v})$ (see [H-S]). Hence, the covering transformation group of $\widetilde{\mathcal{L}M} \to \mathcal{L}M$ is the direct sum $\Gamma = \Gamma_1 \oplus \Gamma_2$:

$$(g_1 \oplus g_2)[\tilde{x}, \tilde{u}] = [g_1 \cdot \tilde{x}, A_2 \# g_1 \cdot \tilde{u}], \tag{2.7}$$

where A_2 is any representative of g_2 in $\pi_2(M)$. (Geometrically, u is a disk bounded by $x \in \mathcal{L}M$. By the homotopy lifting property, there exists a unique disk $\tilde{u} \in \widetilde{M}$ bounded by $\tilde{x}$. The second homotopy groups of M and $\widetilde{M}$ are same, so we consider g_2 as a sphere in $\widetilde{M}$ and $\#$ denotes the connected sum of 2-spheres with the bounding disk (see [H-S])). Namely we get the following

Lemma 2.2. *The group* Γ *is commutative. We have*

$$(g_1 \circ g_2)[\tilde{x}, \tilde{u}] = (g_2 \circ g_1)[\tilde{x}, \tilde{u}],$$

where g_i *denotes its image in the group* Γ.

Since the 1-forms θ_t, $t \in S^1$, satisfy the condition in Deformation Lemma there exists a periodic Hamiltonian $\widetilde{H}_t$ on $\widetilde{M}$ such that $d\widetilde{H}_t = \pi^*\theta_t$. Clearly, the time-dependent Hamiltonian flow on $\widetilde{M}$ generated by $\widetilde{H}_t$ is the pull-back from the original symplectic flow on M. In particular, the set of contractible periodic solutions $\mathcal{P}(\widetilde{H}_t)$ is the set $\pi^{-1}(\mathcal{P}(\theta_t))$. Furthermore, $\tilde{\mathcal{P}}(\widetilde{H}) = \tilde{j}^{-1}\mathcal{P}(\widetilde{H}_t)$ is the critical point set of the following functional

$$\mathcal{A}_{\widetilde{H}}([\tilde{x}, \tilde{u}]) = -\int_D \tilde{u}^*\omega + \int_0^1 \widetilde{H}(t, \tilde{x}(t))\, dt. \tag{2.8}$$

We now consider on $\widetilde{\mathcal{L}M}$ the space of connecting orbits $\tilde{u} : \mathbf{R} \times S^1 \to \widetilde{M}$ satisfying the lifted equation :

$$\partial_{J,\widetilde{H}}(\tilde{u}) = \frac{\partial \tilde{u}}{\partial s} + J(u)(\frac{\partial \tilde{u}}{\partial t} - X_{\widetilde{H}}(\tilde{u})) = 0, \tag{2.9.1}$$

$$\lim_{s \to \pm\infty} \tilde{u}(s,t) = [\tilde{x}^{\pm}(t), \tilde{u}^{\pm}] \tag{2.9.2}$$

and the condition:

$$[\tilde{x}^+, \tilde{u}^- \sharp \tilde{u}] = [\tilde{x}^+, \tilde{u}^+]. \tag{2.9.3}$$

The paths in $\mathcal{L}\widetilde{M}$ are in one-to-one corerespondence with the paths on the covering space $\tilde{\mathcal{L}}\widetilde{M}$ modulo the action of Γ_2 (see condition (2.9.3)). Consequently, for these connecting orbits we have the following energy identity (cf. [H-S], [S-Z]):

$$E(\tilde{u}(s,t)) = \int_{-\infty}^{\infty}\int_0^1 |\frac{\partial \tilde{u}}{\partial s}|^2\, dtds = \mathcal{A}_{\widetilde{H}}([\tilde{x}^-, \tilde{u}^-]) - \mathcal{A}_{\widetilde{H}}([\tilde{x}^+, \tilde{u}^+]). \tag{2.10}$$

where $\tilde{u}$ is a mapping from $\mathbf{R} \times S^1 \longrightarrow \widetilde{M}$.

§3. Transversality and compactness.

From now on, we will deal with a weekly monotone symplectic manifold M, i.e. M satisfies $\omega(A) \leq 0$ for any $A \in \pi_2(M)$ with $3 - n \leq c_1(A) < 0$ [H-S], and a generic almost complex structure J calibrated by ω. This condition yields the non-existence of J-holomorphic spheres of negative Chern number. Moreover, denote by $\mathbf{M}_k(c; J)$ the set of all points $x \in M$ such that there exists a non-constant J-holomorphic sphere $v : S^2 \to M$ such that $c_1(v) \leq k$, $\omega(v) \leq c$ and $x \in v(S^2)$, then the set $\mathbf{M}_0(\infty; J)$ is a subset of M of codimension 4, and the set $\mathbf{M}_1(\infty; J)$ is a subset of codimension 2 [H-S].

All transversality and compactness theorems in this section can be easily obtained by the same arguments in [H-S].

Given any smooth periodic 1-from $\theta_t = \theta + dH_t$ we denote by $\mathcal{U}_\delta(\theta_t)$ the set of all periodic 1-forms $\theta + dH_t'$ with $||H_t' - H_t||_\epsilon < \delta$, where the norm $||h||_\epsilon$ is defined as follows:

$$||h||_\epsilon = \sum_{k=0}^{\infty} \varepsilon_k ||h||_{C^k(S^1\times M)}.$$

Here $\varepsilon_k > 0$ is a sufficiently rapidly decreasing sequence [F2].

Theorem 3.1. *There is a dense subset $\Theta_0 \subset \mathcal{U}_\delta(\theta_t)$ such that the following holds for $\theta_t \in \Theta_0$.*
(i) Every periodic solution $x \in \mathcal{P}(\theta_t)$ is non-degenerate.
(ii) $x(t) \notin \mathbf{M}_1(\infty; J)$ for every $x \in \mathcal{P}(\theta_t)$ and every $t \in \mathbf{R}$.

Theorem 3.1 is obtained by applying the Sard-Smale theorem to certain Banach manifolds. More precisely, for the proof of (i) we consider the Hilbert manifold $\mathcal{B}$ of contractible $W^{1,2}$ loops $x : S^1 \to M$, and the bundle $\mathcal{E} \to \mathcal{B}$ whose fibre at $x \in \mathcal{B}$ is the Hilbert space of L^2-vector fields along x. Let the section $\mathcal{F} : \mathcal{B} \times \mathcal{U}_\delta(\theta_t) \to \mathcal{E}$ be defined by

$$\mathcal{F}(x, \theta + dH_t') = \dot{x} - X_{\theta+dH_t'}(t, x).$$

The differential $d\mathcal{F}(x, \theta + dH_t')$ is onto [H-S]. Hence $\mathcal{F}$ intersects the zero section of $\mathcal{E}$ transversally. Thus, the set

$$\mathcal{P} = \{(x, \theta') \in \mathcal{B} \times \mathcal{U}_\delta(\theta) \,|\mathcal{F}(x, \theta_t') = 0\}$$

is a separable infinite dimensional Banach manifold. A periodic form $\theta_t' = \theta + dH_t' \in \mathcal{U}_\delta(\theta_t)$ is a regular value of the projection $\mathcal{P} \to \mathcal{U}_\delta(\theta_t)$ onto the second factor if and only if every periodic solution $x \in \mathcal{P}(\theta_t')$ is non-degenerate. By the Sard-Smale theorem the set $\Theta' \in \mathcal{U}_\delta(\theta_t)$ is generic in the sense of Baire. □

To prove (ii) we consider the evaluation map

$$e_{z,t} : \mathcal{M}_s(A; J) \times_G \times S^2 \times S^1 \times \mathcal{P} : \ ([v, z], t, x, \theta_t) \mapsto (v(z), x(t)),$$

where $\mathcal{M}_s(A; J)$ denotes the moduli space of simple J-holomorphic spheres realizing homology class $A \in H_2(M, \mathbf{Z})$, and G is the automorphism group $PGL_2(\mathbf{C})$. It is shown that the evaluation map $e_t : \mathcal{P} \to M : \ (x, \theta_t) \mapsto x(t)$, is a submersion for every $t \in S^1$ [H-S]. Therefore, the map $e_{z,t}$ is transversal to the diagonal Δ_M in $M \times M$. Hence the space

$$\mathcal{N} = \{[v, z], t, x, \theta_t \,|v(z) = x(t), (x, \theta_t) \in \mathcal{P}\}$$

is an infinite dimensional Banach submanifold of $\mathcal{M}_s(A; J) \times_G \times S^2 \times S^1 \times \mathcal{P}$ of codimension $2n$. The projection

$$\mathcal{N} \to \mathcal{U}_\delta(\theta) : \ ([v, z], x, \theta_t) \mapsto \theta_t$$

is a Fredholm map of Fredholm index $2c_1(A) - 3$. Applying the Sard-Smale theorem we obtain that the set $\Theta(A)$ of regular values of the above projection is of the second category in the sense of Baire. Denote Θ_0 the intersection of Θ' with $\cap_{a \in \Gamma} \Theta(A)$, where Γ is countable set of integral (and spherical) 2-homology classes in M, for which $c_1(A) \leq 1$. Then Θ_0 is the desired set in Theorem 3.1. □

By the previous theorem there exists a periodic 1-form $\theta_t = \theta + dH_t$ in a prescribed cohomology class $[\theta]$ such that all contractible 1-periodic solution of (2.4) are non-degenerate and do not intersect the set $M_1(\infty, J)$. Choose disjoint compact neighborhoods $U_1, \dots, U_m \subset S^1 \times M$ of the graphs of the finitely many contractible periodic solutions of (2.4). We denote by $\mathcal{V}_\delta(\theta_t)$ the set of all periodic 1-forms $\theta_t = \theta + dH_t'$ with $||H_t' - H_t||_\varepsilon < \delta$ and $H_t' = H_t$ on U_j for $j = 1, \dots, m$. If $\delta > 0$ is sufficiently small then there are no contractible 1-periodic solution of (2.4) outside the set U_j for $\theta_t' \in \mathcal{V}_\delta(\theta_t)$. For an element

$[\tilde{x},\tilde{u}] \in \tilde{\mathcal{P}}(\widetilde{H})$, we assign an integer $\mu([\tilde{x},\tilde{u}])$, which is called the Conley-Zehnder index [H-S],[S-Z].

Theorem 3.2. *There is a generic set $\Theta_1 \subset \mathcal{V}_\delta(\theta_t)$ containing θ_t such that the following holds for $\theta'_t \in \Theta_1$.*
(i) The space $\mathcal{M}([\tilde{x}^-,\tilde{u}^-],[\tilde{x}^+,\tilde{u}^+];\theta'_t,J)$ of solutions of (2.9.1), (2.9.2) and (2.9.3) is a finite dimensional manifold for all $[\tilde{x}^\pm,\tilde{u}^\pm] \in \mathcal{P}(\theta'_t)$. The dimension is given by $\mu([\tilde{x}^-,\tilde{u}^-])-\mu([\tilde{x}^+,\tilde{u}^+])$.
(ii) $u(s,t) \notin \mathcal{M}_0(\infty;J)$ for every $u \in \mathcal{M}([\tilde{x}^-,\tilde{u}^-],[\tilde{x}^+,\tilde{u}^+];\theta'_t,J)$ with $\mu([\tilde{x}^-,\tilde{u}^-]) - \mu([\tilde{x}^+,\tilde{u}^+]) \le 2$ and every $(s,t) \in \mathbf{R}\times S^1$.

The proof of Theorem 3.2 is also obtained by applying the Sard-Smale theorem to certain Banach manifolds [H-S]. For the proof of (i) we consider the Banach manifold $\mathcal{B}$ of $W^{1,p}$ maps $u : \mathbf{R}\times S^1 \to M, p > 2$, whose limit are periodic solutions $x^\pm$ of (2.4). Let $\mathcal{E} \to \mathcal{B}$ be the bundle whose fibre at $u \in \mathcal{B}$ is the Banach space of L^p-vector fields along u. Define the section $\mathcal{F} : \mathcal{B}\times\mathcal{V}_\delta(\theta_t) \to \mathcal{E}$ as follows

$$\mathcal{F}(u,\theta'_t) = \frac{\partial u}{\partial s} + J(u)(\frac{\partial u}{\partial t} - X_{\theta'_t}(t,u)).$$

The linearized operator of $\mathcal{F}(u,\theta'_t)$ coincides with the one of the similar operator in [H-S]. Further arguments in [H-S] can be repeated word by word here. □

In the following we consider the lifted Hamiltonian system on $\widetilde{M}$. Note that the equation (2.9.1) is invariant under translations in s-variable. We denote the quotient space by $\mathcal{M}([\tilde{x}^-,\tilde{u}^-],[\tilde{x}^+,\tilde{u}^+];\widetilde{H},J)/\mathbf{R}$.

Theorem 3.3. *Suppose that J and θ_t are regular (in the sense of Theorem 3.1 and Theorem 3.2). Then $\mathcal{M}([\tilde{x}^-,\tilde{u}^-],[\tilde{x}^+,\tilde{u}^+];\tilde{H},J)/\mathbf{R}$ is compact, if $\mu([\tilde{x}^-,\tilde{u}^-])-\mu([\tilde{x}^+,\tilde{u}^+]) = 1$. $\mathcal{M}([\tilde{x}^-,\tilde{u}^-],[\tilde{x}^+,\tilde{u}^+];\tilde{H},J)/\mathbf{R}$ is compact up to splitting into two elements in $\mathcal{M}([\tilde{x}^-,\tilde{u}^-],[\tilde{z},\tilde{v}];\tilde{H},J)/\mathbf{R}$ and $\mathcal{M}([\tilde{z},\tilde{v}],[\tilde{x}^+,\tilde{u}^+];\tilde{H},J)/\mathbf{R}$, if $\mu([\tilde{x}^-,\tilde{u}^-]) - \mu([\tilde{x}^+,\tilde{u}^+]) = 2$.*

Remark 3.4. For the proof of Theorem 3.3 (in the case of exact Hamiltonian) Hofer and Salamon use the uniform lower bound of the energy for holomorphic spheres and connecting orbits. They prove the existence of such bounds by bubbling analysis (Gromov's compactness theorem). Alternatively, we can use the (explicit) lower estimate for the volume of (globally) minimal cycles in a compact Riemannian manifold [L] to estimate the energy of holomorphic spheres. Using Cauchy integral inequality one can obtain lower bounds for energy of connecting orbits in terms of the distances between periodic solutions. More precisely, we define a distance $\rho(x(t), y(t))$ between loops $x(t)$ and $y(t)$ by

$$\inf\{ \int_{-\infty}^{\infty} \int_0^1 |\frac{\partial u}{\partial s}| \, dt \, ds \mid u : \mathbf{R} \times S^1 \to M, u(-\infty, t) = x(t), u(\infty, t) = y(t)\}.$$

Clearly $\rho(x(t), y(t)) \geq \int_0^1 \operatorname{dist}(x(t), y(t))\, dt \geq \min_t \operatorname{dist}(x(t), y(t))$, where "dist" denotes the Riemannian distance on M. If C^0-distance $\max_t \operatorname{dist}(x(t), y(t)) \leq R_M$ (injective radius of M), then $\rho(x(t), y(t)) \leq \max_t \operatorname{dist}(x(t), y(t))$. Given a finite number of periodic solutions $x_i(t)$ of (2.4) let $\hbar$ denote the minimum of the distance between $x_i(t)$ and $x_j(t)$, $i \neq j$.

Lemma 3.5. *Given a positive $\varepsilon < \min\{R(M), \hbar/2\}$ there exists a number $c(\varepsilon) > 0$ depending on θ_t such that*

$$E(u) \geq c(\varepsilon)(\hbar - 2\varepsilon)$$

for any connecting orbit u on M, satisfying (2.5), whose limits as s tends to $\pm\infty$ are different periodic solutions of (2.4).

Proof. Denote $B_i(\varepsilon)$ the neighborhood of $x_i(t)$ which consists of loops whose distance ρ to $x_i(t)$ is less than or equal to ε. Let $B(\varepsilon) = \cup B_i(\varepsilon)$. By the Palais-Smale condition (see Lemma 5.1 below) there exists $c(\varepsilon) > 0$ such that the following inequality holds such that outside of $B(\varepsilon)$ we have

$$||\dot{x}(t) - X_{\theta_t}(x(t))||_{L^2} \geq c(\varepsilon)^2. \tag{3.1}$$

Suppose u is a connecting orbit from $x_i(t)$ to $x_j(t)$. Since $\lim_{s \to \pm\infty} u(s, .)$ (in C^0-sense) are periodic solutions $x_i(t)$ and $x_j(t)$, we obtain

Claim 3.6. *There exist number R^- and R^+ such that*
i) $u(R^-, t) \in B_i(\varepsilon)$, $u(s,t) \notin B_i(\varepsilon)$ for all $s > R^-$,
ii) $u(R^+, t) \in B_j(\varepsilon)$ $u(s,t) \notin B_j(\varepsilon)$ for all $s < R^+$.

We observe that

$$E(u) \geq \int_{R^-}^{R^+} \int_0^1 |\frac{\partial u}{\partial s}|^2 \, dt \, ds = \int_{R^-}^{R^+} \left(\sqrt{\int_0^1 |\dot{u}(s,t) - X_{\theta_t}(u(s,t))|^2 \, dt} \right)^2 ds.$$

Applying Cauchy's inequality we get

$$E(u) \geq \frac{1}{R^+ - R^-}\left(\int_{R^-}^{R^+} \sqrt{\int_0^1 |\dot{u}(s,t) - X_{\theta_t}(u(s,t))|^2\, dt}\;\; ds\right)^2 .$$

Taking (3.1) and Claim 3.6 into account we get

$$E(u) \geq c(\varepsilon) \int_{R^-}^{R^+} \sqrt{\int_0^1 |\frac{\partial u}{\partial s}|^2\, dt}\;\; ds.$$

Once again applying Cauchy's inequality, we get

$$E(u) \geq c(\varepsilon) \int_{R^-}^{R^+} \int_0^1 |\frac{\partial u}{\partial s}|\, dt\, ds \geq c(\varepsilon)(\rho(x_i(t), x_j(t)) - 2\varepsilon). \qquad \square$$

§4. Floer homology.

In this section we define Floer homology of fixed points of a symplectomorphism isotopic to the identity and prove its invariance under exact deformations. As a result, if two symplectomorphisms have the same Calabi invariant then the associated Floer homology groups are isomorphic. We will deal with the covering space $\widetilde{\mathcal{L}M}$.

In the paper [S-Z], Salamon and Zehnder use the Conley-Zehnder index (which they also call the Maslov index) of a non-degenerate contractible periodic solution as a natural grading for the Floer complex. Remember that the Conley-Zehnder index of a contractible periodic solution $x(t)$ bounding a disk u depends only on the trivialization of the induced complex bundle u^*TM and the linearized flow along $x(t)$. Therefore, the Conley-Zehnder index of a periodic solution $[\tilde{x}, \tilde{u}] \in \widetilde{\mathcal{L}M}$ is Γ_1 invariant, that is $\mu([\tilde{x}, \tilde{u}]) = \mu(g \cdot [\tilde{x}, \tilde{u}])$ for any $g \in \Gamma_1$. Recall that Γ_1 is the covering transformation group of $\widetilde{M}$. This Conley-Zehnder index satisfies the following identities (cf. [S-Z],[H-S]).

$$\mu([\tilde{x}, A\#\tilde{u}]) - \mu([\tilde{x}, \tilde{u}]) = -2c_1(A), \tag{4.1}$$

$$\dim \mathcal{M}([\tilde{x}^-, \tilde{u}^-], [\tilde{x}^+, \tilde{u}^+]; \widetilde{H}, J) = \mu([\tilde{x}^-, \tilde{u}^-]) - \mu([\tilde{x}^+, \tilde{u}^+]). \tag{4.2}$$

As we have seen, the Conley-Zehnder index of a periodic solution $x(t)$ is well-defined up to modulo $2N$: $\mu(x(t)) \in \mathbf{Z}/2N$, where N is the minimal Chern number. Later on we also say $\mu(x(t)) = k \in \mathbf{Z}$ that means that there is a bounding disk u_x such that $\mu([x, u_x]) = k$.

Denote by $\widetilde{\mathcal{P}}_k(\widetilde{H})$ the subset of all periodic solutions $[\tilde{x}, \tilde{u}]$ with $\mu([\tilde{x}, \tilde{u}]) = k$. Consider the chain complex whose k-th chain group $C_k(\widetilde{H})$ consists of all the formal sums $\sum \xi_{[\tilde{x}, \tilde{u}]} \cdot$

$[\tilde{x},\tilde{u}]$, $[\tilde{x},\tilde{u}] \in \widetilde{\mathcal{P}}_k(\widetilde{H})$, $\xi_{[\tilde{x},\tilde{u}]} \in \mathbf{Z}_2$, satisfying that the set $\{[\tilde{x},\tilde{u}] | \xi_{[\tilde{x},\tilde{u}]} \neq 0, \mathcal{A}_{\widetilde{H}}([\tilde{x},\tilde{u}]) > c\}$ is finite for all $c \in \mathbf{R}$.

Let Γ_0 be the following subgroup of Γ_2,

$$\frac{\ker \phi_{c_1}}{\ker \phi_{c_1} \cap \ker \phi_\omega}.$$

Denote by $\Lambda_{\theta,\omega}$ the completion of the group ring of $\Gamma' \subset \Gamma$ over the field $\mathbf{Z}_2$ with respect to the weight homomorphism $\Psi_{\theta,\omega} = I_\theta \oplus -\phi_\omega : \Gamma' = \Gamma_1 \bigoplus \Gamma_0 \to \mathbf{R}$, i.e. the set of all the formal sum $\sum \lambda_A \cdot \delta_A$, $\lambda_A \in \mathbf{Z}_2$, such that $\{A \in \Gamma' | \lambda_A \neq 0, \Psi_{\theta,\omega}(A) > c\}$ is finite for all $c \in \mathbf{R}$. In fact, $\Lambda_{\theta,\omega}$ is a commutative algebra over $\mathbf{Z}_2$ without zero divisors.

Remark 4.1. If M satisfies the condition in Main Theorem, then Γ_0 is trivial and $\Lambda_{\theta,\omega} \cong \Lambda_\theta$, where Λ_θ is the Novikov ring associated to the closed 1-form θ (see Appendix 3). It is easy to see that in this case the ambiguity of the Conley-Zehnder index of a periodic solution $x(t)$ can be controled. More precisely, if $\mu(x(t)) = k (\text{mod } 2N)$ then there exists a bounding disk u_x, which is unique up to the connected sum of an element in Γ_2, such that $\mu([x(t), u_x]) = k$.

The algebra $\Lambda_{\theta,\omega}$ acts on $\widetilde{\mathcal{P}}\widetilde{M}$ in the following way:

$$(\lambda * \xi)_{[\tilde{x},\tilde{u}]} = \sum_{g \in \Gamma} \lambda_g \xi_{g \circ [\tilde{x},\tilde{u}]}.$$

We easily deduce the following lemma.

Lemma 4.2. *The chain group $C_k(\widetilde{H})$ is a torsion-free module over the algebra $\Lambda_{\theta,\omega}$. The rank of this module is the number of contractible 1-periodic solutions $x \in \mathcal{P}(\theta_t)$ with the Conley-Zehnder index $\mu(x) = k$.*

For a generator $[\tilde{x},\tilde{u}]$ in $C_k(\widetilde{H})$, we define the boundary operator ∂_k as follows:

$$\partial_k([\tilde{x},\tilde{u}]) = \sum_{\mu([\tilde{y},\tilde{v}])=k-1} n_2([\tilde{x},\tilde{u}],[\tilde{y},\tilde{v}])[\tilde{y},\tilde{v}], \tag{4.3}$$

where $n_2([\tilde{x},\tilde{u}],[\tilde{y},\tilde{v}])$ denotes the modulo 2-reduction of the number of elements in the space $\mathcal{M}([\tilde{x},\tilde{u}],[\tilde{y},\tilde{v}];\widetilde{H},J)/\mathbf{R}$. The weak compactness argument yields that ∂_k is well-defined. To verify that $\partial_k([\tilde{x},\tilde{u}]) \in C_{k-1}$ we can use compactness argument as in the proof of Lemma 5.5 below or combine it with Lemma 3.5. In fact, let $\partial_k([\tilde{x},\tilde{u}]) = \sum_{g_{p,i} \in \Gamma'} g_{p,i}[\tilde{x}_i,\tilde{u}_i]$, where $\tilde{x}_i$, $i = \{1,\cdots,K\}$, is an arbitrary chosen lift to $\widetilde{M}$ of a periodic solution $x_i \in \mathcal{P}(\theta_t)$ and $\tilde{u}_i$ is an arbitrary chosen bounding disk of $\tilde{x}_i$. We need to show that for any given c and i the set $S_c = \{g_{p,i} | \mathcal{A}_{\widetilde{H}}(g_{p,i}[\tilde{x}_i, \tilde{u}_i]) > c\}$ is finite. Taking into account the energy identity (2.10) and Lemma 3.5 we get that the distance $\rho(x(t), g \cdot x_i(t))$, $g \in S_c$ is bounded. Hence for each i, there is only finite number of $g \in \Gamma_1$ such that the coefficient of $[g \cdot \tilde{x}_i, \tilde{u}]$ in $\partial_k x(t)$ is not zero and $\mathcal{A}_{\widetilde{H}}([g \cdot x_i(t), \tilde{u}]) > c$. If Γ_0 is trivial (e.g. M satisfies the condition in Main Theorem), then we are done. If not, we use Gromov's compactness theorem to show that there is only finite number of homotopy types of connecting orbits between $x(t)$ and $x_i(t)$ whose energy is bounded. □

Since ∂_k is invariant under the action of Γ', we extend ∂_k as a $\Lambda_{\theta,\omega}$-linear map from $C_k(\widetilde{H})$ to $C_{k-1}(\widetilde{H})$. Using gluing argument and the weak compactness argument, we also deduce that $\partial^2 = 0$. The homology groups

$$HF_k(M, \omega, \theta_t, J; \mathbf{Z}_2) = \frac{\ker \partial_k}{\text{im } \partial_{k+1}}$$

are called the Floer homology groups of the quadruple (M, ω, θ_t, J) with the coefficients in $\mathbf{Z}_2$. Obviously, they are a graded $\Lambda_{\theta,\omega}$-modules. The following theorem shows that Floer homology groups are invariant under exact deformations.

Theorem 4.3. *For generic pairs $(\theta_t^\alpha, J^\alpha)$, $(\theta_t^\beta, J^\beta)$ such that $\theta_t^\alpha = \theta_t^\beta + df_t$ there exists a natural $\Lambda_{\theta,\omega}$-module homomorphism*

$$HF^{\beta,\alpha} : HF_*(\theta_t^\alpha, J^\alpha) \to HF_*(\theta_t^\beta, J^\beta)$$

which preserves the grading by the Conley-Zehnder index. If $(\theta_t^\gamma, J^\gamma)$ is any other such pair then

$$HF^{\gamma\beta} \circ HF^{\beta\alpha} = HF^{\gamma\alpha}, HF^{\alpha\alpha} = Id.$$

In particular, $HF^{\alpha\beta}$ is a $\Lambda_{\theta,\omega}$-module isomorphism.

Proof. The proof of Theorem 4.3 is carried out in the same way as in [F1], [H-S] (see also the section 5 below). Namely, we construct a chain homomorphism with the help of the "chain homomorphism equation" which is a s-depending analogue of the connecting orbit equation (2.9.1), (2.9.2), (2.9.3). Let $\theta_{s,t}$ denote a generic path connecting θ_t^α and θ_t^β in the fixed cohomology class. More precisely, there is a 2-parameter family of functions $H_{s,t}$ on M, such that

$$\theta_{s,t} = \theta_t^\alpha + dH_{s,t}$$

and for a sufficiently large R,

$$H_{s,t} = 0 \quad \text{for } s < -R,$$
$$H_{s,t} = f_t \quad \text{for } s > R.$$

To construct a chain homomorphism $\phi^{\alpha,\beta} : C_*(\theta_t^\alpha, J^\alpha) \to C_*(\theta_t^\beta, J^\beta)$, we consider the following equation

$$\frac{\partial u}{\partial s} + J(u)\{\frac{\partial u}{\partial t} - X_{\theta_t} - X_{H_{s,t}}\} = 0 \qquad (4.4).$$

Here, the key problem is to control the energy of solutions $\tilde{u}^{\beta,\alpha}(s,t)$ "connecting" two periodic solutions $[\tilde{x}^\alpha, \tilde{u}^\alpha]$ and $[\tilde{x}^\beta, \tilde{u}^\beta]$. Let $\widetilde{H}$ be a Hamiltonian function on $\widetilde{M}$ such that $d\widetilde{H} = \pi^*\theta_t^\alpha$. Then $\widetilde{H}' = \widetilde{H} + \pi^* f_t$ satisfies $d\widetilde{H}' = \pi^*\theta_t^\beta$. Let $[\tilde{x}, u^-] \in \widetilde{\mathcal{P}}(\widetilde{H})$ and $[\tilde{y}, u^+] \in \widetilde{\mathcal{P}}(\widetilde{H}')$. We have the following inequality.

$$|E(u) - \{\mathcal{A}_{\widetilde{H}}([\tilde{x}, u^-]) - \mathcal{A}_{\widetilde{H}'}([\tilde{y}, u^+])\}| \leq \int_{-\infty}^{\infty} \max_{x\in M, t\in S^1} |\frac{\partial H_{s,t}}{\partial s}| ds$$

for a solution u of (4.4) with $\lim_{s\to-\infty} u(s,t) = \pi(\tilde{x}) \in \mathcal{P}(\theta_t^\alpha)$, $\lim_{s\to\infty} u(s,t) = \pi(\tilde{y}) \in \mathcal{P}(\theta_t^\beta)$ and $[\tilde{y}, u^- \sharp u] = [\tilde{y}, u^+]$. Hence we have the weak-compactness of $\mathcal{M}(\tilde{x}, \tilde{y}; \theta_t + dH_{s,t}, J_s)$ and the argument in [H-S] yields Theorem 4.3. □

§5. A variant of the Palais-Smale condition and continuation.

To compute the Floer homology groups we need to use deformations which change the Calabi invariant. There arise two difficulties in proving the chain homomorphism between the Floer homologies associated with different Calabi invariants: to control the energy of solutions of the "chain homomorphism equation", and to make it sure that the chain homomorphism preserves the "finiteness condition" which arises in the definition of the chain complexes. We overcome the first one by using a variant of the Palais-Smale condition. We have to restrict ourselves to the case of symplectic manifolds satisfying the condition in Main Theorem in order to avoid the second difficulty. First of all, we show a variant of the Palais-Smale condition as follows:

Lemma 5.1. *Let $x_j : S^1 \to M$ be a sequence of contractible $W^{1,2}$-loops in M. If $\| \dot{x}_j - X_{\theta_t}(x_j) \|_{L^2}$ tends to 0 as $j \to +\infty$, there exists a subsequence, which we also denote by $\{j\}$ such that x_j converges to a contractible periodic solution x_∞ in C^0-sense.*

Proof. Without loss of generality, M is assumed to be embedded in $\mathbf{R}^N$ for a sufficiently large N and $x_j \in W^{1,2}(S^1, \mathbf{R}^N)$ such that $\mathrm{Im}(x_j) \subset M$. Since M is compact and

$\lim_{j\to\infty} \| \dot{x}_j(t)-X_{\theta_t}(x_j(t)) \|_{L^2}= 0$, there exists a constant $C > 0$ such that $\| x_j \|_{W^{1,2}}< C$. By the Rellich lemma, x_j converges to x_∞ in C^α-topology for $0 \leq \alpha < 1/2$, and x_∞ is the weak-limit of x_j in $W^{1,2}(S^1, \mathbf{R}^N)$. In particular, $x_\infty \in W^{1,2}(S^1, \mathbf{R}^N)$.

Since x_j converges to x_∞ in C^0-topology, $X_{\theta_t}(x_j)$ converges to $X_{\theta_t}(x_\infty)$ in C^0-topology. Thus it is easy to see that

$$\int_0^1 \langle x_\infty(t), \dot{\varphi}(t)\rangle dt = -\int_0^1 \langle X_{\theta_t}(x_\infty(t)), \varphi(t)\rangle dt$$

for any $\varphi \in C^\infty(S^1, \mathbf{R}^N)$, i.e. $\dot{x}_\infty(t) - X_{\theta_t}(x_\infty(t)) = 0$ in $L^2(S^1, \mathbf{R}^N)$. By the regularity argument, x_∞ satisfies $\dot{x}_\infty(t) - X_{\theta_t}(x_\infty(t)) = 0$ in classical sense. Contractible loops x_j converges to x_∞ in C^0-topology, therefore x_∞ is a contractible loop. □

For a generic periodic time-depending symplectic vector field X_{θ_t}, the set of periodic solutions is finite. Let $x_1, \cdots, x_l$ be all the contractible periodic solutions and $U_1, \cdots, U_l$ tubular neighborhoods of the graphs of x_j in $M \times S^1$. The following lemma is a direct consequence of Lemma 5.1.

Lemma 5.2. *There exists $c > 0$ such that $\| \dot{x} - X_{\theta_t} \|_{L^2}> c$ for any contractible loop x in M whose graph is not contained in either of U_j.*

Let η be a closed 1-form on M, $p : M \times S^1 \to M$ the projection to the second factor. Since each x_j is contractible, the restriction of $p^*\eta$ to U_j is exact. Hence we can find a periodic family $\{\eta_t\}$ of closed 1-forms on M which are cohomologous to η and vanishes on U_j for any j. Let $\{\theta_t\}$ be a regular periodic family of closed 1-forms on M (see Theorems 3.1 and 3.2). Using perturbation of η_t we also assume that $\theta_t + \epsilon \cdot \eta_t$ is regular. Then we can show the following

Theorem 5.3. *Suppose that M satisfies the condition in Main Theorem. For a periodic family $\{\eta_t\}$ in the cohomology class $\eta = Cal(\theta_t)$, there exists $\hbar > 0$ such that*

$$HF_*(\theta_t, J) \cong HF_*(\theta_t + \epsilon \cdot \eta_t, J')$$

for $\epsilon < \hbar$ and . Moreover the above isomorphism is Λ_θ-linear.

To construct a chain homomorphism, we consider the following equation.

$$\frac{\partial u}{\partial s} + J_s(u)(\frac{\partial u}{\partial t} - X_{\theta_{s,t}}) = 0 \tag{5.1}$$

with

$$(J_s, \theta_{s,t}) = (J, \theta_t) \text{ for } s < -R,$$

$$(J_s, \theta_{s,t}) = (J', \theta_t + \epsilon\eta_t) \text{ for } s > R,$$

and

$$\theta_{s,t} = \theta_t + \phi(s)\epsilon \cdot \eta_t.$$

Here $\phi(s)$ is a monotone smooth function on $[-R, R]$ which vanishes near $-R$ and equals to 1 near R.

Define the energy by

$$E(u) = \int_{-\infty}^{\infty}\int_0^1 |\frac{\partial u}{\partial s}|^2 dtds.$$

If a solution u of (5.1) has finite energy, $\lim_{s\to\pm\infty} u(s,t)$ exist and

$$\left.\begin{array}{rcll} z^- &=& \lim\limits_{s\to-\infty} u(s,t) & \in \mathcal{P}(\theta_t) \\ z^+ &=& \lim\limits_{s\to\infty} u(s,t) & \in \mathcal{P}(\theta_t + \epsilon\eta_t) \end{array}\right\} \tag{5.2}$$

Using perturbation of J_s we assume that the path $(J_s, \theta_{s,t})$ is regular (in the sense as in the section 3), and moreover, we assume that J' is sufficiently close to J such that $|J_s - J|(x) < \delta$ for all s, x and a positive δ small enough (which will be specified later). The following lemma contains a key estimate needed in our compactness argument.

Lemma 5.4. *Let ϵ be a real number such that $|\epsilon \cdot \eta_t| < c/3$ for all t. For a solution $\tilde{u}$ of (5.1) with (5.2) and the boundary condition $[\tilde{z}^+, u^- \sharp \tilde{u}] = [\tilde{z}^+, u^+]$, then $E(\tilde{u}) \leq 3(\mathcal{A}_{\widetilde{H}_t}([\tilde{z}^-, u^-]) - \mathcal{A}_{\widetilde{H}_t}([\tilde{z}^+, u^+]))$, where $\widetilde{H}_t$ is a Hamiltonian function on $\widetilde{M}$ such that $d\widetilde{H}_t = \pi^*\theta_t$.*

Proof.

$$\frac{\partial \mathcal{A}_{\widetilde{H}_t}}{\partial s}(\tilde{u}_s) = \int_0^1 \langle \frac{\partial \tilde{u}}{\partial s}, J(\frac{\partial \tilde{u}}{\partial t} - X_{\theta_t})\rangle dt.$$

If $u_s = \pi \circ \tilde{u}_s : S^1 \to M$ factors through $U_j \to M$ for some j, $X_{\eta_t} = 0$ along u_s, hence we get

$$\frac{\partial \mathcal{A}_{\widetilde{H}_t}}{\partial s}(\tilde{u}_s) = \int_0^1 \langle \frac{\partial u}{\partial s}, J(\frac{\partial u}{\partial t} - X_{\theta_t})\rangle dt = \| \frac{\partial u}{\partial s} \|^2_{L^2}. \tag{5.3}$$

If not, we have

$$\langle \frac{\partial u}{\partial s}, J(\frac{\partial u}{\partial t} - X_{\theta_t})\rangle \geq |\frac{\partial u}{\partial s}|^2 - |\frac{\partial u}{\partial s}| \cdot |\frac{\partial u}{\partial s} - J(\frac{\partial u}{\partial t} - X_{\theta_t})| \tag{5.4}$$

Since u satisfies (5.1) we get from (5.4)

$$\langle\frac{\partial u}{\partial s}, J(\frac{\partial u}{\partial t} - X_{\theta_t})\rangle \geq |\frac{\partial u}{\partial s}|^2 - |\frac{\partial u}{\partial s}| \cdot (|\phi(s) \cdot \epsilon\eta_t| + |J_s - J| \cdot |\frac{\partial u}{\partial t} - X_{\theta_t}|). \quad (5.5)$$

Our assumption that $|\phi(s) \cdot \epsilon\eta_t| \leq c/3$ yields

$$|J_s - J| \cdot |\frac{\partial u}{\partial t} - X_{\theta_t}| = |J_s - J| \cdot |J_s \frac{\partial u}{\partial s} + \phi(s) \cdot \epsilon\eta_t| \leq \delta(|\frac{\partial u}{\partial s}| + c/3). \quad (5.6)$$

Applying the Cauchy inequality to $\int_0^1 |\frac{\partial u}{\partial s}| dt$, we obtain from (5.5) and (5.6)

$$\frac{\partial \mathcal{A}_{\widetilde{H}_t}}{\partial s}(\tilde{u}_s) \geq (1-\delta) \parallel \frac{\partial u}{\partial s} \parallel_{L^2}^2 - \parallel \frac{\partial u}{\partial s} \parallel_{L^2} \cdot |(1+\delta)c/3)|. \quad (5.7)$$

Our assumption and Lemma 5.2 implies $\parallel \partial u/\partial s \parallel_{L^2} \geq c$. Now it is easy to verify the following inequality

$$\parallel \frac{\partial u}{\partial s} \parallel_{L^2} \cdot (\frac{2}{3} \parallel \frac{\partial u}{\partial s} \parallel_{L^2} - \frac{c}{3}) > 0.$$

Therefore, for δ small enough (which depends only on c), we obtain

$$(1-\delta) \parallel \frac{\partial u}{\partial s} \parallel_{L^2}^2 - \parallel \frac{\partial u}{\partial s} \parallel_{L^2} \cdot |(1+\delta)c/3| \geq \frac{1}{3} \parallel \frac{\partial u}{\partial s} \parallel_{L^2}^2 . \quad (5.8)$$

Combining (5.3), (5.7) and (5.8) yields the desired estimate. □

Once we get the uniform bound of the energy functional, the weak-compactness holds. In particular, $\mathcal{M}(\tilde{z}^-, \tilde{z}^+; \theta_{s,t}, J_s)$ is a finite set if $\mu_{\theta_t}(\tilde{z}^-) = \mu_{\theta_t+\epsilon\eta_t}(\tilde{z}^+)$. We define a chain homomorphism $\phi : C_*(\theta_t, J) \to C_*(\theta_t + \epsilon\eta_t)$ as follows:

$$\phi(\tilde{x}) = \sum_{\mu_{\theta_t+\epsilon\eta_t}(\tilde{y})=\mu_{\theta_t}(\tilde{x})} m_2(\tilde{x}, \tilde{y}) \cdot \tilde{y},$$

where $m_2(\tilde{x}, \tilde{y})$ denotes the modulo 2-reduction of the cardinality of $\mathcal{M}(\tilde{x}, \tilde{y}; \theta_{s,t}, J_s)$. Now we show that $\phi(\tilde{x}) \in C_*(\widetilde{H}'_t)$.

Lemma 5.5. *For each c and a fixed periodic solution $\tilde{y} \in \mathcal{P}(\widetilde{H}'_t)$ there is only finite number of elements $g \in \Gamma_1$ such that the coefficient of $g \cdot \tilde{y}$ in $\phi(\tilde{x})$ is not zero and $\mathcal{A}_{\widetilde{H}'_t}(g \cdot \tilde{y}) > c$.*

Proof. The same argument in the proof of Lemma 5.4 yields that $\mathcal{A}_{\widetilde{H}'_t}(\tilde{z}^+) \leq \mathcal{A}_{\widetilde{H}'_t}(\tilde{z}^-) - 1/3 \times E(\tilde{u})$, where $\widetilde{H}'_t$ is a Hamiltonian function for $\pi^*(\theta_t + \epsilon\eta_t)$. In other words, $\mathcal{A}_{\widetilde{H}'_t}$ is also a Liapunov function for the "flow" defined by equation (5.1). This inequality yields that if g satisfies the condition in Lemma 5.5 then the energy of the solution u of (5.1), the lift of which to $\widetilde{M}$ joins $\tilde{x}$ and $g \cdot \tilde{y}$, is uniformly bounded by c'. Assume the contrary, i.e. there exists infinite number $g_l \in \Gamma_1$ satisfying the condition in Lemma 5.5. Then there exists infinite number of solutions $\tilde{u}_i$ of (5.1) on M such that $\lim_{s\to-\infty} \tilde{u}_i = \tilde{x}$, $\lim_{s\to\infty} \tilde{u}_i = g_i \cdot \tilde{y}$ and $E(u_i) < c'$. By Gromov compactness theorem, there exists a limit u_∞. Let g_∞ be an element in Γ_1 such that $\lim_{s\to\infty} \tilde{u}_\infty = g_\infty \cdot \tilde{y}$. Pull back to $\widetilde{M}$ such that all $\tilde{u}_l, \tilde{u}_\infty$ have $\tilde{x}$ as one of the ends. Then the other end of $\tilde{u}_l$, when l is sufficient large, is also $g_\infty \cdot \tilde{y}$. We arrive at the contradiction. □

It is easy to see that ϕ is invariant under the Γ_1-action. By Lemma 5.5 we can extend ϕ as a chain homomorphism of Λ_θ-modules $C_*(\widetilde{H})$ and $C_*(\widetilde{H}'_t)$. The same argument yields that ϕ is a Λ_θ-linear isomorphism. □

§6. Floer homology and Novikov homology.

First of all, we recall fundamental facts on Novikov homology. Let X be a closed manifold and η a closed 1-form on X. Denote $\pi : \tilde{X} \to X$ the covering space associated to the homomorphism $I_\eta : \pi_1(X) \to \mathbf{R}$. Then there exists a function $f : \tilde{X} \to \mathbf{R}$ such that $\pi^*\eta = df$. For a generic Riemannian metric g on X, the gradient flow of f with respect to π^*g is of Morse-Smale type, and the Novikov complex $C_*^{Nov}(\eta, g)$ is defined in the same way as the Morse complex (cf Appendix 3). $C_*^{Nov}(\eta, g)$ is a graded module over the Novikov ring Λ_η. The homology group $Nov_*(\eta, g)$ of $C_*^{Nov}(\eta, g)$ is called the Novikov homology associated to η. In this note we consider only Novikov rings over $\mathbf{Z}_2$.

Fact 6.1. *$Nov_*(\eta, g)$ does not depend on the choice of a Riemannian metric g which makes the gradient flow of f being of Morse-Smale type.*

Fact 6.2. *$Nov_*(\eta) = Nov_*(\eta, g)$ depends only on the projective class of the cohomology class of η, i.e.*

$$Nov_*(\eta) \cong Nov_*(\eta') \textit{ if } [\eta] = \lambda[\eta'] \textit{ in } H^1(X; \mathbf{R}) \textit{ for some } \lambda \neq 0.$$

The following Fact (6.3) tells us that the Novikov homology can be computed from the Morse complex $\tilde{C}_*(h)$ of $\pi^*h : \widetilde{X} \to \mathbf{R}$.

Fact 6.3. $Nov_*(\eta) = H_*(\tilde{C}_*(h) \otimes \Lambda_\eta)$.

The goal of this section is to show the following

Theorem 6.4. *Let (M, ω) be a symplectic manifold of dimension $2n$ satisfying the con-*

dition in Main Theorem. For a generic periodic family $\{\theta_t\}$ of closed 1-forms in a fixed cohomology class $[\eta]$ there exists a natural isomorphism

$$HF_k(\theta_t, J) \cong \bigoplus_{j=k(mod\ 2N)} Nov_{j+n}(\eta)$$

as graded Λ_η-modules.

Proof. For a generic pairs $(\{\theta_t^\alpha\}, J^\alpha)$ and $(\{\theta_t^\beta\}, J^\beta)$ with the same Calabi invariant,

$$HF_*(\theta_t^\alpha, J^\alpha) \cong HF_*(\theta_t^\beta, J^\beta) \text{ by Theorem (5.1).}$$

In other words, the Floer homology does not depends on the choice of a generic pair $(\{\theta_t\}, J)$ with the prescribed Calabi invariant η. We denote it by $HF_*(\eta)$. Theorem (5.4) implies that

$$HF_*(\eta) \cong HF_*((1+\epsilon)\cdot\eta) \text{ for } |\epsilon| < \hbar(\eta).$$

Let h be a Morse function on M. We can consider the action functional $\mathcal{A}_h : \tilde{\mathcal{L}}(\widetilde{M}) \to \mathbf{R}$. Note that this is the pull-back of the action functional on $\tilde{\mathcal{L}}(M)$ in the case of exact symplectomorphisms. We shall define a chain complex $C_*(\pi^*h, J)$. The chain group consists of $\sum a_{[\tilde{x},\tilde{u}]}[\tilde{x},\tilde{u}]$, where sum is taken under the condition that $\tilde{x}$ is a critical point of π^*h and $[\tilde{x},\tilde{u}] \in \tilde{\mathcal{L}}(\widetilde{M})$ satisfying the following finiteness condition: $\{[\tilde{x},\tilde{u}] | a_{[\tilde{x},\tilde{u}]} \neq 0 \text{ and } \mathcal{A}_f([\tilde{x},\tilde{u}]) > c\}$ is a finite set for any c. We choose h to be a sufficiently C^2-small function such that all periodic solution $x(t)$ are precisely the critical points of h. Moreover we assume that the critical points of h and the gradient trajectories with Hessian index difference 1 do not intersect the holomorphic spheres of J with Chern number less than or equal to 1. The following lemma yields that the boundary operator ∂ depends only on the gradient trajectories of h.

Lemma 6.5 [O,Corollary 4.2]. *Let (M,ω) be a closed symplectic manifold of dimension $2n$ which satisfies the condition in Main Theorem or $c_1 \cdot |_{\pi_2(M)} = 0$. Suppose h is a Morse function on M. Then there exists a number $\tau > 0$ such that if u is a solution of (2.9.1), (2.9.2) and (2.9.3) corresponding to the (time-independent) Hamiltonian τh, and besides, $\mu(u) \leq 1$ then u is independent of t.*

We also have the non-degeneracy of the linearized operator for (2.9.1) at the gradient trajectories of h (see Appendix 2). The boundary operator ∂ is defined exactly same as (4.3). We denote the homology group by $HF_*(\pi^*h, J)$. Then as in the proof of Theorem

(5.4), the fact that $\mathcal{A}_f$ is a Liapunov function for (5.1) yields that

$$HF_*(h,J) \cong HF_*(\epsilon \cdot \eta) \text{ for } |\epsilon| < \hbar(0).$$

Therefore $HF_*(\eta) \cong HF_*([dh])$. Now remember that the Conley-Zehnder index of the pair $[x(t), u_x]$, where u_x is the unit element in Γ_2, is given by : $\mu([x(t), u_x]) = \text{ind}_h(x) - n$ [S-Z, H-S]. Therefore the set of all $x(t)$ with Conley-Zehnder index k coincides with the set of all critical points $x = x(t)$ of Morse index $j = k (\text{mod } 2N)$. Taking Fact 6.3 into account, the Lemma 6.5 implies that the Floer complex $C_*(dh, J)$ is isomorphic to the Novikov complex $C_*^{Nov}(dh)$. □

Theorem (6.4) implies our Main Theorem. □

§7. An example.

By Theorem A3.4 the Euler number of the Novikov homology corresponding to a free abelian covering over M is the same of the original manifold M. Hence, our theorem also implies the Lefschetz fixed point formula for symplectomorphisms which are symplectically isotopic to the identity. Here we will give a non-trivial example of symplectic manifolds satisfying the condition in the Main Theorem such that the sum of the Betti numbers of the Novikov homology corresponding to any free abelian covering of M is greater than the Euler number of M. By Theorem A3.2 it suffices to consider the maximal free abelian covering $\widetilde{M}$ of M.

The following example of Fano 3-folds with non-vanishing odd Betti number was pointed out to us by Keiji Oguiso. Let X_k be the hypersurface in $\mathbf{C}P^4$ defined by the equation $\{x_0^k + \cdots + x_4^k = 0\}$. Denote h the generator of $H^2(\mathbf{C}P^4, \mathbf{Z})$. A direct calculation yields

Claim 7.1. *The first Chern class of X_k satisfies $c_1 = (5-k)h_{|X_k}$. If $k \geq 3$ the Betti number $b_3(X^k)$ is non-zero.*

Let Σ_g denote a Riemannian surface of genus $g \neq 0$. We have $\pi_2(\Sigma_g) = 0$ and any non-degenerate 2-form ω_g on Σ_g is a symplectic form.

Claim 7.2. *The product manifold $(X_k \times \Sigma_g, \omega_X \oplus \omega_g)$, $k \neq 5$, is a symplectic manifold which satisfies the condition in Main Theorem. Further suppose that $k \geq 3$ and $g \geq 2$. Then the sum of the Betti numbers of the maximal free abelian covering of $X_k \times \Sigma_g$ is greater than the Euler number of $X_k \times \Sigma_g$.*

Proof. The first statement is trivial. The maximal free abelian covering of $X \times \Sigma_g$ is $X \times \widetilde{\Sigma_g}$ where $\widetilde{\Sigma_g}$ denotes the maximal free abelian covering of Σ_g. Using Theorem A.3.4 it is easy to see that the Betti numbers of $\widetilde{\Sigma_g}$ are $0, 2g-2, 0$. With the help of Claim 7.1 and Theorem A3.3 we obtain the second statement in Claim 7.2. In particular, if $k = 3$ we have the Euler number of $X_3 \times \Sigma_g = 12(g-1)$ while the sum of its Betti numbers equals $28(g-1)$, if $k = 4$ we have the Euler number of $X_4 \times \Sigma_g = 112(g-1)$ while the sum of its Betti numbers equals $128(g-1)$. □

Thus, if $k \geq 3$, $k \neq 5$, and $g \geq 2$, the number of the fixed points of a symplectomorphism $f \in \mathit{Diff}^0_\omega(X_k \times \Sigma_g)$ is greater than the Euler number of $X_k \times \Sigma_g$, provided that all the fixed points are non-degenerate. In particular, if $k = 3$ (or $k = 4$, resp.) this number is at least $28(g-1)$ (or $128(g-1)$, resp.).

§8. Concluding remarks

If $c_1\,|_{\pi_2(M)} = 0$ the Novikov ring $\Lambda_{\omega,\theta}$ changes when the cohomology class θ changes. That is the main obstruction to the control of the "finiteness condition", therefore, to the construction of "chain homomorphisms", and to the computation of the corresponding Floer homology groups. However, if the Calabi invariant is small enough we still have the following theorem.

Theorem 8.1. *Let (M, ω) be a closed symplectic manifold and $c_1\,|_{\pi_2(M)} = 0$. There exists $\varepsilon > 0$ such that if $\|[\theta]\| < \varepsilon$ then*

$$HF_*(\theta) \cong Nov_*(\theta) \otimes_{\Lambda_\theta} \Lambda_{\theta,\omega}.$$

Consequently, the sum of ranks of $HF_*(\theta)$ equals the one of $Nov_*(\theta)$.

Here we suppose that M is provided with some Riemannian metric and the norm $\|[\theta]\|$ is defined as infimum of the norm of 1-form θ in this cohomology class $[\theta]$.

Proof. By Theorem 4.3 (invariance under exact deformations) it suffices to show that for each regular periodic 1-form θ_t on M there exists a positive number τ such that $HF_*(\tau \cdot \theta_t, J) \cong Nov_*(\theta) \otimes_{\Lambda_\theta} \Lambda_{\theta,\omega}$. First we observe that if $c_1\,|_{\pi_2(M)} = 0$ then the Floer homology of a Morse-Smale function h which is C^2 small enough is still well defined, and moreover, it is isomorphic to the Morse homology group of h (that is isomorphic to the homology group of M). The proof of this fact is similar to that one in the section 6 and relies on Lemma 6.5. Now we lift the function h on the covering space $\widetilde{M}$ corresponding to the form θ. By the same way as in the section 6 we define the Floer complex $C_*(\tilde{h}, J)$ for the lifted function $\tilde{h}$. Further we choose τ small enough so that the energy estimate for the "chain homomorphism" solution u between $HF_*(\tilde{h}, J)$ and $HF_*(\tau\cdot\theta, J')$ holds (see Lemma 5.4). Finally we compare the Floer homology $HF_*(\tilde{h})$ with the Novikov homology $Nov_*(\theta)$. Denote $C_*(\tilde{h})$ (without J) the Novikov complex. We have (see Appendix 3)

$$\mathrm{rank}_{\Lambda_\theta} Nov_*(\theta) = \mathrm{rank}_{F(\Lambda)} H_*(C(\tilde{h})) \otimes_\Lambda F(\Lambda).$$

In the same way as in the proof of Theorem 6.4, we have

$$HF_*(\tilde{h}, J) \cong Nov_{*+n}(\theta) \otimes_{\Lambda_\theta} \Lambda_{\omega,\theta}.$$

Therefore, the rank of the Floer homology corresponding to $\tilde{h}$ equals the rank of the Novikov homology corresponding to $\tilde{h}$. Observe that these homology groups are vector spaces over the corresponding fields. (Note that the Novikov rings over $\mathbf{Z}/2$ are fields.) Hence follows Theorem 8.1.

Appendix 1: On the Poincare invariant of symplectomorphisms.

In this section, we prove the converse statement of the Poincare theorem, namely, if two embedded loops on a symplectic manifold M have the same Poincare invariant then there exists an exact Hamiltonian flow on M which sends one loop to the other.

Originally the Poincare theorem was stated for the symplectic manifold $\mathbf{R}^{2n}$ and then for the cotangent bundle T^*M with the canonical 1-form $\alpha = pdq$ and the canonical symplectic form $\omega = d\alpha$ [A]. In the general case of an arbitrary symplectic manifold M we replace the Poincare integral by the action functional $\mathcal{A}$ defined on an appropriate covering space of the space $E\mathcal{L}(M)$ of contractible embedded loops. Namely, this covering space $E\tilde{\mathcal{L}}(M)$ can be identified with the quotient space of pairs $\{[\gamma, u] |\ \partial u = \gamma\}$ by the equivalence relation defined by the homotopy equivalence of the bounding disk u. We have

$$\mathcal{A}([\gamma, u]) = \int u^*\omega \qquad (A1.1)$$

Passing to the base space, the action functional is only defined with modulo of the group $\omega([\pi_2(M)])$. However, it also means that the differential of the action functional is well-defined on the loop space:

$$d\mathcal{A}(\gamma)(X) = \int_\gamma \omega(X, \dot{\gamma}) = -\Phi(X)([\gamma]), \qquad (A1.2)$$

where Φ is defined in (2.1). From (A1.2) we obtain immediately the following well-known theorem.

Theorem A1.1. *Any Hamiltonian flow f_s on M preserves the generalized Poincare invariant - the action functional on the covering space of the space of embedded contractible loops $E\mathcal{L}(M)$*

$$\mathcal{A}([\gamma,u]) = \mathcal{A}([f_s(\gamma), f_s(u)])$$

Note that if the symplectic form ω is exact on M, that is $\omega = d\alpha$, then

$$\mathcal{A}([\gamma,u]) = \int_\gamma \alpha. \tag{A1.3}$$

Theorem A1.2. *Let (M,ω) be a symplectic manifold of dimension $2n \leq 4$. Suppose that two embedded contractible loops $[\gamma_0,u_0]$ and $[\gamma_1,u_1]$ have the same Poincare invariant. Then there exists a Hamiltonian flow f_s such that $f_0 = Id$; $f_1(\gamma_0) = \gamma_1$.*

Proof of Theorem A1.2. We infer Theorem A1.2 from the following propositions.

Proposition A1.3. *Every level surface $\mathcal{A}^{-1}(a) \subset E\tilde{\mathcal{L}}(M)$ is path-connected.*

Proposition A1.4. *Suppose the path $[\gamma_s,u_s]$, $s \in [0,1]$, of embedded loops lies on a level surface $\mathcal{A}^{-1}(a)$. Then there exists a Hamiltonian flow f_s such that $f_s(\gamma_0) = \gamma_s$.*

Proof of Proposition A1.3. It is easy to see that the space $E\tilde{\mathcal{L}}(M)$ is path-connected. Suppose that $[\gamma_s,u_s]$ is a path in $E\tilde{\mathcal{L}}(M)$ which joins two points $[\gamma_0,u_0]$ and $[\gamma_1,u_1]$. Our aim now is to find a deformation of the path $[\gamma_s,u_s]$ to a new path of constant Poincare invariant, and besides, this new path also joins $[\gamma_0,u_0]$ and $[\gamma_1,u_1]$.

We consider the path $u_s(0), s \in [0,1]$, in M of the center of bounding disks u_s. Using Darboux's theorem we can find the number $\varepsilon > 0$ and the smooth family of embeddings $\psi_s : D^2(\varepsilon) \times I \longrightarrow N$ such that the following conditions hold.

1) The restriction of ψ_s on $D^2(\varepsilon) \times \tau$ is a symplectic embedding. Here the disk $D^2(\varepsilon)$ of radius ε carries the standard symplectic structure.

2) $\psi_s(\varepsilon,0) \times 0 = \gamma_s(0)$, where (r,θ) denotes the polar coordinate on the disk $D^2(\varepsilon)$.

By the conditions (1) and (2) we can find the sequence $\tau_1, \ldots, \tau_n \in I$ and construct the embedding $\Psi : [0,1] \times S^1 \longrightarrow N$ with the following properties

3)$\Psi(s, S^1)$ is the connected sum of n circles $\psi(S^1(\chi(s)\cdot\varepsilon)\times\tau_i$. Here χ is a fixed smooth positive function on the interval $(0,1)$ and vanishes at the end points, and $(S^1(r))$ denotes the circle of radius r in $D^2(\varepsilon)$.

4) The Poincare invariant $\mathcal{A}(\Psi(s,S^1)) \geq |\mathcal{A}(\gamma_s) - \mathcal{A}(\gamma_0)|$.

Now we consider the new path γ_s' of the connected sum $(\gamma_t)(s)$ and $\Psi(s,S^1)$. By the condition (4) the Poincare invariant of each circle γ_s' is greater than or equals to $\mathcal{A}(\gamma_0) = \mathcal{A}(\gamma_1)$. Applying the construction of connected sum of γ_s' with $\Psi'(s,S^1)$, we obtain a new path γ_t'' such that the Poincare invariant of each γ_s'' is less than or equal to $\mathcal{A}(\gamma_0) = \mathcal{A}(\gamma_1)$. Using deformation by multiplication with a number less than 1 for $\Psi'(s,S^1)$ we can deform the path γ_s'' to a new path of constant Poincare invariant. This completes the proof of Proposition A1.3.□

Proof of Proposition A1.4. Let ϕ_s be a 1-parameter family of embeddings of S^1 into M with the same Poincare invariant. Differentiating it with respect to s, we get a 1-parameter family V_s of vector fields along ϕ_s. It is sufficient to show that V_s, which is defined on $\gamma_s = \phi_s(S^1)$ can extend to a Hamiltonian vector field of M. Using the isomorphism Φ (cf. (2.1)), we get a cross section $\Phi(V_s)$ of $T^*M|_{\gamma_s}$. We shall extend this section to a closed 1-form on a tubular neighborhood $N(\gamma_s)$. Since symplectic manifolds are orientable, $N(\gamma_s)$ is diffeomorphic to $S^1 \times D^{2n-1}$ such that $S^1 \times \{0\}$ corresponds to γ_s. Let $\{x_1, \cdots, x_{2n-1}\}$ be coordinate functions of D^{2n-1}. Then there exist functions $a(t), b_1(t), \cdots, b_{2n-1}(t)$ on S^1 such that

$$\Phi(V_s) = a(t)dt + \sum_1^{2n-1} b_i(t)dx_i.$$

Put

$$\tilde{a}(t,x_i) = a(t) + \frac{db_i}{dt}\cdot x_i.$$

It is easy to see that

$$\eta = \tilde{a}(t,x_i)dt + \sum_{i=1}^{2n-1} b_i(t)dx_i$$

is a desired extension as a closed 1-form. Since the embedded loops γ_t have the same Poincare invariant, from (A1.2) we conclude that $\Phi(V_t)$ is an exact 1-form on γ_s. Note that $S^1 \times \{0\}$ is a deformation retract of $S^1 \times D^{2n-1}$, hence η is also an exact 1-form, i.e. $\eta = dh$ for some function h.

For a cut off function φ which equals 1 near $S^1 \times \{0\}$, the Hamiltonian vector field of $\varphi \cdot h$ coincides with V_t on γ_t and vanishes near the boundary of the tubular neighborhood. Therefore it naturally extends to a Hamiltonian vector field $\tilde{V}_t$ which vanishes outside of the tubular neighborhood. This completes the proof of Proposition A1.4. □.

Remark A1.5. Theorem A1.2 can be generalized to embedded loops of non-trivial homotopic class by the same line of argument. We can also show the case that $M = \mathbf{R}^2$ with the standard symplectic structure.

Appendix 2: Non-degeneracy of the linearized operator for time independent Hamiltonians.

In this appendix, we shall show the surjectivity of the linearized operator at gradient trajectories (see also [S-Z]). Let f be a C^2-small Morse function on a symplectic manifold M such that the Conley-Zehnder index at critical points with trivial bounding disks coincide with the index of the Hessian of f. We also fix a metric, for which the gradient flow of f is of Morse-Smale type and which is compatible with an almost complex structure calibrated by ω.

Let $\gamma : \mathbf{R} \to M$ be a trajectory of the gradient flow joining two critical points p and q. Denote $u_\gamma : \mathbf{R} \times S^1 \to M$ the mapping defined by $u(s,t) = \gamma(s)$. The linearization operator $D\overline{\partial}_{J,H}$ of $\overline{\partial}_{J,H}$ at u_γ is given by

$$D\overline{\partial}_{J,H}\xi = \nabla_{\frac{\partial}{\partial s}}\xi + J\nabla_{\frac{\partial}{\partial t}}\xi + Hess\ (f)\xi, \qquad (A2.1)$$

where $Hess\ (f)\xi = \nabla_\xi \nabla f$. Since the Hamiltonian f and the connecting orbit u_γ are t-independent, we have a symmetry in t-variable. Hence $D\overline{\partial}_{J,H}$ decomposes into Fredholm operators

$$P^{(k)} : V_k \to W_k,$$

where $W^{1,2}(u_\gamma^*TM) = \oplus_{k\in\mathbf{Z}} V_k$ and $L^2(u_\gamma^*TM) = \oplus_{k\in\mathbf{Z}} W_k$ are decompositions as S^1-modules according to weights. From (A2.1), $P^{(k)}\xi = \nabla_{\frac{\partial}{\partial s}}\xi + (-k + Hess\ (f))\xi$.

For $k = 0$, $P^{(0)}$ is surjective and moreover we have index $P^{(0)} =$ dim ker $P^{(0)} =$ index $Hess_p(f)$ − index $Hess_q(f)$, which follow from the assumption that the gradient flow of f is of Morse-Smale type. ($P^{(0)}$ coincides with the linearization operator for the gradient flow of a Morse function [S,Sch].)

From now on, we assume that f is C^2-small such that $\| Hess\ (f) \| < \delta < 1$ for some $\delta > 0$. Let $\langle\langle \xi, \eta \rangle\rangle(s) = \int_0^1 \langle \xi(s,t), \eta(s,t) \rangle dt$.

For $k > 0$, the solution ξ of $P^{(k)}\xi = 0$ satisfies

$$\begin{aligned}\frac{\partial}{\partial s} \| \xi \|^2 &= 2\langle\langle \nabla_{\frac{\partial}{\partial s}}\xi, \xi\rangle\rangle \\ &= 2\langle\langle (k - Hess\ (f))\xi, \xi\rangle\rangle \\ &> (1-\delta) \| \xi \|^2 .\end{aligned}$$

Hence $\| \xi \|^2$ grows exponentially as s tends to $+\infty$, unless $\xi \equiv 0$. Since ξ is square integrable, we get $\xi \equiv 0$, i.e. ker $P^{(k)} = 0$ for $k > 0$. The same conclusion holds for $k < 0$. Therefore we get index $D\overline{\partial}_{J,H} = \dim\ \ker\ P^{(0)} - \sum_{k\neq 0} \dim\ \text{coker}\ P^{(k)}$. On the other hand, index $D\overline{\partial}_{J,H} = \dim\ \ker\ P^{(0)}$, hence $P^{(k)}$ are surjective for all k, i.e. $D\overline{\partial}_{J,H}$ is surjective.

Appendix 3: Note on Novikov homology theory.

(Collaboration with Lê Tu Quôc Thang)

In this appendix, we shall give simple proofs of some fundamental facts on the Novikov homology theory, which seem folklore to specialists (see also [Pa],[Po]). Floer [F3] interpreted the Morse complex in terms of gradient trajectories for generic Morse functions. Details are carried out by Matthias Schwarz [Sch] (see also [Sa]). For a closed 1-form η on M, there is the smallest covering space on which the pull back of η is exact. We denote this covering space by $\pi : \widetilde{M} \to M$. Let f be a function on $\widetilde{M}$ such that $\pi^*\eta = df$. For a generic Riemannian metric on M, the gradient flow of f with respect to the pull back metric is of Morse-Smale type. An element of the k-th Novikov chain group of η is $\sum a_{\tilde{x}}\tilde{x}$ where the sum is taken under the condition that the index of Hessian of f at $\tilde{x}$ is k and $\{\tilde{x} | a_{\tilde{x}} \neq 0$ and $f(\tilde{x}) > c\}$ is a finite set for any c. The Novikov ring Λ_ξ is defined as the completion of the group ring of the covering transformation group of $\pi : \widetilde{M} \to M$ with respect to the weight homomorphism I_ξ. The argument in Lemma (3.5) implies that there are at most finite many trajectories joining critical points $\tilde{x}, \tilde{y}$ and we can define the boundary operator, which is linear over the Novikov ring, exactly same as (4.3). The Novikov complex, hence the Novikov homology, are finite generated modules over the Novikov ring. Note that the Novikov complexes of η and $\lambda\eta$ with the same Riemannian metric are same for $\lambda \neq 0$. The argument in Theorem 4.3 yields Fact 6.1 and Fact 6.2.

We shall prove Fact 6.3. Our argument is a finite dimensional analogue of the proof of Theorem 5.3. Let h be a Morse function on M and U_i be a contractible neighborhood of a critical point p_i of h. Since M is compact, there exists a number $\epsilon > 0$ such that the norm of the gradient vector field ∇h satisfies $\| \nabla h \| > \epsilon$ outside of $\cup U_i$. Since each U_i is contractible, we can find a closed 1-form η' which is cohomologous to η and vanishes identically on $\cup U_i$. Then we can find $\lambda > 0$ such that $\| \lambda \cdot \eta'^{\sharp} \| < \epsilon/3$, where $\eta'^{\sharp}$ is the vector field associated to η' with respect to the given Riemannian metric. From now on η denotes $\lambda \cdot \eta'$ defined as above. This implies that f and $f + \pi^* h$ are Liapunov functions for both of the gradient flows of f and $f + \pi^* h$. Note that the critical point sets of f and $f + \pi^* h$ coincide. We can define chain homomorphisms between the Morse complex of f and $f + \pi^* h$ by using the following ordinary differential equation.

$$\frac{d\gamma}{ds} + \nabla\{f + \pi^* h_s\}(\gamma(s)) = 0, \tag{A3.1}$$

where h_s is a 1-parameter family of functions on M such that

$$h_s = 0 \text{ for } s < -R \text{ and } h_s = h \text{ for } s > R,$$

for some $R > 0$. The energy of a solution γ of (A3.1) is defined by $E(\gamma) = \int_{-\infty}^{\infty} |\frac{d\gamma}{ds}|^2 ds$. The set of solutions of (A3.1) with bounded energy coincides the set of solutions of (A3.1) satisfying the asymptotic condition, i.e.

$$\lim_{s \to -\infty} \gamma(s) = \tilde{x} \text{ and } \lim_{s \to \infty} \gamma(s) = \tilde{y}$$

for some critical points $\tilde{x}$, $\tilde{y}$ of f. Moreover, we can estimate the energy in terms of $\tilde{x}$, $\tilde{y}$ as follows:

$$E(\gamma) < 3 \cdot (f(\tilde{x}) - f(\tilde{y})),$$

and

$$E(\gamma) < 3 \cdot ((f + \pi^* h)(\tilde{x}) - (f + \pi^* h)(\tilde{y})).$$

The bound of energy implies the weak compactness of the set of solutions and no bubbling phenomena occurs. We define $\phi(\tilde{x}) = \sum m_2(\tilde{x}, \tilde{y}) \cdot \tilde{y}$ where $m_2(\tilde{x}, \tilde{y})$ denotes the modulo 2-reduction of the number of solutions of (A3.1) satisfying the asymptotic condition. To get a chain homomorphism from ϕ, we have to show that ϕ preserves the finiteness condition with respect to $f + \pi^* h$. However this can be derived in the same way as Lemma 3.5. We can also define a chain homomorphism in the other direction and they are inverses each other on homology groups.

We define the rank of a module L over a commutative algebra A with a unit by the dimension of the vector space $L \otimes F(A)$ over the fractional field $F(A)$ of A. The rank of the Novikov homology is defined as the rank of it over the Novikov ring. Now we shall show the following:

Theorem A3.1. *Let ξ and ξ' be closed 1-forms on M whose corresponding abelian coverings are same. Then ranks of the Novikov homology of ξ and the Novikov homology of ξ' are same.*

Theorem A3.2. *Let ξ and ξ' be closed 1-forms such that ξ' vanishes on the kernel of ξ. Then the rank of $Nov_*(\xi)$ is less than or equal to the rank of $Nov_*(\xi')$.*

Proof of Theorem A3.1. Let $\widetilde{M} \to M$ be the abelian covering of M corresponding to a 1-form ξ. Let h be a Morse function on M and $\widetilde{C}_*(h) = C_*(\pi^*h)$ the Morse complex of π^*h. Let Λ be the group ring of the covering transformation group and Λ_ξ the Novikov ring of ξ. Then Fact 6.3 implies that $Nov_*(\xi) \cong H_*(\widetilde{C}_*(h) \otimes_\Lambda \Lambda_\xi)$. Since Λ_ξ is faithfully flat over Λ (this fact is due to Sikorav, see [Pa], but in fact, we need only the flatness , which can be derived from the well-known facts of commutative algebra [A-M,Chap.10]), we get $Nov_*(\xi) \cong H_*(\widetilde{C}_*(h)) \otimes_\Lambda \Lambda_\xi$. Then we have $\mathrm{rank}_{\Lambda_\xi}\ Nov_*(\xi) = \mathrm{rank}_{F(\Lambda_\xi)}\ H_*(\widetilde{C}_*(h)) \otimes_\Lambda F(\Lambda_\xi) = \mathrm{rank}_{F(\Lambda_\xi)}\{H_*(\widetilde{C}_*(h)) \otimes_\Lambda F(\Lambda)\} \otimes_{F(\Lambda)} F(\Lambda_\xi) = \mathrm{rank}_{F(\Lambda)} H_*(\widetilde{C}_*(h)) \otimes_\Lambda F(\Lambda)$. Therefore the rank of the Novikov homology depends only on the covering space $\tilde{M}$, i.e. the ranks of $Nov_*(\xi)$ and $Nov_*(\xi')$ are same. □

Proof of Theorem A3.2. Let $\widetilde{M} \to M$ and $\overline{M} \to M$ be the covering spaces corresponding to ξ and ξ' with the covering transformation groups Γ_1 and Γ_2 respectively. We can choose generators of covering transformation groups such that abelian groups Γ_1 and Γ_2 are generated by $t_1, \cdots, t_k$ and $t'_1, \cdots, t'_l$ $(l < k)$ respectively. Here t'_i is the image of t_i by the quotient homomorphism $\Gamma_1 \to \Gamma_2$. Let Λ_1 and Λ_2 be group rings of Γ_1 and Γ_2 respectively. Then there is a natural homomorphism $\phi : \Lambda_1 \to \Lambda_2$ which maps t_i to 1 for $l + 1 \leq i \leq k$. We may assume that $l = k - 1$. By Theorem A3.1 and Fact 6.3, it is enough to show that $\mathrm{rank}_{\Lambda_1}\ H_*(\widetilde{C}_*(h)) \leq \mathrm{rank}_{\Lambda_2}\ H_*(\widetilde{C}_*(h) \otimes_{\Lambda_1} \Lambda_2)$.

Claim. The natural map $H_*(\widetilde{C}_*(h)) \otimes_{\Lambda_1} \Lambda_2 \to H_*(\widetilde{C}_*(h) \otimes_{\Lambda_1} \Lambda_2)$ is injective. Therefore we get $\mathrm{rank}_{F(\Lambda_2)}\ \{H_*(\widetilde{C}_*(h)) \otimes_{\Lambda_1} \Lambda_2\} \otimes_{\Lambda_2} F(\Lambda_2) \geq \mathrm{rank}_{F(\Lambda_2)}\ \{H_*(\widetilde{C}_*(h)) \otimes_{\Lambda_1} \Lambda_2\} \otimes_{\Lambda_2} F(\Lambda_2)$.

This Claim follows from the observation that $B_*(\widetilde{C}_*(h)) \otimes \Lambda_2 = B_*(\widetilde{C}_*(h) \otimes \Lambda_2)$, where $B_*(*)$ denotes the submodule of boundary cycles.

In order to complete the proof of Theorem A3.2, it is sufficient to show that

$$\mathrm{rank}_{F(\Lambda_2)}\ H_*(\widetilde{C}_*(f)) \otimes_{\Lambda_2} F(\Lambda_2) \geq \mathrm{rank}_{\Lambda_1}\ H_*(\widetilde{C}_*(f)).$$

Put $R = F(\Lambda_2)[t_k^{\pm}]$. Denote $L = H_*(\tilde{C}_*(f))$. Then R is a principal ideal domain and has the same fractional field as Λ_1. Hence for the Λ_1-module L, we have $\mathrm{rank}_{\Lambda_1} L = \mathrm{rank}_R L \otimes_{\Lambda_1} R$. Since R is a principal ideal domain, we have $L \otimes_{\Lambda_1} R \cong R^p \oplus Torsion$, which yields that $\{L \otimes_{\Lambda_1} R\} \otimes_R F(\Lambda_2) \supset (F(\Lambda_2))^p$. Hence we get $\mathrm{rank}_{F(\Lambda_2)} L \otimes_{\Lambda_2} F(\Lambda_2) \geq \mathrm{rank}_{\Lambda_1} L$. We get the desired inequality. □

We have the Künneth formula in Novikov homology theory.

Theorem A3.3. *Let M_1 and M_2 be closed manifolds and ξ and η closed 1-forms on M_1 and M_2 respectively. If the kernel of the weight homomorphism for $\pi_1^*\xi + \pi_2^*\eta$ is the direct sum of the kernels of the weight homomorphism for ξ and η, we have*

$$Nov_*(M_1 \times M_2; \pi_1^*\xi + \pi_2^*\eta) \cong (Nov_*(M_1;\xi) \otimes_{\mathbf{Z}_2} Nov_*(M_2;\eta)) \otimes_{\Lambda_\xi \otimes \Lambda_\eta} \Lambda_{\pi_1^*\xi+\pi_2^*\eta}.$$

Proof. We denote by Γ_1 and Γ_2 the group ring of the covering transformation groups corresponding to ξ and η. The Künneth formula for the corresponding covering spaces yields that

$$H_*(\tilde{C}_*(\pi_1^*h_1 + \pi_2^*h_2)) \cong H_*(\tilde{C}_*(\pi_1^*h_1)) \otimes H_*(\tilde{C}_*(\pi_2^*h_2)),$$

where h_1 and h_2 are Morse functions on M_1 and M_2 respectively. Since the kernel of the weight homomorphism for $\pi_1^*\xi + \pi_2^*\eta$ is the direct sum of the kernels of the weight homomorphisms for ξ and η, the Novikov ring for $\pi_1^*\xi + \pi_2^*\eta$ is the completion of the tensor product of the Novikov rings for ξ and η. Note that $Nov_*(M_1 \times M_2; \pi_1^*\xi + \pi_2^*\eta)$ is isomorphic to $H_*(\tilde{C}_*(\pi_1^*h_1 + \pi_2^*h_2)) \otimes_{\mathbf{Z}_2[\Gamma_1 \oplus \Gamma_2]} \Lambda_{\pi_1^*\xi+\pi_2^*\eta}$. Combining these facts, we get the fact that $Nov_*(M_1 \times M_2; \pi_1^*\xi + \pi_2^*\eta)$ is isomorphic to the completion of the tensor product of

$$Nov_*(M_1;\xi) \cong H_*(\tilde{C}_*(h_1)) \otimes_{\mathbf{Z}_2[\Gamma_1]} \Lambda_\xi$$

and

$$Nov_*(M_2;\eta) \cong H_*(\tilde{C}_*(h_2)) \otimes_{\mathbf{Z}_2[\Gamma_2]} \Lambda_\eta$$

with respect to the weight homomorphism for $\pi_1^*\xi + \pi_2^*\eta$, i.e.

$$\{Nov_*(M_1;\xi) \otimes Nov_*(M_2;\eta)\} \otimes_{\Lambda_\xi \otimes \Lambda_\eta} \Lambda_{\pi_1^*\xi+\pi_2^*\eta}. \quad \square$$

Since the Euler number of the homology equals the alternating sum of ranks of the complex, we get

Theorem A3.4. *The Euler number of the Novikov homology, i.e. the alternating sum of the ranks of the Novikov homology groups, equals to the Euler number of the ordinary homology of M.*

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