THE FANO SURFACE OF THE FERMAT CUBIC THREEFOLD, THE DEL PEZZO SURFACE OF DEGREE 5 AND A BALL QUOTIENT

XAVIER ROULLEAU

Abstract. We prove that the Fano surface of the Fermat cubic threefold is a degree 81 abelian cover of the degree 5 del Pezzo surface branched over the 10 lines and that the complementary of the union of 12 disjoint elliptic curves of this surface is a ball quotient. The lattice of this ball quotient is linked with a congruence sub-group of the lattice of the Eisenstein integers.

Let us recall a classical construction of surfaces due to Hirzebruch [4]: The configuration of 6 lines $L_1, ..., L_6$ going through 4 points $p_1, ..., p_4$ in general position on the plane is called the complete quadrilateral. Let $\ell_i \in H^0(\mathbb{P}^2, \mathcal{O}(1))$ be a linear form defining L_i and let $n > 1$ be an integer. The field

$$\mathbb{C}(\mathcal{H}_n) = \mathbb{C}(\mathbb{P}^2)((\frac{\ell_2}{\ell_1})^{\frac{1}{n}}, ..., (\frac{\ell_6}{\ell_1})^{\frac{1}{n}})$$

determine a normal algebraic surface, $\mathcal{H}'_n$, that is a branched cover, $\pi : \mathcal{H}'_n \to \mathbb{P}^2$, of $\mathbb{P}^2$ of degree n^5 with the complete quadrilateral as the branching locus. Let $\tau : \mathcal{H}_1 \to \mathbb{P}^2$ denotes the blow-up map above the 4 points $p_1, .., p_4$. The surface $\mathcal{H}_1$ is called the del Pezzo surface of degree 5 and contains exactly 10 (-1)-curves : these curves are the proper transform of the lines L_i and the 4 exceptional divisors. Let be $\mathcal{H}_n$ the fibre product of $\mathcal{H}_1$ and $\mathcal{H}'_n$ over $\mathbb{P}^2$:

$$\begin{array}{ccc} \mathcal{H}_n & \xrightarrow{\iota} & \mathcal{H}'_n \\ \downarrow \eta_n & & \downarrow \pi \\ \mathcal{H}_1 & \xrightarrow{\tau} & \mathbb{P}^2 \end{array}$$

The surface $\mathcal{H}_n$ is smooth of general type ; the cover η_n is branched exactly over the 10 (-1)-curves, and with order n. Hirzebruch proves that:

Theorem 0.1. *The Chern numbers of $\mathcal{H}_5$ satifies: $c_1^2(\mathcal{H}_5) = 3c_2(\mathcal{H}_5) > 0$.*

Few examples of surfaces with Chern ratio $\frac{c_1^2}{c_2}$ equals 3 have been constructed algebraically i.e. by ramified covers of known surfaces. The following result formulated by Kobayashi [7], that generalizes the works of Miyaoka, Yau, Hirzebruch and Sakai, gives an analytic characterization of (log-)surfaces with Chern ratio 3:

Theorem 0.2. *Let S be a smooth projective surface with canonical bundle K and let D be a reduced simple normal crossing divisor on S (may be 0). Suppose that $K + D$ is nef and big. Then the following inequality:*

$$3\bar{c}_2 - \bar{c}_1^2 \geq 0$$

holds, where $\bar{c}_1^2, \bar{c}_2$ are the logarithmic Chern numbers of $S - D$.
The equality occurs if and only if the universal covering of S minus D and the union

of the (-2)-curves is biholomorphic to the open unit ball $\mathbb{B}_2$ minus a discrete set of points.
If a compact surface X contains a rational curve and $c_1^2(X) = 3c_2(X) > 0$ holds, then X is the projective plane.

The first algebraic construction of a surface $\mathbb{S}$ which is a ball quotient (ie $\mathbb{S} \neq \mathbb{P}^2$ and $c_1^2(\mathbb{S}) = 3c_2(\mathbb{S}) > 0$) was done independently by Inoue and Livné as a cyclic cover of the Shioda modular surface of level 5 (for a reference see [1]). Ishida [6] has then proved that :

Proposition 0.3. *There is a étale map $\mathcal{H}_5 \to \mathbb{S}$ that is a quotient of $\mathcal{H}_5$ by an automorphism group of order 25.*

Having recalling these facts, we can state the results of this paper, the remainder being the proof of this Theorem:
Let $F \hookrightarrow \mathbb{P}^4 = \mathbb{P}(\mathbb{C}^5)$ be the Fermat cubic threefold:

$$F = \{x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0\}.$$

The variety that parametrizes the lines on F is a smooth complex surface S called the Fano surface of lines of F [3].

Theorem 0.4. *A) There is a étale map $\kappa : \mathcal{H}_3 \to S$ that is a quotient of $\mathcal{H}_3$ by an automorphism of order 3.*
B) There is an open subvariety $S' \subset S$ such that S' is a ball quotient i.e. $\bar{c}_1(S')^2 = 3\bar{c}_2(S')$.
C) Let $\mathbb{B}^2$ be the 2-dimensional ball with respect to the Hermitian form represented by the diagonal matrix H with entries $(1, 1, -1)$. Let $\mathcal{T}$ be the inverse image of S' by κ. The ball lattice of the ball quotient $\mathcal{T}$ is the commutator group of the congruence group:

$$\Gamma = \{T \in GL(\mathbb{Z}[\alpha])/T \equiv I\, modulo\, (1-\alpha)\ and\ {}^t\bar{T}HT = H\}$$

where α is a primitive third root of unity and I is the identity matrix.

Let us prove Theorem 0.4.
Let $A(3, 3, 5) \subset GL_5(\mathbb{C})$ be the group of diagonal matrices of determinant 1 whose diagonal elements are in $\mu_3 := \{x \in \mathbb{C}/x^3 = 1\}$. The group $A(3, 3, 5) \simeq (\mathbb{Z}/3\mathbb{Z})^4$ acts faithfully on F. An automorphism f of F preserves the lines and induce an automorphism on the Fano surface S denoted by $\rho(f)$. Let G be the group $\rho(A(3, 3, 5))$.

Proposition 0.5. *Let X be the quotient of S by the group G and let $\eta : S \to X$ be the quotient map. The surface X is (isomorphic to) the del Pezzo surface of degree 5 and the cover is branched with index 3 over the 10 (-1)-curves of X.*

Let us prove this Proposition.
Let s be a point of S. Let us denote by $T_{S,s}$ the tangent space of S at s, by $L_s \hookrightarrow F$ the line on F corresponding to s and by

$$P_s \subset \mathbb{C}^5$$

the subjacent plane to the line L_s. The following Proposition is a consequence of the tangent bundle Theorem [3] (see also [9]).

Proposition 0.6. *Let s be a fixed point of an automorphism $\rho(f)$ ($f \in A(3,3,5)$). The plane P_s is stable by the action of f and the eigenvalues of*

$$d\rho(f) : T_{S,s} \to T_{S,s}$$

are equal to the eigenvalues of the restriction of $f \in A(3,3,5)$ to the plane $P_s \subset \mathbb{C}^5$.

Hence we know the action of the differential $d\rho(f)$ on the fixed points of $\rho(f)$. Recall ([9]):

Proposition 0.7. *For $1 \leq i < j \leq 5$, $\beta \in \mu_3$, the hyperplane $\{x_i + \beta x_j = 0\}$ cuts out a cone on F. The curve that parametrizes the lines on this cone is an elliptic curve E_{ij}^{β} that is naturally embedded in the Fano surface S. The configuration of these 30 elliptic curves is:*

$$E_{ij}^{\beta} E_{st}^{\gamma} = \begin{cases} 1 & \text{if } \{i,j\} \cap \{s,t\} = \emptyset \\ -3 & \text{if } E_{ij}^{\beta} = E_{st}^{\gamma} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\alpha \in \mu_3$ be a primitive root. The orbit by G of the curve E_{ij}^1 is $E_{ij}^1 + E_{ij}^{\alpha} + E_{ij}^{\alpha^2}$. Let be $\{i,j\} \cap \{s,t\} = \emptyset$ and let s be the intersection point of E_{ij}^1 and E_{st}^1. The orbit of s by G is the set of the 9 intersection points of the curves E_{ij}^{β} and E_{st}^{γ} ($\beta, \gamma \in \mu_3$). Let I be the set of the 135 intersection points of the 30 elliptic curves and let s be a point of I. The group

$$G_s = \{g \in G / g(s) = s \text{ and } s \text{ is a fixed isolated point of } g\}$$

is isomorphic to μ_3^2 and, by the Proposition 0.6, its representation on the space $T_{S,s}$ is isomorphic to the representation:

$$(\alpha_1, \alpha_2) \in \mu_3^2 \quad (\alpha_1, \alpha_2).(x,y) = (\alpha_1 x, \alpha_2 y) \in \mathbb{C}^2$$

on $\mathbb{C}^2$. The quotient of S by this action is a smooth point [2]. This implies that the surface X is smooth. The ramification index of $\eta : S \to X$ at the points of I is 9 and the ramification index of η on the curve E_{ij}^{β} is 3.
Let us denote by K_V the canonical divisor of a surface V. Let be $\Sigma = \sum_{i,j,\beta} E_{ij}^{\beta}$; the ramification divisor of $\eta : S \to X$ is 2Σ and

$$K_S = \eta^* K_X + 2\Sigma.$$

By [3], we know moreover: $\Sigma = 2K_S$, hence $3^4 (K_X)^2 = (\eta^* K_X)^2 = (-3K_S)^2 = 9.45$ and $(K_X)^2 = 5$.

The stabilisator of an elliptic curve $E_{ij}^{\beta} \hookrightarrow S$ contains 27 elements, the group that fixes each points of E_{ij}^{β} has 3 elements. Let $\eta_{ij}^{\beta} : E_{ij}^{\beta} \to X_{ij}$ be the restriction on E_{ij}^{β} of η. This is a cover of degree 9 ramified over 3 points with ramification index 3. Hence

$$0 = e(E_{ij}^{\beta}) = 9(e(X_{ij}) - 3) + 3.3$$

(here e is the Euler characteristic) and $e(X_{ij}) = 2 :$ X_{ij} is a smooth rational curve. We known moreover that:

$$\eta^* X_{ij} = 3(E_{ij}^1 + E_{ij}^{\alpha} + E_{ij}^{\alpha^2})$$

We deduce that the 10 curves X_{ij} have the following configuration:

$$X_{ij}X_{st}=\begin{cases} 1 & \text{if } \{i,j\}\cap\{s,t\}=\emptyset \\ -1 & \text{if } X_{ij}=X_{st} \\ 0 & \text{otherwise.}\end{cases}$$

Let I' be the 15 points on X image of the 135 points of I and let be $\Sigma'=\sum X_{ij}$. We have

$$3^3=e(S)=3^4e(X-\Sigma')+3^3e(\Sigma'-I')+3^2e(I').$$

As we can verify, $e(\Sigma')=5$ and we obtain $e(X)=7$. We can blow down 4 (-1)-curves among the 10 curves X_{ij} and we obtain a surface with Chern numbers

$$c_1^2=3c_2=9$$

but this surface contains 6 rational curves. Hence, by Theorem 0.2, it cannot be a ball quotient and this is the plane : X is the blow-up of the plane at four points. These points are in general position because of the intersection numbers of the X_{ij}. Hence X is the degree 5 del Pezzo surface $\mathcal{H}_1$ and the X_{ij} are its 10 (-1)-curves.

Moreover, we have proved that the quotient map $S\to X$ is an abelian cover branched over the ten (-1)-curves of X with ramification index 3. By the work of Namba [8] on abelian covers, $\mathcal{H}_3$ is universal among finite abelian covers with such properties. That means that :

Corollary 0.8. *There exists a map $\kappa:\mathcal{H}_3\to S$ of degree 3 that is a quotient of $\mathcal{H}_3$ by a group of order 3.*

Now, let us consider $S'\subset S$ be the complementary of 12 disjoints elliptic curves on S (there are 5 such sets of 12 elliptic curves).

Corollary 0.9. *The logarithmic Chern ratio of S' is 3 : S' is a ball quotient.*

Proof. A canonical divisor K_S of S is ample, moreover $K_S^2=45$ and $K_SE=3$ for an elliptic curve $E\hookrightarrow S$ [3], [9]. As $\bar{c}_2(S')=e(S-D)=e(S)=27>0$ and $(K_S+D)^2=45+2.12.3-12.3=81$, the logarithmic Chern ratio of S' satisfies:

$$\frac{(K_S+D)^2}{e(S-D)}=3.$$

Thus S' is a ball quotient. $\square$

Let us recall the notations

$$\begin{array}{ccc} \mathcal{H}_3 & \stackrel{\iota}{\to} & \mathcal{H}'_3 \\ \downarrow\eta_3 & & \downarrow\pi \\ \mathcal{H}_1 & \stackrel{\tau}{\to} & \mathbb{P}^2. \end{array}$$

The composite of $\kappa:\mathcal{H}_3\to S$ and $\eta:S\to\mathcal{H}_1$ is the map η_3. As this map η_3 is branched with order 3 over the 10 (-1)-curves of $\mathcal{H}_1$, the map κ is étale. Let S' be the complementary of a set of 12 disjoint elliptic curves on S. As S' is a ball quotient and κ is étale, the surface $\mathcal{T}=\kappa^{-1}S'$ is a ball quotient. It remains to find the lattice corresponding to $\mathcal{T}$. To this aim, we take ideas in [10], where Yamazaki and Yoshida computed the lattice of the Ball quotient surface $\mathcal{H}_5$ and we use Namba's results as follows:

Let $b : \mathbb{P}^2 \to \mathbb{N}$ be the function such that $b(p) = 1$ outside the complete quadrilateral, $b(p) = 3$ on the complete quadrilateral minus the 4 triple points $p_1, .., p_4$, and $b(p) = \infty$ on these 4 points. The pair $(\mathbb{P}^2, b)$ is an orbifold that has been studied by Holzapfel and Shiga. The universal cover of that orbifold is $\mathbb{B}_2$ with the transformation group:

$$\Gamma = \{T \in GL(\mathbb{Z}[\alpha])/T \equiv I \,\text{modulo}\, (1-\alpha) \,\text{and}\, {}^t\bar{T}HT = H\}$$

([12], chapter 10, [5], chapter 5). A cover $Z \to \mathbb{P}^2$ with branching index 3 over the complete quadrilateral corresponds to a normal sub group K of Γ and Γ/K is isomorphic to the group of transformation of the covering $Z \to \mathbb{P}^2$. In particular, if $Z \to \mathbb{P}^2$ is an abelian cover, the group K contains the commutator group $[\Gamma, \Gamma]$. By the work of Namba, $\pi : \mathcal{H'}_3 \to \mathbb{P}^2$ is universal among abelian covers of $(\mathbb{P}^2, b)$, thus the lattice of the ball quotient $\mathcal{T}$ is the commutator $[\Gamma, \Gamma]$.

Acknowledgement. I wish to thank Amir Dzambic for stimulating discussions on this paper, and the Max Planck Institute of Bonn, where this research was done.

References

[1] Barth, W., Hulek, K. "Projective models of Shioda modular surfaces", Manus. Math. 50 (1985) 73-132.
[2] Cartan H., "Quotients d'un espace analytique par un groupe d'automorphismes", Algebraic geometry and topology, Princeton Univ. Pr. (1957), 90-102.
[3] Clemens H., Griffiths P., "The intermediate Jacobian of the cubic threefold", Annals of Math. 95 (1972), 281-356.
[4] Hirzebruch F., "Arrangement of lines and algebraic surfaces", Arithmetic and geometry, Vol II, Progress in Math. Vol. 36. pp. 113-140, Birkhauser 1983.
[5] Holzaphel R. "Ball and Surface Arithmetics", Aspects of mathematics, Vieweg, 1998.
[6] Ishida M-N., "Hirzebruch's examples of surfaces of general type with $c_1^2 = 3c_2$", Algebraic Geometry, Tokyo/Kyoto, 1982, 412-431, LNM 1016, Springer, Berlin, 1983.
[7] Kobayashi, R., "Einstein-Kaehler metrics on open algebraic surfaces of general type", Tohoku Math. J. (2) 37 (1985), no. 1, 43-77.
[8] Namba M., "On Branched Coverings of Projective Manifolds", Proc. Japan Acad, 61, Ser. A. (1985).
[9] Roulleau X., "Elliptic curve configurations on Fano surfaces", arXiv, 2008.
[10] Yamazaki, T., Yoshida M., "On Hirzebruch examples of surfaces with $c_1^2 = 3c_2$", Math. Ann. 266, 421-431 (1984).
[11] Sakaï F., "Semi-stable curves on algebraic surfaces and logarithmic pluricanonical maps", Math. Ann. 254, (1980) 89-120.
[12] Yoshida M. "Fuchian Differential equations", Aspects of mathematics, Vieweg, 1987.

roulleau@mpim-bonn.mpg.de

Xavier Roulleau, Max Planck Institute für Mathematik, Vivatgasse 7, 53111 Bonn, Germany.