THE ALGEBRA OF SECONDARY COHOMOLOGY OPERATIONS AND ITS DUAL

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We describe a modification $\mathscr{B}^{\mathbb{F}}$ of the algebra $\mathscr{B}$ of secondary cohomology operations in [1] which is suitable for dualization. The resulting dual object $\mathscr{B}_{\mathbb{F}}$ will be used to give an alternative description of the E_3-term of the Adams spectral sequence, which complements the algorithm described in [1, 2]. All triple Massey products in the Steenrod algebra can be deduced from $\mathscr{B}^{\mathbb{F}}$ or $\mathscr{B}_{\mathbb{F}}$.

In Part I we first recall the notions of pair modules and pair algebras from [2] and give the corresponding dual notions. Next we define the concept of M-algebras and N-coalgebras, where M is a folding system and N an unfolding system. An M-algebra is a variation on the notion of a $[p]$-algebra from [1]. We show that the algebra $\mathscr{B}$ of secondary cohomology operations in [1] gives rise to a comonoid $\mathscr{B}^{\mathbb{F}}$ in the monoidal category of M-algebras, and we describe the dual object $\mathscr{B}_{\mathbb{F}}$, which is a monoid in the monoidal category of N-coalgebras.

In Part II we study the algebraic objects $\mathscr{B}^{\mathbb{F}}$ and $\mathscr{B}_{\mathbb{F}}$ in terms of generators. This way we obtain explicit descriptions which can be used for computations. In particular we characterize algebraically multiplication maps A_ϕ and comultiplication maps A^ψ which determine $\mathscr{B}^{\mathbb{F}}$ and $\mathscr{B}_{\mathbb{F}}$ completely, see sections 15, 16, 17. For the dual object $\mathscr{B}_{\mathbb{F}}$ the inclusion of polynomial algebras $\mathscr{A}_* \subset \mathscr{F}_*$ is crucial. Here $\mathscr{A}_*$ is the Milnor dual of the Steenrod algebra and $\mathscr{F}_*$ is the dual of a free associative algebra.

Part I. Hopf pair algebras and Hopf pair coalgebras representing the algebra of secondary cohomology operations

1. P

We here recall from [1, 2] the notion of pair modules and pair algebras.

Let k be a field (usually it will be actually a prime field $\mathbb{F} = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for some prime p) and let **Mod** be the category of finite dimensional k-modules (i. e. k-vector spaces) and k-linear maps. This is a symmetric monoidal category via the tensor product $A \otimes B$ over k of k-modules A and B. A *pair module* is a homomorphism

$$X = \left(X_1 \xrightarrow{\partial} X_0 \right) \tag{1.1}$$

in **Mod**. We write $\pi_0(X) = \operatorname{coker} \partial$ and $\pi_1(X) = \ker \partial$. A *morphism* $f : X \to Y$ of pair modules is a commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial} \\ X_0 & \xrightarrow{f_0} & Y_0. \end{array}$$

Evidently pair modules with these morphisms form a category $\mathbf{Mod}_*$ and one has functors

$$\pi_0, \pi_1 : \mathbf{Mod}_* \to \mathbf{Mod}.$$

A morphism of pair modules is called a *weak equivalence* if it induces isomorphisms on π_0 and π_1.

Clearly a pair module is the same as a chain complex concentrated in degrees 0 and 1. For two pair modules X and Y the tensor product of the complexes corresponding to them is concentrated in degrees in 0, 1 and 2 and is given by

$$X_1 \otimes Y_1 \xrightarrow{\partial_1} X_1 \otimes Y_0 \oplus X_0 \otimes Y_1 \xrightarrow{\partial_0} X_0 \otimes Y_0 \tag{1.2}$$

with $\partial_0 = (\partial \otimes 1, 1 \otimes \partial)$ and $\partial_1 = \binom{-1\otimes\partial}{\partial\otimes 1}$. Truncating this chain complex we get the pair module

$$X\bar{\otimes}Y = \Big((X\bar{\otimes}Y)_1 = \mathrm{coker}(\partial_1) \xrightarrow{\partial} X_0 \otimes Y_0 = (X\bar{\otimes}Y)_0\Big)$$

with ∂ induced by ∂_0.

(1.3) Remark. Note that the full embedding of the category of pair modules into the category of chain complexes induced by the above identification has a left adjoint Tr_* given by truncation: for a chain complex

$$C_* = \Big(... \to C_2 \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0 \xrightarrow{\partial_{-1}} C_{-1} \to ...\Big),$$

one has

$$\mathrm{Tr}_*(C_*) = \Big(\mathrm{coker}(\partial_1) \xrightarrow{\underline{\partial}_0} C_0\Big),$$

with $\underline{\partial}_0$ induced by ∂_0. Then clearly one has

$$X\bar{\otimes}Y = \mathrm{Tr}_*(X \otimes Y).$$

Using the fact that Tr_* is a reflection onto a full subcategory, one easily checks that the category $\mathbf{Mod}_*$ together with the tensor product $\bar{\otimes}$ and unit $k_* = (0 \to k)$ is a symmetric monoidal category, and Tr_* is a monoidal functor.

Clearly one has $\pi_0(X\bar{\otimes}Y) \cong \pi_0(X) \otimes \pi_0(Y)$. We will also need the following

(1.4) Lemma. *For any pair modules X, Y there is a natural isomorphism*

$$\pi_1(X\bar{\otimes}Y) \cong \pi_1(X)\otimes\pi_0(Y) \oplus \pi_0(X)\otimes\pi_1(Y).$$

Proof. By the Künneth formula, $\pi_1(X)\otimes\pi_0(Y) \oplus \pi_0(X)\otimes\pi_1(Y)$ is isomorphic to the middle homology of the complex (1.2) above. But it is clear that for any homomorphisms $\partial_1 : P \to Q$, $\partial_0 : Q \to R$ with $\partial_0\partial_1 = 0$ the homology $\ker\partial_0/\operatorname{im}\partial_1$ is isomorphic to the kernel of the induced map $\mathrm{coker}(\partial_1) \to R$. In our case this kernel is precisely $\pi_1(X\bar{\otimes}Y)$. □

We next consider the category $\mathbf{Mod}^{\cdot}$ of *graded* modules, i. e. graded objects in **Mod** (graded k-vector spaces $A^{\cdot} = (A^n)_{n\in\mathbb{Z}}$ with upper indices, which in each degree have finite dimension). For graded modules $A^{\cdot}$, $B^{\cdot}$ we define their graded tensor product $A^{\cdot} \otimes B^{\cdot}$ in the usual way, i. e. by

$$(A^{\cdot} \otimes B^{\cdot})^n = \bigoplus_{i+j=n} A^i \otimes B^j.$$

This tensor product has an interchange

$$T_{A^{\cdot},B^{\cdot}} : A^{\cdot} \otimes B^{\cdot} \xrightarrow{\cong} B^{\cdot} \otimes A^{\cdot} \tag{1.5}$$

given on homogeneous elements by $T_{A^{\cdot},B^{\cdot}}(a \otimes b) = (-1)^{\deg(a)\deg(b)} b \otimes a$.

A *graded pair module* is a graded object of $\mathbf{Mod}_*$, i. e. a sequence $X^n = (\partial^n : X_1^n \to X_0^n)$ with $n \in \mathbb{Z}$ of pair modules. We can also identify such a graded pair module $X^{\cdot}$ with the underlying morphism ∂ of degree 0 between graded modules

$$X^{\cdot} = \Big(X_1^{\cdot} \xrightarrow{\partial^{\cdot}} X_0^{\cdot}\Big).$$

Now the tensor product $X^{\cdot}\bar{\otimes}Y^{\cdot}$ of graded pair modules $X^{\cdot}$, $Y^{\cdot}$ is defined by

$$(X^{\cdot}\bar{\otimes}Y^{\cdot})^n = \bigoplus_{i+j=n} X^i\bar{\otimes}Y^j. \tag{1.6}$$

This defines a monoidal structure on the category $\mathbf{Mod}_*^{\cdot}$ of graded pair modules, with morphisms maps of degree 0. Again $\bar{\otimes}$ is symmetric.

For two morphisms $f, g : X^{\cdot} \to Y^{\cdot}$ between graded pair modules, a *homotopy* $H : f \Rightarrow g$ is a morphism $H : X^{\cdot}_0 \to Y^{\cdot}_1$ of degree 0 as in the diagram

(1.7)
$$\begin{array}{ccc} X^{\cdot}_1 & \overset{f_1}{\underset{g_1}{\rightrightarrows}} & Y^{\cdot}_1 \\ {\scriptstyle\partial}\downarrow & \overset{H}{\nearrow} & \downarrow{\scriptstyle\partial} \\ X^{\cdot}_0 & \underset{g_0}{\overset{f_0}{\rightrightarrows}} & Y^{\cdot}_0, \end{array}$$

satisfying $f_0 - g_0 = \partial H$ and $f_1 - g_1 = H\partial$.

A *pair algebra* $B^{\cdot}$ is a monoid in the monoidal category of graded pair modules, with multiplication

$$\mu : B^{\cdot} \bar{\otimes} B^{\cdot} \to B^{\cdot}.$$

We assume that $B^{\cdot}$ is concentrated in nonnegative degrees, that is $B^n = 0$ for $n < 0$.

More explicitly pair algebras can be described as follows.

(1.8) Definition. A *pair algebra* $B^{\cdot}$ is a graded pair module, i. e. an object

$$\partial^{\cdot} : B^{\cdot}_1 \to B^{\cdot}_0$$

in $\mathbf{Mod}^{\cdot}_*$ with $B^n_1 = B^n_0 = 0$ for $n < 0$ such that $B^{\cdot}_0$ is a graded algebra in $\mathbf{Mod}^{\cdot}$, $B^{\cdot}_1$ is a graded $B^{\cdot}_0$-$B^{\cdot}_0$-bimodule, and $\partial^{\cdot}$ is a bimodule homomorphism. Moreover for $x, y \in B^{\cdot}_1$ the equality

$$\partial(x)y = x\partial(y) \tag{1.9}$$

holds in $B^{\cdot}_1$.

It is easy to see that a graded pair algebra $B^{\cdot}$ yields an exact sequence of graded $B^{\cdot}_0$-$B^{\cdot}_0$-bimodules

$$0 \to \pi_1 B^{\cdot} \to B^{\cdot}_1 \xrightarrow{\partial} B^{\cdot}_0 \to \pi_0 B^{\cdot} \to 0 \tag{1.10}$$

where in fact $\pi_0 B^{\cdot}$ is a graded k-algebra, $\pi_1 B^{\cdot}$ is a graded $\pi_0 B^{\cdot}$-$\pi_0 B^{\cdot}$-bimodule, and $B^{\cdot}_0 \to \pi_0 B^{\cdot}$ is a homomorphism of graded k-algebras.

The tensor product of pair algebras has a natural pair algebra structure, as it happens in any symmetric monoidal category.

We are mainly interested in two examples of pair algebras defined below in sections 5 and 6 respectively: the $\mathbb{G}$-*relation pair algebra* $\mathscr{R}$ *of the Steenrod algebra* $\mathscr{A}$ and the *pair algebra* $\mathscr{B}$ *of secondary cohomology operations* deduced from [1, 5.5.2].

By the work of Milnor [7] it is well known that the dual of the Steenrod algebra $\mathscr{A}$ is a polynomial algebra and this fact yields important algebraic properties of $\mathscr{A}$. For this reason we also consider the dual of the $\mathbb{G}$-relation pair algebra $\mathscr{R}$ of $\mathscr{A}$ and the dual of the pair algebra $\mathscr{B}$ of secondary cohomology operations. The duality functor D is studied in the next section.

2. P

This section is exactly dual to the previous one. There is a contravariant self-equivalence of categories

$$D = \mathrm{Hom}_k(_, k) : \mathbf{Mod}^{\mathrm{op}} \to \mathbf{Mod}$$

which carries a vector space V in **Mod** to its dual

$$DV = \mathrm{Hom}_k(V, k).$$

We also denote the dual of V by $V_* = DV$, for example, the dual of the Steenrod algebra $\mathscr{A}$ is $\mathscr{A}_* = D(\mathscr{A})$. We can apply the functor $\mathrm{Hom}_k(_, k)$ to dualize straightforwardly all notions of section 1. Explicitly, one gets:

A *pair comodule* is a homomorphism

$$X = \left(X^1 \xleftarrow{d} X^0\right) \tag{2.1}$$

in **Mod**. We write $\pi^0(X) = \ker d$ and $\pi^1(X) = \operatorname{coker} d$. The dual of a pair module X is a pair comodule

$$\begin{aligned} DX &= \mathrm{Hom}_k(X, k) \\ &= (D\partial : DX_0 \to DX_1) \end{aligned}$$

with $(DX)^i = D(X_i)$. A *morphism* $f : X \to Y$ of pair comodules is a commutative diagram

$$\begin{array}{ccc} X^1 & \xrightarrow{f^1} & Y^1 \\ {\scriptstyle d}\uparrow & & \uparrow{\scriptstyle d} \\ X^0 & \xrightarrow{f_0} & Y^0. \end{array}$$

Evidently pair comodules with these morphisms form a category $\mathbf{Mod}^*$ and one has functors

$$\pi^0, \pi^1 : \mathbf{Mod}^* \to \mathbf{Mod}.$$

which are compatible with the duality functor D, that is, for any pair module X one has

$$\pi_i(DX) = D(\pi_i X) \text{ for } i = 0, 1.$$

A morphism of pair comodules is called a *weak equivalence* if it induces isomorphisms on π^0 and π^1.

Clearly a pair comodule is the same as a cochain complex concentrated in degrees 0 and 1. For two pair comodules X and Y the tensor product of the cochain complexes is concentrated in degrees in 0, 1 and 2 and is given by

$$X^1 \otimes Y^1 \xleftarrow{d^1} X^1 \otimes Y^0 \oplus X^0 \otimes Y^1 \xleftarrow{d^0} X^0 \otimes Y^0$$

with $d^0 = \binom{d\otimes 1}{1\otimes d}$ and $d^1 = (-1 \otimes d, d \otimes 1)$. Cotruncating this cochain complex we get the pair comodule

$$X \bar{\bar{\otimes}} Y = \left((X \bar{\bar{\otimes}} Y)^1 = \ker(d^1) \xleftarrow{d} X^0 \otimes Y^0 = (X \bar{\bar{\otimes}} Y)^0 \right)$$

with d induced by d_0. One readily checks the natural isomorphism

$$D(X \bar{\otimes} Y) \cong DX \bar{\bar{\otimes}} DY. \tag{2.2}$$

(2.3) Remark. Note that the full embedding of the category of pair comodules into the category of cochain complexes induced by the above identification has a right adjoint Tr^* given by cotruncation: for a cochain complex

$$C^* = \left(\ldots \leftarrow C^2 \xleftarrow{d^1} C^1 \xleftarrow{d^0} C^0 \xleftarrow{d^{-1}} C^{-1} \leftarrow \ldots \right),$$

one has

$$\mathrm{Tr}^*(C^*) = \left(\ker(d^1) \xleftarrow{\bar{d}^0} C^0 \right),$$

with $\bar{d}^0$ induced by d^0. Then clearly one has

$$X \bar{\bar{\otimes}} Y = \mathrm{Tr}^*(X \otimes Y).$$

Using the fact that Tr^* is a coreflection onto a full subcategory, one easily checks that the category $\mathbf{Mod}^*$ together with the tensor product $\bar{\bar{\otimes}}$ and unit $k^* = (0 \leftarrow k)$ is a symmetric monoidal category, and Tr^* is a monoidal functor.

We next consider the category $\mathbf{Mod}_\cdot$ of *graded* modules, i. e. graded objects in **Mod** (graded k-vector spaces $A_\cdot = (A_n)_{n\in\mathbb{Z}}$ with lower indices which in each degree have finite dimension). For graded modules $A_\cdot$, $B_\cdot$ we define their graded tensor product $A_\cdot \otimes B_\cdot$ again in the usual way, i. e. by

$$(A_\cdot \otimes B_\cdot)_n = \bigoplus_{i+j=n} A_i \otimes B_j.$$

A *graded pair comodule* is a graded object of $\mathbf{Mod}^*$, i. e. a sequence $X_n = (d_n : X_n^0 \to X_n^1)$ of pair comodules. We can also identify such a graded pair comodule $X_\cdot$ with the underlying morphism d of degree 0 between graded modules

$$X_\cdot = \left(X_\cdot^1 \xleftarrow{d_\cdot} X_\cdot^0 \right).$$

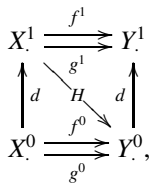
Now the tensor product $X_\cdot \bar{\bar{\otimes}} Y_\cdot$ of graded pair comodules $X_\cdot$, $Y_\cdot$ is defined by

$$(X_\cdot \bar{\bar{\otimes}} Y_\cdot)_n = \bigoplus_{i+j=n} X_i \bar{\bar{\otimes}} Y_j. \tag{2.4}$$

This defines a monoidal structure on the category $\mathbf{Mod}_\cdot$ of graded pair comodules. Morphisms in this category are of degree 0.

For two morphisms $f, g : X_\cdot \to Y_\cdot$ between graded pair comodules, a *homotopy* $H : f \Rightarrow g$ is a morphism $H : X_\cdot^1 \to Y_\cdot^0$ of degree 0 as in the diagram

(2.5)



satisfying $f^0 - g^0 = Hd$ and $f^1 - g^1 = dH$.

A *pair coalgebra* $B_\cdot$ is a comonoid in the monoidal category of graded pair comodules, with the diagonal

$$\delta : B_\cdot \to B_\cdot \bar{\bar{\otimes}} B_\cdot.$$

We assume that $B_\cdot$ is concentrated in nonnegative degrees, that is $B_n = 0$ for $n < 0$.

Of course the duality functor D yields a duality functor

$$D : (\mathbf{Mod}_*^\cdot)^{\mathrm{op}} \to \mathbf{Mod}_\cdot^*$$

which is compatible with the monoidal structure, i. e.

$$D(X^\cdot \bar{\otimes} Y^\cdot) \cong (DX^\cdot) \bar{\bar{\otimes}} (DY^\cdot).$$

We also write $D(X^\cdot) = X_\cdot$.

More explicitly pair coalgebras can be described as follows.

(2.6) Definition. A *pair coalgebra* $B_\cdot$ is a graded pair comodule, i. e. an object

$$d_\cdot : B_\cdot^0 \to B_\cdot^1$$

in $\mathbf{Mod}_\cdot^*$ with $B_n^1 = B_n^0 = 0$ for $n < 0$ such that $B_\cdot^0$ is a graded coalgebra in $\mathbf{Mod}_\cdot$, $B_\cdot^1$ is a graded $B_\cdot^0$-$B_\cdot^0$-bicomodule, and $d_\cdot$ is a bicomodule homomorphism. Moreover the diagram

$$\begin{array}{ccc} B_\cdot^1 & \xrightarrow{\lambda} & B_\cdot^0 \otimes B_\cdot^1 \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle d_\cdot \otimes 1} \\ B_\cdot^1 \otimes B_\cdot^0 & \xrightarrow{1 \otimes d_\cdot} & B_\cdot^1 \otimes B_\cdot^1 \end{array}$$

commutes, where λ, resp. ρ is the left, resp. right coaction.

It is easy to see that there results an exact sequence of graded $B_\cdot^0$-$B_\cdot^0$-bicomodules dual to (1.10)

$$0 \leftarrow \pi^1 B_\cdot \leftarrow B_\cdot^1 \xleftarrow{d_\cdot} B_\cdot^0 \leftarrow \pi^0 B_\cdot \leftarrow 0 \tag{2.7}$$

where in fact $\pi^0 B_\cdot$ is a graded k-coalgebra, $\pi^1 B_\cdot$ is a graded $\pi^0 B_\cdot$-$\pi^0 B_\cdot$-bicomodule, and $B_\cdot^0 \leftarrow \pi^0 B_\cdot$ is a homomorphism of graded k-coalgebras.

One sees easily that the notions in this section correspond to those in the previous section under the duality functor $D = \mathrm{Hom}_k(_, k)$. In particular, D carries (graded) pair algebras to (graded) pair coalgebras.

3. F

In this section we associate to a "right module system" M a category of M-algebras $\mathbf{Alg}_M^r$ which is a monoidal category if M is a "folding system". Our main examples given by the $\mathbb{G}$-relation pair algebra $\mathscr{R}$ of the Steenrod algebra $\mathscr{A}$ and by the pair algebra $\mathscr{B}$ of secondary cohomology operations are in fact comonoids in monoidal categories of such type, see sections 5 and 6. This generalizes the well known fact that the Steenrod algebra $\mathscr{A}$ is a Hopf algebra, i. e. a comonoid in the category of algebras.

(3.1) Definition. Let $\mathbf{A}$ be a subcategory of the category of graded k-algebras. A *right module system* M over $\mathbf{A}$ is an assignment, to each $A \in \mathbf{A}$, of a right A-module $M(A)$, and, to each homomorphism $f : A \to A'$ in $\mathbf{A}$, of a homomorphism $f_* : M(A) \to M(A')$ which is f-equivariant, i. e.

$$f_*(xa) = f_*(x)f(a)$$

for any $a \in A$, $x \in M(A)$. The assignment must be functorial, i. e. one must have $(\mathrm{id}_A)_* = \mathrm{id}_{M(A)}$ for all A and $(fg)_* = f_*g_*$ for all composable f, g.

There are the obvious similar notions of a *left module system* and a *bimodule system* on a category of graded k-algebras $\mathbf{A}$. Clearly any bimodule system can be considered as a left module system and a right module system by forgetting part of the structure.

(3.2) Examples. One obvious example is the bimodule system $\mathbb{1}$ given by $\mathbb{1}(A) = A$, $f_* = f$ for all A and f. Another example is the bimodule system Σ given by the suspension. That is, ΣA is given by the shift

$$\Sigma : A^{n-1} = (\Sigma A)^n$$

($n \in \mathbb{Z}$) which is the identity map denoted by Σ. The bimodule structure for $a, m \in A$ is given by

$$a(\Sigma m) = (-1)^{\deg(a)}\Sigma(am),$$
$$(\Sigma m)a = \Sigma(ma).$$

We shall need the *interchange* of Σ which for graded modules U, V, W is the isomorphism

$$\sigma_{U,V,W} : U \otimes (\Sigma V) \otimes W \xrightarrow{\cong} \Sigma(U \otimes V \otimes W) \tag{3.3}$$

which carries $u \otimes \Sigma v \otimes w$ to $(-1)^{\deg(u)}\Sigma(u \otimes v \otimes w)$.

Clearly a direct sum of module systems is again a module system of the same kind, so that in particular we get a bimodule system $\mathbb{1} \oplus \Sigma$ with $(\mathbb{1} \oplus \Sigma)(A) = A \oplus \Sigma A$.

We are mainly interested in the bimodule system $\mathbb{1}$ and the bimodule system $\mathbb{1} \oplus \Sigma$ which are in fact both folding systems, see (3.15) below.

(3.4) Definition. For a right module system M on the category of algebras $\mathbf{A}$ and an algebra A from $\mathbf{A}$, an *M-algebra of type A* is a pair $D_* = (\partial : D_1 \to D_0)$ with $\pi_0(D_*) = A$ and $\pi_1(D_*) = M(A)$, such that D_0 is a k-algebra, the quotient homomorphism $D_0 \twoheadrightarrow \pi_0 D_* = A$ is a homomorphism of algebras, D_1 is a right D_0-module, ∂ is a homomorphism of right D_0-modules, and the induced structure of a right $\pi_0(D_*)$-module on $\pi_1(D_*)$ conicides with the original right A-module structure on M. For A, A' in $\mathbf{A}$, an M-algebra D_* of type A, and another one D'_* of type A', a morphism $D_* \to D'_*$ of M-pair algebras is defined to be a commutative diagram of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M(A) & \longrightarrow & D_1 & \xrightarrow{\partial} & D_0 & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & M(A') & \longrightarrow & D'_1 & \xrightarrow[\partial']{} & D'_0 & \longrightarrow & A' & \longrightarrow & 0
\end{array}$$

where f_0 is a homomorphism of algebras and f_1 is a right f_0-equivariant k-linear map. It is clear how to compose such morphisms, so that M-algebras form a category which we denote $\mathbf{Alg}^r_M$.

With obvious modifications, we also get notions of M-algebra of type A when M is a left module system or a bimodule system; the corresonding categories of algebras will be denoted by $\mathbf{Alg}^\ell_M$ and $\mathbf{Alg}^b_M$, respectively. Moreover, for a bimodule system M there is also a further full subcategory

$$\mathbf{Alg}^{pair}_M \subset \mathbf{Alg}^b_M$$

whose objects, called *M-pair algebras* are those M-algebras which satisfy the pair algebra equation $(\partial x)y = x\partial y$ for all $x, y \in D_1$.

(3.5) Remark. Note that if $\mathbf{A}$ contains k, then $\mathbf{Alg}^?_M$ has an initial object given by the M-algebra $I = (0 : M(k) \to k)$ of type k. Moreover if $\mathbf{A}$ contains the trivial algebra 0, then $\mathbf{Alg}^?_M$ also has a terminal object — the M-algebra $0 = M(0) \to 0$ of type 0. Here ? stands for ℓ, r or b if M is a left- right- or bimodule system, respectively.

(3.6) Definition. Let **A** be a category of graded algebras as above which in addition is closed under tensor product, i. e. k belongs to **A** and for any A, A' from **A** the algebra $A \otimes_k A'$ also belongs to **A**. A *right folding system* on **A** is then defined to be a right module system M on **A** together with the collection of right $A \otimes_k A'$-module homomorphisms

$$\lambda_{A,A'} : A \otimes_k M(A') \to M(A \otimes_k A'),$$
$$\rho_{A,A'} : M(A) \otimes_k A' \to M(A \otimes_k A')$$

for all A, A' in **A** which are natural in the sense that for any homomorphisms $f : A \to A_1$, $f' : A' \to A'_1$ in **A** the diagrams

(3.7)
$$\begin{array}{ccc} A \otimes_k M(A') & \xrightarrow{\lambda_{A,A'}} & M(A \otimes_k A') \\ {\scriptstyle f\otimes f'_*}\downarrow & & \downarrow{\scriptstyle (f\otimes f')_*} \\ A_1 \otimes_k M(A'_1) & \xrightarrow{\lambda_{A_1,A'_1}} & M(A_1 \otimes_k A'_1) \end{array} \quad , \qquad \begin{array}{ccc} M(A) \otimes_k A' & \xrightarrow{\rho_{A,A'}} & M(A \otimes_k A') \\ {\scriptstyle f_*\otimes f'}\downarrow & & \downarrow{\scriptstyle (f\otimes f')_*} \\ M(A_1) \otimes_k A'_1 & \xrightarrow{\rho_{A_1,A'_1}} & M(A_1 \otimes_k A'_1) \end{array}$$

commute. Moreover the homomorphisms

(3.8)
$$\lambda_{k,A} : k \otimes_k M(A) \to M(k \otimes_k A),$$
$$\rho_{A,k} : M(A) \otimes_k k \to M(A \otimes_k k)$$

must coincide with the obvious isomorphisms and the diagrams

(3.9)
$$\begin{array}{ccc} & A \otimes_k M(A' \otimes_k A'') & \\ {\scriptstyle 1\otimes\lambda_{A',A''}}\nearrow & & \searrow{\scriptstyle \lambda_{A,A'\otimes_k A''}} \\ A \otimes_k A' \otimes_k M(A'') & \xrightarrow{\lambda_{A\otimes_k A',A''}} & M(A \otimes_k A' \otimes_k A''), \end{array}$$

(3.10)
$$\begin{array}{ccc} & M(A \otimes_k A') \otimes_k A'' & \\ {\scriptstyle \rho_{A,A'}A\otimes 1}\nearrow & & \searrow{\scriptstyle \rho_{A\otimes_k A',A''}} \\ M(A) \otimes_k A' \otimes_k A'' & \xrightarrow{\rho_{A,A'\otimes_k A''}} & M(A \otimes_k A' \otimes_k A''), \end{array}$$

(3.11)
$$\begin{array}{ccccc} & & M(A \otimes_k A') \otimes_k A'' & & \\ & {\scriptstyle \lambda_{A,A'}\otimes 1}\nearrow & & \searrow{\scriptstyle \rho_{A\otimes_k A',A''}} & \\ A \otimes_k M(A') \otimes_k A'' & & & & M(A \otimes_k A' \otimes_k A'') \\ & {\scriptstyle 1\otimes\rho_{A',A''}}\searrow & & \nearrow{\scriptstyle \lambda_{A,A'\otimes_k A''}} & \\ & & A \otimes_k M(A' \otimes_k A'') & & \end{array}$$

must commute for all A, A', A'' in **A**. A folding system is called *symmetric* if in addition the diagrams

$$\begin{array}{ccc} A \otimes_k M(A') & \xrightarrow{\lambda_{A,A'}} & M(A \otimes_k A') \\ {\scriptstyle T_{A,M(A')}}\downarrow & & \downarrow{\scriptstyle M(T_{A,A'})} \\ M(A') \otimes_k A & \xrightarrow{\rho_{A',A}} & M(A' \otimes_k A) \end{array}$$

commute for all A, A', where T is the graded interchange operator given in (1.5).

Once again, we have the corresponding obvious notions of a left folding system and a bifolding system.

For a right folding system M, the category $\mathbf{Alg}^r_M$ has a monoidal structure given by the *folding product* $\hat{\otimes}$ below. Given an M-algebra D of type A and another one, D' of type A', we define an M-pair algebra

$D\hat{\otimes}D'$ of type $A \otimes A'$ as the lower row in the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A\bar{\otimes}M(A') \oplus M(A)\bar{\otimes}A' & \longrightarrow & (D\bar{\otimes}D')_1 & \xrightarrow{\partial_{\bar{\otimes}}} & (D\bar{\otimes}D')_0 & \longrightarrow & A \otimes A' \longrightarrow 0 \\
 & & \downarrow{\scriptstyle (\lambda_{A,A'},\rho_{A,A'})} & & \downarrow & & \| & & \| \\
0 & \longrightarrow & M(A \otimes A') & \longrightarrow & (D\hat{\otimes}D')_1 & \xrightarrow[\partial_{\hat{\otimes}}]{} & D_0 \otimes D'_0 & \longrightarrow & A \otimes A' \longrightarrow 0.
\end{array}
\tag{3.12}
$$

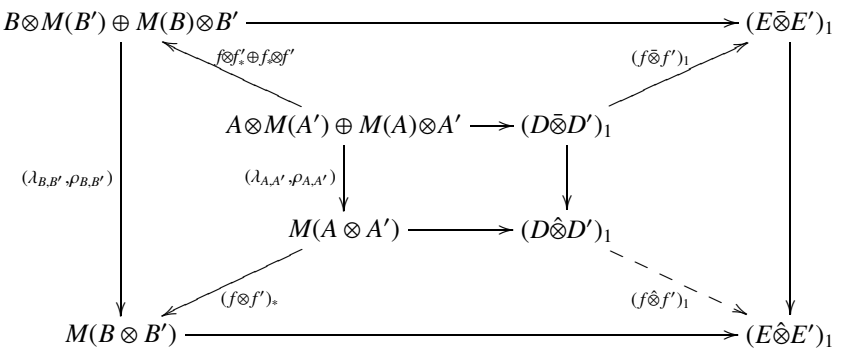
(push)

Here the leftmost square is required to be pushout, and the upper row is exact by (1.4).

(3.13) Proposition. *For any right (resp. left, bi-) folding system M, the folding product defines a monoidal structure on $\mathbf{Alg}^r_M$ (resp. $\mathbf{Alg}^\ell_M$, $\mathbf{Alg}^b_M$, $\mathbf{Alg}^{pair}_M$), with unit object $I = (0 : M(k) \to k)$. If moreover the folding system is symmetric, then this monoidal structure is symmetric.*

We only will use the monoidal categories $\mathbf{Alg}^r_{\mathbb{1}\oplus\Sigma}$ and $\mathbf{Alg}^{pair}_{\mathbb{1}}$.

Proof. To begin with, let us show that $\hat{\otimes}$ is functorial, i. e. let us for any morphisms $f : D \to E$, $f' : D' \to E'$ in $\mathbf{Alg}_M$, define a morphism $f\hat{\otimes}f' : D\hat{\otimes}E \to D'\hat{\otimes}E'$ in a way compatible with identities and composition. We put $(f\hat{\otimes}f')_0 = f_0\hat{\otimes}f'_0$ and define $(f\hat{\otimes}f')_1$ as the unique homomorphism making the following diagram commute:



where the left hand trapezoid commutes by (3.7). Using universality of pushout it is clear that right equivariance of f_1 and f'_1 implies that of $(f\hat{\otimes}f')_1$ so that this indeed defines a morphism in $\mathbf{Alg}_M$. The same universality implies compatibility with composition.

Next to show that $I = (0 : M(k) \to k)$ is a unit object first note that for an M-algebra D by (1.3) one has

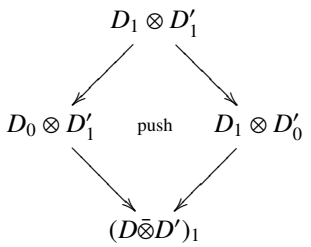
$$
I\bar{\otimes}D = \mathrm{Tr}_*\left(M(k) \otimes D_1 \xrightarrow{\binom{0}{1\otimes\partial}} D_1 \oplus M(k)\otimes D_0 \xrightarrow{(\partial,0)} D_0\right) \cong \left(D_1 \oplus M(k)\otimes A \xrightarrow{(\partial,0)} D_0\right).
$$

From this using (3.8) it is easy to see that $(I\hat{\otimes}D)_1$ is given by the pushout

$$
\begin{array}{ccc}
M(A) \oplus M(k)\otimes A & \xrightarrow{\mathrm{incl}\oplus 1} & D_1 \oplus M(k)\otimes A \\
\downarrow{\scriptstyle \mathrm{proj}} & & \downarrow \\
M(A) & \longrightarrow & (I\hat{\otimes}D)_1
\end{array}
$$

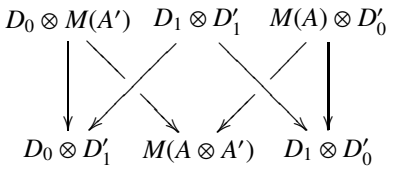
so that there is a canonical isomorphism $(I\hat{\otimes}D)_1 \cong D_1$ compatible with the canonical isomorphism $k\otimes D_0 \cong D_0$. Symmetrically one constructs the isomorphism $D\hat{\otimes}I \cong D$.

Turning now to associativity, first note that the tensor product (1.2) can be equivalently stated as defining $(D\bar{\otimes}D')_1$ by the requirement that the diagram

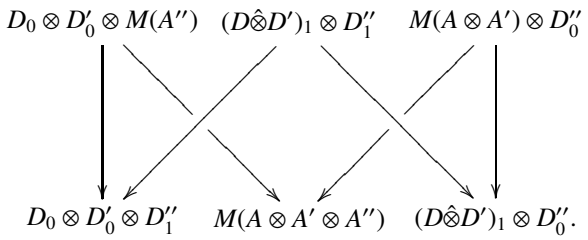


be pushout. Then combining diagrams we see that $(D\hat{\otimes}D')_1$ can be equivalently defined as the colimit of the following diagram:

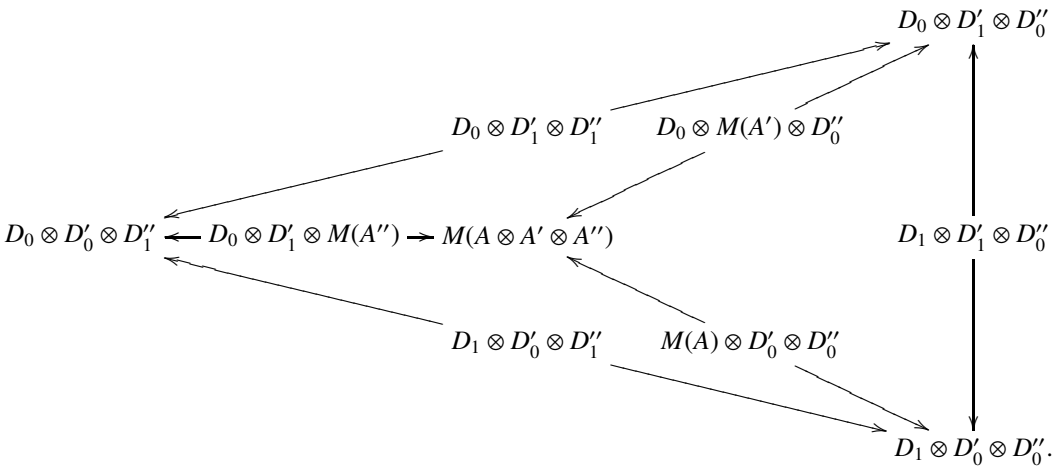
(3.14)



where the map $D_0 \otimes M(A') \to M(A \otimes A')$ is the composite $D_0 \otimes M(A') \to A \otimes M(A') \to M(A \otimes A')$ and similarly for $M(A) \otimes D'_0 \to M(A \otimes A')$. Hence $((D\hat{\otimes}D')\hat{\otimes}D'')_1$ is given by the colimit of the diagram

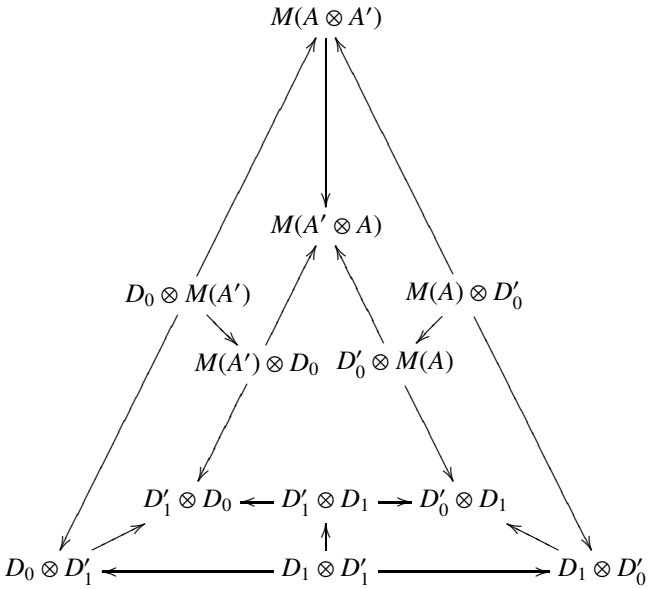


Substituting here the diagram for $(D\hat{\otimes}D')_1$ we obtain that this is the same as colimit of a diagram of the form



Treating now $(D\hat{\otimes}(D'\hat{\otimes}D''))_1$ in the same way we obtain that it is colimit of a diagram with same objects; then, using (3.9), (3.11), and (3.10), one can see that also morphisms in these diagrams are the same.

Finally suppose that M is a symmetric folding system. Then for any M-algebras D, D' of type A, A' respectively, there is a commutative diagram



which induces a map from the colimit of the outer triangle to that of the inner one, i. e. by (3.14) a map $(D\hat{\otimes}D')_1 \to (D'\hat{\otimes}D)_1$. It is then straightforward to check that this defines an interchange for the monoidal structure. □

(3.15) Examples. The bimodule system $\mathbb{1}$ above clearly has a structure of a folding system, with λ and ρ both identity maps. Also the bimodule system $\mathbb{1} \oplus \Sigma$ is a folding system via the obvious isomorphisms

$$\lambda_{A,A'} : A \otimes (A' \oplus \Sigma A') \cong A\otimes A' \oplus A\otimes\Sigma A' \xrightarrow{1\oplus\sigma} A\otimes A' \oplus \Sigma(A\otimes A'), \tag{3.16}$$

$$\rho_{A,A'} : (A \oplus \Sigma A) \otimes A' \cong A\otimes A' \oplus (\Sigma A)\otimes A' \cong A\otimes A' \oplus \Sigma(A\otimes A') \tag{3.17}$$

where in (3.16), the interchange (3.3) for Σ is used.

(3.18) Lemma. *The isomorphisms* (3.16)*,* (3.17) *equip the bimodule system* $\mathbb{1} \oplus \Sigma$ *with the structure of a symmetric folding system on any category* **A** *of algebras closed under tensor products.*

Proof. It is obvious that $\mathbb{1}$ with the identity maps is a folding system, and that a direct sum of folding systems is a folding system again, so it suffices to show that Σ is a folding system.

The right diagram in (3.7) is trivially commutative, while commutativity of the left one follows from

$$\begin{aligned}\sigma_{A_1,M(A'_1)}(f(a) \otimes \Sigma f'(a')) &= (-1)^{\deg(a)}\Sigma(f(a) \otimes f'(a')) \\ &= \Sigma(f \otimes f')((-1)^{\deg(a)}\Sigma(a \otimes a')) = \Sigma(f \otimes f')\sigma_{A,M(A')}(a \otimes \Sigma a')\end{aligned}$$

for any $a \in A$, $a' \in A'$, $f : A \to A_1$, $f' : A' \to A'_1$. Next, the diagrams (3.8) commute since k is concentrated in degree 0.

The diagrams (3.10) commute trivially, as only right actions are involved. Commutativity of (3.9) follows from the obvious equality

$$(-1)^{\deg(a)}\Sigma(a \otimes (-1)^{\deg(a')}a' \otimes a'') = (-1)^{\deg(a\otimes a')}\Sigma(a \otimes a' \otimes a'')$$

and that of (3.11) is also obvious from

$$\begin{array}{ccc} & (-1)^{\deg(a)}\Sigma(a\otimes a')\otimes a'' & \\ \nearrow & & \searrow \\ a\otimes\Sigma(a')\otimes a'' & & (-1)^{\deg(a)}\Sigma(a\otimes a'\otimes a'') \\ \searrow & & \nearrow \\ & a\otimes\Sigma(a'\otimes a'') & \end{array}$$

□

Thus by (3.13) the folding system $\mathbb{1}\oplus\Sigma$ yields a well-defined monoidal category $\mathbf{Alg}^r_{\mathbb{1}\oplus\Sigma}$ of $\mathbb{1}\oplus\Sigma$-*algebras* as in (3.4). The initial object and at the same time the unit for the monoidal structure of $\mathbf{Alg}^r_{\mathbb{1}\oplus\Sigma}$ is by (3.5) and (3.13)

$$I_{\mathbb{1}\oplus\Sigma} = \left(\mathbb{F}\oplus\Sigma\mathbb{F}\xrightarrow{0}\mathbb{F}\right).$$

For $\mathbf{Alg}^r_{\mathbb{1}}$ it is

$$I_{\mathbb{1}} = \left(\mathbb{F}\xrightarrow{0}\mathbb{F}\right).$$

The projections $q : A\oplus\Sigma A\to A$ can be used to construct a monoidal functor

$$q : \mathbf{Alg}^r_{\mathbb{1}\oplus\Sigma}\to\mathbf{Alg}^r_{\mathbb{1}} \tag{3.19}$$

carrying an object D in $\mathbf{Alg}^r_{\mathbb{1}\oplus\Sigma}$ to the pushout in the following diagram

$$\begin{array}{ccccccc} A\oplus\Sigma A & \rightarrowtail & D_1 & \longrightarrow & D_0 & \twoheadrightarrow & A \\ {\scriptstyle q}\downarrow & \text{push} & \downarrow & & \| & & \| \\ A & \rightarrowtail & q(D)_1 & \longrightarrow & q(D)_0 & \twoheadrightarrow & A. \end{array}$$

Evidently $q(I_{\mathbb{1}\oplus\Sigma}) = I_{\mathbb{1}}$.

4. U

It is clear how to dualize the constructions from the previous section along the lines of section 2. We will not give detailed definitions but only briefly indicate the underlying structures.

We thus consider a category $\mathbf{C}$ of graded k-coalgebras, and define a right comodule system N on $\mathbf{C}$ as an assignment, to each coalgebra C in $\mathbf{C}$, of a C-comodule $N(C)$, and to each homomorphism $f : C\to C'$ of coalgebras of an f-equivariant homomorphism $f_* : N(C)\to N(C')$, i. e. the diagram

$$\begin{array}{ccc} N(C) & \xrightarrow{\text{coaction}} & N(C)\otimes C \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*\otimes f} \\ N(C') & \xrightarrow{\text{coaction}} & N(C')\otimes C' \end{array}$$

is required to commute. Similarly one defines left comodule systems and bicomodule systems. As before, we have a bicomodule system $\mathbb{1}$ given by $\mathbb{1}(C) = C$ and also Σ, $\mathbb{1}\oplus\Sigma$ defined dually to (3.2).

Then further for a right comodule system N on $\mathbf{C}$ and for a coalgebra C from $\mathbf{C}$ one defines an N-coalgebra of type C by dualizing (3.4). It is thus a pair $D^* = (d : D^0\to D^1)$ where D^0 is a coalgebra, D^1 is a right D^0-comodule and d is a comodule homomorphism. Moreover one must have $\pi^0(D^*) = C$, $\pi^1(D^*) = N(C)$, and the C-comodule structure on $N(C)$ induced by this must be the one coming from the comodule system N. With morphisms defined dually to (3.4), the N-coalgebras form a category $\mathbf{Coalg}^r_N$. Similarly one defines categories $\mathbf{Coalg}^\ell_N$ and $\mathbf{Coalg}^{pair}_N\subset\mathbf{Coalg}^b_N$ for a left, resp. bicomodule system N. These categories have the initial object $0 : 0\to N(0)$ and the terminal object $0 : k\to N(k)$.

Also dually to (3.6) one defines *unfolding systems* as comodule systems N equipped with $C \otimes C'$-comodule homomorphisms

$$l^{C,C'} : N(C \otimes C') \to C \otimes N(C')$$
$$r^{C,C'} : N(C \otimes C') \to N(C) \otimes C'$$

for all $C, C' \in \mathbf{C}$ required to satisfy obvious duals to the diagrams (3.7) – (3.11). Also there is an obvious notion of a symmetric unfolding system.

Then for an unfolding system N we can dualize (3.12) to obtain definition of the *unfolding product* $D \check{\otimes} D'$ of N-coalgebras via the upper row in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C \otimes C' & \longrightarrow & D^0 \otimes D'^0 & \xrightarrow{d^{\check{\otimes}}} & (D \check{\otimes} D')^1 & \longrightarrow & N(C \otimes C') & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow & \text{pull} & \downarrow{\scriptstyle \binom{l^{C,C'}}{r^{C,C'}}} & & \\
0 & \longrightarrow & C \otimes C' & \longrightarrow & (D \bar{\check{\otimes}} D')^0 & \xrightarrow{d^{\bar{\check{\otimes}}}} & (D \bar{\check{\otimes}} D')^1 & \longrightarrow & C \otimes N(C') \oplus N(C) \otimes C' & \longrightarrow & 0
\end{array}$$

where now the rightmost square is required to be pullback and the lower row is exact by the dual of (1.4).

It is then straightforward to dualize (3.13), so we conclude that for any unfolding system N the unfolding product equips the category $\mathbf{Coalg}^{?}_{N}$ with the structure of a monoidal category, symmetric if N is symmetric. Here, "?" stands for "r", "l", "b" or "pair", according to the type of N. Obviously also the dual of (3.18) holds, so that the categories $\mathbf{Coalg}^{pair}_{\mathbb{1}}$ and $\mathbf{Coalg}^{r}_{\mathbb{1}\oplus\Sigma}$ have monoidal structures given by the unfolding product.

5. T $\mathbb{G}$- S

Fix a prime p and let $\mathbb{G} = \mathbb{Z}/p^2\mathbb{Z}$ be the ring of integers mod p^2, with the quotient map $\mathbb{G} \twoheadrightarrow \mathbb{F} = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Let $\mathscr{A}$ be the mod p Steenrod algebra and let

$$\mathrm{E}_{\mathscr{A}} = \begin{cases} \left\{\mathrm{Sq}^1, \mathrm{Sq}^2, ...\right\} & \text{for } p = 2, \\ \left\{\mathcal{P}^1, \mathcal{P}^2, ...\right\} \cup \left\{\beta, \beta\mathcal{P}^1, \beta\mathcal{P}^2, ...\right\} & \text{for odd } p \end{cases}$$

be the set of generators of the algebra $\mathscr{A}$. We consider the following algebras and homomorphisms

$$\begin{array}{ccccc}
q : \mathscr{B}_0 & \twoheadrightarrow & \mathscr{F}_0 & \xrightarrow{q_{\mathscr{F}}} & \mathscr{A} \\
\| & & \| & & \\
T_{\mathbb{G}}(\mathrm{E}_{\mathscr{A}}) & & T_{\mathbb{F}}(\mathrm{E}_{\mathscr{A}}) & &
\end{array} \quad . \tag{5.1}$$

Here $T_k(S)$ for a commutative ring k denotes the free associative k-algebra with unit generated by the set S, i. e. the tensor algebra of the free k-module on S. The map $q_{\mathscr{F}}$ is the algebra homomorphism which is the identity on $E_{\mathscr{A}}$. For $f \in \mathscr{F}_0$ we denote the element $q_{\mathscr{F}}(f) \in \mathscr{A}$ by

$$\bar{f} = q_{\mathscr{F}}(f).$$

Let $R_{\mathscr{B}}$ denote the kernel of q, i. e. there is a short exact sequence

$$R_{\mathscr{B}} \rightarrowtail \mathscr{B}_0 \xrightarrow{q} \mathscr{A}.$$

This short exact sequence gives a long exact sequence

$$\mathrm{Tor}(R_{\mathscr{B}}, \mathbb{F}) \rightarrowtail \mathrm{Tor}(\mathscr{B}_0, \mathbb{F}) \longrightarrow \mathrm{Tor}(\mathscr{A}, \mathbb{F}) \xrightarrow{i} R_{\mathscr{B}} \otimes \mathbb{F} \longrightarrow \mathscr{B}_0 \otimes \mathbb{F} \twoheadrightarrow \mathscr{A} \otimes \mathbb{F}.$$

Here $A \otimes \mathbb{F} \cong A/pA$ and $\mathrm{Tor}(A, \mathbb{F})$ is just the p-torsion part of A for an abelian group A, so the connecting homomorphism i sends $a = q(b) + p\mathscr{B}_0$ to $pb + pR_{\mathscr{B}}$. It follows that the second homomorphism in the above sequence is zero. Moreover clearly we can identify $\mathscr{B}_0 \otimes \mathbb{F} = \mathscr{F}_0$ and $\mathrm{Tor}(\mathscr{A}, \mathbb{F}) = \mathscr{A}$, so that there

is an exact sequence

$$\begin{array}{ccccccc} \mathscr{A} & \xrightarrow{i} & \mathscr{R}_1^{\mathbb{F}} & \xrightarrow{\partial} & \mathscr{R}_0^{\mathbb{F}} & \twoheadrightarrow & \mathscr{A} \\ & & \| & & \| & & \\ & & R_{\mathscr{B}} \otimes \mathbb{F} & & \mathscr{F}_0 & & \end{array} \tag{5.2}$$

One has

(5.3) Lemma. *The pair $\mathscr{R}^{\mathbb{F}} = (\partial : \mathscr{R}_1^{\mathbb{F}} \to \mathscr{R}_0^{\mathbb{F}})$ above has a pair algebra structure compatible with the standard bimodule structure of $\mathscr{A}$ on itself, so that $\mathscr{R}^{\mathbb{F}}$ yields an object in $\mathbf{Alg}_{\mathbb{1}}^{pair}$, see* (3.4)*.*

Proof. Clearly mod p reduction of any pair algebra over $\mathbb{G}$ is a pair algebra over $\mathbb{F}$. Then let $\mathscr{R}^{\mathbb{F}}$ be the mod p reduction of the pair algebra $R_{\mathscr{B}} \rightarrowtail \mathscr{B}_0$. Thus the $\mathscr{F}_0$-$\mathscr{F}_0$-bimodule structure on $\mathscr{R}_1^{\mathbb{F}} = R_{\mathscr{B}}/pR_{\mathscr{B}}$ is just the mod p reduction of the $\mathscr{B}_0$-$\mathscr{B}_0$-bimodule structure on $R_{\mathscr{B}}$, i. e. $b' + p\mathscr{B}_0 \in \mathscr{R}_0^{\mathbb{F}} = \mathscr{B}_0/p\mathscr{B}_0$ acts on $r + pR_{\mathscr{B}} \in \mathscr{R}_1^{\mathbb{F}} = R_{\mathscr{B}}/pR_{\mathscr{B}}$ via

$$(b' + p\mathscr{B}_0)(r + pR_{\mathscr{B}}) = b'r + pR_{\mathscr{B}}.$$

Moreover the above inclusion $\mathscr{A} \rightarrowtail R_{\mathscr{B}}/pR_{\mathscr{B}}$ sends an element $q(b)$ to $pb + pR_{\mathscr{B}}$. Then the action of $a' = q(b') \in \mathscr{A}$ on $i(a) = pb + pR_{\mathscr{B}} \in i(\mathscr{A}) = \ker \partial$ induced by this pair algebra is given as follows:

$$a'i(a) = q_{\mathscr{F}}(b' + p\mathscr{B}_0)(pb + pR_{\mathscr{B}}) = pb'b + pR_{\mathscr{B}} = iq(b'b) = i(a'a)$$

and similarly for the right action. □

We call the object $\mathscr{R}^{\mathbb{F}}$ of the category $\mathbf{Alg}_{\mathbb{1}}^{pair}$ the $\mathbb{G}$-*relation pair algebra of* $\mathscr{A}$.

(5.4) Theorem. *The $\mathbb{1}$-pair algebra $\mathscr{R}^{\mathbb{F}}$ has a structure of a cocommutative comonoid in the symmetric monoidal category $\mathbf{Alg}_{\mathbb{1}}^{pair}$.*

Proof. For $n \geqslant 0$, let $R_{\mathscr{B}}^{(n)}$ denote the kernel of the map $q^{\otimes n}$, so that there is a short exact sequence

$$R_{\mathscr{B}}^{(n)} \rightarrowtail \mathscr{B}_0^{\otimes n} \xrightarrow{q^{\otimes n}} \mathscr{A}^{\otimes n}$$

and similarly to (5.3) there is a pair algebra of the form

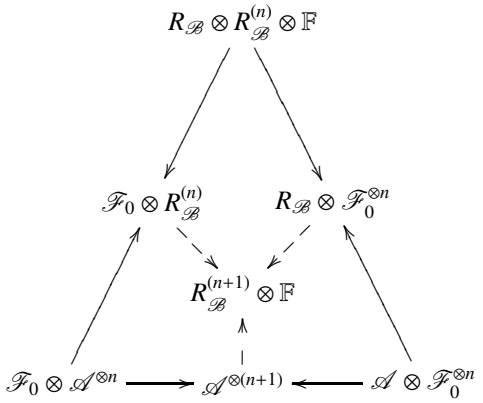
$$\mathscr{A}^{\otimes n} \rightarrowtail R_{\mathscr{B}}^{(n)} \otimes \mathbb{F} \longrightarrow \mathscr{F}_0^{\otimes n} \xrightarrow{q_{\mathscr{F}}^{\otimes n}} \mathscr{A}^{\otimes n}$$

determining an object $\mathscr{R}^{(n)}$ in $\mathbf{Alg}_{\mathbb{1}}^{pair}$. Then one has the following lemma which yields natural examples of folding products in $\mathbf{Alg}_{\mathbb{1}}^{pair}$.

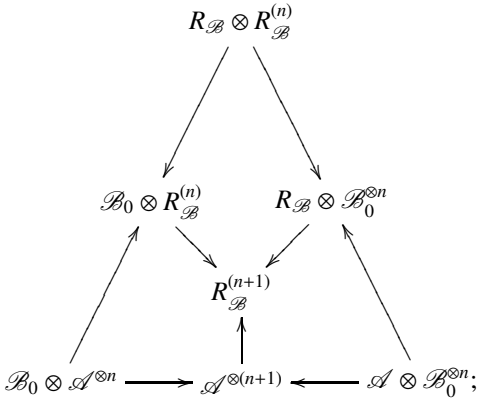
(5.5) Lemma. *There is a canonical isomorphism $\mathscr{R}^{(n)} \cong (\mathscr{R}^{\mathbb{F}})^{\hat{\otimes} n}$ in $\mathbf{Alg}_{\mathbb{1}}^{pair}$.*

Proof. Using induction, we will assume given an isomorphism $\alpha_n : (\mathscr{R}^{\mathbb{F}})^{\hat{\otimes} n} \cong \mathscr{R}^{(n)}$ and construct α_{n+1} in a canonical way. To do this it clearly suffices to construct a canonical isomorphism $\mathscr{R}^{\mathbb{F}} \hat{\otimes} \mathscr{R}^{(n)} \cong \mathscr{R}^{(n+1)}$ as then its composite with $\mathscr{R}^{\mathbb{F}} \hat{\otimes} \alpha_n$ will give α_{n+1}.

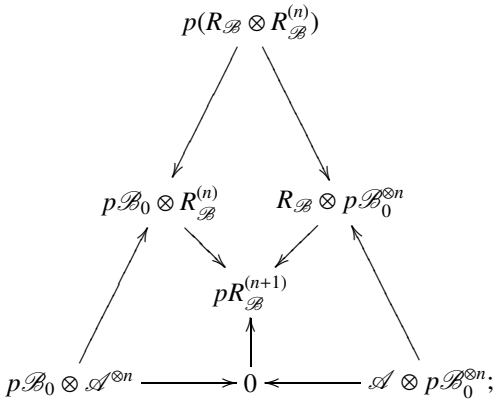
To construct a map $(\mathscr{R}^{\mathbb{F}}\hat{\otimes}\mathscr{R}^{(n)})_1 \to \mathscr{R}_1^{(n+1)}$ means by (3.14) the same as to find three dashed arrows making the diagram



commute. For this we use the commutative diagram

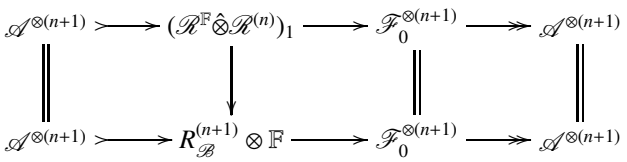


This diagram has a commutative subdiagram



It is obvious that quotient by this subdiagram gives us a diagram of the kind we need.

We thus obtain a map $(\mathscr{R}^{\mathbb{F}}\hat{\otimes}\mathscr{R}^{(n)})_1 \to R^{(n+1)}_{\mathscr{B}} \otimes \mathbb{F}$. Moreover by its construction this map fits into the commutative diagram



with exact rows, hence by the five lemma it is an isomorphism. □

Using the lemma we next construct the diagonal of $\mathscr{R}^{\mathbb{F}}$ given by

$$\begin{array}{ccccccccc}
R_{\mathscr{B}} \otimes \mathbb{F} & = & \mathscr{R}_1^{\mathbb{F}} & \xrightarrow{\partial} & \mathscr{R}_0^{\mathbb{F}} & = & \mathscr{F}_0 \\
\downarrow{\scriptstyle \Delta^{\mathbb{G}}\otimes 1} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} \\
R_{\mathscr{B}}^{(2)} \otimes \mathbb{F} & \xrightarrow{\cong} & (\mathscr{R}^{\mathbb{F}} \hat{\otimes} \mathscr{R}^{\mathbb{F}})_1 & \xrightarrow{\partial_{\otimes}} & \mathscr{R}_0^{\mathbb{F}} \otimes \mathscr{R}_0^{\mathbb{F}} & = & \mathscr{F}_0 \otimes \mathscr{F}_0.
\end{array}$$

Here $\Delta^{\mathbb{G}}$ is defined by the commutative diagram

$$\begin{array}{ccc}
R_{\mathscr{B}} & \hookrightarrow & \mathscr{B}_0 \\
\downarrow{\scriptstyle \Delta^{\mathbb{G}}} & & \downarrow{\scriptstyle \Delta^{\mathbb{G}}} \\
R_{\mathscr{B}}^{(2)} & \hookrightarrow & \mathscr{B}_0 \otimes \mathscr{B}_0,
\end{array} \tag{5.6}$$

where the diagonal $\Delta^{\mathbb{G}}$ on $\mathscr{B}_0$ is defined on generators by

$$\left.\begin{aligned}
\Delta^{\mathbb{G}}(\mathrm{Sq}^n) &= \sum_{i=0}^{n} \mathrm{Sq}^i \otimes \mathrm{Sq}^{n-i} && \text{for } p = 2, \\
\Delta^{\mathbb{G}}(\beta) &= \beta \otimes 1 + 1 \otimes \beta, \\
\Delta^{\mathbb{G}}(\mathcal{P}^n) &= \sum_{i+j=n} \mathcal{P}^i \otimes \mathcal{P}^j, \\
\Delta^{\mathbb{G}}(\mathcal{P}^n_\beta) &= \sum_{i+j=n} (\mathcal{P}^i_\beta \otimes \mathcal{P}^j + \mathcal{P}^i \otimes \mathcal{P}^j_\beta)
\end{aligned}\right\} \text{for odd } p$$

(with $\mathrm{Sq}^0 = 1$, $\mathcal{P}^0 = 1$ as usual) and extended to the whole $\mathscr{B}_0$ as the unique algebra homomorphism with respect to the algebra structure on $\mathscr{B}_0 \otimes \mathscr{B}_0$ given by the nonstandard interchange formula

$$\begin{array}{ccccc}
 & & \mathscr{B}_0 \otimes \mathscr{B}_0 \otimes \mathscr{B}_0 \otimes \mathscr{B}_0 & & \\
 & \nearrow{\scriptstyle 1\otimes T^{\mathbb{G}}\otimes 1} & & \searrow{\scriptstyle \mu\otimes\mu} & \\
\mathscr{B}_0 \otimes \mathscr{B}_0 \otimes \mathscr{B}_0 \otimes \mathscr{B}_0 & & \xrightarrow{\mu_{\otimes}} & & \mathscr{B}_0 \otimes \mathscr{B}_0
\end{array}$$

with

$$\begin{aligned}
T^{\mathbb{G}} &: \mathscr{B}_0 \otimes \mathscr{B}_0 \xrightarrow{\cong} \mathscr{B}_0 \otimes \mathscr{B}_0 \\
T^{\mathbb{G}}(x \otimes y) &= (-1)^{p \deg(x) \deg(y)} y \otimes x.
\end{aligned}$$

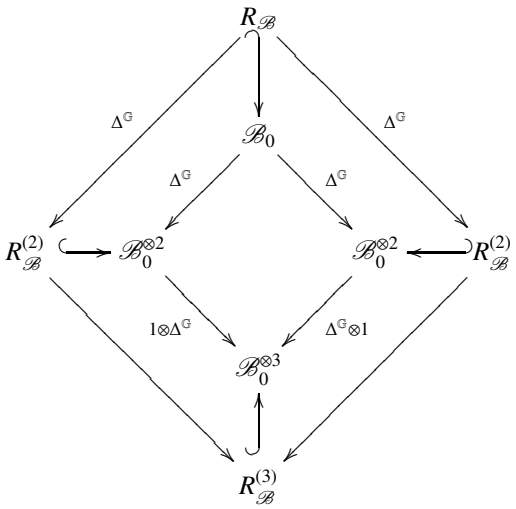
In particular, clearly for all p one has $T^{\mathbb{G}}\Delta^{\mathbb{G}} = \Delta^{\mathbb{G}}$, i. e. the coalgebra structure on $\mathscr{B}_0$ is cocommutative.

The counit for $\mathscr{R}^{\mathbb{F}}$ is given by the diagram

$$\begin{array}{ccccccc}
\mathscr{A} & \rightarrowtail & R_{\mathscr{B}} \otimes \mathbb{F} & \longrightarrow & \mathscr{F}_0 & \twoheadrightarrow & \mathscr{A} \\
\downarrow{\scriptstyle \epsilon} & & \vdots & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon} \\
\mathbb{F} & = & \mathbb{F} & \xrightarrow{0} & \mathbb{F} & = & \mathbb{F}
\end{array} \tag{5.7}$$

where the map $R_{\mathscr{B}} \otimes \mathbb{F} \to \mathbb{F}$ sends the generator $(p1_{\mathscr{B}_0}) \otimes 1$ in degree 0 to 1 and all elements in higher degrees to zero. It is then clear from the formula for $\Delta^{\mathbb{G}}$ that this indeed gives a counit for this diagonal.

Finally to prove coassociativity, by the lemma it suffices to consider the diagram



□

6. T

Let us next consider a derivation of degree 0 of the form

$$\varkappa : \mathscr{A} \to \Sigma\mathscr{A},$$

uniquely determined by

$$\left.\begin{aligned} \varkappa \mathrm{Sq}^n &= \Sigma \mathrm{Sq}^{n-1} \quad \text{for } p = 2, \\ \varkappa\beta &= \Sigma 1, \\ \varkappa(\mathcal{P}^i) &= 0, i \geqslant 0 \end{aligned}\right\} \text{for odd } p. \tag{6.1}$$

We will use $\varkappa$ to define an $\mathscr{A}$-$\mathscr{A}$-bimodule

$$\mathscr{A} \oplus_\varkappa \Sigma\mathscr{A}$$

as follows. The right $\mathscr{A}$-module structure is the same as on $\mathscr{A} \oplus \Sigma\mathscr{A}$ above, i. e. one has $(x, \Sigma y)a = (xa, \Sigma ya)$. As for the left $\mathscr{A}$-module structure, it is given by

$$a(x, \Sigma y) = (ax, (-1)^{\deg(a)}\Sigma ay + \varkappa(a)x).$$

There is a short exact sequence of $\mathscr{A}$-$\mathscr{A}$-bimodules

$$0 \to \Sigma\mathscr{A} \to \mathscr{A} \oplus_\varkappa \Sigma\mathscr{A} \to \mathscr{A} \to 0$$

given by the standard inclusion and projection.

(6.2) Remark. The above short exact sequence of bimodules and the derivation $\varkappa$ correspond to each other under the well known description of the first Hochschild cohomology group by bimodule extensions and derivations, respectively. Indeed, more generally recall that for a graded k-algebra A and an A-A-bimodule M, one of the possible definitions of the Hochschild cohomology of A with coefficients in M is

$$HH^n(A; M) = \mathrm{Ext}^n_{A\otimes_k A^\circ}(A, M).$$

On the other hand, $HH^1(A; M)$ can be also described in terms of derivations. Recall that an M-valued derivation on A is a k-linear map $\varkappa : A \to M$ of degree 0 satisfying

$$\varkappa(xy) = \varkappa(x)y + (-1)^{\deg(x)}x\varkappa(y)$$

for any $x, y \in A$. Such derivations form a k-vector space $\mathrm{Der}(A; M)$. A derivation $\varkappa = \iota_m$ is called inner if there is an $m \in M$ such that

$$\varkappa(x) = mx - (-1)^{\deg(x)}xm = \iota_m(x)$$

for all $x \in A$. These form a subspace $\mathrm{Ider}(A;M) \subset \mathrm{Der}(A;M)$ and one has an isomorphism $HH^1(A;M) \cong \mathrm{Der}(A;M)/\mathrm{Ider}(A;M)$. Moreover there is an exact sequence

$$0 \to HH^0(A;M) \to M \xrightarrow{\iota_\cdot} \mathrm{Der}(A;M) \to HH^1(A;M) \to 0.$$

Explicitly, the isomorphism

$$\mathrm{Der}(A;M)/\mathrm{Ider}(A;M) \cong \mathrm{Ext}^1_{A\otimes A^\circ}(A,M),$$

can be described by assigning to a class of a derivation $\varkappa : A \to M$ the class of the extension

$$0 \to M \to A \oplus_\varkappa M \to A \to 0$$

where as a vector space, $A \oplus_\varkappa M = A \oplus M$, the maps are the canonical inclusion and projection and the bimodule structure is given by

$$a(x,m) = (ax, am + \varkappa(a)x),$$
$$(x,m)a = (xa, ma).$$

Obviously the $\mathscr{A} \oplus_\varkappa \Sigma\mathscr{A}$ above is an example of this construction.

(6.3) Definition. A *Hopf pair algebra* $\mathscr{V}$ (associated to $\mathscr{A}$) is a pair algebra $\partial : \mathscr{V}_1 \to \mathscr{V}_0$ over $\mathbb{F}$ together with the following commutative diagram in the category of $\mathscr{F}_0$-$\mathscr{F}_0$-bimodules

(6.4)
$$\begin{array}{ccccccc}
\Sigma\mathscr{A} & = & \Sigma\mathscr{A} & & & & \\
\downarrow & & \downarrow & & & & \\
\mathscr{A} \oplus_\varkappa \Sigma\mathscr{A} & \rightarrowtail & \mathscr{V}_1 & \xrightarrow{\partial} & \mathscr{V}_0 & \twoheadrightarrow & \mathscr{A} \\
\downarrow q & & \downarrow q & & \| & & \| \\
\mathscr{A} & \rightarrowtail & \mathscr{R}^{\mathbb{F}}_1 & \longrightarrow & \mathscr{R}^{\mathbb{F}}_0 & \twoheadrightarrow & \mathscr{A}
\end{array}$$

with exact rows and columns. The pair morphism $q : \mathscr{V} \to \mathscr{R}^{\mathbb{F}}$ will be called the $\mathbb{G}$-*structure* of $\mathscr{V}$. Moreover $\mathscr{V}$ has a structure of a comonoid in $\mathbf{Alg}^r_{\mathbb{1}\oplus\Sigma}$ and q is compatible with the $\mathbf{Alg}^{pair}_{\mathbb{1}}$-comonoid structure on $\mathscr{R}^{\mathbb{F}}$ in (5.4), in the sense that the diagrams

(6.5)
$$\begin{array}{ccc}
\mathscr{V}_1 & \xrightarrow{\Delta_{\mathscr{V}}} & (\mathscr{V}\hat{\otimes}\mathscr{V})_1 \\
\downarrow q & & \downarrow q\hat{\otimes}q \\
\mathscr{R}^{\mathbb{F}}_1 & \xrightarrow{\Delta_{\mathscr{R}}} & (\mathscr{R}^{\mathbb{F}}\hat{\otimes}\mathscr{R}^{\mathbb{F}})_1
\end{array}$$

and

(6.6)
$$\begin{array}{ccc}
\mathscr{V}_1 & \xrightarrow{\epsilon_{\mathscr{V}}} & \mathbb{F}\oplus\Sigma\mathbb{F} \\
\downarrow q & & \downarrow \\
\mathscr{R}^{\mathbb{F}}_1 & \xrightarrow{\epsilon_{\mathscr{R}}} & \mathbb{F}
\end{array}$$

commute.

We next observe that the following diagrams commute:

$$\begin{array}{ccccc}
\mathscr{A} & & \xrightarrow{\quad\varkappa\quad} & & \Sigma\mathscr{A} \\
\downarrow \delta & & & & \downarrow \Sigma\delta \\
\mathscr{A}\otimes\mathscr{A} & \xrightarrow{\varkappa\otimes 1} & \Sigma\mathscr{A}\otimes\mathscr{A} & = & \Sigma(\mathscr{A}\otimes\mathscr{A}),
\end{array}
\qquad
\begin{array}{ccccc}
\mathscr{A} & & \xrightarrow{\quad\varkappa\quad} & & \Sigma\mathscr{A} \\
\downarrow \delta & & & & \downarrow \Sigma\delta \\
\mathscr{A}\otimes\mathscr{A} & \xrightarrow{1\otimes\varkappa} & \mathscr{A}\otimes\Sigma\mathscr{A} & \xrightarrow{\sigma} & \Sigma(\mathscr{A}\otimes\mathscr{A})
\end{array}$$

where σ is the interchange for Σ in (3.3). Or, on elements,

$$\sum \varkappa(a_\ell)\otimes a_r = \sum \varkappa(a)_\ell \otimes \varkappa(a)_r = \sum \sigma(a_\ell \otimes \varkappa(a_r)), \tag{6.7}$$

where we use the Sweedler notation for the diagonal

$$\delta(x) = \sum x_\ell \otimes x_r.$$

(6.8) Remark. The above identities have a simple explanation using dualization. We will see in (9.7) below that the map dual to $\varkappa$ is the map $\Sigma\mathscr{A}_* \to \mathscr{A}_*$ given, for $p = 2$, by multiplication with the degree 1 generator $\zeta_1 \in \mathscr{A}_*$ and for odd p by the degree 1 generator τ_0. Then the duals of (6.7) are the obvious identities for any $x, y \in \mathscr{A}_*$

$$(\zeta_1 x)y = \zeta_1(xy) = x(\zeta_1 y)$$

for $p = 2$ and

$$(\tau_0 x)y = \tau_0(xy) = (-1)^{\deg(x)} x(\tau_0 y)$$

for odd p (recall that $\mathscr{A}_*$ is graded commutative).

Using (6.7) we prove:

(6.9) Lemma. *For a Hopf pair algebra $\mathscr{V}$ there is a unique left action of $\mathscr{F}_0$ on $(\mathscr{V}\hat{\otimes}\mathscr{V})_1$ such that the quotient map*

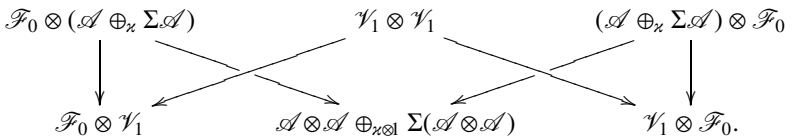
$$(\mathscr{V}\bar{\otimes}\mathscr{V})_1 \twoheadrightarrow (\mathscr{V}\hat{\otimes}\mathscr{V})_1$$

is $\mathscr{F}_0$-equivariant. Here we use the pair algebra structure on $\mathscr{V}\bar{\otimes}\mathscr{V}$ to equip $(\mathscr{V}\bar{\otimes}\mathscr{V})_1$ with an $\mathscr{F}_0 \otimes \mathscr{F}_0$-bimodule structure and then turn it into a left $\mathscr{F}_0$-module via restriction of scalars along $\Delta : \mathscr{F}_0 \to \mathscr{F}_0 \otimes \mathscr{F}_0$.

Proof. Uniqueness is clear as the module structure on the quotient of any module M by a submodule is clearly uniquely determined by the module structure on M.

For the existence, consider the diagram

(6.10)



whose colimit, by (3.14), is $(\mathscr{V}\hat{\otimes}\mathscr{V})_1$, with the right $\mathscr{F}_0 \otimes \mathscr{F}_0$-module structure coming from the category $\mathbf{Alg}^r_{\mathbb{1}\oplus\Sigma}$. It then suffices to show that all maps in this diagram are also left $\mathscr{F}_0$-equivariant, if one uses the left $\mathscr{F}_0$-module structure by restricting scalars along the diagonal $\mathscr{F}_0 \to \mathscr{F}_0 \otimes \mathscr{F}_0$.

This is trivial except possibly for two of the maps involved. For the map

$$\Phi : \mathscr{F}_0 \otimes (\mathscr{A} \oplus_\varkappa \Sigma\mathscr{A}) \to \mathscr{A}\otimes\mathscr{A} \oplus_{\varkappa\otimes\mathscr{A}} \Sigma(\mathscr{A}\otimes\mathscr{A})$$

given by

$$\Phi(f' \otimes (x, \Sigma y)) = (\bar{\bar{f}}' \otimes x, (-1)^{\deg(f')}\Sigma\bar{\bar{f}}' \otimes y),$$

this amounts to checking that for any $f, f' \in \mathscr{F}_0$ and $x, y \in \mathscr{A}$ one must have

$$\sum (f_\ell \otimes f_r)(\bar{\bar{f}}' \otimes x, (-1)^{\deg(f')}\Sigma\bar{\bar{f}}' \otimes y) = \Phi((-1)^{\deg(f_r)\deg(f')} \sum f_\ell f' \otimes (\bar{\bar{f}}_r x, (-1)^{\deg(f_r)}\Sigma\bar{\bar{f}}_r y + \varkappa(\bar{\bar{f}}_r)x)),$$

where again the above Sweedler notation

$$\Delta(f) = \sum f_\ell \otimes f_r,$$

is used for the diagonal of $\mathscr{F}_0$ too, and $\bar{\bar{f}}'$ denotes $q_{\mathscr{F}}(f')$ by the notation in (5.1).

The left hand side expression then expands as

$$\sum ((-1)^{\deg(f_r)\deg(f')}\bar{\bar{f}}_\ell\bar{\bar{f}}' \otimes \bar{\bar{f}}_r x,$$
$$(-1)^{\deg(f_r)\deg(f')}(-1)^{\deg(f)}(-1)^{\deg(f')}\Sigma\bar{\bar{f}}_\ell\bar{\bar{f}}' \otimes \bar{\bar{f}}_r y + (-1)^{\deg(f_r)\deg(f')}\varkappa(\bar{\bar{f}}_\ell)\bar{\bar{f}}' \otimes \bar{\bar{f}}_r x)$$

and the right hand side expands as

$$(-1)^{\deg(f_r)\deg(f')} \sum (\bar{\bar{f}}_\ell\bar{\bar{f}}' \otimes \bar{\bar{f}}_r x, (-1)^{\deg(f_\ell f')}((-1)^{\deg(f_r)}\Sigma\bar{\bar{f}}_\ell\bar{\bar{f}}' \otimes \bar{\bar{f}}_r y + \bar{\bar{f}}_\ell\bar{\bar{f}}' \otimes \varkappa(\bar{\bar{f}}_r)x)).$$

Thus left equivariance of Φ is equivalent to the equality

$$\sum \varkappa(\bar{\bar{f}}_\ell)\bar{\bar{f}}' \otimes \bar{\bar{f}}_r x = \sum (-1)^{\deg(f_\ell f')}\bar{\bar{f}}_\ell\bar{\bar{f}}' \otimes \varkappa(\bar{\bar{f}}_r)x.$$

This is easily deduced from

$$\sum \varkappa(\bar{\bar{f}}_\ell) \otimes \bar{\bar{f}}_r = \sum (-1)^{\deg(f_\ell)} \bar{\bar{f}}_\ell \otimes \varkappa(\bar{\bar{f}}_r),$$

which is an instance of (6.7).

For another map

$$\Psi : (\mathscr{A} \oplus_\varkappa \Sigma\mathscr{A}) \otimes \mathscr{F}_0 \to \mathscr{A} \otimes \mathscr{A} \oplus_{\varkappa\otimes\mathscr{A}} \Sigma(\mathscr{A} \otimes \mathscr{A})$$

given by

$$\Psi((x, \Sigma y) \otimes f') = (x \otimes \bar{\bar{f}}', \Sigma y \otimes \bar{\bar{f}}')$$

the equality to check is

$$\sum (f_\ell \otimes f_r)(x \otimes \bar{\bar{f}}', \Sigma y \otimes \bar{\bar{f}}') = \Psi((-1)^{\deg(f_r)\deg(x,\Sigma y)} \sum (\bar{\bar{f}}_\ell x, (-1)^{\deg(f_\ell)} \Sigma \bar{\bar{f}}_\ell y + \varkappa(\bar{\bar{f}}_\ell)x) \otimes f_r f').$$

Here the left hand side expands as

$$\sum ((-1)^{\deg(f_r)\deg(x)} \bar{\bar{f}}_\ell x \otimes \bar{\bar{f}}_r \bar{\bar{f}}', (-1)^{\deg(f_r)\deg(\Sigma y)} (-1)^{\deg(f_\ell)} \bar{\bar{f}}_\ell y \otimes \bar{\bar{f}}_r \bar{\bar{f}}' + (-1)^{\deg(f_r)\deg(x)} \varkappa(\bar{\bar{f}}_\ell)x \otimes \bar{\bar{f}}_r \bar{\bar{f}}')$$

and the right hand side expands as

$$(-1)^{\deg(f_r)\deg(x,\Sigma y)} \sum (\bar{\bar{f}}_\ell x \otimes \bar{\bar{f}}_r \bar{\bar{f}}', (-1)^{\deg(f_\ell)} \Sigma \bar{\bar{f}}_\ell y \otimes \bar{\bar{f}}_r \bar{\bar{f}}' + \varkappa(\bar{\bar{f}}_\ell)x \otimes \bar{\bar{f}}_r \bar{\bar{f}}');$$

these two expressions are visibly the same. □

Given this left module structure on $(\mathscr{V}\hat{\otimes}\mathscr{V})_1$, one can measure the deviation from left equivariance of the diagonal $\Delta_\mathscr{V} : \mathscr{V}_1 \to (\mathscr{V}\hat{\otimes}\mathscr{V})_1$. For that, consider the map $\hat{L} : \mathscr{V}_0 \otimes \mathscr{V}_1 \to (\mathscr{V}\hat{\otimes}\mathscr{V})_1$ given by

$$\hat{L}(f \otimes x) := \Delta_\mathscr{V}(fx) - f \cdot \Delta_\mathscr{V}(x),$$

for any $f \in \mathscr{F}_0 = \mathscr{V}_0$, $x \in \mathscr{V}_1$, where $\cdot$ denotes the left $\mathscr{F}_0$-module action defined in (6.9). Since the diagonal $\Delta_\mathscr{R}$ of $\mathscr{R}^\mathbb{F}$ is left equivariant, it follows from (6.5) that the image of $\hat{L}$ lies in the kernel of the map $q\hat{\otimes}q$, i. e. in $\Sigma\mathscr{A} \otimes \mathscr{A}$. Moreover if $f = \partial v_1$ for some $v_1 \in \mathscr{V}_1$, then one has

$$\Delta_\mathscr{V}(\partial(v_1)x) = \Delta_\mathscr{V}(v_1\partial x) = \Delta_\mathscr{V}(v_1)\Delta_\mathscr{F}(\partial x) = \Delta_\mathscr{V}(v_1)\partial_{\hat{\otimes}}\Delta_\mathscr{V}(x) = \partial_{\hat{\otimes}}\Delta_\mathscr{V}(v_1)\Delta_\mathscr{V}(x) = \Delta_\mathscr{F}(\partial v_1)\Delta_\mathscr{V}(x),$$

so that the image of $\partial \otimes \mathscr{V}_1 : \mathscr{V}_1 \otimes \mathscr{V}_1 \to \mathscr{V}_0 \otimes \mathscr{V}_1$ lies in the kernel of $\hat{L}$. Similarly commutativity of

(6.11)
$$\begin{array}{ccc} \mathscr{V}_1 & \xrightarrow{\Delta_\mathscr{V}} & (\mathscr{V}\hat{\otimes}\mathscr{V})_1 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial_{\hat{\otimes}}} \\ \mathscr{V}_0 & \xrightarrow{\Delta_\mathscr{F}} & \mathscr{V}_0 \otimes \mathscr{V}_0 \end{array}$$

implies that $\mathscr{V}_0 \otimes \ker\partial$ is in the kernel of $\hat{L}$. It then follows that L factors uniquely through a map

$$\mathscr{A} \otimes R_\mathscr{F} = (\mathscr{V}_0/\operatorname{im}\partial) \otimes (\mathscr{V}_1/\ker\partial) \to \ker(q\hat{\otimes}q) = \Sigma\mathscr{A} \otimes \mathscr{A}.$$

(6.12) Definition. The map

$$L_\mathscr{V} : \mathscr{A} \otimes R_\mathscr{F} \to \Sigma\mathscr{A} \otimes \mathscr{A}$$

given by the unique factorization of the map $\hat{L}$ above is characterized by the deviation of the diagonal $\Delta_\mathscr{V}$ of the Hopf pair algebra $\mathscr{V}$ from left equivariance. That is, one has

$$\Delta_\mathscr{V}(fx) = f \cdot \Delta_\mathscr{V}(x) + L_\mathscr{V}(\bar{\bar{f}} \otimes \partial x)$$

for any $f \in \mathscr{F}_0 = \mathscr{V}_0$, $x \in \mathscr{V}_1$ and the action $\cdot$ from (6.9).

Similarly one can measure deviation of $\Delta_\mathscr{V} : \mathscr{V}_1 \to (\mathscr{V}\hat{\otimes}\mathscr{V})_1$ from cocommutativity by means of the map $\hat{S} : \mathscr{V}_1 \to (\mathscr{V}\hat{\otimes}\mathscr{V})_1$ given by

$$\hat{S}(x) := \Delta_\mathscr{V}(x) - T\Delta_\mathscr{V}(x),$$

where $T : (\mathscr{V}\hat{\otimes}\mathscr{V})_1 \to (\mathscr{V}\hat{\otimes}\mathscr{V})_1$ is the interchange operator for $\mathbf{Alg}^r_{\mathbb{1}\oplus\Sigma}$ as constructed in (3.13). Then similarly to $\hat{L}$ above, $\hat{S}$ admits a factorization in the following way. First, by commutativity of (6.5) one has

$$(q\hat{\otimes}q)T\Delta_\mathscr{V} = T(q\hat{\otimes}q)\Delta_\mathscr{V} = T\Delta_\mathscr{R}q = \Delta_\mathscr{R}q = (q\hat{\otimes}q)\Delta_\mathscr{V},$$

since the $\mathbf{Alg}_{\mathbb{1}}^{pair}$-comonoid $\mathscr{R}^{\mathbb{F}}$ is cocommutative. Thus the image of $\hat{S}$ is contained in $\ker(q\hat{\otimes}q) = \Sigma\mathscr{A}\otimes\mathscr{A}$. Next, commutativity of (6.11) implies that $\ker\partial$ is contained in the kernel of $\hat{S}$. Hence $\hat{S}$ factors uniquely as follows

$$R_{\mathscr{F}} = \mathscr{V}_1/\ker\partial \to \ker(q\hat{\otimes}q) = \Sigma\mathscr{A}\otimes\mathscr{A}.$$

(6.13) Definition. The map

$$S_{\mathscr{V}} : R_{\mathscr{F}} \to \Sigma\mathscr{A}\otimes\mathscr{A}$$

given by the unique factorization of the map $\hat{S}$ above is characterized by the deviation of the diagonal $\Delta_{\mathscr{V}}$ of the Hopf pair algebra $\mathscr{V}$ from cocommutativity. That is, one has

$$T\Delta_{\mathscr{V}}(x) = \Delta_{\mathscr{V}}(x) + S_{\mathscr{V}}(\partial x)$$

for any $x \in \mathscr{V}_1$.

It is clear from these definitions that $L_{\mathscr{V}}$ and $S_{\mathscr{V}}$ are well defined maps by the Hopf pair algebra $\mathscr{V}$. Below in (12.4) we define the left action operator $L : \mathscr{A}\otimes R_{\mathscr{F}} \to \Sigma\mathscr{A}\otimes\mathscr{A}$ and the symmetry operator $S : R_{\mathscr{F}} \to \Sigma\mathscr{A}\otimes\mathscr{A}$ with $L = 0$ and $S = 0$ if p is odd. For $p = 2$ these operators are quite intricate but explicitly given. We also will study the dualization of S and L.

The next two results are essentially reformulations of the main results in the book [1].

(6.14) Theorem (Existence)**.** *There exists a Hopf pair algebra $\mathscr{V}$ with $L_{\mathscr{V}} = L$ and $S_{\mathscr{V}} = S$.*

(6.15) Theorem (Uniqueness)**.** *The Hopf pair algebra $\mathscr{V}$ satisfying $L_{\mathscr{V}} = L$ and $S_{\mathscr{V}} = S$ is unique up to an isomorphism over the $\mathbb{G}$-structure $\mathscr{V} \to \mathscr{R}^{\mathbb{F}}$ and under the kernel $\mathscr{A}\oplus_{\varkappa}\Sigma\mathscr{A} \rightarrowtail \mathscr{V}$.*

The Hopf pair algebra appearing in these theorems is the *algebra of secondary cohomology operations* over $\mathbb{F}$, denoted by $\mathscr{B}^{\mathbb{F}} = (\mathscr{B}_1^{\mathbb{F}} \to \mathscr{B}_0^{\mathbb{F}}) = \mathscr{B}\otimes\mathbb{F}$. The algebra $\mathscr{B}$ has been defined over $\mathbb{G}$ in [1].

Proof of (6.14)*.* Recall that in [1, 12.1.8] a folding product $\hat{\otimes}$ is defined for pair $\mathbb{G}$-algebras in such a way that $\mathscr{B}$ has a comonoid structure with respect to it, i. e. a *secondary Hopf algebra* structure. Let

$$\Delta_1 : \mathscr{B}_1 \to (\mathscr{B}\hat{\otimes}\mathscr{B})_1$$

be the corresponding *secondary diagonal* from [1, (12.2.2)]. It is proved in [1, 14.4] that the left action operator L satisfies

$$\Delta_1(bx) = b\Delta_1(x) + L(q(b)\otimes(\partial x\otimes 1))$$

for $b \in \mathscr{B}_0$, $x \in \mathscr{B}_1$, $\partial x\otimes 1 \in R_{\mathscr{B}}\otimes\mathbb{F} = \mathscr{R}_1^{\mathbb{F}}$. Also in [1, 14.5] it is proved that the symmetry operator S satisfies

$$T\Delta_1(x) = \Delta_1(x) + S(\partial x\otimes 1)$$

for $x \in \mathscr{B}_1$. Moreover it is proved in [1, 15.3.13] that the secondary Hopf algebra $\mathscr{B}$ is determined uniquely up to isomorphism by the maps $\varkappa$, L and S.

Consider now the diagram

$$\begin{array}{ccccccc}
\Sigma\mathscr{A} & = & \Sigma\mathscr{A} & & & & \\
\downarrow & & \downarrow & & & & \\
\mathscr{A}\oplus_{\varkappa}\Sigma\mathscr{A} & \xrightarrow{i_{\varkappa}} & \mathscr{B}_1\otimes\mathbb{F} & \xrightarrow{\partial\otimes 1} & \mathscr{B}_0\otimes\mathbb{F} & \twoheadrightarrow & \mathscr{A} \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q=\partial\otimes 1} & & \| & & \| \\
\mathscr{A} & \rightarrowtail & R_{\mathscr{B}}\otimes\mathbb{F} & \longrightarrow & \mathscr{F}_0 & \twoheadrightarrow & \mathscr{A}.
\end{array}$$

Here the inclusion $i_{\varkappa} : \mathscr{A}\oplus_{\varkappa}\Sigma\mathscr{A} \rightarrowtail \mathscr{B}_1\otimes\mathbb{F}$ is given by the inclusion $\Sigma\mathscr{A} \subset \mathscr{B}_1$ and by the map

$$\mathscr{A} \to \mathscr{B}_1\otimes\mathbb{F}$$

which assigns to an element $q(b) \in \mathscr{A}$, for $b \in \mathscr{B}_0$, the element $[p]\cdot b\otimes 1$. Then it is clear that $i_{\varkappa}$ is a right $\mathscr{A}$-module homomorphism. Moreover it also is a left $\mathscr{A}$-module homomorphism since for $b \in \mathscr{B}_0$ the following identity holds in $\mathscr{B}_1$:

$$b\cdot[p] - [p]\cdot b = \varkappa(b).$$

Compare [1, A20 in the introduction]. Now one can check that the properties of $\mathscr{B}$ established in [1] yield the result. □

(6.16) Remark. For elements $\alpha, \beta, \gamma \in \mathscr{A}$ with $\alpha\beta = 0$ and $\beta\gamma = 0$ the *triple Massey product*

$$\langle \alpha, \beta, \gamma \rangle \in \mathscr{A}/(\alpha\mathscr{A} + \mathscr{A}\gamma)$$

is defined. Here the degree of elements in $\langle \alpha, \beta, \gamma \rangle$ is $\deg(\alpha) + \deg(\beta) + \deg(\gamma) - 1$. We can compute $\langle \alpha, \beta, \gamma \rangle$ by use of the Hopf pair algebra $\mathscr{B}^{\mathbb{F}}$ above as follows. For this we consider the maps

$$\mathscr{A} \xleftarrow{q_{\mathscr{B}}} \mathscr{B}_0 \supset R_{\mathscr{B}} \xrightarrow{q_R} R_{\mathscr{B}} \otimes \mathbb{F}.$$

We choose elements $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in \mathscr{B}_0$ which $q_{\mathscr{B}}$ carries to α, β, γ respectively. Then we know that the products $\bar{\alpha}\bar{\beta}, \bar{\beta}\bar{\gamma}$ are elements in $R_{\mathscr{B}}$ for which we can choose elements $x, y \in \mathscr{B}_1 \otimes \mathbb{F}$ with

$$q(x) = q_R(\bar{\alpha}\bar{\beta}),$$
$$q(y) = q_R(\bar{\beta}\bar{\gamma}).$$

Then the bimodule structure of $\mathscr{B}_1 \otimes \mathbb{F}$ yields the element $\bar{\alpha}y - x\bar{\gamma}$ in the kernel $\Sigma\mathscr{A}$ of $q : \mathscr{B}_1 \otimes \mathbb{F} \to R_{\mathscr{B}} \otimes \mathbb{F}$. Now $\bar{\alpha}y - x\bar{\gamma} \in \Sigma\mathscr{A}$ represents $\langle \alpha, \beta, \gamma \rangle$, see [1].

7. T $\mathbb{G}$-

We next turn to the dualization of the $\mathbb{G}$-relation pair algebra of the Steenrod algebra from section 5. For this we just apply the duality functor D to (5.2). There results an exact sequence

$$\mathscr{A}_* \rightarrowtail \mathscr{R}^0_{\mathbb{F}} \xrightarrow{d} \mathscr{R}^1_{\mathbb{F}} \twoheadrightarrow \mathscr{A}_*,$$

i. e. the sequence

$$\begin{array}{ccccccc} \mathscr{A}_* & \rightarrowtail & D(\mathscr{R}^{\mathbb{F}}_0) & \xrightarrow{D(\partial)} & D(\mathscr{R}^{\mathbb{F}}_1) & \twoheadrightarrow & \mathscr{A}_* \\ & & \| & & \| & & \\ & & \mathrm{Hom}(\mathscr{F}_0, \mathbb{F}) & & \mathrm{Hom}(R_{\mathscr{B}}, \mathbb{F}). & & \end{array} \tag{7.1}$$

In particular, by the dual of (5.3) one has

(7.2) Lemma. *The pair $\mathscr{R}_{\mathbb{F}} = (d : \mathscr{R}^0_{\mathbb{F}} \to \mathscr{R}^1_{\mathbb{F}})$ has a pair coalgebra structure compatible with the standard bicomodule structure of $\mathscr{A}_*$ over itself, so that $\mathscr{R}_{\mathbb{F}}$ yields an object in* $\mathbf{Coalg}^{pair}_{\mathbb{1}}$*, see section 4.*

Moreover the dual of (5.4) takes place, i. e. one has

(7.3) Theorem. *The pair coalgebra $\mathscr{R}_{\mathbb{F}}$ has a structure of a commutative monoid in the category* $\mathbf{Coalg}^{pair}_{\mathbb{1}}$ *with respect to the unfolding product $\check{\otimes}$.*

□

The proof uses the duals of the pair algebras $\mathscr{R}^{(n)}$, $n \geqslant 0$, from (5.4). Namely, applying to the short exact sequence

$$R^{(n)}_{\mathscr{B}} \rightarrowtail \mathscr{B}^{\otimes n}_0 \xrightarrow{q^{\otimes n}} \mathscr{A}^{\otimes n}$$

the functor $D = \mathrm{Hom}(_, \mathbb{F})$ gives, similarly to (7.2), a pair coalgebra

$$\mathscr{R}^{(n)}_* = \left(\mathscr{A}^{\otimes n}_* \rightarrowtail \mathscr{F}^{\otimes n}_* \longrightarrow R^{(n)}_{\mathscr{B}*} \twoheadrightarrow \mathscr{A}^{\otimes n}_* \right)$$

such that the following dual of (5.5) holds:

(7.4) Lemma. *There is a canonical isomorphism $\mathscr{R}^{(n)}_* \cong (\mathscr{R}_{\mathbb{F}})^{\check{\otimes} n}$ in* $\mathbf{Coalg}^{pair}_{\mathbb{1}}$.

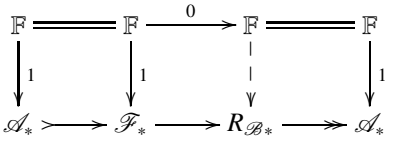
□

Using this lemma one constructs the $\check{\otimes}$-monoid structure on $\mathscr{R}_{\mathbb{F}}$ by the diagram

$$\begin{array}{ccccccc} \mathscr{F}_* \otimes \mathscr{F}_* & = & \mathscr{R}^0_{\mathbb{F}} \otimes \mathscr{R}^0_{\mathbb{F}} & \xrightarrow{d_{\check{\otimes}}} & (\mathscr{R}_{\mathbb{F}} \check{\otimes} \mathscr{R}_{\mathbb{F}})^1 & \xrightarrow{\cong} & R^{(2)}_{\mathscr{B}*} \\ \downarrow{\scriptstyle \Delta_*} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \Delta^{\mathbb{G}}_*} \\ \mathscr{F}_* & = & \mathscr{R}^0_{\mathbb{F}} & \xrightarrow{d} & \mathscr{R}^1_{\mathbb{F}} & = & R_{\mathscr{B}*} \end{array}$$

with $\Delta^{\mathbb{G}}$ as in (5.6).

Moreover the unit of $\mathscr{R}_{\mathbb{F}}$ is given by the dual of (5.7), i. e. by the diagram

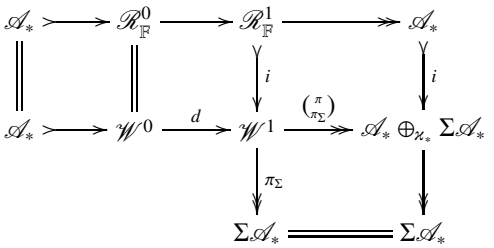


so that the unit element of $R_{\mathscr{B}*}$ is the map $R_{\mathscr{B}} \to \mathbb{F}$ sending the generator $p1_{\mathscr{B}_0}$ in degree 0 to 1 and all elements in higher degrees to zero.

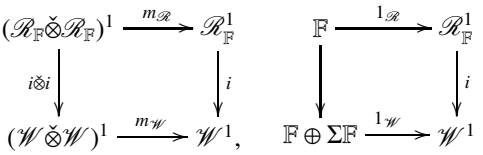
8. H

We next turn to the dualization of the notion of a Hopf pair algebra from (6.3), using the dual $\mathscr{R}_{\mathbb{F}}$ of $\mathscr{R}^{\mathbb{F}}$ from the previous section.

(8.1) Definition. A *Hopf pair coalgebra* $\mathscr{W}$ (associated to $\mathscr{A}_*$) is a pair coalgebra $d : \mathscr{W}^0 \to \mathscr{W}^1$ over $\mathbb{F}$ together with the following commutative diagram in the category of $\mathscr{F}_*$-$\mathscr{F}_*$-bicomodules



with exact rows and columns. The pair morphism $i : \mathscr{R}_{\mathbb{F}} \to \mathscr{W}$ will be called the $\mathbb{G}$-*structure of* $\mathscr{W}$. Moreover $\mathscr{W}$ must be equipped with a structure of a monoid $(m_{\mathscr{W}}, 1_{\mathscr{W}})$ in $\mathbf{Coalg}^r_{\mathbb{1}\oplus\Sigma}$ such that i is compatible with the $\mathbf{Coalg}^{pair}_{\mathbb{1}}$-monoid structure on $\mathscr{R}_{\mathbb{F}}$ from (7.3), i. e. diagrams dual to (6.5) and (6.6)



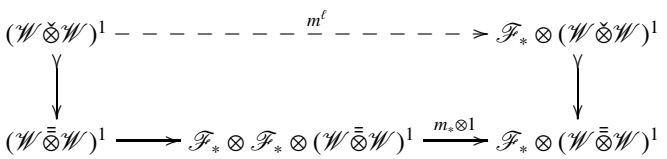
commute.

We next note that the dual of (6.9) holds; more precisely, one has

(8.2) Lemma. *For a Hopf pair coalgebra* $\mathscr{W}$ *the subspace*

$$(\mathscr{W}\check{\otimes}\mathscr{W})^1 \subset (\mathscr{W}\bar{\bar{\otimes}}\mathscr{W})^1$$

is closed under the left coaction of the coalgebra $\mathscr{F}_*$ *on* $(\mathscr{W}\bar{\bar{\otimes}}\mathscr{W})^1$ *given by the corestriction of scalars along the multiplication* $m_* : \mathscr{F}_* \otimes \mathscr{F}_* \to \mathscr{F}_*$ *of the left* $\mathscr{F}_* \otimes \mathscr{F}_* = (\mathscr{W}\bar{\bar{\otimes}}\mathscr{W})^0$*-comodule structure given by the pair coalgebra* $\mathscr{W}\bar{\bar{\otimes}}\mathscr{W}$. *In other words, there is a unique map* $m^\ell : (\mathscr{W}\check{\otimes}\mathscr{W})^1 \to \mathscr{F}_* \otimes (\mathscr{W}\check{\otimes}\mathscr{W})^1$ *making the diagram*



commute.

□

Given this left coaction, one can define the dual of the left action operator in (6.12) by measuring deviation of the multiplication $(\mathscr{W}\check{\otimes}\mathscr{W})^1 \to \mathscr{W}^1$ from being a left comodule homomorphism. For that, one first observes the map $\check{L} : (\mathscr{W}\check{\otimes}\mathscr{W})^1 \to \mathscr{F}_* \otimes \mathscr{W}^1$ given by the difference of two composites in the diagram

$$\begin{array}{ccc} (\mathscr{W}\check{\otimes}\mathscr{W})^1 & \xrightarrow{m^\ell} & \mathscr{F}_* \otimes (\mathscr{W}\check{\otimes}\mathscr{W})^1 \\ \downarrow{\scriptstyle m_{\mathscr{W}}} & & \downarrow{\scriptstyle 1\otimes m_{\mathscr{W}}} \\ \mathscr{W}^1 & \xrightarrow{m^\ell} & \mathscr{F}_* \otimes \mathscr{W}^1. \end{array}$$

Then by the argument dual to that before (6.12) one sees that the map $\check{L}$ factors uniquely through $\operatorname{coker}(i\check{\otimes}i) = \left((\mathscr{W}\check{\otimes}\mathscr{W})^1 \twoheadrightarrow \Sigma\mathscr{A}_* \otimes \mathscr{A}_*\right)$ and into $\ker(d) \otimes \operatorname{im}(d) = \left(\mathscr{A}_* \otimes R_{\mathscr{F}*} \rightarrowtail \mathscr{W}^0 \otimes \mathscr{W}^1\right)$ to yield a map $\Sigma\mathscr{A}_* \otimes \mathscr{A}_* \to \mathscr{A}_* \otimes R_{\mathscr{F}*}$. We thus can make, dually to (6.12), the following

(8.3) Definition. The map

$$L_{\mathscr{W}} : \Sigma\mathscr{A}_* \otimes \mathscr{A}_* \to \mathscr{A}_* \otimes R_{\mathscr{F}*}$$

given by the unique factorization of the map $\check{L}$ above is characterized by the deviation of the multiplication $m_{\mathscr{W}}$ of the Hopf pair coalgebra $\mathscr{W}$ from being a left $\mathscr{F}_*$-comodule homomorphism. That is, for any $t \in (\mathscr{W}\check{\otimes}\mathscr{W})^1$ one has

$$(1 \otimes m_{\mathscr{W}})m^\ell(t) = m^\ell m_{\mathscr{W}}(t) + L_{\mathscr{W}}(\pi_\Sigma\check{\otimes}\pi_\Sigma)(t).$$

Next, we define a map $S_{\mathscr{W}}$ in a manner dual to (6.13), measuring noncommutativity of the $\mathbf{Coalg}^r_{\mathbb{1}\oplus\Sigma}$-monoid structure on $\mathscr{W}$. For that, we first consider the map $\check{S} : (\mathscr{W}\check{\otimes}\mathscr{W})^1 \to \mathscr{W}^1$ given by

$$\check{S}(t) = m_{\mathscr{W}}T(t) - m_{\mathscr{W}}(t)$$

for $t \in (\mathscr{W}\check{\otimes}\mathscr{W})^1$ and then observe that, dually to (6.13), this map factors uniquely through $\operatorname{coker}(i\check{\otimes}i) = \left((\mathscr{W}\check{\otimes}\mathscr{W})^1 \twoheadrightarrow \Sigma\mathscr{A}_* \otimes \mathscr{A}_*\right)$ and into $\operatorname{im}(d) = \left(R_{\mathscr{F}*} \rightarrowtail \mathscr{W}^1\right)$ so we have

(8.4) Definition. The map

$$S_{\mathscr{W}} : \Sigma\mathscr{A}_* \otimes \mathscr{A}_* \to R_{\mathscr{F}*}$$

given by the unique factorization of the map $\check{S}$ above is characterized by being the graded commutator map with respect to the $\check{\otimes}$-monoid structure on the Hopf pair coalgebra $\mathscr{W}$. That is, for any $t \in (\mathscr{W}\check{\otimes}\mathscr{W})^1$ one has

$$m_{\mathscr{W}}T(t) = m_{\mathscr{W}}(t) + S_{\mathscr{W}}(\pi_\Sigma\check{\otimes}\pi_\Sigma)(t).$$

We now dualize the left action operator (12.4) and the symmetry operator (13.1).

(8.5) Definition. The *left coaction operator*

$$L_* : \mathscr{A}_* \otimes \mathscr{A}_* \to \mathscr{A}_* \otimes R_{\mathscr{F}*}$$

of degree +1 is the graded dual of the left action operator (12.4).

(8.6) Definition. The *cosymmetry operator*

$$S_* : \mathscr{A}_* \otimes \mathscr{A}_* \to R_{\mathscr{F}*}$$

of degree +1 is the graded dual of the symmetry operator (13.1).

It is clear that the duals of (6.14) and (6.15) hold. Let us state these explicitly.

(8.7) Theorem (Existence)**.** *There exists a Hopf pair coalgebra $\mathscr{W}$ with $L_{\mathscr{W}} = L_*$ and $S_{\mathscr{W}} = S_*$.*

(8.8) Theorem (Uniqueness)**.** *The Hopf pair coalgebra $\mathscr{W}$ satisfying $L_{\mathscr{W}} = L_*$ and $S_{\mathscr{W}} = S_*$ is unique up to an isomorphism over $\mathscr{W} \twoheadrightarrow \mathscr{A}_* \oplus_{\varkappa_*} \Sigma\mathscr{A}_*$ and under $\mathscr{R}_{\mathbb{F}} \rightarrowtail \mathscr{W}$.*

The Hopf pair coalgebra appearing in these theorems will be denoted by $\mathscr{B}_{\mathbb{F}} = (\mathscr{B}^0_{\mathbb{F}} \to \mathscr{B}^1_{\mathbb{F}}) = D(\mathscr{B}^{\mathbb{F}})$.

Part II. Description of $\mathscr{B}^{\mathbb{F}}$ and $\mathscr{B}_{\mathbb{F}}$ in terms of generators

9. T M S

Here we recall the needed facts from [7]. The graded dual of the Hopf algebra $\mathscr{A}$ is the Milnor Hopf algebra $\mathscr{A}_* = \mathrm{Hom}(\mathscr{A}, \mathbb{F}) = D(\mathscr{A})$. It is proved in [7] that for odd p as an algebra $\mathscr{A}_*$ is a graded polynomial algebra, i. e. it is isomorphic to a tensor product of an exterior algebra on generators of odd degree and a polynomial algebra on generators of even degree; for $p = 2$ the algebra $\mathscr{A}_*$ is a polynomial algebra. Moreover, in [7], explicit generators are given in terms of the admissible basis.

First recall that the admissible basis for $\mathscr{A}$ is given by the following monomials: for odd p they are of the form

$$M = \beta^{\epsilon_0}\mathcal{P}^{s_1}\beta^{\epsilon_1}\mathcal{P}^{s_2}\cdots\mathcal{P}^{s_n}\beta^{\epsilon_n}$$

where $\epsilon_k \in \{0, 1\}$ and

$$s_1 \geqslant \epsilon_1 + ps_2,\ s_2 \geqslant \epsilon_2 + ps_3, \ldots, s_{n-1} \geqslant \epsilon_{n-1} + ps_n,\ s_n \geqslant 1.$$

Then let $\xi_k \in \mathscr{A}_{2(p^k-1)} = \mathrm{Hom}(\mathscr{A}^{2(p^k-1)}, \mathbb{F})$, $k \geqslant 1$ and $\tau_k \in \mathscr{A}_{2p^k-1} = \mathrm{Hom}(\mathscr{A}^{2p^k-1}, \mathbb{F})$, $k \geqslant 0$ be given on this basis by

$$\xi_k(M) = \begin{cases} 1, & M = \mathcal{P}^{p^{k-1}}\mathcal{P}^{p^{k-2}}\cdots\mathcal{P}^{p}\mathcal{P}^{1}, \\ 0 & \text{otherwise} \end{cases} \tag{9.1}$$

and

$$\tau_k(M) = \begin{cases} 1, & M = \mathcal{P}^{p^{k-1}}\mathcal{P}^{p^{k-2}}\cdots\mathcal{P}^{p}\mathcal{P}^{1}\beta, \\ 0 & \text{otherwise.} \end{cases} \tag{9.2}$$

As proved in [7], $\mathscr{A}_*$ is a graded polynomial algebra on these elements, i. e. it is generated by the elements ξ_k and τ_k with the defining relations

$$\begin{aligned} \xi_i\xi_j &= \xi_j\xi_i, \\ \xi_i\tau_j &= \tau_j\xi_i, \\ \tau_i\tau_j &= -\tau_j\tau_i \end{aligned}$$

only.

For $p = 2$, the admissible basis for $\mathscr{A}$ is given by the monomials

$$M = \mathrm{Sq}^{s_1}\mathrm{Sq}^{s_2}\cdots\mathrm{Sq}^{s_n}$$

with

$$s_1 \geqslant 2s_2,\ s_2 \geqslant 2s_3, \ldots, s_{n-1} \geqslant 2s_n,\ s_n \geqslant 1$$

and the polynomial generators of $\mathscr{A}_*$ are elements $\zeta_k \in \mathscr{A}_{2^k-1} = \mathrm{Hom}(\mathscr{A}^{2^k-1}, \mathbb{F})$ given by

$$\zeta_k(M) = \begin{cases} 1, & M = \mathrm{Sq}^{2^{k-1}}\mathrm{Sq}^{2^{k-2}}\cdots\mathrm{Sq}^{2}\mathrm{Sq}^{1}, \\ 0 & \text{otherwise.} \end{cases} \tag{9.3}$$

In terms of these generators, also the coalgebra structure $m_* : \mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_*$ dual to the multiplication m of $\mathscr{A}$ is determined in [7]. Namely, for odd p one has

$$\begin{aligned} m_*(\xi_k) &= \xi_k \otimes 1 + \xi_{k-1}^{p} \otimes \xi_1 + \xi_{k-2}^{p^2} \otimes \xi_2 + \cdots + \xi_1^{p^{k-1}} \otimes \xi_{k-1} + 1 \otimes \xi_k, \\ m_*(\tau_k) &= \xi_k \otimes \tau_0 + \xi_{k-1}^{p} \otimes \tau_1 + \xi_{k-2}^{p^2} \otimes \tau_2 + \cdots + \xi_1^{p^{k-1}} \otimes \tau_{k-1} + 1 \otimes \tau_k + \tau_k \otimes 1. \end{aligned} \tag{9.4}$$

For $p = 2$ one has

$$m_*(\zeta_k) = \zeta_k \otimes 1 + \zeta_{k-1}^{2} \otimes \zeta_1 + \zeta_{k-2}^{4} \otimes \zeta_2 + \cdots + \zeta_1^{2^{k-1}} \otimes \zeta_{k-1} + 1 \otimes \zeta_k. \tag{9.5}$$

We will need expression of the dual $\mathrm{Sq}^1_* : \mathscr{A}_* \to \Sigma\mathscr{A}_*$ to the map $\mathrm{Sq}^1\cdot : \Sigma\mathscr{A} \to \mathscr{A}$ given by multiplication with Sq^1 from the left.

(9.6) Lemma. *The map* Sq^1_* *is equal to* $\frac{\partial}{\partial\zeta_1}$. *That is, on the monomial basis it is given by*

$$\mathrm{Sq}^1_*(\zeta_1^{n_1}\zeta_2^{n_2}\cdots) = \begin{cases} \zeta_1^{n_1-1}\zeta_2^{n_2}\cdots, & n_1 \equiv 1 \mod 2 \\ 0, & n_1 \equiv 0 \mod 2. \end{cases}$$

Proof. Note that Sq^1_* is a derivation, since $\mathrm{Sq}^1\cdot$ is a coderivation, i.e. the diagram

$$\begin{array}{ccccc}
\Sigma\mathscr{A} & & \xrightarrow{\qquad\qquad\mathrm{Sq}^1\cdot\qquad\qquad} & & \mathscr{A} \\
\Big\downarrow{\scriptstyle \Sigma\delta} & & & & \Big\downarrow{\scriptstyle \delta} \\
\Sigma\mathscr{A}\otimes\mathscr{A} & & \xrightarrow{\binom{\mathrm{Sq}^1\cdot\otimes 1}{1\otimes\mathrm{Sq}^1\cdot}} & \mathscr{A}\otimes\mathscr{A}\times\mathscr{A}\otimes\mathscr{A} & \xrightarrow{+} \mathscr{A}\otimes\mathscr{A}
\end{array}$$

commutes: indeed for any $x \in \mathscr{A}$ one has

$$\delta(\mathrm{Sq}^1 x) = \delta(\mathrm{Sq}^1)\delta(x) = (\mathrm{Sq}^1\otimes 1 + 1\otimes\mathrm{Sq}^1)\delta(x) = (\mathrm{Sq}^1\otimes 1)\delta(x) + (1\otimes\mathrm{Sq}^1)\delta(x).$$

On the other hand, on the Milnor generators the derivation Sq^1_* acts as follows:

$$\mathrm{Sq}^1_*(\zeta_n)(x) = \zeta_n(\mathrm{Sq}^1 x) = \begin{cases} 1, & \mathrm{Sq}^1 x = \mathrm{Sq}^{2^{n-1}}\mathrm{Sq}^{2^{n-2}}\cdots\mathrm{Sq}^1, \\ 0, & \mathrm{Sq}^1 x \neq \mathrm{Sq}^{2^{n-1}}\mathrm{Sq}^{2^{n-2}}\cdots\mathrm{Sq}^1. \end{cases}$$

It follows that $\mathrm{Sq}^1_*(\zeta_1) = 1$; on the other hand for $n > 1$ the equation $\mathrm{Sq}^1 x = \mathrm{Sq}^{2^{n-1}}\mathrm{Sq}^{2^{n-2}}\cdots\mathrm{Sq}^1$ has no solutions, since it would imply $\mathrm{Sq}^1\mathrm{Sq}^{2^{n-1}}\mathrm{Sq}^{2^{n-2}}\cdots\mathrm{Sq}^1 = \mathrm{Sq}^1\mathrm{Sq}^1 x = 0$, whereas actually

$$\mathrm{Sq}^1\mathrm{Sq}^{2^{n-1}}\mathrm{Sq}^{2^{n-2}}\cdots\mathrm{Sq}^1 = \mathrm{Sq}^{1+2^{n-1}}\mathrm{Sq}^{2^{n-2}}\cdots\mathrm{Sq}^1 \neq 0.$$

But $\frac{\partial}{\partial\zeta_1}$ is the unique derivation sending ζ_1 to 1 and all other ζ_n's to 0. □

We will also need expression of the dual $\varkappa_*$ of the derivation $\varkappa$ from (6.1) in terms of the above generators.

(9.7) Lemma. *The map $\varkappa_* : \Sigma\mathscr{A}_* \to \mathscr{A}_*$ is equal to the left multiplication by τ_0 for odd p and by ζ_1 for $p = 2$.*

Proof. For any linear map $\phi : \mathscr{A}^n \to \mathbb{F}$ the map $\varkappa_*(\phi) : \mathscr{A}_{n+1} \to \mathbb{F}$ is the composite of ϕ with $\varkappa : \mathscr{A}_{n+1} \to \mathscr{A}_n$. Thus for p odd one has

$$\varkappa_*(\phi)(\beta^{\epsilon_0}\mathcal{P}^{s_1}\beta^{\epsilon_1}\mathcal{P}^{s_2}\cdots\mathcal{P}^{s_n}\beta^{\epsilon_n}) = \sum_{\epsilon_k=1}(-1)^{\epsilon_0+\epsilon_1+\cdots+\epsilon_{k-1}}\phi(\beta^{\epsilon_0}\mathcal{P}^{s_1}\beta^{\epsilon_1}\cdots\beta^{\epsilon_{k-1}}\mathcal{P}^{s_k}\mathcal{P}^{s_{k+1}}\beta^{\epsilon_{k+1}}\cdots\mathcal{P}^{s_n}\beta^{\epsilon_n}). \tag{9.8}$$

On the other hand, one has for M as above

$$(\tau_0\phi)(M) = \sum \tau_0(M_\ell)\phi(M_r) = \sum_{\substack{M_\ell = c\beta \\ 0\neq c\in\mathbb{F}}} c\phi(M_r),$$

if

$$\delta(M) = \sum M_\ell \otimes M_r.$$

On the other hand one evidently has

$$\delta(\beta^{\epsilon_0}\mathcal{P}^{s_1}\beta^{\epsilon_1}\mathcal{P}^{s_2}\cdots\mathcal{P}^{s_n}\beta^{\epsilon_n}) = \sum_{\substack{0\leqslant\iota_0\leqslant\epsilon_0 \\ 0\leqslant i_1\leqslant s_1 \\ 0\leqslant\iota_1\leqslant\epsilon_1 \\ \cdots \\ 0\leqslant i_n\leqslant s_n \\ 0\leqslant\iota_n\leqslant\epsilon_n}} (-1)^{\sum_{0\leqslant\mu<\nu\leqslant n}(\epsilon_\mu-\iota_\mu)\iota_\nu}\beta^{\iota_0}\mathcal{P}^{i_1}\beta^{\iota_1}\cdots\mathcal{P}^{i_n}\beta^{\iota_n}\otimes\beta^{\epsilon_0-\iota_0}\mathcal{P}^{s_1-i_1}\beta^{\epsilon_1-\iota_1}\cdots\mathcal{P}^{s_n-i_n}\beta^{\epsilon_n-\iota_n}$$

so that for $M = \beta^{\epsilon_0}\mathcal{P}^{s_1}\beta^{\epsilon_1}\cdots\mathcal{P}^{s_n}\beta^{\epsilon_n}$ one has

$$\sum_{\substack{M_\ell = c\beta \\ 0\neq c\in\mathbb{F}}} c\phi(M_r) = \sum_{\epsilon_k=1}\sum_{\substack{\iota_0=0 \\ i_1=0 \\ \cdots \\ i_k=0 \\ \iota_k=1 \\ i_{k+1}=0 \\ \cdots \\ i_n=0 \\ \iota_n=0}} (-1)^{\sum_{0\leqslant\mu<\nu\leqslant n}(\epsilon_\mu-\iota_\mu)\iota_\nu}\phi(\beta^{\epsilon_0-\iota_0}\mathcal{P}^{s_1-i_1}\beta^{\epsilon_1-\iota_1}\cdots\mathcal{P}^{s_n-i_n}\beta^{\epsilon_n-\iota_n})$$

$$= \sum_{\epsilon_k=1}(-1)^{\sum_{0\leqslant\mu<k}\epsilon_\mu}\phi(\beta^{\epsilon_0}\mathcal{P}^{s_1}\beta^{\epsilon_1}\cdots\mathcal{P}^{s_k}\mathcal{P}^{s_{k+1}}\beta^{\epsilon_{k+1}}\cdots\mathcal{P}^{s_n}\beta^{\epsilon_n})$$

which is the same as (9.8) above.

Similarly for $p = 2$ the map $\varkappa_*(\phi)$ is given by

$$\varkappa_*(\phi)(\mathrm{Sq}^{s_1}\cdots\mathrm{Sq}^{s_n}) = \phi(\varkappa(\mathrm{Sq}^{s_1}\cdots\mathrm{Sq}^{s_n})) = \sum_{k=1}^{n}\phi(\mathrm{Sq}^{s_1}\cdots\mathrm{Sq}^{s_k-1}\cdots\mathrm{Sq}^{s_n}) \tag{9.9}$$

and the map $\zeta_1\phi$ is given by

$$(\zeta_1\phi)(M) = \sum \zeta_1(M_\ell)\phi(M_r) = \sum_{M_\ell=\mathrm{Sq}^1}\phi(M_r).$$

On the other hand one has

$$\delta(\mathrm{Sq}^{s_1}\cdots\mathrm{Sq}^{s_n}) = \sum_{\substack{0\leqslant i_1\leqslant s_1\\ \cdots\\ 0\leqslant i_n\leqslant s_n}}\mathrm{Sq}^{i_1}\cdots\mathrm{Sq}^{i_n}\otimes\mathrm{Sq}^{s_1-i_1}\cdots\mathrm{Sq}^{s_n-i_n},$$

so that for $M = \mathrm{Sq}^{s_1}\cdots\mathrm{Sq}^{s_n}$ one has

$$\sum_{M_\ell=\mathrm{Sq}^1}\phi(M_r) = \sum_{k=1}^{n}\sum_{\substack{i_1=0\\ \cdots\\ i_{k-1}=0\\ i_k=1\\ i_{k+1}=0\\ \cdots\\ i_n=0}}\phi(\mathrm{Sq}^{s_1-i_1}\cdots\mathrm{Sq}^{s_n-i_n})$$

which is equal to (9.9). □

It is clear that with respect to the coalgebra structure on $\mathscr{A}_*$ the map $\varkappa_*$ is a coderivation, i. e. the diagram

$$\begin{array}{ccccc}
\Sigma\mathscr{A}_* & & \xrightarrow{\qquad\varkappa_*\qquad} & & \mathscr{A}_*\\
\downarrow{\scriptstyle \Sigma m_*} & & & & \downarrow{\scriptstyle m_*}\\
\Sigma(\mathscr{A}_*\otimes\mathscr{A}_*) & \xrightarrow{\binom{1}{\sigma}} & \Sigma\mathscr{A}_*\otimes\mathscr{A}_* \oplus \mathscr{A}_*\otimes\Sigma\mathscr{A}_* & \xrightarrow{(\varkappa_*\otimes 1, 1\otimes\varkappa_*)} & \mathscr{A}_*\otimes\mathscr{A}_*
\end{array}$$

is commutative. Here σ is the interchange of Σ as in (3.3). Then using dual of the construction mentioned in (6.2) one may equip the vector space $\mathscr{A}_* \oplus \Sigma\mathscr{A}_*$ with a structure of an $\mathscr{A}_*$-$\mathscr{A}_*$-bicomodule, in such a way that one has a short exact sequence of $\mathscr{A}_*$-$\mathscr{A}_*$-bicomodules

$$0 \to \mathscr{A}_* \to \mathscr{A}_* \oplus_{\varkappa_*} \Sigma\mathscr{A}_* \to \Sigma\mathscr{A}_* \to 0. \tag{9.10}$$

Explicitly, one defines the right coaction of $\mathscr{A}_*$ on $\mathscr{A}_* \oplus_{\varkappa_*} \Sigma\mathscr{A}_*$ as the direct sum of standard coactions on $\mathscr{A}_*$ and on $\Sigma\mathscr{A}_*$, whereas the left coaction is given by the composite

$$\mathscr{A}_* \oplus \Sigma\mathscr{A}_* \xrightarrow{m_*\oplus\Sigma m_*} \mathscr{A}_*\otimes\mathscr{A}_* \oplus \Sigma\mathscr{A}_*\otimes\mathscr{A}_* \xrightarrow{\begin{pmatrix}1 & \varkappa_*\otimes 1\\ 0 & \sigma\end{pmatrix}} \mathscr{A}_*\otimes\mathscr{A}_* \oplus \mathscr{A}_*\otimes\Sigma\mathscr{A}_* \cong \mathscr{A}_*\otimes(\mathscr{A}_* \oplus \Sigma\mathscr{A}_*).$$

10. T $\qquad\qquad \mathscr{F}_0 = T_{\mathbb{F}}(E_{\mathscr{A}}) \qquad p = 2$

We begin by recalling the constructions from [5] relevant to our case.

The *Leibniz-Hopf algebra* is the free graded associative ring with unit $1 = Z_0$

$$\mathscr{Z} = T_{\mathbb{Z}}\{Z_1, Z_2, ...\} \tag{10.1}$$

on generators Z_n, one for each degree $n \geqslant 1$. Here we use notation as in (5.1). Then $\mathscr{Z}$ is a cocommutative Hopf algebra with respect to the diagonal

$$\Delta(Z_n) = \sum_{i=0}^{n} Z_i \otimes Z_{n-i}.$$

Of course for $p = 2$ we have $\mathscr{Z}\otimes\mathbb{F} = \mathscr{F}_0 = T_{\mathbb{Z}}(E_{\mathscr{A}})$ by identifying $Z_i = \mathrm{Sq}^i$, and moreover the diagonal Δ corresponds to $\Delta^{\mathbb{G}}\otimes\mathbb{F}$ in (5.6). The graded dual of $\mathscr{Z}$ over integers is denoted by $\mathscr{M}$; it is proved in [5] that it is a polynomial algebra. There also a certain set of elements of $\mathscr{M}$ is given; it is still a conjecture (first formulated by Ditters) that these elements form a set of polynomial generators for $\mathscr{M}$. If, however, one localizes at any prime p, then there is another set of elements, defined using the so called *p-elementary words*, which, as proved in [5], is a set of polynomial generators for the localized algebra $\mathscr{M}$. This in

particular gives a polynomial generating set for $\mathscr{F}_* = \mathrm{Hom}(\mathscr{F}_0, \mathbb{F}_2) \cong \mathscr{M}/2\mathscr{M}$. Moreover it turns out that the embedding $\mathscr{A}_* \rightarrowtail \mathscr{F}_*$ given by $\mathrm{Hom}(\mathscr{A}, \mathbb{F}_2) \rightarrowtail \mathrm{Hom}(\mathscr{F}_0, \mathbb{F}_2)$ (dual to the quotient map $\mathscr{F}_0 \twoheadrightarrow \mathscr{A}$) carries the Milnor generators of $\mathscr{A}_*$ to a subset of these generators.

One chooses a basis in $\mathscr{M}$ which is dual to the (noncommutative) monomial basis in $\mathscr{Z}$: for any sequence $\alpha = (d_1, ..., d_n)$ of positive integers, let $M_\alpha = M_{d_1,...,d_n}$ be the element of the free abelian group $\mathscr{M}^{d_1+...+d_n} = \mathrm{Hom}(\mathscr{Z}^{d_1+...+d_n}, \mathbb{Z})$ determined by

$$M_{d_1,...,d_n}(Z_{k_1} \cdots Z_{k_m}) = \begin{cases} 1, & (k_1, ..., k_m) = (d_1, ..., d_n), \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathscr{Z}$ is a free algebra, dually $\mathscr{M}$ is a cofree coalgebra, i. e. the diagonal is given by deconcatenation:

$$\Delta(M_{d_1,...,d_n}) = \sum_{i=0}^{n} M_{d_1,...,d_i} \otimes M_{d_{i+1},...,d_n}. \tag{10.2}$$

It is noted in [5] (and easy to check) that in this basis the multiplication in $\mathscr{M}$ is given by the so called *overlapping shuffle product*. Rather than defining this rigorously, we will give some examples.

$$\begin{aligned} M_5 M_{2,4,1,9} &= M_{5,2,4,1,9} + M_{7,4,1,9} + M_{2,5,4,1,9} + M_{2,9,1,9} + M_{2,4,5,1,9} + M_{2,4,6,9} \\ &\quad + M_{2,4,1,5,9} + M_{2,4,1,14} + M_{2,4,1,9,5}; \\ M_{8,5} M_{1,2} &= M_{8,5,1,2} + M_{8,6,2} + M_{8,1,5,2} + M_{9,5,2} + M_{8,1,7} + M_{9,7} + M_{1,8,5,2} \\ &\quad + M_{1,8,7} + M_{1,8,2,5} + M_{9,2,5} + M_{1,2,8,5} + M_{1,10,5} + M_{8,1,2,5} \end{aligned}$$

Thus in general, whereas the ordinary shuffle product of the elements, say, M_{a_1,a_2,a_3} and M_{b_1,b_2,b_3,b_4,b_5} contains all possible summands like $M_{b_1,a_1,a_2,b_2,b_3,a_3,b_4,b_5}$, the overlapping shuffle product contains together with each such summand also in addition the summands of the form $M_{b_1+a_1,a_2,b_2,b_3,a_3,b_4,b_5}$, $M_{b_1,a_1,a_2+b_2,b_3,a_3,b_4,b_5}$, $M_{b_1,a_1,a_2,b_2,b_3+a_3,b_4,b_5}$, $M_{b_1,a_1,a_2,b_2,b_3,a_3+b_4,b_5}$, $M_{b_1+a_1,a_2+b_2,b_3,a_3,b_4,b_5}$ and so on, obtained by replacing an a_i and a b_j standing one next to other with their sum, in all possible positions.

Note that the algebra of ordinary shuffles is also a polynomial algebra, but over rationals; it is *not* a polynomial algebra until at least one prime number remains uninverted. On the other hand, over rationals $\mathscr{M}$ becomes isomorphic to the algebra of ordinary shuffles.

To define a polynomial generating set for $\mathscr{M}$, we need some definitions. To conform with the admissible basis in the Steenrod algebra, which consists of monomials with decreasing indices, we will reverse the order of indices in the definitions from [5], where the indices go in the increasing order. Thus in our case statements about some $M_{d_1,...,d_n}$ will be equivalent to the corresponding ones in [5] about $M_{d_n,...,d_1}$.

(10.3) Definitions. The *lexicographic order* on the basis $M_{d_1,...,d_n}$ of $\mathscr{M}$ is defined by declaring $M_{d_1,...,d_n} > M_{e_1,...,e_m}$ if either there is an i with $1 \leqslant i \leqslant \min(n, m)$ and $d_i > e_i$, $d_n = e_m$, $d_{n-1} = e_{m-1}$, ..., $d_{n-i+1} = e_{m-i+1}$, $d_{n-i} > e_{m-i}$ or $n > m$ and $d_{n-m+1} = e_1$, $d_{n-m+2} = e_2$, ..., $d_n = e_m$.

A basis element $M_{d_1,...,d_n}$ is *Lyndon* if with respect to this ordering one has $M_{d_1,...,d_n} < M_{d_1,...,d_i}$ for all $1 < i \leqslant n$. For example, $M_{3,2,3,2,2}$ and $M_{2,2,1,2,1,2,1}$ are Lyndon but $M_{3,2,2,3,2}$ and $M_{2,1,2,1,2,1}$ are not.

A basis element $M_{d_1,...,d_n}$ is *$\mathbb{Z}$-elementary* if no number > 1 divides all of the d_i, i. e. $\gcd(d_1, ..., d_n) = 1$. The set $\mathrm{ESL}(\mathbb{Z})$ is the set of elementary basis elements of the form $M_{d_1,...,d_n,d_1,...,d_n,...,d_1,...,d_n}$ (i. e. $d_1, ..., d_n$ repeated any number of times), where $M_{d_1,...,d_n}$ is a Lyndon element.

For a prime p, a basis element $M_{d_1,...,d_n}$ is called *p-elementary* if there is a d_i not divisible by p, i. e. $p \nmid \gcd(d_1, ..., d_n)$. The set $\mathrm{ESL}(p)$ is defined as the set of p-elementary basis elements of the form

$$M_{\underbrace{d_1,...,d_n,d_1,...,d_n,...,d_1,...,d_n}_{p^r \text{ times}}}$$

with $d_1, ..., d_n$ repeated p^r times for some r, where $M_{d_1,...,d_n}$ is required to be Lyndon.

For example, $M_{15,6,15,6,15,6,15,6}$ is in $\mathrm{ESL}(2)$ but not in $\mathrm{ESL}(\mathbb{Z})$ or in $\mathrm{ESL}(p)$ for any other p, whereas $M_{30,6,6}$ is in $\mathrm{ESL}(p)$ for any $p \neq 2, 3$ but not in $\mathrm{ESL}(2)$, not in $\mathrm{ESL}(3)$ and not in $\mathrm{ESL}(\mathbb{Z})$.

One then has

(10.4) Theorem ([5])**.** *The algebra $\mathscr{M}$ is a polynomial algebra.*

(10.5) Conjecture (Ditters, [5])**.** *The set* $\mathrm{ESL}(\mathbb{Z})$ *is the set of polynomial generators for $\mathscr{M}$.*

(10.6) Theorem ([5]). *For each prime p, the set* ESL(p) *is a set of polynomial generators for $\mathscr{M}_{(p)} = \mathscr{M} \otimes \mathbb{Z}_{(p)}$, i. e. if one inverts all primes except p.*

In particular, it follows that ESL(p) is a set of polynomial generators for $\mathscr{M}/p^n$ over $\mathbb{Z}/p^n$ for all n.

Here are the polynomial generators in low degrees, over $\mathbb{Z}$ and over few first primes. Note that the numbers of generators in each degree are the same (as it should be since all these algebras become isomorphic over $\mathbb{Q}$).

	1	2	3	4	5
$\mathbb{Z}$	M_1	$M_{1,1}$	$M_{2,1}, M_{1,1,1}$	$M_{3,1}, M_{2,1,1}, M_{1,1,1,1}$	$M_{4,1}, M_{3,2}, M_{3,1,1}, M_{2,2,1}, M_{2,1,1,1}, M_{1,1,1,1,1}$
$p = 2$	M_1	$M_{1,1}$	$M_3, M_{2,1}$	$M_{3,1}, M_{2,1,1}, M_{1,1,1,1}$	$M_5, M_{4,1}, M_{3,2}, M_{3,1,1}, M_{2,2,1}, M_{2,1,1,1}$
$p = 3$	M_1	M_2	$M_{2,1}, M_{1,1,1}$	$M_4, M_{3,1}, M_{2,1,1}$	$M_5, M_{4,1}, M_{3,2}, M_{3,1,1}, M_{2,2,1}, M_{2,1,1,1}$
$p = 5$	M_1	M_2	$M_3, M_{2,1}$	$M_4, M_{3,1}, M_{2,1,1}$	$M_{4,1}, M_{3,2}, M_{3,1,1}, M_{2,2,1}, M_{2,1,1,1}, M_{1,1,1,1,1}$

It is easy to calculate the numbers of polynomial generators in each degree. Let these numbers be m_1, $m_2, \cdots$. Then the Poincaré series for the algebra $\mathscr{M}$ (or $\mathscr{Z}$, or $\mathscr{F}$, or $\mathscr{F}_*$, it does not matter) is

$$\sum_n \dim(\mathscr{M}_n)t^n = (1-t)^{-m_1}(1-t^2)^{-m_2}(1-t^3)^{-m_3}\cdots;$$

on the other hand, we know that it is a tensor coalgebra with one generator in each degree $n \geqslant 1$; this implies that $\dim(\mathscr{M}_n) = 2^{n-1}$ for $n \geqslant 1$ (and $\dim(M_0) = 1$). Thus we have equality of power series

$$\prod_{k=1}^{\infty}(1-t^k)^{-m_k} = 1 + t + 2t^2 + 4t^3 + 8t^4 + \cdots = 1 + t(1 + 2t + (2t)^2 + (2t)^3 + \cdots) = 1 + t\frac{1}{1-2t} = \frac{1-t}{1-2t}.$$

Then taking logarithmic derivatives one obtains

$$\sum_{k=1}^{\infty}\frac{km_kt^k}{1-t^k} = \frac{2t}{1-2t} - \frac{t}{1-t} = t + 3t^2 + 7t^3 + \cdots + (2^n-1)t^n + \cdots.$$

It follows that for all n one has

$$\sum_{d|n} dm_d = 2^n - 1,$$

which by the Möbius inversion formula gives

$$m_n = \frac{1}{n}\sum_{d|n}\mu(d)(2^{\frac{n}{d}} - 1).$$

The latter expression is well known in the literature on combinatorics; it equals the number of aperiodic bicolored necklaces consisting of n beads, and also the dimension of the nth homogeneous component of the free Lie algebra on two generators. See e. g. [8].

11. T $R_{\mathscr{F}}$

We now turn to the algebra $\mathscr{F}_* = \mathrm{Hom}(\mathscr{F}, \mathbb{F}_2) \cong \mathscr{M}/2$. By the above, we know that it, as well as $\mathscr{M}_{(2)}$, is a polynomial algebra on the set of generators ESL(2). As an illustration, we will give some expressions of the M-basis elements in terms of sums of overlapping shuffle products of elements from ESL(2). We will give these in $\mathscr{M}_{(2)}$ and then their images in $\mathscr{F}_*$.

$$
\begin{aligned}
M_2 &= M_1^2 - 2M_{1,1}\\
&\equiv M_1^2 \mod 2\\
M_{1,2} &= M_1^3 - M_3 - M_{2,1} - 2M_1M_{1,1}\\
&\equiv M_1^3 + M_3 + M_{2,1} \mod 2\\
M_{1,1,1} &= M_1M_{1,1} - \frac{1}{3}M_1^3 + \frac{1}{3}M_3\\
&\equiv M_1M_{1,1} + M_1^3 + M_3 \mod 2\\
M_4 &= \frac{4}{3}M_1M_3 - \frac{1}{3}M_1^4 + 2M_{1,1}^2 - 4M_{1,1,1,1}\\
&\equiv M_1^4 \mod 2\\
M_{2,2} &= M_{1,1}^2 - 2M_1^2M_{1,1} - \frac{2}{3}M_1M_3 + \frac{2}{3}M_1^4 + 2M_{1,1,1,1}\\
&\equiv M_{1,1}^2 \mod 2\\
M_{1,3} &= \frac{1}{3}M_1^4 - \frac{1}{3}M_1M_3 - 2M_{1,1}^2 - M_{3,1} + 4M_{1,1,1,1}\\
&\equiv M_1^4 + M_1M_3 + M_{3,1} \mod 2\\
M_{1,2,1} &= M_1M_{2,1} - M_{3,1} - M_{1,1}^2 + 2M_1^2M_{1,1} + \frac{2}{3}M_1M_3 - \frac{2}{3}M_1^4 - 2M_{1,1,1,1} - 2M_{2,1,1}\\
&\equiv M_1M_{2,1} + M_{3,1} + M_{1,1}^2 \mod 2\\
M_{1,1,2} &= M_{1,1}^2 - M_1^2M_{1,1} - \frac{1}{3}M_1M_3 + \frac{1}{3}M_1^4 - 2M_{1,1,1,1} + M_{3,1} - M_1M_{2,1} + M_{2,1,1}\\
&\equiv M_{1,1}^2 + M_1^2M_{1,1} + M_1M_3 + M_1^4 + M_{3,1} + M_1M_{2,1} + M_{2,1,1} \mod 2
\end{aligned}
$$

Moreover it is straightforward to calculate the diagonal in terms of these generators. For example, in $\mathscr{F}_*$ one has

$$
\begin{aligned}
\Delta(M_1) &= 1\otimes M_1 + M_1\otimes 1,\\
\Delta(M_{1,1}) &= 1\otimes M_{1,1} + M_1\otimes M_1 + M_{1,1}\otimes 1,\\
\Delta(M_3) &= 1\otimes M_3 + M_3\otimes 1\\
\Delta(M_{2,1}) &= 1\otimes M_{2,1} + M_1^2\otimes M_1 + M_{2,1}\otimes 1\\
\Delta(M_{3,1}) &= 1\otimes M_{3,1} + M_3\otimes M_1 + M_{3,1}\otimes 1\\
\Delta(M_{2,1,1}) &= 1\otimes M_{2,1,1} + M_1^2\otimes M_{1,1} + M_{2,1}\otimes M_1 + M_{2,1,1}\otimes 1\\
\Delta(M_{1,1,1,1}) &= 1\otimes M_{1,1,1,1} + M_1\otimes M_1M_{1,1} + M_1\otimes M_1^3 + M_1\otimes M_3 + M_{1,1}\otimes M_{1,1} + M_1M_{1,1}\otimes M_1\\
&\quad + M_1^3\otimes M_1 + M_3\otimes M_1 + M_{1,1,1,1}\otimes 1\\
\Delta(M_{4,1}) &= 1\otimes M_{4,1} + M_1^4\otimes M_1 + M_{4,1}\otimes 1\\
\Delta(M_{3,2}) &= 1\otimes M_{3,2} + M_3\otimes M_1^2 + M_{3,2}\otimes 1\\
\Delta(M_{2,1,1,1}) &= 1\otimes M_{2,1,1,1} + M_1^2\otimes M_1M_{1,1} + M_1^2\otimes M_1^3 + M_1^2\otimes M_3 + M_{2,1}\otimes M_{1,1} + M_{2,1,1}\otimes M_1\\
&\quad + M_{2,1,1,1}\otimes 1\\
\Delta(M_5) &= 1\otimes M_5 + M_5\otimes 1\\
\Delta(M_{3,1,1}) &= 1\otimes M_{3,1,1} + M_3\otimes M_{1,1} + M_{3,1}\otimes M_1 + M_{3,1,1}\otimes 1\\
\Delta(M_{2,2,1}) &= 1\otimes M_{2,2,1} + M_1^2\otimes M_{2,1} + M_{1,1}^2\otimes M_1 + M_{2,2,1}\otimes 1.
\end{aligned}
$$

Also it follows from the results in [5] that one has

(11.1) Lemma. *For any prime p, in $\mathscr{M}_{(p)}$ one has*

$$M_{pd_1,\dots,pd_n} \equiv M_{d_1,\dots,d_n}^p \mod p.$$

To identify the elements to which the Milnor generators ζ_k of $\mathscr{A}_*$ go under the isomorphism $\mathscr{F}_* \cong \mathscr{M}/2$, we first identify $\mathscr{A}_*$ with the graded dual of $\mathscr{A}$; then ζ_k corresponds to a linear form $\mathscr{A}_{2^k-1} \to \mathbb{F}$ given by (9.3).

(11.2) Proposition. *Under the embedding $\mathscr{A}_* \rightarrowtail \mathscr{M}/2$, the Milnor generator ζ_k maps to the generator $M_{2^{k-1},2^{k-2},\ldots,2,1}$. In particular, this generator is in* ESL(2)*, i. e. is one of the polynomial generators of $\mathscr{F}_*$.*

Note how this together with (10.2) and (11.1) implies the Milnor formula (9.5) for the diagonal in $\mathscr{A}_*$. Identifying ζ_k with its image in $\mathscr{M}/2$ by (11.2), one obtains

$$m_*(\zeta_k) = \Delta(M_{2^{k-1},2^{k-2},\ldots,2,1}) = \sum_{i=0}^{k} M_{2^{k-1},2^{k-2},\ldots,2^i} \otimes M_{2^{i-1},\ldots,2,1} = \sum_{i=0}^{k} M^{2^i}_{2^{k-1-i},2^{k-2-i},\ldots,2,1} \otimes M_{2^{i-1},\ldots,2,1}$$
$$= \sum_{i=0}^{k} \zeta_{k-i}^{2^i} \otimes \zeta_i.$$

Thus the set $\{\zeta_1, \zeta_2, \ldots\}$ of polynomial generators for $\mathscr{A}_*$ can be identified with the subset

$$Q = \{M_1, M_{2,1}, M_{4,2,1}, M_{8,4,2,1}, \ldots\}$$

of the set of polynomial generators ESL(2) for $\mathscr{M}/2 \cong \mathscr{F}_*$. This in particular gives an explicit basis for $R_{\mathscr{F}*}$: it is in one-to-one correspondence with those monomials in the generators $M_{d_1,\ldots,d_n}$ from ESL(2) not all of whose variables belong to Q. For example, in first few low dimensions this basis contains the following monomials:

$M_{1,1}$,

$M_1M_{1,1}, M_3$,

$M_1^2M_{1,1}, M_1M_3, M_{1,1}^2, M_{3,1}, M_{2,1,1}, M_{1,1,1,1}$,

$M_1^3M_{1,1}, M_1^2M_3, M_1M_{1,1}^2, M_1M_{3,1}, M_1M_{2,1,1}, M_1M_{1,1,1,1}, M_{1,1}M_3, M_{1,1}M_{2,1}, M_5, M_{4,1}, M_{3,2}, M_{3,1,1}, M_{2,2,1}$, $M_{2,1,1,1}$.

We next note that obviously the embedding $\mathscr{A}_* \rightarrowtail \mathscr{F}_*$ identifies $\mathscr{F}_*$ with a polynomial algebra over $\mathscr{A}_*$, namely one has a canonical isomorphism

$$\mathscr{F}_* \cong \mathscr{A}_*[\mathrm{ESL}(2) \setminus Q]. \tag{11.3}$$

In particular, as an $\mathscr{A}_*$-module $\mathscr{F}_*$ is free on the generating set $\mathbb{N}^{(\mathrm{ESL}(2)\setminus Q)}$ (= the free commutative monoid on ESL(2) $\setminus$ Q). Then obviously the quotient module $R_{\mathscr{F}*}$ is a free $\mathscr{A}_*$-module with the generating set $\mathbb{N}^{(\mathrm{ESL}(2)\setminus Q)} \setminus \{1\}$.

We will need the dual $\mathscr{F}_*^{\leqslant 2}$ of the subspace $\mathscr{F}_0^{\leqslant 2} \subset \mathscr{F}_0$ spanned by the monomials of length $\leqslant 2$ in the generators Sq^i. Observe that $\mathscr{F}_0^{\leqslant 2}$ is a subcoalgebra of $\mathscr{F}_0$, so that dually $\mathscr{F}_* \twoheadrightarrow \mathscr{F}_*^{\leqslant 2}$ is a quotient algebra. We have

(11.4) Proposition. *The algebra $\mathscr{F}_*^{\leqslant 2}$ is a quotient of the polynomial algebra on three generators M_1, $M_{1,1}$, $M_{2,1}$ by a single relation*

$$M_1M_{1,1}M_{2,1} + M_{1,1}^3 + M_{2,1}^2 = 0.$$

Proof. First of all, it is straightforward to calculate in $\mathscr{F}_*$ the sum of the overlapping shuffle products

$$M_1M_{1,1}M_{2,1} + M_{1,1}^3 + M_{2,1}^2 =$$
$$M_{1,4,1} + M_{2,2,2} + M_{2,3,1} + M_{3,1,2} + M_{3,2,1} + M_{2,1,2,1} + M_{3,1,1,1} + M_{1,1,3,1} + M_{1,2,1,1,1} + M_{1,2,1,2} + M_{1,1,1,2,1}$$

so that indeed this gives zero in $\mathscr{F}_*^{\leqslant 2}$. Let

$$X = \mathbb{F}[x_1, x_2, x_3]/(x_1x_2x_3 + x_2^3 + x_3^2)$$

be the graded algebra with $\deg(x_i) = i$, $i = 1, 2, 3$, so that there is a homomorphism of algebras $f : X \to \mathscr{F}_*^{\leqslant 2}$ sending $x_1 \mapsto M_1$, $x_2 \mapsto M_{1,1}$, $x_3 \mapsto M_{2,1}$. It is straightforward to calculate the Hilbert function of X, i. e. the formal power series

$$\sum_n \dim(X_n)t^n;$$

it is equal to

$$\frac{1-t^6}{(1-t)(1-t^2)(1-t^3)}.$$

On the other hand $\mathscr{F}_*^{\leqslant 2}$ is dual to $\mathscr{F}_0^{\leqslant 2}$ and it is straightforward also to calculate dimensions of homogeneous components of this space. One then simply checks that these dimensions coincide for X and for $\mathscr{F}_*^{\leqslant 2}$. Thus it suffices to show that f is surjective, i. e. that $\mathscr{F}_*^{\leqslant 2}$ is generated by (the images of) M_1, $M_{1,1}$ and $M_{2,1}$.

We will show by induction on degree that every M_n and $M_{i,j}$ can be obtained as a polynomial in these three elements. In degree 1, M_1 is the only nonzero element. In degree 2, besides $M_{1,1}$ we have M_2 which is equal to M_1^2 by (11.1). In degree 3, we have

$$M_1 M_{1,1} = M_{1,2} + M_{2,1} + M_{1,1,1} \equiv M_{1,2} + M_{2,1} \mod \mathscr{F}_*^{>2}$$

and

$$M_1^3 = M_3 + M_{1,2} + M_{2,1},$$

so that in $\mathscr{F}_*^{\leqslant 2}$ we may solve

$$M_{1,2} \equiv M_1 M_{1,1} + M_{2,1}$$

and

$$M_3 \equiv M_1^3 + M_1 M_{1,1}.$$

Given now any degree $n > 3$, we can obtain any element $M_{i,j}$ with $i > 1$, $j > 1$, $i + j = n$ from elements of lower degree since

$$M_{i,j} \equiv M_{1,1} M_{i-1,j-1}.$$

Next we also can obtain the element $M_{n-1,1}$ from

$$M_{n-1,1} + M_{2,n-2} \equiv M_{2,1} M_{n-3}.$$

Then we can obtain $M_{1,n-1}$ from

$$M_{1,n-1} + M_{n-1,1} \equiv M_{1,1} M_{n-2},$$

and finally we can obtain M_n from

$$M_n + M_{1,n-1} + M_{n-1,1} \equiv M_1 M_{n-1}.$$

□

Let us also identify the dual of the product map

$$\mathscr{F}_0^{\leqslant 1} \otimes \mathscr{F}_0^{\leqslant 1} \to \mathscr{F}_0^{\leqslant 2}$$

in terms of the above generators. By dualizing it is clear that this dual is the unique factorization in the diagram

$$\begin{array}{ccc} \mathscr{F}_* & \xrightarrow{m_*} & \mathscr{F}_* \otimes \mathscr{F}_* \\ \downarrow & & \downarrow \\ \mathscr{F}_*^{\leqslant 2} & \dashrightarrow & \mathscr{F}_*^{\leqslant 1} \otimes \mathscr{F}_*^{\leqslant 1}. \end{array}$$

In particular, it is an algebra homomorphism. Moreover the algebra $\mathscr{F}_*^{\leqslant 1}$ may be identified with the polynomial algebra on a single generator $M_1 = \zeta_1$, with the quotient map $\mathscr{F}_* \to \mathscr{F}_*^{\leqslant 1}$ given by sending M_1 to itself and all other polynomial generators from ESL(2) to zero. From this it is straightforward to identify the map $\mathscr{F}_*^{\leqslant 2} \to \mathscr{F}_*^{\leqslant 1} \otimes \mathscr{F}_*^{\leqslant 1}$ with the algebra homomorphism

$$\mathbb{F}[x_1, x_2, x_3]/(x_1 x_2 x_3 + x_2^3 + x_3^2) \to \mathbb{F}[y_1, z_1]$$

given by

$$\begin{aligned} x_1 &\mapsto y_1 + z_1 \\ x_2 &\mapsto y_1 z_1 \\ x_3 &\mapsto y_1^2 z_1. \end{aligned} \tag{11.5}$$

Let us identify in these terms the map $\mathscr{F}_*^{\leqslant 2} \twoheadrightarrow R_{\mathscr{F}*}^{\leqslant 2}$. One clearly has

$$R_{\mathscr{F}}^{\leqslant 2} = R_{\mathscr{F}} \cap \mathscr{F}_0^{\leqslant 2}$$

in $\mathscr{F}_0$, so that dually one has that the diagram

$$\begin{array}{ccc} \mathscr{F}_* & \twoheadrightarrow & R_{\mathscr{F}*} \\ \downarrow & & \downarrow \\ \mathscr{F}_*^{\leqslant 2} & \twoheadrightarrow & R_{\mathscr{F}*}^{\leqslant 2} \end{array}$$

is pushout. Thus $R_{\mathscr{F}*}^{\leqslant 2}$ is isomorphic to the quotient of $\mathscr{F}_*^{\leqslant 2}$ by the image of the composite $\mathscr{A}_* \rightarrowtail \mathscr{F}_* \twoheadrightarrow \mathscr{F}_*^{\leqslant 2}$. That image is clearly the subalgebra generated by M_1 and $M_{2,1}$.

We can alternatively describe $R_{\mathscr{F}*}^{\leqslant 2}$ in terms of linear forms on $R_{\mathscr{F}}^{\leqslant 2} \subset \mathscr{F}_0^{\leqslant 2}$. It is clear that the latter subspace is spanned by all Adem relations $[n, m]$, $n < 2m$. The map $\pi : \mathscr{F}_*^{\leqslant 2} \twoheadrightarrow R_{\mathscr{F}*}^{\leqslant 2}$ assigns to a linear form on $\mathscr{F}_0^{\leqslant 2}$ its restriction to $R_{\mathscr{F}}^{\leqslant 2}$. One then clearly has

$$\pi(M_1^k) = \pi(M_{2,1}^k) = 0 \tag{11.6}$$

for all $k \geqslant 0$; moreover $\pi(M_{1,1})$ is dual to $[1, 1]$ in the basis given by the elements $[n, m]$, i. e. $M_{1,1}([1, 1]) = 1$ and $M_{1,1}([n, m]) = 0$ for all other n, m. Moreover for $x, y \in \mathscr{F}_*^{\leqslant 2}$ we have

$$(xy)([n, m]) = \sum x([n, m]_\ell) y([n, m]_r) \tag{11.7}$$

in the Sweedler notation

$$\Delta([n, m]) = \sum [n, m]_\ell \otimes [n, m]_r.$$

For example, we have

$$\Delta([1, 2]) = (1 + T)(1 \otimes [1, 2] + \mathrm{Sq}^1 \otimes [1, 1])$$

which implies that $M_1 M_{1,1}$ is dual to $[1, 2]$ in this basis, i. e. $(M_1 M_{1,1})[1, 2] = 1$ and $(M_1 M_{1,1})[n, m] = 0$ for all other n, m. Similarly

$$\Delta([1, 3]) = (1 + T)(1 \otimes [1, 3] + \mathrm{Sq}^1 \otimes [1, 2] + \mathrm{Sq}^2 \otimes [1, 1])$$

and

$$\Delta([2, 2]) = (1 + T)(1 \otimes [2, 2] + \mathrm{Sq}^1 \otimes [1, 2] + \mathrm{Sq}^2 \otimes [1, 1]) + [1, 1] \otimes [1, 1]$$

imply that $M_{1,1}^2$ is dual to $[2, 2]$ whereas $(M_1^2 M_{1,1})[1, 3] = (M_1^2 M_{1,1})[2, 2] = 1$, so that dual to $[1, 3]$ is $M_1^2 M_{1,1} + M_{1,1}^2$.

We will also need a description of the dual $\bar{R}_*$ of $\bar{R} = R_{\mathscr{F}}/(R_{\mathscr{F}} \cdot R_{\mathscr{F}})$. For this first note that similarly to the above $\mathscr{F}_* \otimes \mathscr{F}_*$ is a free $\mathscr{A}_* \otimes \mathscr{A}_*$-module on $\mathbb{N}^{(\mathrm{ESL}(2)\setminus Q)} \times \mathbb{N}^{(\mathrm{ESL}(2)\setminus Q)}$ and $R_{\mathscr{F}*} \otimes R_{\mathscr{F}*}$ is a free $\mathscr{A}_* \otimes \mathscr{A}_*$-module on $\left(\mathbb{N}^{(\mathrm{ESL}(2)\setminus Q)} \setminus \{1\}\right) \times \left(\mathbb{N}^{(\mathrm{ESL}(2)\setminus Q)} \setminus \{1\}\right)$. Moreover the diagonal $\Delta_{\mathscr{F}} : \mathscr{F}_* \to \mathscr{F}_* \otimes \mathscr{F}_*$ and its factorization $\Delta_R : R_{\mathscr{F}*} \to R_{\mathscr{F}*} \otimes R_{\mathscr{F}*}$ through the quotient maps $\mathscr{F}_* \twoheadrightarrow R_{\mathscr{F}*}$, $\mathscr{F}_* \otimes \mathscr{F}_* \twoheadrightarrow R_{\mathscr{F}*} \otimes R_{\mathscr{F}*}$ are obviously both equivariant with respect to the diagonal $\delta : \mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_*$, i. e. one has

$$\begin{aligned} \Delta_{\mathscr{F}}(af) &= \delta(a)\Delta_{\mathscr{F}}(f), \\ \Delta_R(ar) &= \delta(a)\Delta_R(r) \end{aligned} \tag{11.8}$$

for any $a \in \mathscr{A}_*$, $f \in \mathscr{F}_*$, $r \in R_{\mathscr{F}*}$.

12. T L

We next recall constructions of certain maps L and S from [1, 14.4,14.5] of the same kind as the operators in (6.12) and (6.13) respectively. For that, we first introduce the following notation:

$$\bar{R} := R_{\mathscr{F}}/(R_{\mathscr{F}} \cdot R_{\mathscr{F}}), \tag{12.1}$$

with the quotient map $R_{\mathscr{F}} \twoheadrightarrow \bar{R}$ denoted by $r \mapsto \bar{r}$. There is a well-defined $\mathscr{A}$-$\mathscr{A}$-bimodule structure on $\bar{R}$ given by

$$\bar{\bar{f}}\bar{r} = \overline{fr}, \quad \bar{r}\bar{\bar{f}} = \overline{rf}$$

for $f \in \mathscr{F}_0$, $r \in R_{\mathscr{F}}$. As we show below $\bar{R}$ is free both as a left and as a right $\mathscr{A}$-module (but not as a bimodule). A basis for $\bar{R}$ as a right $\mathscr{A}$-module can be found using the set PAR $\subset R_{\mathscr{F}}$ of *preadmissible relations* as defined in [1, 16.5]. These are the elements of $R_{\mathscr{F}}$ of the form

$$\mathrm{Sq}^{n_1} \cdots \mathrm{Sq}^{n_k}[n, m]$$

where $[n, m]$, $n < 2m$, is an Adem relation, the monomial $\mathrm{Sq}^{n_1} \cdots \mathrm{Sq}^{n_k}$ is admissible (i. e. $n_1 \geqslant 2n_2$, $n_2 \geqslant 2n_3$, ..., $n_{k-1} \geqslant 2n_k$), and moreover $n_k \geqslant 2n$. It is then proved in [1, 16.5.2] that PAR is a basis of $R_{\mathscr{F}}$ as a free right $\mathscr{F}_0$-module. Using this, one also has

(12.2) Lemma. *As a right $\mathscr{A}$-module, $\bar{R}$ is free. Moreover, the images $\bar{\rho}$ of the preadmissible relations $\rho \in$ PAR under the quotient map $R_{\mathscr{F}} \twoheadrightarrow \bar{R}$ form a basis of this free right $\mathscr{A}$-module.*

Proof. One has

$$\begin{aligned}
R_{\mathscr{F}}/(R_{\mathscr{F}} \cdot R_{\mathscr{F}}) &\cong \left(\bigoplus_{\rho \in \mathrm{PAR}} \rho \mathscr{F}_0\right) \Big/ \left(\left(\bigoplus_{\rho \in \mathrm{PAR}} \rho \mathscr{F}_0\right) \cdot R_{\mathscr{F}}\right) \\
&= \left(\bigoplus_{\rho \in \mathrm{PAR}} \rho \mathscr{F}_0\right) \Big/ \left(\bigoplus_{\rho \in \mathrm{PAR}} \rho\,(\mathscr{F}_0 \cdot R_{\mathscr{F}})\right) \\
&= \left(\bigoplus_{\rho \in \mathrm{PAR}} \rho \mathscr{F}_0\right) \Big/ \left(\bigoplus_{\rho \in \mathrm{PAR}} \rho R_{\mathscr{F}}\right) \\
&\cong \bigoplus_{\rho \in \mathrm{PAR}} \bar{\rho}\,(\mathscr{F}_0/R_{\mathscr{F}}) \\
&= \bigoplus_{\rho \in \mathrm{PAR}} \bar{\rho}\mathscr{A};
\end{aligned}$$

moreover the above equalities/isomorphisms are easily seen to be compatible with the right $\mathscr{A}$-module structure. □

In particular we see that every element of $R_{\mathscr{F}}$ can be written in a unique way in the form

$$\rho^{(2)} + \sum_i \alpha_i[n_i, m_i]\beta_i \tag{12.3}$$

with $\rho^{(2)} \in R_{\mathscr{F}} \cdot R_{\mathscr{F}}$, $\alpha_i[n_i, m_i] \in$ PAR and β_i an admissible monomial.

(12.4) Definition. The *left action operator*

$$L : \mathscr{A} \otimes R_{\mathscr{F}} \to \mathscr{A} \otimes \mathscr{A}$$

of degree -1 is defined as follows. For odd p let L be the zero map. For $p = 2$, let first the additive map $L_{\mathscr{F}} : \mathscr{F}_0^{\leqslant 2} \to \mathscr{A} \otimes \mathscr{A}$ be given by the formula

$$L_{\mathscr{F}}(\mathrm{Sq}^n \mathrm{Sq}^m) = \sum_{\substack{n_1+n_2=n \\ m_1+m_2=m \\ m_1,\, n_2 \text{ odd}}} \mathrm{Sq}^{n_1}\mathrm{Sq}^{m_1} \otimes \mathrm{Sq}^{n_2}\mathrm{Sq}^{m_2}$$

($n, m \geqslant 0$; remember that $\mathrm{Sq}^0 = 1$). Equivalently, using the algebra structure on $\mathscr{A} \otimes \mathscr{A}$ one may write

$$L_{\mathscr{F}}(\mathrm{Sq}^n \mathrm{Sq}^m) = (1 \otimes \mathrm{Sq}^1)\delta(\mathrm{Sq}^{n-1})(\mathrm{Sq}^1 \otimes 1)\delta(\mathrm{Sq}^{m-1}).$$

Restricting this map to $R_{\mathscr{F}}^{\leqslant 2} \subset \mathscr{F}_0^{\leqslant 2}$ gives a map $L_R : R_{\mathscr{F}}^{\leqslant 2} \to \mathscr{A} \otimes \mathscr{A}$. It is thus an additive map given on the Adem relations $[n, m]$, for $0 < n < 2m$, by

$$L_R[n, m] = L_{\mathscr{F}}(\mathrm{Sq}^n \mathrm{Sq}^m) + \sum_{k=\max\{0,n-m+1\}}^{\min\{n/2,m-1\}} \binom{m-k-1}{n-2k} L_{\mathscr{F}}(\mathrm{Sq}^{n+m-k}\mathrm{Sq}^k).$$

Next we define the map

$$\bar{L} : \mathscr{A} \otimes \bar{R} \to \mathscr{A} \otimes \mathscr{A}$$

as the right $\mathscr{A}$-module homomorphism which satisfies

$$\bar{L}(a \otimes \overline{\alpha[n,m]}) = \delta(\varkappa(a)\bar{\bar{\alpha}})L_R[n,m] \tag{12.5}$$

with $\alpha[n,m] \in \mathrm{PAR}$; by (12.2) such a homomorphism exists and is unique.

Finally, $\bar{L}$ yields a unique linear map $L : \mathscr{A} \otimes R_{\mathscr{F}} \to \mathscr{A} \otimes \mathscr{A}$ by composing $\bar{L}$ with the quotient map $\mathscr{A} \otimes R_{\mathscr{F}} \twoheadrightarrow \mathscr{A} \otimes \bar{R}$. Thus one has

$$L(\mathscr{A} \otimes (R_{\mathscr{F}} \cdot R_{\mathscr{F}})) = 0.$$

The map L is the left action operator in [1, 14.4] where the following lemma is proved (see [1, 14.4.3]):

(12.6) Lemma. *The map $\bar{L}$ satisfies the equalities*

$$\begin{aligned}
\bar{L}(a \otimes [n,m]) &= \varkappa(a)L_R[n,m] \\
\bar{L}(a \otimes br) &= \bar{L}(ab \otimes r) + \delta(a)\bar{L}(b \otimes r) \\
\bar{L}(a \otimes rb) &= \bar{L}(a \otimes r)\delta(b)
\end{aligned}$$

for any $a, b \in \mathscr{A}$, $r \in \bar{R}$.

□

We observe that L can be alternatively constructed as follows. Let

$$\tilde{L} : \bar{R} \to \mathscr{A} \otimes \mathscr{A}$$

be the map given by

$$\tilde{L}(\bar{r}) = \bar{L}(\mathrm{Sq}^1 \otimes \bar{r}).$$

Then one has

(12.7) Proposition. *For any $a \in \mathscr{A}$, $r \in R_{\mathscr{F}}$ one has*

$$L(a \otimes r) = \delta(\varkappa(a))\tilde{L}(\bar{r});$$

moreover $\tilde{L}$ is a homomorphism of $\mathscr{A}$-$\mathscr{A}$-bimodules, hence uniquely determined by its values on the Adem relations, which are

$$\tilde{L}([n,m]) = L_R[n,m].$$

Proof. For any $a \in \mathscr{A}$, $\alpha[n,m] \in \mathrm{PAR}$ and β admissible we have

$$\begin{aligned}
L(a \otimes \alpha[n,m]\beta) &= \bar{L}(a \otimes \overline{\alpha[n,m]})\bar{\bar{\beta}} \\
&= \delta(\varkappa(a)\bar{\bar{\alpha}})L_R[n,m]\delta\bar{\bar{\beta}} \\
&= \delta\varkappa(a)\delta\bar{\bar{\alpha}}L_R[n,m]\delta\bar{\bar{\beta}} \\
&= \delta\varkappa(a)\delta(\varkappa(\mathrm{Sq}^1)\bar{\bar{\alpha}})L_R[n,m]\delta\bar{\bar{\beta}} \\
&= \delta\varkappa(a)L(\mathrm{Sq}^1 \otimes \alpha[n,m]\beta) \\
&= \delta\varkappa(a)\tilde{L}(\alpha[n,m]\beta).
\end{aligned}$$

Then using (12.3) we see that the same identity holds for $L(a \otimes r)$ with any $r \in R_{\mathscr{F}}$.

Next for any $a \in \mathscr{A}$, $r \in R_{\mathscr{F}}$ we have by (12.6) and $\varkappa\mathrm{Sq}^1 = \mathrm{Sq}^0 = 1$,

$$\begin{aligned}
\tilde{L}(a\bar{r}) &= \bar{L}(\mathrm{Sq}^1 \otimes a\bar{r}) \\
&= \bar{L}(\mathrm{Sq}^1 a \otimes \bar{r}) + \delta(\mathrm{Sq}^1)\bar{L}(a \otimes \bar{r}) \\
&= \delta(\varkappa(\mathrm{Sq}^1 a))\tilde{L}(\bar{r}) + \delta(\mathrm{Sq}^1\varkappa(a))\tilde{L}(\bar{r}) \\
&= \delta(\varkappa(\mathrm{Sq}^1 a) + \mathrm{Sq}^1\varkappa(a))\tilde{L}(\bar{r}) \\
&= \delta(a)\tilde{L}(\bar{r}).
\end{aligned}$$

Thus $\tilde{L}$ is a left $\mathscr{A}$-module homomorphism. It is also clearly a right $\mathscr{A}$-module homomorphism since $\bar{L}$ is.

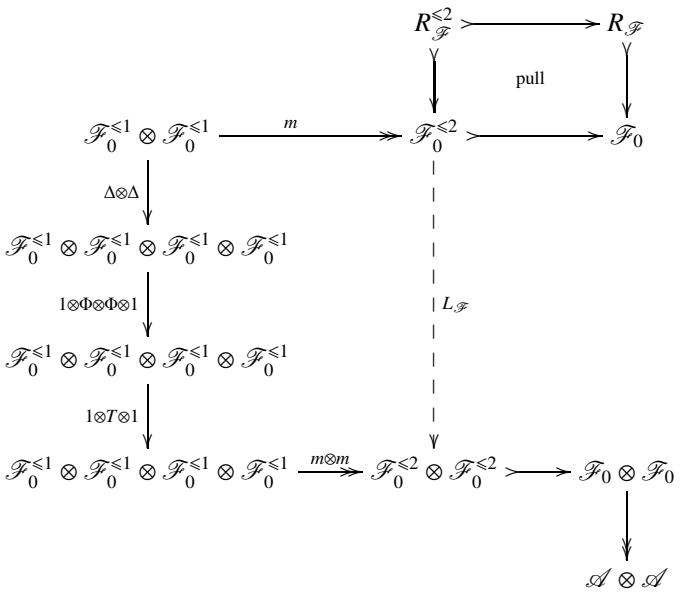
Finally by (12.5) we have

$$L_R[n,m] = \delta(\varkappa(\mathrm{Sq}^1))L_R[n,m] = \bar{L}(\mathrm{Sq}^1 \otimes \overline{[n,m]}) = \tilde{L}([n,m]).$$

□

Explicit calculation of the left coaction operator L_* is as follows. For odd p it is the zero map, and for $p = 2$ we first define the additive map $L_{R*} : \mathscr{A}_* \otimes \mathscr{A}_* \to R_{\mathscr{F}_*}^{\leqslant 2}$. It is dual to the composite map $R_{\mathscr{F}}^{\leqslant 2} \to \mathscr{A} \otimes \mathscr{A}$ in the diagram

(12.8)



where Φ is restriction $\mathscr{F}_0^{\leqslant 1} \to \mathscr{F}_0^{\leqslant 1}$ of the map $\mathscr{F}_0 \to \mathscr{F}_0$ given by

$$\Phi(x) = \mathrm{Sq}^1 \varkappa(x),$$

so that one has

$$\Phi(\mathrm{Sq}^n) = \begin{cases} \mathrm{Sq}^n, & n \equiv 1 \mod 2 \\ 0, & n \equiv 0 \mod 2. \end{cases}$$

Indeed by (12.4) we have

$$L_{\mathscr{F}}(\mathrm{Sq}^n \mathrm{Sq}^m) = (1 \otimes \mathrm{Sq}^1)\Delta(\mathrm{Sq}^{n-1})(\mathrm{Sq}^1 \otimes 1)\Delta(\mathrm{Sq}^{m-1}) = (1 \otimes \mathrm{Sq}^1)\Delta\varkappa(\mathrm{Sq}^n)(\mathrm{Sq}^1 \otimes 1)\Delta\varkappa(\mathrm{Sq}^m);$$

on the other hand we saw in (6.7) that one has

$$\Delta\varkappa = (\varkappa \otimes 1)\Delta = (1 \otimes \varkappa)\Delta,$$

so that we can write

$$L_{\mathscr{F}}(\mathrm{Sq}^n \mathrm{Sq}^m) = (1 \otimes \mathrm{Sq}^1\varkappa)\Delta(\mathrm{Sq}^n)(\mathrm{Sq}^1\varkappa \otimes 1)\Delta(\mathrm{Sq}^m) = (1 \otimes \Phi)\Delta(\mathrm{Sq}^n)(\Phi \otimes 1)\Delta(\mathrm{Sq}^m).$$

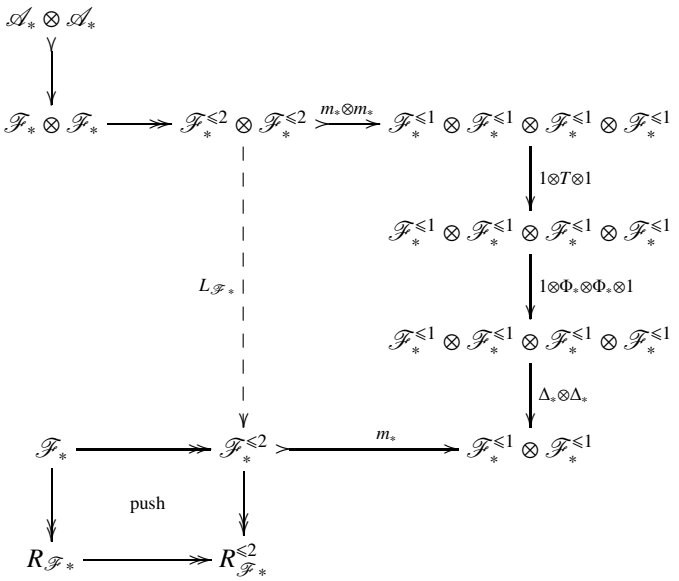
So the map dual of Φ is the map $\Phi_* : \mathbb{F}[\zeta_1] \to \mathbb{F}[\zeta_1]$ given by factorization through $\mathscr{A}_* \twoheadrightarrow \mathbb{F}[\zeta_1]$ of the map $\Phi_* : \mathscr{A}_* \to \mathscr{A}_*$ given on the monomial basis by

$$\Phi_*(\zeta_1^{n_1}\zeta_2^{n_2}\cdots) = \begin{cases} \zeta_1^{n_1}\zeta_2^{n_2}\cdots, & n_1 \equiv 1 \mod 2 \\ 0, & n_1 \equiv 0 \mod 2. \end{cases}$$

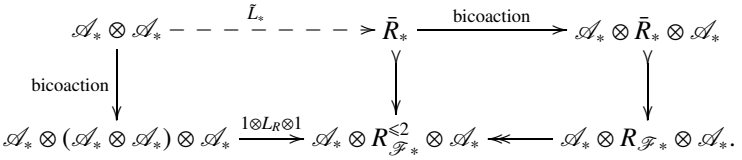
Equivalently, by (9.6) and (9.7), $\Phi_* = \varkappa_* \mathrm{Sq}^1_*$ is the map $\zeta_1 \frac{\partial}{\partial \zeta_1}$.

Thus the map L_{R*} is the composite $\mathscr{A}_* \otimes \mathscr{A}_* \to R^{\leqslant 2}_{\mathscr{F}*}$ in the diagram

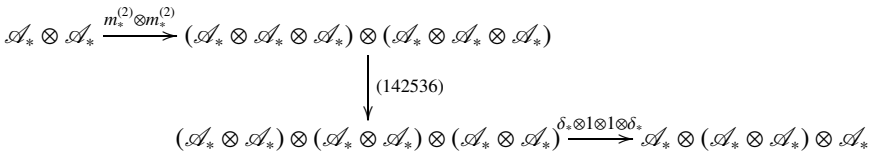
(12.9)



Now by (12.7) we know that $\tilde{L}$ is a bimodule homomorphism, and moreover $\bar{R}$ is generated by $R^{\leqslant 2}_{\mathscr{F}} \cong \bar{R}^{\leqslant 2} \subset \bar{R}$ as an $\mathscr{A}$-$\mathscr{A}$-bimodule, so knowledge of L_R (actually already of $L_{\mathscr{F}}$ whose restriction it is) determines $\tilde{L}$ and, by (12.7), also L. Dually, one can reconstruct $\tilde{L}_*$ and then L_* from $L_{\mathscr{F}*}$ via the diagram



Here the bicoaction $\mathscr{A}_* \otimes \mathscr{A}_* \to \mathscr{A}_* \otimes (\mathscr{A}_* \otimes \mathscr{A}_*) \otimes \mathscr{A}_*$ is the composite



We next note the following

(12.10) Lemma. *The map $\tilde{L}_*$ is a biderivation, i. e.*

$$\tilde{L}_*(x_1x_2, y) = x_1\tilde{L}_*(x_2, y) + x_2\tilde{L}_*(x_1, y),$$
$$\tilde{L}_*(x, y_1y_2) = y_1\tilde{L}_*(x, y_2) + y_2\tilde{L}_*(x, y_1)$$

for any $x, x_1, x_2, y, y_1, y_2 \in \mathscr{A}_*$.

Proof.

□

It thus follows that $\tilde{L}_*$ is fully determined by its values $\tilde{L}_*(\zeta_n \otimes \zeta_{n'})$ on the Milnor generators. To calculate the bicoaction on these first note that we have

$$m_*^{(2)}(\zeta_n) = (1 \otimes m_*)m_*(\zeta_n) = \sum_{i+i'=n} \zeta_i^{2^{i'}} \otimes m_*(\zeta_{i'}) = \sum_{i+j+k=n} \zeta_i^{2^{j+k}} \otimes \zeta_j^{2^k} \otimes \zeta_k,$$

where as always $\zeta_0 = 1$. For the coaction on $\zeta_n \otimes \zeta_{n'}$ this then gives in succession

$$\begin{aligned}\zeta_n \otimes \zeta_{n'} &\mapsto \sum_{\substack{i+j+k=n\\ i'+j'+k'=n'}} \zeta_i^{2^{j+k}} \otimes \zeta_j^{2^k} \otimes \zeta_k \otimes \zeta_{i'}^{2^{j'+k'}} \otimes \zeta_{j'}^{2^{k'}} \otimes \zeta_{k'}\\ &\mapsto \sum_{\substack{i+j+k=n\\ i'+j'+k'=n'}} \zeta_i^{2^{j+k}} \otimes \zeta_{i'}^{2^{j'+k'}} \otimes \zeta_j^{2^k} \otimes \zeta_{j'}^{2^{k'}} \otimes \zeta_k \otimes \zeta_{k'}\\ &\mapsto \sum_{\substack{i+j+k=n\\ i'+j'+k'=n'}} \zeta_i^{2^{j+k}} \zeta_{i'}^{2^{j'+k'}} \otimes \zeta_j^{2^k} \otimes \zeta_{j'}^{2^{k'}} \otimes \zeta_k \zeta_{k'},\end{aligned}$$

so that for the values of $\tilde{L}_*$ we have the equation

$$\iota\tilde{L}_*(\zeta_n \otimes \zeta_{n'}) = \sum_{\substack{i+j+k=n\\ i'+j'+k'=n'}} \zeta_i^{2^{j+k}} \zeta_{i'}^{2^{j'+k'}} \otimes L_{R*}(\zeta_j^{2^k} \otimes \zeta_{j'}^{2^{k'}}) \otimes \zeta_k \zeta_{k'}$$

where ι is the above embedding $\bar{R}_* \rightarrowtail \mathscr{A}_* \otimes R^{\leqslant 2}_{\mathscr{F}*} \otimes \mathscr{A}_*$. Thus we only have to know the values of $L_{\mathscr{F}*}$ on the elements of the form $\zeta_j^{2^k} \otimes \zeta_{j'}^{2^{k'}}$ for $j \geqslant 0$, $k \geqslant 0$. Obviously these values are zero for $j > 2$ or $j' > 2$. They are also zero for $j = 0$ or $j' = 0$ since $\Phi_*(1) = 0$. There thus remain four cases $j = j' = 1$, $j = j' = 2$, $j = 1$, $j' = 2$, and $j = 2$, $j' = 1$. We then have under $L_{\mathscr{F}*}$

$$\begin{aligned}\zeta_1^{2^k} \otimes \zeta_1^{2^{k'}} &\xrightarrow{m_*\otimes m_*} (\zeta_1 \otimes 1 + 1 \otimes \zeta_1)^{2^k} \otimes (\zeta_1 \otimes 1 + 1 \otimes \zeta_1)^{2^{k'}} =\\ &\qquad \zeta_1^{2^k} \otimes 1 \otimes \zeta_1^{2^{k'}} \otimes 1 + \zeta_1^{2^k} \otimes 1 \otimes 1 \otimes \zeta_1^{2^{k'}} + 1 \otimes \zeta_1^{2^k} \otimes \zeta_1^{2^{k'}} \otimes 1 + 1 \otimes \zeta_1^{2^k} \otimes 1 \otimes \zeta_1^{2^{k'}}\\ &\xrightarrow{1\otimes T\otimes 1} \zeta_1^{2^k} \otimes \zeta_1^{2^{k'}} \otimes 1 \otimes 1 + \zeta_1^{2^k} \otimes 1 \otimes 1 \otimes \zeta_1^{2^{k'}} + 1 \otimes \zeta_1^{2^{k'}} \otimes \zeta_1^{2^k} \otimes 1 + 1 \otimes 1 \otimes \zeta_1^{2^k} \otimes \zeta_1^{2^{k'}}\\ &\xrightarrow{1\otimes\Phi_*\otimes\Phi_*\otimes 1} 0 + 0 + 1 \otimes \Phi_*\zeta_1^{2^{k'}} \otimes \Phi_*\zeta_1^{2^k} \otimes 1 + 0\\ &\xrightarrow{\Delta_*\otimes\Delta_*} \Phi_*\zeta_1^{2^{k'}} \otimes \Phi_*\zeta_1^{2^k}.\end{aligned}$$

We thus have

$$L_{\mathscr{F}*}(\zeta_1^{2^k} \otimes \zeta_1^{2^{k'}}) = \begin{cases} M_{1,1}, & k = k' = 0\\ 0 & \text{otherwise.}\end{cases}$$

We next take $j = j' = 2$; then

$$\begin{aligned}\zeta_2^{2^k} \otimes \zeta_2^{2^{k'}} &\xrightarrow{m_*\otimes m_*} (\zeta_1^2 \otimes \zeta_1)^{2^k} \otimes (\zeta_1^2 \otimes \zeta_1)^{2^{k'}} = \zeta_1^{2^{k+1}} \otimes \zeta_1^{2^k} \otimes \zeta_1^{2^{k'+1}} \otimes \zeta_1^{2^{k'}}\\ &\xrightarrow{1\otimes T\otimes 1} \zeta_1^{2^{k+1}} \otimes \zeta_1^{2^{k'+1}} \otimes \zeta_1^{2^k} \otimes \zeta_1^{2^{k'}}\\ &\xrightarrow{1\otimes\Phi_*\otimes\Phi_*\otimes 1} \zeta_1^{2^{k+1}} \otimes \Phi_*\zeta_1^{2^{k'+1}} \otimes \Phi_*\zeta_1^{2^k} \otimes \zeta_1^{2^{k'}} = 0\\ &\xrightarrow{\Delta_*\otimes\Delta_*} 0,\end{aligned}$$

so that

$$L_{\mathscr{F}*}(\zeta_2^{2^k} \otimes \zeta_2^{2^{k'}}) = 0$$

for all k and k'. Next for $j = 2$, $j' = 1$ we have

$$\begin{aligned}\zeta_2^{2^k} \otimes \zeta_1^{2^{k'}} &\xrightarrow{m_*\otimes m_*} (\zeta_1^2 \otimes \zeta_1)^{2^k} \otimes (\zeta_1 \otimes 1 + 1 \otimes \zeta_1)^{2^{k'}} = \zeta_1^{2^{k+1}} \otimes \zeta_1^{2^k} \otimes \zeta_1^{2^{k'}} \otimes 1 + \zeta_1^{2^{k+1}} \otimes \zeta_1^{2^k} \otimes 1 \otimes \zeta_1^{2^{k'}}\\ &\xrightarrow{1\otimes T\otimes 1} \zeta_1^{2^{k+1}} \otimes \zeta_1^{2^{k'}} \otimes \zeta_1^{2^k} \otimes 1 + \zeta_1^{2^{k+1}} \otimes 1 \otimes \zeta_1^{2^k} \otimes \zeta_1^{2^{k'}}\\ &\xrightarrow{1\otimes\Phi_*\otimes\Phi_*\otimes 1} \zeta_1^{2^{k+1}} \otimes \Phi_*\zeta_1^{2^{k'}} \otimes \Phi_*\zeta_1^{2^k} \otimes 1 + 0\\ &\xrightarrow{\Delta_*\otimes\Delta_*} \zeta_1^{2^{k+1}}\Phi_*\zeta_1^{2^{k'}} \otimes \Phi_*\zeta_1^{2^k},\end{aligned}$$

hence

$$L_{\mathscr{F}*}(\zeta_2^{2^k} \otimes \zeta_1^{2^{k'}}) = \begin{cases} M_{1,1}^2 + M_1 M_{2,1}, & k = k' = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Finally for $j = 1$, $j' = 2$ we get

$$\begin{aligned} \zeta_1^{2^k} \otimes \zeta_2^{2^{k'}} &\xrightarrow{m_*\otimes m_*} (\zeta_1 \otimes 1 + 1 \otimes \zeta_1)^{2^k} \otimes (\zeta_1^2 \otimes \zeta_1)^{2^{k'}} = \zeta_1^{2^k} \otimes 1 \otimes \zeta_1^{2^{k'+1}} \otimes \zeta_1^{2^{k'}} + 1 \otimes \zeta_1^{2^k} \otimes \zeta_1^{2^{k'+1}} \otimes \zeta_1^{2^{k'}} \\ &\xrightarrow{1\otimes T\otimes 1} \zeta_1^{2^k} \otimes \zeta_1^{2^{k'+1}} \otimes 1 \otimes \zeta_1^{2^{k'}} + 1 \otimes \zeta_1^{2^{k'+1}} \otimes \zeta_1^{2^k} \otimes \zeta_1^{2^{k'}} \\ &\xrightarrow{1\otimes\Phi_*\otimes\Phi_*\otimes 1} 0 + 0 \\ &\xrightarrow{\Delta_*\otimes\Delta_*} 0, \end{aligned}$$

so that

$$L_{\mathscr{F}*}(\zeta_1^{2^k} \otimes \zeta_2^{2^{k'}}) = 0$$

for all k and k'.

To pass to L_{R*} from these values means just nullifying all monomials which do not contain $M_{1,1}$; we thus obtain

$$\begin{aligned} L_{R*}(\zeta_1 \otimes \zeta_1) &= M_{1,1}, \\ L_{R*}(\zeta_2 \otimes \zeta_1) &= M_{1,1}^2, \end{aligned}$$

and $L_{R*}(\zeta_j^{2^k} \otimes \zeta_{j'}^{2^{k'}}) = 0$ in all other cases.

From this we easily obtain

(12.11) Proposition. $\iota\tilde{L}_*(\zeta_n \otimes \zeta_{n'}) = \zeta_{n-1}^2 \zeta_{n'-1}^2 \otimes M_{1,1} \otimes 1 + \zeta_{n-2}^4 \zeta_{n'-1}^2 \otimes M_{1,1}^2 \otimes 1$

□

where now $\zeta_{n-2} = 0$ for $n = 1$ is understood. Solving $\tilde{L}_*(\zeta_n, \zeta_{n'})$ from these equations is then straightforward. In this way we obtain

$$\begin{aligned}
\tilde{L}_*(\zeta_1,\zeta_1) &= M_{1,1}\\
\tilde{L}_*(\zeta_1,\zeta_2) &= M_{2,1,1}\\
\tilde{L}_*(\zeta_2,\zeta_1) &= M_{2,1,1} + M_{1,1}^2\\
\tilde{L}_*(\zeta_2,\zeta_2) &= M_{4,1,1} + M_{2,3,1} + M_{2,1,2,1}\\
\tilde{L}_*(\zeta_1,\zeta_3) &= M_{4,2,1,1}\\
\tilde{L}_*(\zeta_3,\zeta_1) &= M_{4,2,1,1} + M_{2,1,1}^2\\
\tilde{L}_*(\zeta_2,\zeta_3) &= M_{6,2,1,1} + M_{4,4,1,1} + M_{4,2,3,1} + M_{4,2,1,2,1} + M_{2,4,2,1,1}\\
\tilde{L}_*(\zeta_3,\zeta_2) &= M_{6,2,1,1} + M_{4,4,1,1} + M_{4,2,3,1} + M_{4,2,1,2,1} + M_{2,4,2,1,1}\\
&\quad + M_5^2 + M_{4,1}^2 + M_{3,2}^2 + M_{2,1,1,1}^2 + M_1^2 M_{2,1,1}^2 + M_1^4 M_3^2\\
\tilde{L}_*(\zeta_3,\zeta_3) &= M_{8,4,1,1} + M_{8,2,3,1} + M_{8,2,1,2,1} + M_{4,6,3,1} + M_{4,6,1,2,1} + M_{4,2,5,2,1} + M_{4,2,4,3,1} + M_{4,2,4,1,2,1}\\
&\quad + M_{4,2,1,4,2,1}\\
\tilde{L}_*(\zeta_1,\zeta_4) &= M_{8,4,2,1,1}\\
\tilde{L}_*(\zeta_4,\zeta_1) &= M_{8,4,2,1,1} + M_{4,2,1,1}^2\\
\tilde{L}_*(\zeta_2,\zeta_4) &= M_{10,4,2,1,1} + M_{8,6,2,1,1} + M_{8,4,4,1,1} + M_{8,4,2,3,1} + M_{8,4,2,1,2,1} + M_{8,2,4,2,1,1} + M_{2,8,4,2,1,1}\\
\tilde{L}_*(\zeta_4,\zeta_2) &= M_{10,4,2,1,1} + M_{8,6,2,1,1} + M_{8,4,4,1,1} + M_{8,4,2,3,1} + M_{8,4,2,1,2,1} + M_{8,2,4,2,1,1} + M_{2,8,4,2,1,1}\\
&\quad + M_9^2 + M_{7,2}^2 + M_{5,4}^2 + M_{6,2,1}^2 + M_{4,4,1}^2 + M_{4,3,2}^2 + M_{4,2,1,1,1}^2 + M_{3,4,2}^2 + M_{2,4,2,1}^2\\
&\quad + M_1^2 M_{4,2,1,1}^2 + M_1^8 M_5^2 + M_{2,1}^4 M_3^2\\
\tilde{L}_*(\zeta_3,\zeta_4) &= M_{12,6,2,1,1} + M_{12,4,4,1,1} + M_{12,4,2,3,1} + M_{12,4,2,1,2,1} + M_{12,2,4,2,1,1} + M_{8,8,4,1,1} + M_{8,8,2,3,1}\\
&\quad + M_{8,8,2,1,2,1} + M_{8,4,6,3,1} + M_{8,4,6,1,2,1} + M_{8,4,2,5,2,1} + M_{8,4,2,4,3,1} + M_{8,4,2,4,1,2,1} + M_{8,4,2,1,4,2,1}\\
&\quad + M_{4,10,4,2,1,1} + M_{4,8,6,2,1,1} + M_{4,8,4,4,1,1} + M_{4,8,4,2,3,1} + M_{4,8,4,2,1,2,1} + M_{4,8,2,4,2,1,1} + M_{4,2,8,4,2,1,1},
\end{aligned}$$

etc.

Having $\tilde{L}_*$ we then can obtain L_* by the dual of (12.7) as

$$L_*(x,y) = \sum \zeta_1 x_\ell y_{\ell'} \otimes \tilde{L}_*(x_r, y_{r'}) \tag{12.12}$$

for $x, y \in \mathscr{A}_*$, with

$$m_*(x) = \sum x_\ell \otimes x_r,\ m_*(y) = \sum y_{\ell'} \otimes y_{r'}.$$

13. T S

(13.1) Definition. The *symmetry operator*

$$S : R_{\mathscr{F}} \to \mathscr{A} \otimes \mathscr{A}$$

of degree -1 is defined as follows. For odd p, let S be the zero map. For $p = 2$ let the elements $S_n \in \mathscr{A}\otimes\mathscr{A}$, $n \geqslant 0$, be given by

$$S_n = \sum_{\substack{n_1+n_2=n-1\\ n_1,\, n_2 \text{ odd}}} \mathrm{Sq}^{n_1} \otimes \mathrm{Sq}^{n_2} = (\mathrm{Sq}^1 \otimes \mathrm{Sq}^1)\delta(\mathrm{Sq}^{n-3}),$$

i. e.

$$\begin{aligned}
S_{2k} &= 0,\\
S_{2k+1} &= \sum_{0\leqslant i<k} \mathrm{Sq}^{2i+1} \otimes \mathrm{Sq}^{2(k-i)-1},
\end{aligned}$$

$k \geqslant 0$. Then let the linear map $S_{\mathscr{F}} : \mathscr{F}_0^{\leqslant 2} \to \mathscr{A} \otimes \mathscr{A}$ be given by

$$\begin{aligned} S_{\mathscr{F}}(\mathrm{Sq}^n\mathrm{Sq}^m) &= S_n\delta(\mathrm{Sq}^m) + \delta(\mathrm{Sq}^n)S_m + \delta(\mathrm{Sq}^{n-1})S_{m+1} \\ &= (\mathrm{Sq}^1 \otimes \mathrm{Sq}^1)\delta(\mathrm{Sq}^{n-3}\mathrm{Sq}^m) + \delta(\mathrm{Sq}^n)(\mathrm{Sq}^1 \otimes \mathrm{Sq}^1)\delta(\mathrm{Sq}^{m-3}) + \delta(\mathrm{Sq}^{n-1})(\mathrm{Sq}^1 \otimes \mathrm{Sq}^1)\delta(\mathrm{Sq}^{m-2}), \end{aligned}$$

$n, m \geqslant 0$. Next define the map $S_R : R_{\mathscr{F}}^{\leqslant 2} \to \mathscr{A} \otimes \mathscr{A}$ by restriction to $R_{\mathscr{F}}^{\leqslant 2} \subset \mathscr{F}_0^{\leqslant 2}$. Thus on the Adem relations this map is given by

$$S_R[n,m] = S_{\mathscr{F}}(\mathrm{Sq}^n\mathrm{Sq}^m) + \sum_{k=\max\{0,n-m+1\}}^{\min\{n/2,m-1\}} \binom{m-k-1}{n-2k} S_{\mathscr{F}}(\mathrm{Sq}^{n+m-k}\mathrm{Sq}^k). \tag{13.2}$$

Now let us define the map

$$\bar{S} : \bar{R} \to \mathscr{A} \otimes \mathscr{A}$$

as a unique right $\mathscr{A}$-module homomorphism satisfying

$$\bar{S}(\overline{\alpha[n,m]}) = \delta(\alpha)S_R[n,m] + (1+T)\bar{L}(\alpha \otimes \overline{[n,m]})$$

for $\alpha[n,m] \in \mathrm{PAR}$. Then finally this determines a unique linear map $S : R_{\mathscr{F}} \to \mathscr{A} \otimes \mathscr{A}$ by composing with the quotient map $R_{\mathscr{F}} \twoheadrightarrow \bar{R}$.

The map S is the symmetry operator in [1, 14.5.2] where the following lemma is proved.

(13.3) Lemma. *The map $\bar{S}$ satisfies the equations*

$$\begin{aligned} \bar{S}([n,m]) &= S_R[n,m] \\ \bar{S}(ar) &= \delta(a)\bar{S}(r) + (1+T)\bar{L}(a \otimes r) \\ \bar{S}(ra) &= \bar{S}(r)\delta(a) \end{aligned}$$

for any $0 < n < 2m$, $a \in \mathscr{A}$ and $r \in \bar{R}$.

We now turn to the dual $S_* : \mathscr{A}_* \otimes \mathscr{A}_* \to R_{\mathscr{F}*}$ of S (dually to the above, the image of this operator actually lies in $\bar{R}_* \subset R_{\mathscr{F}*}$ and so defines the operator $\bar{S}_* : \mathscr{A}_* \otimes \mathscr{A}_* \to \bar{R}_*$). Since we know that S_* is a biderivation, it suffices to compute the values $S_*(\zeta_n \otimes \zeta_{n'})$. Now dually to the equation

$$S(a[n,m]b) = \delta(a)S_R([n,m])\delta(b) + (1+T)L(a \otimes [n,m]b) = \delta(a)S_R([n,m])\delta(b) + (1+T)\left(\delta\varkappa(a)L_R([n,m])\delta(b)\right)$$

we have

$$\begin{aligned} &\iota S_*(\zeta_n \otimes \zeta_{n'}) = \\ &\sum_{\substack{i+j+k=n \\ i'+j'+k'=n'}} \left(\zeta_i^{2^{j+k}}\zeta_{i'}^{2^{j'+k'}} \otimes S_{R*}(\zeta_j^{2^k} \otimes \zeta_{j'}^{2^{k'}}) \otimes \zeta_k\zeta_{k'} + \zeta_1\zeta_i^{2^{j+k}}\zeta_{i'}^{2^{j'+k'}} \otimes \left(L_{R*}(\zeta_j^{2^k} \otimes \zeta_{j'}^{2^{k'}}) + L_{R*}(\zeta_{j'}^{2^{k'}} \otimes \zeta_j^{2^k})\right) \otimes \zeta_k\zeta_{k'}\right) \\ &\quad = \sum_{\substack{i+j+k=n \\ i'+j'+k'=n'}} \zeta_i^{2^{j+k}}\zeta_{i'}^{2^{j'+k'}} \otimes S_{R*}(\zeta_j^{2^k} \otimes \zeta_{j'}^{2^{k'}}) \otimes \zeta_k\zeta_{k'} + \zeta_1\zeta_{n-2}^4\zeta_{n'-1}^2 \otimes M_{1,1}^2 \otimes 1 + \zeta_1\zeta_{n-1}^2\zeta_{n'-2}^4 \otimes M_{1,1}^2 \otimes 1, \end{aligned}$$

with $\zeta_0 = 1$ and $\zeta_n = 0$ for $n < 0$, as before.

It thus remains to find the values $S_{R*}(\zeta_j^{2^k} \otimes \zeta_{j'}^{2^{k'}})$ — which in turn are images of the corresponding values of $S_{\mathscr{F}*}$ under the map $\mathscr{F}_* \twoheadrightarrow R_{\mathscr{F}*}$. To find the latter, let us first define another intermediate operator

$$S^1 : \mathscr{F}_0^{\leqslant 1} \to \mathscr{A} \otimes \mathscr{A}$$

by the equation

$$S^1(\mathrm{Sq}^n) = S_{n+1} = (\mathrm{Sq}^1 \otimes \mathrm{Sq}^1)\delta\varkappa\varkappa(\mathrm{Sq}^n) = \sum_{\substack{n_1+n_2=n \\ n_1,\, n_2 \text{ odd}}} \mathrm{Sq}^{n_1} \otimes \mathrm{Sq}^{n_2},$$

so that we have

$$S_{\mathscr{F}}m(\mathrm{Sq}^n \otimes \mathrm{Sq}^m) = S_{\mathscr{F}}(\mathrm{Sq}^n\mathrm{Sq}^m) = S^1\varkappa(\mathrm{Sq}^n)\delta(\mathrm{Sq}^m) + \delta(\mathrm{Sq}^n)S^1\varkappa(\mathrm{Sq}^m) + \delta\varkappa(\mathrm{Sq}^n)S^1(\mathrm{Sq}^m).$$

We have the dual operator

$$S^1_* : \mathscr{A}_* \otimes \mathscr{A}_* \to \mathscr{F}_*^{\leqslant 1}$$

such that dual

$$S_{\mathscr{F}*} : \mathscr{A}_* \otimes \mathscr{A}_* \to \mathscr{F}_*^{\leqslant 2}$$

of $S_{\mathscr{F}}$ is given by

$$\begin{aligned} &m_* S_{\mathscr{F}*}(x \otimes y) = \\ &\sum \left(\zeta_1 S^1_*(x_\ell \otimes y_{\ell'}) \otimes (x_r y_{r'})^{\leqslant 1} + (x_\ell y_{\ell'})^{\leqslant 1} \otimes \zeta_1 S^1_*(x_r \otimes y_{r'}) + (\zeta_1 x_\ell y_{\ell'})^{\leqslant 1} \otimes S^1_*(x_r \otimes y_{r'}) \right) \end{aligned} \tag{13.4}$$

where as before we use the Sweedler notation

$$m_*(x) = \sum x_\ell \otimes x_r, \quad m_*(y) = \sum y_{\ell'} \otimes y_{r'}$$

and

$$(_)^{\leqslant 1} : \mathscr{A}_* \to \mathscr{F}_*^{\leqslant 1}$$

sends ζ_1 to M_1 and all other Milnor generators to 0. Thus we have

$$\begin{aligned} &m_* S_{\mathscr{F}*}(\zeta_j^{2^k} \otimes \zeta_{j'}^{2^{k'}}) = \\ &\sum_{\substack{\ell+r=j \\ \ell'+r'=j'}} \zeta_1 S^1_*(\zeta_\ell^{2^{r+k}} \otimes \zeta_{\ell'}^{2^{r'+k'}}) \otimes (\zeta_r^{2^k} \zeta_{r'}^{2^{k'}})^{\leqslant 1} + (\zeta_\ell^{2^{r+k}} \zeta_{\ell'}^{2^{r'+k'}})^{\leqslant 1} \otimes \zeta_1 S^1_*(\zeta_r^{2^k} \otimes \zeta_{r'}^{2^{k'}}) + (\zeta_1 \zeta_\ell^{2^{r+k}} \zeta_{\ell'}^{2^{r'+k'}})^{\leqslant 1} \otimes S^1_*(\zeta_r^{2^k} \otimes \zeta_{r'}^{2^{k'}}) \end{aligned}$$

Now the operator S^1_* is obviously given by

$$S^1_*(x \otimes y) = \begin{cases} xy, & x = \zeta_1^{n_1}, y = \zeta_1^{n_2}, n_1, n_2 \text{ odd}, \\ 0 & \text{otherwise}, \end{cases} \tag{13.5}$$

so that $S_{\mathscr{F}*}(\zeta_j^{2^k} \otimes \zeta_{j'}^{2^{k'}}) = 0$ whenever $k > 0$ or $k' > 0$. And among the remaining values $S_{\mathscr{F}*}(\zeta_j \otimes \zeta_{j'})$ the only nonzero ones are given by

$$\begin{aligned} S_{\mathscr{F}*}(\zeta_1 \otimes \zeta_1) &= M_3 + M_{1,2} = M_1^3 + M_{2,1}, \\ S_{\mathscr{F}*}(\zeta_1 \otimes \zeta_2) = S_{\mathscr{F}*}(\zeta_2 \otimes \zeta_1) &= M_{2,3} + M_{3,2} = M_1 M_{1,1}^2, \\ S_{\mathscr{F}*}(\zeta_2 \otimes \zeta_2) &= M_{5,2} + M_{4,3} = M_1 M_{2,1}^2. \end{aligned}$$

Then further passing to S_{R*} means, as before, to remove the monomials not containing $M_{1,1}$, so that the only nonzero values of the form $S_{R*}(\zeta_j^{2^k} \otimes \zeta_{j'}^{2^{k'}})$ are

$$S_{R*}(\zeta_1 \otimes \zeta_2) = S_{R*}(\zeta_2 \otimes \zeta_1) = M_1 M_{1,1}^2.$$

Hence we obtain

(13.6) Proposition.

$$\iota S_*(\zeta_n \otimes \zeta_{n'}) = \zeta_{n-1}^2 \zeta_{n'-2}^4 \otimes M_1 M_{1,1}^2 \otimes 1 + \zeta_{n-2}^4 \zeta_{n'-1}^2 \otimes M_1 M_{1,1}^2 \otimes 1 + \zeta_1 \zeta_{n-2}^4 \zeta_{n'-1}^2 \otimes M_{1,1}^2 \otimes 1 + \zeta_1 \zeta_{n-1}^2 \zeta_{n'-2}^4 \otimes M_{1,1}^2 \otimes 1.$$

□

As for $\tilde{L}_*$ above, we then solve these equations obtaining e. g.

$$\begin{aligned} S_*(\zeta_1, \zeta_1) &= 0, \\ S_*(\zeta_1, \zeta_2) = S_*(\zeta_2, \zeta_1) &= M_{2,2,1} + M_1 M_{1,1}^2, \\ S_*(\zeta_2, \zeta_2) &= 0, \\ S_*(\zeta_1, \zeta_3) = S_*(\zeta_3, \zeta_1) &= M_{4,2,2,1} + M_1 M_{2,1,1}^2, \\ S_*(\zeta_2, \zeta_3) = S_*(\zeta_3, \zeta_2) &= M_{6,2,2,1} + M_{4,4,2,1} + M_{2,4,2,2,1} \\ &\quad + M_1 M_5^2 + M_1 M_{4,1}^2 + M_1 M_{3,2}^2 + M_1 M_{2,1,1,1}^2 + M_1^3 M_{2,1,1}^2 + M_1^5 M_3^2, \\ S_*(\zeta_3, \zeta_3) &= 0, \\ S_*(\zeta_1, \zeta_4) = S_*(\zeta_4, \zeta_1) &= M_{8,4,2,2,1} + M_1 M_{4,2,1,1}^2, \\ S_*(\zeta_2, \zeta_4) = S_*(\zeta_4, \zeta_2) &= M_{10,4,2,2,1} + M_{8,6,2,2,1} + M_{8,4,4,2,1} + M_{8,2,4,2,2,1} + M_{2,8,4,2,2,1} \\ &\quad + M_1 M_9^2 + M_1 M_{7,2}^2 + M_1 M_{6,2,1}^2 + M_1 M_{5,4}^2 + M_1 M_{4,4,1}^2 + M_1 M_{4,3,2}^2 + M_1 M_{4,2,1,1,1}^2 \\ &\quad + M_1 M_{3,4,2}^2 + M_1 M_{2,4,2,1}^2 + M_1 M_{2,1}^2 M_3^2 + M_1^3 M_{4,2,1,1}^2 + M_1^9 M_5^2, \end{aligned}$$

etc.

14. T S

In this section we introduce a singular extension $\hat{\mathscr{A}}$ of the Steenrod algebra $\mathscr{A}$ which is determined by the symmetry operator S.

(14.1) Definition. A *singular extension* of a Hopf algebra A is a direct sum diagram

$$R \underset{q}{\overset{i}{\rightleftarrows}} \hat{A} \underset{s}{\overset{p}{\rightleftarrows}} A,$$

i. e. one has $ps = \mathrm{id}_A$, $qi = \mathrm{id}_R$ and $sp+iq = \mathrm{id}_{\hat{A}}$, such that $\hat{A}$ is an algebra with multiplication $\mu : \hat{A}\otimes\hat{A} \to \hat{A}$ and $\hat{A}$ is also a coalgebra with diagonal $\hat{\delta} : \hat{A} \to \hat{A} \otimes \hat{A}$. (Here we do not assume that $\hat{\delta}$ is a homomorphism of algebras, or equivalently that μ is a homomorphism of coalgebras, so that in general $\hat{A}$ is not a Hopf algebra). In addition p is an algebra homomorphism, and s is a coalgebra homomorphism. Moreover (i, p) must be a singular extension of algebras and (q, s) must be a singular extension of coalgebras. This means that the ideal $R = \ker i$ of the algebra $\hat{A}$ is a square zero ideal, i. e. $xy = 0$ for any $x, y \in R$, and the coideal $R = \mathrm{coker}\, s$ of the coalgebra $\hat{A}$ is a square zero coideal, i. e. the composite

$$\hat{A} \xrightarrow{\hat{\delta}} \hat{A} \otimes \hat{A} \xrightarrow{q\otimes q} R \otimes R$$

is zero.

It follows that the $\hat{A}$-$\hat{A}$-bimodule and $\hat{A}$-$\hat{A}$-bicomodule structures on R descend to an A-A-bimodule and A-A-bicomodule structures respectively.

Our basic example of a singular Hopf algebra extension is as follows. We have seen that $\bar{R}$ from (12.1) has an $\mathscr{A}$-$\mathscr{A}$-bimodule structure. Now it also has an $\mathscr{A}$-$\mathscr{A}$-bicomodule structure as follows. On the one hand, there is a diagonal $\Delta_R : R_{\mathscr{F}} \to R^{(2)}_{\mathscr{F}} = \ker(q_{\mathscr{F}} \otimes q_{\mathscr{F}})$ induced in the commutative diagram

$$\begin{array}{ccccc} R_{\mathscr{F}} & \rightarrowtail & \mathscr{F}_0 & \xrightarrow{q_{\mathscr{F}}} & \mathscr{A} \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\delta} \\ R^{(2)}_{\mathscr{F}} & \rightarrowtail & \mathscr{F}_0 \otimes \mathscr{F}_0 & \xrightarrow{q_{\mathscr{F}}\otimes q_{\mathscr{F}}} & \mathscr{A} \otimes \mathscr{A} \end{array}$$

with short exact rows. Moreover there is a short exact sequence

$$R_{\mathscr{F}} \otimes R_{\mathscr{F}} \xrightarrow{i^{(2)}=\binom{i_{\mathscr{F}}\otimes 1}{-1\otimes i_{\mathscr{F}}}} \mathscr{F}_0\otimes R_{\mathscr{F}} \oplus R_{\mathscr{F}}\otimes\mathscr{F}_0 \twoheadrightarrow R^{(2)}_{\mathscr{F}},$$

where $i_{\mathscr{F}} : R_{\mathscr{F}} \hookrightarrow \mathscr{F}_0$ is the inclusion. Since the composite of the quotient map

$$\mathscr{F}_0\otimes R_{\mathscr{F}} \oplus R_{\mathscr{F}}\otimes\mathscr{F}_0 \twoheadrightarrow \mathscr{A}\otimes\bar{R}\oplus\bar{R}\otimes\mathscr{A}$$

with $i^{(2)}$ is obviously zero, we get the induced map

$$R^{(2)}_{\mathscr{F}} \to \mathscr{A}\otimes\bar{R}\oplus\bar{R}\otimes\mathscr{A}.$$

Moreover the diagonal of $\mathscr{F}_0$ factors through this map as follows

$$\begin{array}{ccccccc} \bar{R} & \longleftarrow & R_{\mathscr{F}} & \xrightarrow{i_{\mathscr{F}}} & \mathscr{F}_0 & \xrightarrow{q_{\mathscr{F}}} & \mathscr{A} \\ \downarrow{\scriptstyle\binom{\Delta_\ell}{\Delta_r}} & & \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\delta} \\ \mathscr{A}\otimes\bar{R}\oplus\bar{R}\otimes\mathscr{A} & \longleftarrow & R^{(2)} & \rightarrowtail & \mathscr{F}_0 \otimes \mathscr{F}_0 & \xrightarrow{q_{\mathscr{F}}\otimes q_{\mathscr{F}}} & \mathscr{A} \otimes \mathscr{A} \end{array} \tag{14.2}$$

giving the left, resp. right coaction Δ_ℓ, resp. Δ_r of the desired $\mathscr{A}$-$\mathscr{A}$-bicomodule structure on $\bar{R}$.

Note that the above construction is actually precisely dual to the standard procedure for equipping the kernel of a singular extension with a structure of a bimodule over a base. In particular we could use the

dual diagram

$$
\begin{array}{ccccccc}
\mathscr{A}\otimes\bar{R}\oplus\bar{R}\otimes\mathscr{A} & \longleftarrow & R^{(2)} & \hookrightarrow & \mathscr{F}_0\otimes\mathscr{F}_0 & \xrightarrow{q_{\mathscr{F}}\otimes q_{\mathscr{F}}} & \mathscr{A}\otimes\mathscr{A} \\
\downarrow{\scriptstyle \binom{m_\ell}{m_r}} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\
\bar{R} & \longleftarrow & R_{\mathscr{F}} & \xhookrightarrow{i_{\mathscr{F}}} & \mathscr{F}_0 & \xrightarrow{q_{\mathscr{F}}} & \mathscr{A}
\end{array}
\tag{14.3}
$$

to give $\bar{R}$ via m_ℓ and m_r the structure of $\mathscr{A}$-$\mathscr{A}$-bimodule.

(14.4) Theorem. *There is a unique singular extension of Hopf algebras*

$$\Sigma^{-1}\bar{R} \underset{q}{\overset{i}{\rightleftarrows}} \hat{\mathscr{A}} \underset{s}{\overset{p}{\rightleftarrows}} \mathscr{A},$$

where $\hat{\mathscr{A}}$ is the split singular extension of algebras, that is, as an algebra

$$\hat{\mathscr{A}} = \mathscr{A} \oplus \Sigma^{-1}\bar{R}$$

is the semidirect product with multiplication

$$(a, r)(a', r') = (aa', ar' + ra')$$

and the following conditions are satisfied.

The induced $\mathscr{A}$-$\mathscr{A}$-bimodule and $\mathscr{A}$-$\mathscr{A}$-bicomodule structures on $\Sigma^{-1}\bar{R}$ are given by the ones indicated in (14.2) *above, and the diagonal $\hat{\delta}$ of the coalgebra $\hat{\mathscr{A}}$ fits into the commutative diagram*

$$
\begin{array}{ccccc}
\hat{\mathscr{A}} & & \xrightarrow{\hat{\delta}} & & \hat{\mathscr{A}}\otimes\hat{\mathscr{A}} \\
\downarrow & & & & \downarrow{\scriptstyle 1+T} \\
\Sigma^{-1}\bar{R} & \xrightarrow{S} & \mathscr{A}\otimes\mathscr{A} & \hookrightarrow & \hat{\mathscr{A}}\otimes\hat{\mathscr{A}}
\end{array}
\tag{14.5}
$$

where S is the symmetry operator in (13.1)*.*

We will prove this theorem together with the dual statement. Note that clearly the dual of a singular extension of any Hopf algebra A is a singular extension of the dual Hopf algebra A_*. Clearly then the above theorem is equivalent to

(14.6) Theorem. *There is a unique singular extension of Hopf algebras*

$$\Sigma^{-1}\bar{R}_* \underset{i_*}{\overset{q_*}{\rightleftarrows}} \hat{\mathscr{A}}_* \underset{p_*}{\overset{s_*}{\rightleftarrows}} \mathscr{A}_*,$$

where $\hat{\mathscr{A}}_*$ is the split singular extension of coalgebras, that is, as a coalgebra

$$\hat{\mathscr{A}}_* = \mathscr{A}_* \oplus \Sigma^{-1}\bar{R}_*$$

with diagonal

$$\mathscr{A}_* \oplus \Sigma^{-1}\bar{R}_* \xrightarrow{\begin{pmatrix} m_* & 0 \\ 0 & m_{\ell *} \\ 0 & m_{r*} \\ 0 & 0 \end{pmatrix}} \mathscr{A}_*\otimes\mathscr{A}_* \oplus \mathscr{A}_*\otimes\Sigma^{-1}\bar{R}_* \oplus \Sigma^{-1}\bar{R}_*\otimes\mathscr{A}_* \oplus \Sigma^{-1}\bar{R}_*\otimes\Sigma^{-1}\bar{R}_*$$

where the diagonal m_* is dual to the multiplication $m : \mathscr{A}\otimes\mathscr{A} \to \mathscr{A}$ and $m_{\ell *}$, m_{r*} are the $\mathscr{A}_*$-$\mathscr{A}_*$-bicomodule structure maps dual to the $\mathscr{A}$-$\mathscr{A}$-bimodule structure maps $m_\ell : \mathscr{A}\otimes\Sigma^{-1}\bar{R} \to \Sigma^{-1}\bar{R}$, $m_r : \Sigma^{-1}\bar{R}\otimes\mathscr{A} \to \Sigma^{-1}\bar{R}$ in (14.3)*, where the induced $\mathscr{A}_*$-$\mathscr{A}_*$-bimodule structure on $\bar{R}_*$ is dual to the $\mathscr{A}$-$\mathscr{A}$-bicomodule structure indicated in* (14.2) *above, and where the multiplication $\hat{\delta}_*$ of the algebra $\hat{\mathscr{A}}_*$ satisfies the commutation rule*

$$p_*(y)p_*(x) = p_*(x)p_*(y) + S_*(x\otimes y)$$

for any $x, y \in \mathscr{A}_*$, where

$$S_* : \mathscr{A}_*\otimes\mathscr{A}_* \to \Sigma^{-1}\bar{R}_*$$

is the cosymmetry operator from (8.6)*.*

Proof of (14.4) *and* (14.6). The diagonal $\hat{\delta}$ can be written as follows

$$\mathscr{A} \oplus \Sigma^{-1}\bar{R} \xrightarrow{\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \\ \phi_{31} & \phi_{32} \\ \phi_{41} & \phi_{42} \end{pmatrix}} \mathscr{A}\otimes\mathscr{A} \oplus \mathscr{A}\otimes\Sigma^{-1}\bar{R} \oplus \Sigma^{-1}\bar{R}\otimes\mathscr{A} \oplus \Sigma^{-1}\bar{R}\otimes\Sigma^{-1}\bar{R}.$$

Then the condition that $s : \mathscr{A} \rightarrowtail \mathscr{A} \oplus \Sigma^{-1}\bar{R}$ is a coalgebra homomorphism implies $\phi_{11} = \delta$ and $\phi_{21} = 0$, $\phi_{31} = 0$, $\phi_{41} = 0$. Moreover the condition that the $\mathscr{A}$-$\mathscr{A}$-bicomodule structure induced on $\Sigma^{-1}\bar{R}$ coincides with the one given in (14.2) implies $\phi_{22} = \Delta_\ell$, $\phi_{32} = \Delta_r$. Next the condition that (s, q) is a singular extension of coalgebras, i. e. the coideal $\bar{R}$ has zero comultiplication, implies $\phi_{42} = 0$. Finally, let us look at the diagram (14.5). The lower composite in this diagram sends $(a, r) \in \mathscr{A} \oplus \Sigma^{-1}\bar{R}$ to

$$(S(r), 0, 0, 0) \in \mathscr{A}\otimes\mathscr{A} \oplus \mathscr{A}\otimes\Sigma^{-1}\bar{R} \oplus \Sigma^{-1}\bar{R}\otimes\mathscr{A} \oplus \Sigma^{-1}\bar{R}\otimes\Sigma^{-1}\bar{R}.$$

The upper composite sends it to

$$\begin{aligned}(1+T)\hat{\delta}(a, r) &= (1+T)(\delta(a) + \phi_{12}(r), \Delta_\ell(r), \Delta_r(r), 0) \\ &= ((1+T)\delta(a) + (1+T)\phi_{12}(r), \Delta_\ell(r) + T\Delta_r(r), \Delta_r(r) + T\Delta_\ell(r), 0).\end{aligned}$$

Since δ is cocommutative, one has $(1+T)\delta = 0$. Moreover cocommutativity of $\Delta : \mathscr{F}_0 \to \mathscr{F}_0\otimes\mathscr{F}_0$ implies $T\Delta_\ell = \Delta_r$, $T\Delta_r = \Delta_\ell$. Thus commutativity of (14.5) is equivalent to the condition

$$(1+T)\phi_{12} = S : \Sigma^{-1}\bar{R} \to \mathscr{A}\otimes\mathscr{A}. \tag{14.7}$$

Equivalently, passing to the dual we see that the dual map $\xi_* = \phi_{12*} : \mathscr{A}_*\otimes\mathscr{A}_* \to \Sigma^{-1}\bar{R}_*$ must satisfy

$$\xi_*(1+T) = S_*.$$

Now it is easy to see that ξ_* is in fact the algebra cocycle determining the algebra extension

$$\bar{R}_* \xrightarrow{q_*} \hat{\mathscr{A}_*} \xrightarrow{s_*} \mathscr{A}_*,$$

that is, in $\hat{\mathscr{A}_*} = \mathscr{A}_* \oplus \Sigma^{-1}\bar{R}_*$ one has

$$(\alpha, \beta)(\alpha', \beta') = (\alpha\alpha', \alpha\beta' + \beta\alpha' + \xi_*(\alpha\otimes\alpha')).$$

Hence by (14.7) one has

$$(\alpha, \beta)(\alpha', \beta') - (\alpha', \beta')(\alpha, \beta) = (0, S_*(\alpha\otimes\alpha')).$$

Now recall that $\mathscr{A}_*$ is actually a polynomial algebra. Using this fact it has been shown in [1, 16.2] that the algebra structure of any of its singular extensions such as $\hat{\mathscr{A}_*}$ above is completely determined by its commutator map, i. e. by S_*. Thus ϕ_{12*} and hence the whole ϕ_{ij} matrix is uniquely determined. It is then straightforward to check that indeed this matrix yields a coalgebra structure on $\hat{\mathscr{A}}$ with desired properties. □

It follows immediately from (14.6) (and actually this was also deduced during its proof) that one has

(14.8) Corollary. *For the cosymmetry operator S_* from* (8.6) *there exists a map*

$$\xi_* : \mathscr{A}_*\otimes\mathscr{A}_* \to \Sigma^{-1}\bar{R}_*$$

which is a 2-cocycle, i. e. for any $x, y, z \in \mathscr{A}_*$ one has

$$x\xi_*(y, z) + \xi_*(x, yz) = z\xi_*(x, y) + \xi_*(xy, z)$$

and such that its symmetrization is equal to S_*, i. e. for any $x, y \in \mathscr{A}_*$ one has

$$\xi_*(x, y) + \xi_*(y, x) = S_*(x, y).$$

Proof. This follows since any extension

$$M \xrightarrow{i} A' \xrightarrow{p} A$$

of a commutative algebra A by a symmetric A-module M is determined by a 2-cocycle $c : A\otimes A \to M$ such that for any $x, y \in A'$ one has

$$xy - yx = i(c(px, py) - c(py, px)),$$

i. e. the commutator map for A' is given by the antisymmetrization of c. Of course for $p = 2$ there is no difference between symmetrization and antisymmetrization. □

(14.9) Remark. The above corollary is easily seen to be exactly dual to [1, Theorem 16.1.5].

Using the extended Steenrod algebra we can next compute the deviation of the cocycle ξ_* from being an $\mathscr{A}_*$-comodule homomorphism. Namely, let

$$\nabla_{\xi_*} : \mathscr{A}_* \otimes \mathscr{A}_* \to \mathscr{A}_* \otimes \Sigma^{-1}\bar{R}_*$$

be the difference between the upper and lower composites in the diagram

$$\begin{array}{ccc} \mathscr{A}_* \otimes \mathscr{A}_* & \xrightarrow{\text{coaction}} & \mathscr{A}_* \otimes \mathscr{A}_* \otimes \mathscr{A}_* \\ {\scriptstyle \xi_*}\downarrow & & \downarrow{\scriptstyle 1\otimes\xi_*} \\ \Sigma^{-1}\bar{R}_* & \xrightarrow{\text{coaction}} & \mathscr{A}_* \otimes \Sigma^{-1}\bar{R}_*. \end{array} \tag{14.10}$$

Thus on elements we have

$$\nabla_{\xi_*}(x,y) = \sum \xi_*(x,y)_{\mathscr{A}} \otimes \xi_*(x,y)_R - \sum x_\ell y_{\ell'} \otimes \xi_*(x_r, y_{r'}),$$

where again the Sweedler notation is used,

$$m_*(x) = \sum x_\ell \otimes x_r$$

for the diagonal

$$m_* : \mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_*$$

and

$$a_*(x) = \sum x_{\mathscr{A}} \otimes x_C$$

for the coaction

$$a_* : C \to \mathscr{A}_* \otimes C$$

of a left $\mathscr{A}_*$-comodule C.

Let us also denote by ∇_{S_*} the similar operator but with S_* in place of ξ_*. That is, we define

$$\nabla_{S_*}(x,y) = \sum S_*(x,y)_{\mathscr{A}} \otimes S_*(x,y)_R - \sum x_\ell y_{\ell'} \otimes S_*(x_r, y_{r'}).$$

We then obviously have

$$\nabla_{\xi_*}(x,y) + \nabla_{\xi_*}(y,x) = \nabla_{S_*}(x,y) \tag{14.11}$$

for any $x, y \in \mathscr{A}_*$.

(14.12) Lemma. *The map ∇_{ξ_*} above is a 2-cocycle, i. e. for any $x, y, z \in \mathscr{A}_*$ one has*

$$m_*(x)\nabla_{\xi_*}(y,z) + \nabla_{\xi_*}(x,yz) = \nabla_{\xi_*}(x,y)m_*(z) + \nabla_{\xi_*}(xy,z).$$

Proof. First note that the diagram

$$\begin{array}{ccccc} \mathscr{A}_* \otimes \bar{R}_* & \xrightarrow{m_*\otimes\text{coaction}} & \mathscr{A}_* \otimes \mathscr{A}_* \otimes \mathscr{A}_* \otimes \bar{R}_* & & \\ & & & \searrow^{1\otimes T\otimes 1} & \\ {\scriptstyle \text{action}}\downarrow & & & & \mathscr{A}_* \otimes \mathscr{A}_* \otimes \mathscr{A}_* \otimes \bar{R}_* \\ & & & & \downarrow{\scriptstyle \delta_*\otimes\text{action}} \\ \bar{R}_* & & \xrightarrow{\qquad\text{coaction}\qquad} & & \mathscr{A}_* \otimes \bar{R}_* \end{array}$$

commutes — this follows from the fact that the action and coaction of $\mathscr{A}_*$ on $\bar{R}_*$ are induced from the multiplication and comultiplication in $\mathscr{F}_*$ which is a Hopf algebra.

We thus conclude that the coaction map

$$\bar{R}_* \to \mathscr{A}_* \otimes \bar{R}_*$$

is a homomorphism of $\mathscr{A}_*$-modules, so that its compliste with the cocycle ξ_* is a cocycle. It thus remains to show that the composite

$$\mathscr{A}_* \otimes \mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_* \otimes \mathscr{A}_* \to \mathscr{A}_* \otimes \bar{R}_*$$

in the diagram (14.10) is also a cocycle. Let us denote this composite by ϕ.

Now observe that the Hopf algebra diagram for $\mathscr{A}_*$ expressing interchange of the multiplication and diagonal can be written on elements as follows:

$$\sum (xy)_\ell \otimes (xy)_r = \sum x_\ell y_{\ell'} \otimes x_r y_{r'}.$$

Using this identity we then have for any $x, y, z \in \mathscr{A}_*$

$$\begin{aligned}
m_*(x)\phi(y,z) &= \sum x_\ell y_{\ell'} z_{\ell''} \otimes x_r \xi_*(y_{r'}, z_{r''});\\
\phi(x,yz) &= \sum x_\ell (yz)_{\ell'} \otimes \xi_*(x_r, (yz)_{r'}) = (\delta_* \otimes \xi_*)\Big(\sum x_\ell \otimes (yz)_{\ell'} \otimes x_r \otimes (yz)_{r'}\Big)\\
&= (\delta_* \otimes \xi_*)\Big(\sum x_\ell \otimes y_{\ell'} z_{\ell''} \otimes x_r \otimes y_{r'} z_{r''}\Big) = \sum x_\ell y_{\ell'} z_{\ell''} \otimes \xi_*(x_r, y_{r'} z_{r''});\\
\phi(xy,z) &= \sum (xy)_\ell z_{\ell'} \otimes \xi_*((xy)_r, z_{r'}) = (\delta_* \otimes \xi_*)\Big(\sum (xy)_\ell \otimes z_{\ell'} \otimes (xy)_r \otimes z_{r'}\Big)\\
&= (\delta_* \otimes \xi_*)\Big(\sum x_\ell y_{\ell'} \otimes z_{\ell''} \otimes x_r y_{r'} \otimes z_{r''}\Big) = \sum x_\ell y_{\ell'} z_{\ell''} \otimes \xi_*(x_r y'_r, z_{r''});\\
\phi(x,y)m_*(z) &= \sum x_\ell y_{\ell'} z_{\ell''} \otimes \xi_*(x_r, y_{r'}) z_{r''}.
\end{aligned}$$

These indentities readily imply that ϕ is a cocycle as required. □

We next use the fact the cocycle ∇_{ξ_*} is defined on a polynomial algebra and hence can be expressed by its values on generators and by its (anti)symmetrization ∇_{S*}. Indeed the proof of [1, 16.2.3] works in this generality, i. e. one has

(14.13) Proposition. *Let $P = k[\zeta_1, \zeta_2, ...]$ be a polynomial algebra over a commutative ring k, let M be a P-module, let*

$$\gamma : P \otimes P \to M$$

be a Hochschild 2-cocycle, i. e. one has

$$x\gamma(y,z) - \gamma(xy,z) + \gamma(x,yz) - z\gamma(x,y) = 0$$

for all $x, y, z \in P$, and let σ be the antisymmetrization of γ, i. e.

$$\sigma(x,y) = \gamma(x,y) - \gamma(y,x).$$

Then, up to coboundaries, γ can be recovered from σ, i. e. there is a cocycle γ_σ cohomologous to γ which depends only on σ.

Proof. To γ corresponds a singular extension of k-algebras

$$M \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} P$$

whose isomorphism class uniquely determines the cohomology class of γ. Let us choose for each polynomial generator $\zeta_n \in P$ an element $s(\zeta_n) \in E$ with $ps(\zeta_n) = \zeta_n$. Furthermore let us choose an ordering on the polynomial generators of P, $\zeta_1 < \zeta_2 < ...$; these data determine uniquely a k-linear section of p, by the formula

$$s(\zeta_{n_1}\zeta_{n_2}\cdots) = s(\zeta_{n_1})s(\zeta_{n_2})\cdots$$

for any finite sequence $n_1 \leqslant n_2 \leqslant \cdots$ of positive integers. Then we can use s to construct the cocycle γ_σ cohomologous to γ determined by

$$s(xy) = s(x)s(y) + i\gamma_\sigma(x,y).$$

But if x and y are monomials, then $s(xy)$ and $s(x)s(y)$ differ only by the order of terms, so that $i\gamma_\sigma(x,y)$ is contained in the ideal generated by commutators

$$\gamma_\sigma(\zeta_i, \zeta_j) = s(\zeta_i)s(\zeta_j) - s(\zeta_j)s(\zeta_i) = \sigma(\zeta_i, \zeta_j)$$

for $i > j$. So in fact one can express each $\gamma_\sigma(x,y)$ by a linear combination of elements of M of the form $z\sigma(\zeta_i, \zeta_j)$ for $z \in P$. □

(14.14) Remark. Obviously the above proof actually contains an algorithm for expressing the cocycle γ_σ in terms of σ. For $x = \zeta_{n_1}\zeta_{n_2}\cdots\zeta_{n_k}$ and $y = \zeta_{m_1}\zeta_{m_2}\cdots\zeta_{m_l}$, with $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$, $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_l$, either one has $n_k \leqslant m_1$, in which case $\gamma_\sigma(x, y) = 0$ since $s(x)s(y) = s(xy)$, or one has $n_k > m_1$, in which case one can write

$$s(x)s(y) = s(\zeta_{n_1})\cdots s(\zeta_{n_{k-1}})s(\zeta_{m_1})s(\zeta_{n_k})s(\zeta_{m_2})\cdots s(\zeta_{m_l}) + \zeta_{n_1}\cdots\zeta_{n_{k-1}}\zeta_{m_2}\cdots\zeta_{m_l}\sigma(\zeta_{m_1}, \zeta_{n_k}).$$

Applying the same procedure again several times one finally arrives at $s(xy)$+(a sum of elements of the form $z\sigma(\zeta_i, \zeta_j)$). In fact it is easy to see that one has

$$\gamma_\sigma(\zeta_{n_1}\zeta_{n_2}\cdots\zeta_{n_k}, \zeta_{m_1}\zeta_{m_2}\cdots\zeta_{m_l}) = \sum_{n_i > m_j} \zeta_{n_1}\cdots\zeta_{n_{i-1}}\zeta_{n_{i+1}}\cdots\zeta_{n_k}\zeta_{m_1}\cdots\zeta_{m_{j-1}}\zeta_{m_{j+1}}\cdots\zeta_{m_l}\sigma(\zeta_{m_j}, \zeta_{n_i}).$$

In the characteristic $p > 0$ case further obvious simplifications occur. In particular we can choose the cocycle ξ_* in (14.8) in such a way that the formula

$$\begin{aligned}&\xi_*(\zeta_1^{d_1}\zeta_2^{d_2}\cdots, \zeta_1^{e_1}\zeta_2^{e_2}\cdots) = \\ &\sum_{\substack{i<j \\ e_i,\, d_j \text{ odd}}} \zeta_1^{d_1+e_1}\cdots\zeta_{i-1}^{d_{i-1}+e_{i-1}}\zeta_i^{d_i+e_i-1}\zeta_{i+1}^{d_{i+1}+e_{i+1}}\cdots\zeta_{j-1}^{d_{j-1}+e_{j-1}}\zeta_j^{d_j+e_j-1}\zeta_{j+1}^{d_{j+1}+e_{j+1}}\cdots S_*(\zeta_i, \zeta_j)\end{aligned} \tag{14.15}$$

holds, and similarly for ξ_* and S_* replaced by ∇_{ξ_*} and ∇_{S*} respectively:

$$\begin{aligned}&\nabla_{\xi_*}(\zeta_1^{d_1}\zeta_2^{d_2}\cdots, \zeta_1^{e_1}\zeta_2^{e_2}\cdots) = \\ &\sum_{\substack{i<j \\ e_i,\, d_j \text{ odd}}} \zeta_1^{d_1+e_1}\cdots\zeta_{i-1}^{d_{i-1}+e_{i-1}}\zeta_i^{d_i+e_i-1}\zeta_{i+1}^{d_{i+1}+e_{i+1}}\cdots\zeta_{j-1}^{d_{j-1}+e_{j-1}}\zeta_j^{d_j+e_j-1}\zeta_{j+1}^{d_{j+1}+e_{j+1}}\cdots\nabla_{S*}(\zeta_i, \zeta_j).\end{aligned} \tag{14.16}$$

The operator ∇_{S*} is readily computable. It is a symmetric biderivation, with $\nabla_{S*}(x, x) = 0$ for all x, thus uniquely determined by its values of the form $\nabla_{S*}(\zeta_n, \zeta_m)$ for $n < m$, which are expressed easily from the corresponding values of S_*. For example, one has

$$\begin{aligned}
\nabla_{S*}(\zeta_1, \zeta_2) &= \zeta_1 \otimes M^2_{1,1},\\
\nabla_{S*}(\zeta_1, \zeta_3) &= \zeta_1^5 \otimes M^2_{1,1} + \zeta_1 \otimes M^2_{2,1,1},\\
\nabla_{S*}(\zeta_2, \zeta_3) &= \left(\zeta_1^7 + \zeta_1\zeta_2^2\right) \otimes M^2_{1,1} + \zeta_1^3 \otimes M^2_{2,1,1} + \zeta_1 \otimes \left(M_1^2M_3 + M_1M_{2,1,1} + M_5 + M_{4,1} + M_{3,2} + M_{2,1,1,1}\right)^2,\\
\nabla_{S*}(\zeta_1, \zeta_4) &= \zeta_1\zeta_2^4 \otimes M^2_{1,1} + \zeta_1^9 \otimes M^2_{2,1,1} + \zeta_1 \otimes M^2_{4,2,1,1},\\
\nabla_{S*}(\zeta_2, \zeta_4) &= \left(\zeta_1^3\zeta_2^4 + \zeta_1\zeta_3^2\right) \otimes M^2_{1,1} + \zeta_1^{11} \otimes M^2_{2,1,1}\\
&\quad + \zeta_1^9 \otimes \left(M_1^2M_3 + M_1M_{2,1,1} + M_5 + M_{4,1} + M_{3,2} + M_{2,1,1,1}\right)^2 + \zeta_1^3 \otimes M^2_{4,2,1,1}\\
&\quad + \zeta_1 \otimes \big(M_1^4M_5 + M_{2,1}^2M_3 + M_1^2M_{4,2,1,1} + M_9 + M_{7,2} + M_{6,2,1} + M_{5,4} + M_{3,4,2} + M_{4,3,2} + M_{4,4,1}\\
&\qquad + M_{2,4,2,1} + M_{4,2,1,1,1}\big)^2,
\end{aligned}$$

etc.

We will next use certain elements defined in [6, Theorem 3.3] to derive more explicit expressions for ξ_*, hence for S_*, ∇_{S*} and ∇_{ξ_*}. We recall that Kristensen defines

$$A[a, b] = (\mathrm{Sq}^1 \otimes \mathrm{Sq}^{0,1})\delta\left(\mathrm{Sq}^{a-3}\mathrm{Sq}^{b-2} + \mathrm{Sq}^{a-2}\mathrm{Sq}^{b-3} + \sum_j \binom{b-1-j}{a-2j}(\mathrm{Sq}^{a+b-j-3}\mathrm{Sq}^{j-2} + \mathrm{Sq}^{a+b-j-2}\mathrm{Sq}^{j-3})\right),$$

for natural numbers a, b. Obviously one has

$$A[a, b] = (\mathrm{Sq}^1 \otimes \mathrm{Sq}^{0,1})\delta k([a, b]),$$

where k is the operator determined by

$$k(xy) = \varkappa(\varkappa\varkappa(x)\varkappa\varkappa(y))$$

for $x, y \in \mathscr{F}_0^{\leqslant 1}$. We then interpret $A[a, b]$ as an $\mathbb{F}$-linear operator of the form

$$K : \mathscr{F}_0^{\leqslant 1} \otimes \mathscr{F}_0^{\leqslant 1} \to \mathscr{A} \otimes \mathscr{A}$$

given by

$$K(x\otimes y) = (\mathrm{Sq}^1\otimes \mathrm{Sq}^{0,1})\delta\varkappa(\varkappa\varkappa(x)\varkappa\varkappa(y))$$

which is factored through $\mathscr{F}_0^{\leqslant 1}\otimes\mathscr{F}_0^{\leqslant 1}\twoheadrightarrow \mathscr{F}_0^{\leqslant 2}$ and then restricted to $R_{\mathscr{F}}^{\leqslant 2}\rightarrowtail \mathscr{F}_0^{\leqslant 2}$. We then can dualize K to get

(14.17) Definition. We define an $\mathbb{F}$-linear operator

$$K_*: \mathscr{A}_*\otimes\mathscr{A}_*\to R_{\mathscr{F}*}^{\leqslant 2}$$

as composite with the quotient map $\mathscr{F}_*^{\leqslant 2}\twoheadrightarrow R_{\mathscr{F}*}^{\leqslant 2}$ of the dual of K above (whose image lies in that of $m_*: \mathscr{F}_*^{\leqslant 2}\rightarrowtail \mathscr{F}_*^{\leqslant 1}\otimes\mathscr{F}_*^{\leqslant 1}$.

Thus explicitly, K_* is the composite

$$\mathscr{A}_*\otimes\mathscr{A}_* \xrightarrow{\mathrm{Sq}^1\cdot_*\otimes\mathrm{Sq}^{0,1}\cdot_*} \mathscr{A}_*\otimes\mathscr{A}_* \xrightarrow{\delta_*} \mathscr{A}_* \xrightarrow{\zeta_1} \mathscr{A}_* \rightarrowtail \mathscr{F}_* \xrightarrow{m_*} \mathscr{F}_*\otimes\mathscr{F}_* \twoheadrightarrow \mathscr{F}_*^{\leqslant 1}\otimes\mathscr{F}_*^{\leqslant 1} \xrightarrow{M_1^2\otimes M_1^2} \mathscr{F}_*^{\leqslant 1}\otimes\mathscr{F}_*^{\leqslant 1}$$

landing in $\mathscr{F}_*^{\leqslant 2}\rightarrowtail \mathscr{F}_*^{\leqslant 1}\otimes\mathscr{F}_*^{\leqslant 1}$ and precomposed with $\mathscr{F}_*^{\leqslant 2}\twoheadrightarrow R_{\mathscr{F}*}^{\leqslant 2}$. Or on elements,

$$K_*(x\otimes y) = (M_1^2\otimes M_1^2)\left(m_*(\zeta_1\frac{\partial x}{\partial\zeta_1}\frac{\partial y}{\partial\zeta_2})\right)^{\leqslant 1} = m_*(M_{1,1}^2 M_1\frac{\partial x}{\partial\zeta_1}\frac{\partial y}{\partial\zeta_2})^{\leqslant 1}.$$

One thus has

$$K_*(\zeta_1^{n_1}\zeta_2^{n_2}\cdots\otimes\zeta_1^{m_1}\zeta_2^{m_2}\cdots) = \begin{cases} M_1^{n_1+m_1}M_{2,1}^{n_2+m_2-1}M_{1,1}^2, & n_1, m_2 \text{ odd}, n_i = m_i = 0 \text{ for } i>2,\\ 0 & \text{otherwise.}\end{cases} \tag{14.18}$$

We have

(14.19) Proposition. *Symmetrization of the operator K_* dual to the operator S_R in* (13.2)*, i. e. is given by precomposing $S_{\mathscr{F}*}$ given in* (13.4) *with the restriction map $\mathscr{F}_*^{\leqslant 2}\twoheadrightarrow R_{\mathscr{F}*}^{\leqslant 2}$.*

Proof. From the above formula (14.18), for monomials $x=\zeta_1^{n_1}\zeta_2^{n_2}\zeta_3^{n_3}\cdots$ and $y=\zeta_1^{m_1}\zeta_2^{m_2}\zeta_3^{m_3}\cdots$ we have

$$K_*(x\otimes y)+K_*(y\otimes x) = \begin{cases} M_1^{n_1+m_1}M_{2,1}^{n_2+m_2-1}M_{1,1}^2, & n_1m_2+m_1n_2 \text{ odd and } n_i=m_i=0 \text{ for } i>2,\\ 0 & \text{otherwise.}\end{cases}$$

On the other hand, using the explicit expression (13.4) and the expression for the operator S_*^1 in (13.5) we can write

$$m_*S_{\mathscr{F}*}(x\otimes y) = \sum_{\substack{x_\ell=\zeta_1^{2n-1}\\ y_{\ell'}=\zeta_1^{2n'-1}}} \zeta_1^{2(n+n')-1}\otimes(x_ry_{r'})^{\leqslant 1} + (\zeta_1\otimes 1+1\otimes\zeta_1)\sum_{\substack{x_r=\zeta_1^{2n-1}\\ y_{r'}=\zeta_1^{2n'-1}}}(x_\ell y_{\ell'})^{\leqslant 1}\otimes\zeta_1^{2(n+n'-1)}.$$

From the expression (9.5) for the Milnor diagonal we thus see that for monomials $x=\zeta_1^{n_1}\zeta_2^{n_2}\zeta_3^{n_3}\cdots$ and $y=\zeta_1^{m_1}\zeta_2^{m_2}\zeta_3^{m_3}\cdots$ one has $S_{\mathscr{F}*}(x\otimes y)=0$ unless $n_i=m_i=0$ for $i>2$, whereas in the remaining cases one has

$$\begin{aligned} m_*S_{\mathscr{F}*}(\zeta_1^{n_1}\zeta_2^{n_2}\otimes\zeta_1^{m_1}\zeta_2^{m_2}) = &\sum_{\substack{0\leqslant i\leqslant n_1\\ 0\leqslant j\leqslant m_1\\ i,\,j \text{ odd}}}\binom{n_1}{i}\binom{m_1}{j}\zeta_1^{i+j+2(n_2+m_2)+1}\otimes\zeta_1^{n_1+m_1-i-j+n_2+m_2}\\ &+(\zeta_1\otimes 1+1\otimes\zeta_1)\sum_{\substack{0\leqslant i\leqslant n_1\\ 0\leqslant j\leqslant m_1\\ n_1-i+n_2,\,m_1-j+m_2 \text{ odd}}}\binom{n_1}{i}\binom{m_1}{j}\zeta_1^{i+j+2(n_2+m_2)}\otimes\zeta_1^{n_1+m_1-i-j+n_2+m_2}.\end{aligned}$$

Let us now turn back to the symmetrization of K_*. We compute its image under the map m_*; by (11.5) it sends M_1 to $\zeta_1\otimes 1+1\otimes\zeta_1$, $M_{1,1}$ to $\zeta_1\otimes\zeta_1$ and $M_{2,1}$ to $\zeta_1^2\otimes\zeta_1$. Thus the nonzero values of this image are, for $n_1m_2+m_1n_2$ odd,

$$m_*K_*(1+T)(\zeta_1^{n_1}\zeta_2^{n_2}\otimes\zeta_1^{m_1}\zeta_2^{m_2}) = (\zeta_1\otimes 1+1\otimes\zeta_1)^{n_1+m_1}(\zeta_1^2\otimes\zeta_1)^{n_2+m_2-1}(\zeta_1^2\otimes\zeta_1^2).$$

Then expanding $(\zeta_1 \otimes 1 + 1 \otimes \zeta_1)^{n_1+m_1} = (\zeta_1 \otimes 1 + 1 \otimes \zeta_1)^{n_1}(\zeta_1 \otimes 1 + 1 \otimes \zeta_1)^{m_1}$ via binomials we obtain

$$m_* K_*(1+T)(\zeta_1^{n_1}\zeta_2^{n_2} \otimes \zeta_1^{m_1}\zeta_2^{m_2}) = \sum_{\substack{0\leqslant i\leqslant n_1\\ 0\leqslant j\leqslant m_1}} \binom{n_1}{i}\binom{m_1}{j}\zeta_1^{i+j+2(n_2+m_2)} \otimes \zeta_1^{n_1+m_1-i-j+n_2+m_2+1}.$$

It follows that nonzero values of the difference $m_*(S_{\mathscr{F}*} - K_*(1+T))$ on monomials in Milnor generators are equal to

$$\sum_{\substack{0\leqslant i\leqslant n_1\\ 0\leqslant j\leqslant m_1\\ i,\ j \text{ odd}}} \binom{n_1}{i}\binom{m_1}{j}\zeta_1^{i+j+2(n_2+m_2)+1} \otimes \zeta_1^{n_1+m_1-i-j+n_2+m_2}$$

$$+ \sum_{\substack{0\leqslant i\leqslant n_1\\ 0\leqslant j\leqslant m_1\\ n_1-i+n_2,\ m_1-j+m_2 \text{ odd}}} \binom{n_1}{i}\binom{m_1}{j}\zeta_1^{i+j+2(n_2+m_2)+1} \otimes \zeta_1^{n_1+m_1-i-j+n_2+m_2}$$

$$+ \sum_{\substack{0\leqslant i\leqslant n_1\\ 0\leqslant j\leqslant m_1\\ n_1-i+n_2,\ m_1-j+m_2 \text{ even}}} \binom{n_1}{i}\binom{m_1}{j}\zeta_1^{i+j+2(n_2+m_2)} \otimes \zeta_1^{n_1+m_1-i-j+n_2+m_2+1}$$

for $n_1m_2 + m_1n_2$ odd and $m_*S_{\mathscr{F}*}(\zeta_1^{n_1}\zeta_2^{n_2} \otimes \zeta_1^{m_1}\zeta_2^{m_2})$ for $n_1m_2 + m_1n_2$ even.

The first expression can be rewritten as

$$(\zeta_1^2 \otimes \zeta_1)^{n_2+m_2} \sum_k \zeta_1^{k+1} \otimes \zeta_1^{n_1+m_1-k}$$

$$\left(\sum_{\substack{0\leqslant i\leqslant n_1\\ 0\leqslant k-i\leqslant m_1\\ i,\ k-i \text{ odd}}} \binom{n_1}{i}\binom{m_1}{k-i} + \sum_{\substack{0\leqslant i\leqslant n_1\\ 0\leqslant k-i\leqslant m_1\\ n_1-i+n_2,\ m_1-k+i+m_2 \text{ odd}}} \binom{n_1}{i}\binom{m_1}{k-i} + \sum_{\substack{0\leqslant i\leqslant n_1\\ 0\leqslant k+1-i\leqslant m_1\\ n_1-i+n_2,\ m_1-k-1+i+m_2 \text{ even}}} \binom{n_1}{i}\binom{m_1}{k+1-i} \right)$$

and in the second case we may write

$$m_*S_{\mathscr{F}*}(\zeta_1^{n_1}\zeta_2^{n_2} \otimes \zeta_1^{m_1}\zeta_2^{m_2}) = (\zeta_1^2 \otimes \zeta_1)^{n_2+m_2} \sum_k \zeta_1^{k+1} \otimes \zeta_1^{n_1+m_1-k}$$

$$\left(\sum_{\substack{0\leqslant i\leqslant n_1\\ 0\leqslant k-i\leqslant m_1\\ i,\ k-i \text{ odd}}} \binom{n_1}{i}\binom{m_1}{k-i} + \sum_{\substack{0\leqslant i\leqslant n_1\\ 0\leqslant k-i\leqslant m_1\\ n_1-i+n_2,\ m_1-k+i+m_2 \text{ odd}}} \binom{n_1}{i}\binom{m_1}{k-i} + \sum_{\substack{0\leqslant i\leqslant n_1\\ 0\leqslant k+1-i\leqslant m_1\\ n_1-i+n_2,\ m_1-k-1+i+m_2 \text{ odd}}} \binom{n_1}{i}\binom{m_1}{k+1-i} \right).$$

One then shows that these expressions lie in the subalgebra of $\mathscr{F}_*^{\leqslant 1} \otimes \mathscr{F}_*^{\leqslant 1}$ generated by $\zeta_1^2 \otimes \zeta_1$ and $\zeta_1 \otimes 1 + 1 \otimes \zeta_1$, without involvement of $\zeta_1 \otimes \zeta_1$. This means that the image of the difference $S_{\mathscr{F}*} - K_*(1+T)$ under the restriction map $\mathscr{F}_*^{\leqslant 2} \twoheadrightarrow R_{\mathscr{F}*}^{\leqslant 2}$ is zero. □

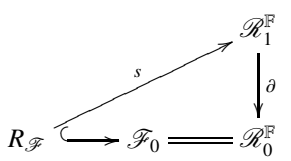
15. C $\mathscr{R}^{\mathbb{F}}$ $\mathscr{R}_{\mathbb{F}}$

Let us fix a function $\chi : \mathbb{F} \to \mathbb{G}$ which splits the projection $\mathbb{G} \to \mathbb{F}$, namely, take

$$\chi(k \mod p) = k \mod p^2,\ 0 \leqslant k < p. \tag{15.1}$$

We will use χ to define splittings of $\mathscr{R}^{\mathbb{F}} = \left(\mathscr{R}_1^{\mathbb{F}} \xrightarrow{\partial} \mathscr{R}_0^{\mathbb{F}}\right)$. Here a *splitting* s of $\mathscr{R}^{\mathbb{F}}$ is an $\mathbb{F}$-linear map for which the diagram

(15.2)



commutes with $R_{\mathscr{F}} = \mathrm{im}(\partial) = \ker(q_{\mathscr{F}} : \mathscr{F}_0 \to \mathscr{A})$. We only consider the case $p = 2$.

(15.3) Definition (The right equivariant splitting of $\mathscr{R}^{\mathbb{F}}$)**.** Using χ, all *Adem relations*

$$[a,b] := \mathrm{Sq}^a\mathrm{Sq}^b + \sum_{k=0}^{\left[\frac{a}{2}\right]} \binom{b-k-1}{a-2k} \mathrm{Sq}^{a+b-k}\mathrm{Sq}^k$$

for $a, b > 0$, $a < 2b$, can be lifted to elements $[a,b]_\chi \in R_{\mathscr{B}}$ by applying χ to all coefficients, i. e. by interpreting $[a,b]$ as an element of $\mathscr{B}$. As shown in [1, 16.5.2], $R_{\mathscr{F}}$ is a free right $\mathscr{F}_0$-module with a basis consisting of *preadmissible relations*. For $p = 2$ these are elements of the form

$$\mathrm{Sq}^{a_1} \cdots \mathrm{Sq}^{a_{k-1}}[a_k, a] \in R_{\mathscr{F}}$$

satisfying $a_1 \geqslant 2a_2$, ..., $a_{k-2} \geqslant 2a_{k-1}$, $a_{k-1} \geqslant 2a_k$, $a_k < 2a$. Sending such an element to

$$\mathrm{Sq}^{a_1} \cdots \mathrm{Sq}^{a_{k-1}}[a_k, a]_\chi \in R_{\mathscr{B}}$$

determines then a unique right $\mathscr{F}_0$-equivariant splitting ϕ in the pair algebra $\mathscr{R}^{\mathbb{F}}$; that is, we get a commutative diagram

$$\begin{array}{ccc} R_{\mathscr{F}} & \xrightarrow{\phi} & R_{\mathscr{B}} \otimes \mathbb{F} = \mathscr{R}^{\mathbb{F}}_1 \\ \big\downarrow & & \big\downarrow \partial \\ \mathscr{F}_0 & = & \mathscr{R}^{\mathbb{F}}_0. \end{array}$$

For a splitting s of $\mathscr{R}^{\mathbb{F}}$ the map $s\otimes 1 \oplus 1\otimes s$ induces the map $s_\#$ in the diagram

$$\begin{array}{ccccc} \mathscr{R}^{\mathbb{F}}_1 & \xrightarrow{\Delta} & (\mathscr{R}^{\mathbb{F}}\hat{\otimes}\mathscr{R}^{\mathbb{F}})_1 & \longleftarrow & \mathscr{R}^{\mathbb{F}}_1\otimes\mathscr{F}_0 \oplus \mathscr{F}_0\otimes\mathscr{R}^{\mathbb{F}}_1 \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle\partial} & & {\scriptstyle s_\#}\uparrow\downarrow{\scriptstyle\partial_{\hat{\otimes}}} & & \uparrow{\scriptstyle s\otimes 1\oplus 1\otimes s} \\ R_{\mathscr{F}} & \xrightarrow{\Delta_R} & R^{(2)}_{\mathscr{F}} & \longleftarrow & R_{\mathscr{F}}\otimes\mathscr{F}_0 \oplus \mathscr{F}_0\otimes R_{\mathscr{F}} \\ \big\downarrow & & \big\downarrow & & \\ \mathscr{F}_0 & \xrightarrow{\Delta} & \mathscr{F}_0 \otimes \mathscr{F}_0. & & \end{array} \tag{15.4}$$

Then the difference $U = s_\#\Delta_R - \Delta s : R_{\mathscr{F}} \to (\mathscr{R}^{\mathbb{F}}\hat{\otimes}\mathscr{R}^{\mathbb{F}})_1$ satisfies $\partial_{\hat{\otimes}} U = 0$ since

$$\partial_{\hat{\otimes}} s_\#\Delta_R = \Delta_R = \Delta_R\partial s = \partial_{\hat{\otimes}}\Delta s.$$

Thus U lifts to $\ker\partial_{\hat{\otimes}} \cong \mathscr{A}\otimes\mathscr{A}$ and gives an $\mathbb{F}$-linear map

$$U_s : R_{\mathscr{F}} \to \mathscr{A}\otimes\mathscr{A}. \tag{15.5}$$

If we use the splitting s to identify $\mathscr{R}^{\mathbb{F}}_1$ with the direct sum $\mathscr{A}\oplus R_{\mathscr{F}}$, then it is clear that knowledge of the map U_s determines the diagonal $\mathscr{R}^{\mathbb{F}}_1 \to (\mathscr{R}^{\mathbb{F}}\hat{\otimes}\mathscr{R}^{\mathbb{F}})_1$ completely. Indeed $s_\#$ yields the identification $(\mathscr{R}^{\mathbb{F}}\hat{\otimes}\mathscr{R}^{\mathbb{F}})_1 \cong \mathscr{A}\otimes\mathscr{A} \oplus R^{(2)}_{\mathscr{F}}$, and under these identifications $\Delta : \mathscr{R}^{\mathbb{F}}_1 \to (\mathscr{R}^{\mathbb{F}}\hat{\otimes}\mathscr{R}^{\mathbb{F}})_1$ corresponds to a map which by commutativity of (15.4) must have the form

$$\mathscr{A}\oplus R_{\mathscr{F}} \xrightarrow{\left(\begin{smallmatrix}\Delta_{\mathscr{A}} & U_s\\ 0 & \Delta_R\end{smallmatrix}\right)} \mathscr{A}\otimes\mathscr{A} \oplus R^{(2)}_{\mathscr{F}} \tag{15.6}$$

and is thus determined by U_s.

One readily checks that the map U_s for $s = \phi$ in (15.3) coincides with the map U defined in [1, 16.4.3] in terms of the algebra $\mathscr{B}$.

Given the splitting s and the map U_s, the only piece of structure remaining to determine the $\mathbf{Alg}^{pair}_{\mathbb{1}}$-comonoid structure of $\mathscr{R}^{\mathbb{F}}$ completely is the $\mathscr{F}_0$-$\mathscr{F}_0$-bimodule structure on $\mathscr{R}^{\mathbb{F}}_1 \cong \mathscr{A}\oplus R_{\mathscr{F}}$. Consider for $f \in \mathscr{F}_0$, $r \in R_{\mathscr{F}}$ the difference $s(fr) - fs(r)$. It belongs to the kernel of ∂ since

$$\partial s(fr) = fr = f\partial s(r) = \partial(fs(r)).$$

Thus we obtain the *left multiplication map*

$$a_s : \mathscr{F}_0 \otimes R_{\mathscr{F}} \to \mathscr{A}. \tag{15.7}$$

Similarly we obtain the *right multiplication map*

$$b_s : R_{\mathscr{F}} \otimes \mathscr{F}_0 \to \mathscr{A}$$

by the difference $s(rf) - s(r)f$.

(15.8) Lemma. *For $s = \phi$ in* (15.3) *the right multiplication map b_ϕ is trivial, that is ϕ is right equivariant, and the left multiplication map factors through $q_{\mathscr{F}} \otimes 1$ inducing the map*

$$a_\phi : \mathscr{A} \otimes R_{\mathscr{F}} \to \mathscr{A}.$$

Proof. Right equivariance holds by definition. As for the factorization, $R_{\mathscr{F}} \otimes R_{\mathscr{F}} \rightarrowtail \mathscr{F}_0 \otimes R_{\mathscr{F}}$ is in the kernel of $a_\phi : \mathscr{F}_0 \otimes R_{\mathscr{F}} \to \mathscr{A}$, since by right equivariance of s and by the pair algebra property (1.9) for $\mathscr{R}^{\mathbb{F}}$ one has for any $r, r' \in R_{\mathscr{F}}$

$$s(rr') = s(r)r' = s(r)\partial s(r') = (\partial s(r))s(r') = rs(r').$$

Hence factoring the above map through $(\mathscr{F}_0 \otimes R_{\mathscr{F}})/(R_{\mathscr{F}} \otimes R_{\mathscr{F}}) \cong \mathscr{A} \otimes R_{\mathscr{F}}$ we obtain a map

$$\mathscr{A} \otimes R_{\mathscr{F}} \to \mathscr{A}.$$

□

Summarizing the above, we thus have proved

(15.9) Proposition. *Using the splitting $s = \phi$ of $\mathscr{R}^{\mathbb{F}}$ in* (15.3) *the comonoid $\mathscr{R}^{\mathbb{F}}$ in the category* $\mathbf{Alg}_{\mathbb{1}}^{pair}$ *described in* (5.4) *is completely determined by the maps*

$$U_\phi : R_{\mathscr{F}} \to \mathscr{A} \otimes \mathscr{A}$$

and

$$a_\phi : \mathscr{A} \otimes R_{\mathscr{F}} \to \mathscr{A}$$

given in (15.5) *and* (15.7) *respectively.*

□

We next introduce another splitting $s = \psi$ for which $U_s = 0$. For this we use the fact that $\mathscr{A}_* = \mathrm{Hom}(\mathscr{A}, \mathbb{F})$ and

$$\mathscr{B}_\# = \mathrm{Hom}(\mathscr{B}_0, \mathbb{G}) \tag{15.10}$$

with $\mathscr{B}_0 = T_{\mathbb{G}}(E_{\mathscr{A}})$ both are polynomial algebras in such a way that generators of $\mathscr{A}_*$ are also (part of the) generators of $\mathscr{B}_\#$.

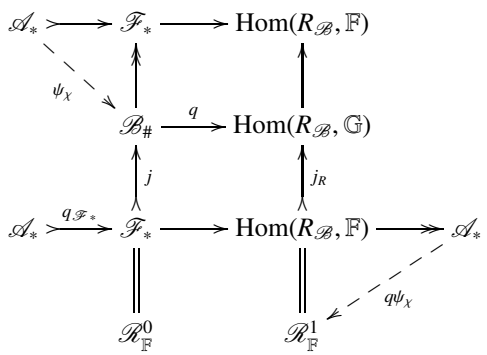
Using χ in (15.1) we obtain the function

$$\psi_\chi : \mathscr{A}_* \to \mathscr{B}_\# \tag{15.11}$$

(which is not $\mathbb{F}$-linear) as follows. Each element x in $\mathscr{A}_*$ is uniquely an $\mathbb{F}$-linear combination $x = \sum_\alpha n_\alpha \alpha$ where α runs through the monomials in Milnor generators. Such a monomial can be also considered as an element in $\mathscr{B}_\#$ by (10.6) so that we can define

$$\psi_\chi(x) = \sum_\alpha \chi(n_\alpha)\alpha \in \mathscr{B}_\#.$$

(15.12) Definition (The comultiplicative splitting of $\mathscr{R}^{\mathbb{F}}$)**.** Consider the following commutative diagram with exact rows and columns



with the columns induced by the short exact sequence $\mathbb{F} \rightarrowtail \mathbb{G} \twoheadrightarrow \mathbb{F}$ and the rows induced by (7.1). In particular q is induced by the inclusion $R_{\mathscr{B}} \subset \mathscr{B}_0$. Now it is clear that ψ_χ yields a map $q\psi_\chi$ which lifts to $\mathrm{Hom}(R_{\mathscr{B}}, \mathbb{F})$ so that we get the map

$$q\psi_\chi : \mathscr{A}_* \to \mathscr{R}^1_{\mathbb{F}}$$

which splits the projection $\mathscr{R}^1_{\mathbb{F}} \twoheadrightarrow \mathscr{A}_*$. Moreover $q\psi_\chi$ is $\mathbb{F}$-linear since for all $x, y \in \mathscr{A}_*$ the elements $\psi_\chi(x) + \psi_\chi(y) - \psi_\chi(x+y) \in \mathscr{B}_\#$ are in the image of the inclusion $jq_{\mathscr{F}_*} : \mathscr{A}_* \rightarrowtail \mathscr{B}_\#$ and thus go to zero under q.

The dual of $q\psi_\chi$ is thus a retraction $(q\psi_\chi)^*$ in the short exact sequence

$$\begin{array}{ccccc} R_{\mathscr{F}} & \xleftarrow{\pi} & R_{\mathscr{B}} \otimes \mathbb{F} & \xleftarrow{\iota} & \mathscr{A} \\ & {\scriptstyle (q\psi_\chi)^*_\perp} \searrow & \| & \nearrow {\scriptstyle (q\psi_\chi)^*} & \\ & & \mathscr{R}^{\mathbb{F}}_1 & & \end{array}$$

which induces the splitting $\psi = (q\psi_\chi)^*_\perp$ of $\mathscr{R}^{\mathbb{F}}$ determined by

$$\psi(\pi(x)) = x - \iota((q\psi_\chi)^*(x)).$$

(15.13) Lemma. *For the splitting $s = \psi$ of $\mathscr{R}^{\mathbb{F}}$ we have $U_\psi = 0$.*

Proof. We must show that the following diagram commutes:

$$\begin{array}{ccc} R_{\mathscr{F}} & \xrightarrow{\Delta_R} & R^{(2)}_{\mathscr{F}} \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \psi_\#} \\ R_{\mathscr{B}} \otimes \mathbb{F} & \xrightarrow{\Delta} & (\mathscr{R}^{\mathbb{F}} \hat{\otimes} \mathscr{R}^{\mathbb{F}})_1. \end{array}$$

Obviously this is equivalent to commutativity of the dual diagram

$$\begin{array}{ccc} (\mathscr{R}_{\mathbb{F}} \check{\otimes} \mathscr{R}_{\mathbb{F}})^1 & \xrightarrow{\Delta_*} & \mathrm{Hom}(R_{\mathscr{B}}, \mathbb{F}) \\ {\scriptstyle (\psi_\#)_*}\downarrow & & \downarrow{\scriptstyle \psi_*} \\ R^{(2)}_{\mathscr{F}_*} & \xrightarrow{\Delta_{R*}} & R_{\mathscr{F}_*} \end{array}$$

which in turn is equivalent to commutativity of

$$\begin{array}{ccc} \mathscr{A}_* \otimes \mathscr{A}_* & \xrightarrow{\delta_*} & \mathscr{A}_* \\ {\scriptstyle (\psi_\#)^\perp_*}\downarrow & & \downarrow{\scriptstyle q\psi_\chi} \\ (\mathscr{R}_{\mathbb{F}} \check{\otimes} \mathscr{R}_{\mathbb{F}})^1 & \xrightarrow{\Delta_*} & \mathrm{Hom}(R_{\mathscr{B}}, \mathbb{F}). \end{array} \tag{15.14}$$

On the other hand, the left hand vertical map in the latter diagram can be included into another commutative diagram

$$\begin{array}{ccc} \mathscr{F}_* \otimes \mathscr{A}_* \oplus \mathscr{A}_* \otimes \mathscr{F}_* & \xleftarrow{\binom{i\otimes 1}{1\otimes i}} & \mathscr{A}_* \otimes \mathscr{A}_* \\ {\scriptstyle 1\otimes q\psi_\chi \oplus q\psi_\chi \otimes 1}\downarrow & & \downarrow{\scriptstyle (\psi_\#)^\perp_*} \\ \mathscr{F}_* \otimes \mathscr{R}^1_{\mathbb{F}} \oplus \mathscr{R}^1_{\mathbb{F}} \otimes \mathscr{F}_* & \longleftarrow & (\mathscr{R}_{\mathbb{F}} \otimes \mathscr{R}_{\mathbb{F}})^1 \end{array}$$

It follows that on elements, commutativity of (15.14) means that the equality

$$q\psi_\chi(xy) = i(x)q\psi_\chi(y) + q\psi_\chi(x)i(y)$$

holds for any $x, y \in \mathscr{A}_*$. By linearity, it is clearly enough to prove this when x and y are monomials in Milnor generators.

For this observe that for any $x \in \mathscr{A}_* = \mathrm{Hom}(\mathscr{A}, \mathbb{F})$, the element $q\psi_\chi(x) \in \mathrm{Hom}(R_{\mathscr{B}}, \mathbb{F})$ is the unique $\mathbb{F}$-linear map making the diagram

$$\begin{array}{ccccc} R_{\mathscr{B}} & \rightarrowtail & \mathscr{B}_0 & \twoheadrightarrow & \mathscr{A} \\ {\scriptstyle q\psi_\chi(x)}\downarrow & & \downarrow{\scriptstyle \psi_\chi(x)} & & \downarrow{\scriptstyle x} \\ \mathbb{F} & \rightarrowtail & \mathbb{G} & \twoheadrightarrow & \mathbb{F} \end{array}$$

commute. This uniqueness implies the equality we need in view of the following commutative diagram with exact columns:

$$\begin{array}{ccccccc} R_{\mathscr{B}} & \longrightarrow & R^{(2)}_{\mathscr{B}} & \dashrightarrow & \mathbb{F}\otimes\mathbb{F} & \xrightarrow{\cong} & \mathbb{F} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathscr{B}_0 & \xrightarrow{\Delta} & \mathscr{B}_0\otimes\mathscr{B}_0 & \xrightarrow{\psi_\chi(x)\otimes\psi_\chi(y)} & \mathbb{G}\otimes\mathbb{G} & \xrightarrow{\cong} & \mathbb{G} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathscr{A} & \xrightarrow{\delta} & \mathscr{A}\otimes\mathscr{A} & \xrightarrow{x\otimes y} & \mathbb{F}\otimes\mathbb{F} & \xrightarrow{\cong} & \mathbb{F}, \end{array} \qquad \mathscr{A} \xrightarrow{xy} \mathbb{F}$$

since when x and y are monomials in Milnor generators, one has $\psi_\chi(xy) = \psi_\chi(x)\psi_\chi(y)$. □

Therefore we call ψ the comultiplicative splitting of $\mathscr{R}^{\mathbb{F}}$. We now want to compute the left and right multiplication maps a_ψ and b_ψ defined in (15.7). The dual maps $(a_\psi)_*$ and $(b_\psi)_*$ can be described by the diagrams

$$\begin{array}{ccccc} & & (R_{\mathscr{B}})_* & \xrightarrow{q\psi_\chi} & \mathscr{A}_* \\ & & {\scriptstyle m^\ell_*}\downarrow & & \downarrow{\scriptstyle m_*} \\ \mathscr{F}_*\otimes(R_{\mathscr{F}})_* & \rightarrowtail & \mathscr{F}_*\otimes(R_{\mathscr{B}})_* & \twoheadrightarrow & \mathscr{A}_*\otimes\mathscr{A}_* \end{array} \qquad \mathscr{A}_*\otimes\mathscr{A}_* \xrightarrow{i\otimes q\psi_\chi} \mathscr{F}_*\otimes(R_{\mathscr{B}})_* \tag{15.15}$$

and

$$\begin{array}{ccccc} & & (R_{\mathscr{B}})_* & \xrightarrow{q\psi_\chi} & \mathscr{A}_* \\ & & {\scriptstyle m^r_*}\downarrow & & \downarrow{\scriptstyle m_*} \\ (R_{\mathscr{F}})_*\otimes\mathscr{F}_* & \rightarrowtail & (R_{\mathscr{B}})_*\otimes\mathscr{F}_* & \twoheadrightarrow & \mathscr{A}_*\otimes\mathscr{A}_*. \end{array} \qquad \mathscr{A}_*\otimes\mathscr{A}_* \xrightarrow{q\psi_\chi\otimes i} (R_{\mathscr{B}})_*\otimes\mathscr{F}_* \tag{15.16}$$

Here m_* is dual to the multiplication in $\mathscr{A}$ and m^ℓ_* and m^r_* are induced by the $\mathscr{F}_0$-$\mathscr{F}_0$-bimodule structure of $R_{\mathscr{B}} \otimes \mathbb{F}$. One readily checks

$$(a_\psi)_* = m^\ell_* q\psi_\chi - (i \otimes q\psi_\chi)m_*$$
$$(b_\psi)_* = m^r_* q\psi_\chi - (q\psi_\chi \otimes i)m_*.$$

We now consider the diagram

$$\begin{array}{ccccc} & & \mathscr{B}_\# & \xleftarrow{\psi_\chi} & \mathscr{A}_* \\ & & {\scriptstyle m^{\mathbb{G}}_*}\downarrow & & \downarrow{\scriptstyle m_*} \\ \mathscr{F}_*\otimes\mathscr{F}_* & \rightarrowtail & \mathscr{B}_\#\otimes\mathscr{B}_\# & \xleftarrow{\psi^\otimes_\chi} & \mathscr{A}_*\otimes\mathscr{A}_* \end{array}$$

Here $\psi^{\otimes}_{\chi}$ is defined similarly as ψ_{χ} in (15.11) by the formula

$$\psi^{\otimes}_{\chi}\left(\sum_{\alpha,\beta} n_{\alpha\beta}\alpha\otimes\beta\right) = \sum_{\alpha,\beta}\chi(n_{\alpha\beta})\alpha\otimes\beta$$

where α,β run through the monomials in Milnor generators. Moreover $m^{\mathbb{G}}_*$ is the dual of the multiplication map $m^{\mathbb{G}}$ of $\mathscr{B}_0 = T_{\mathbb{G}}(E_{\mathscr{A}})$.

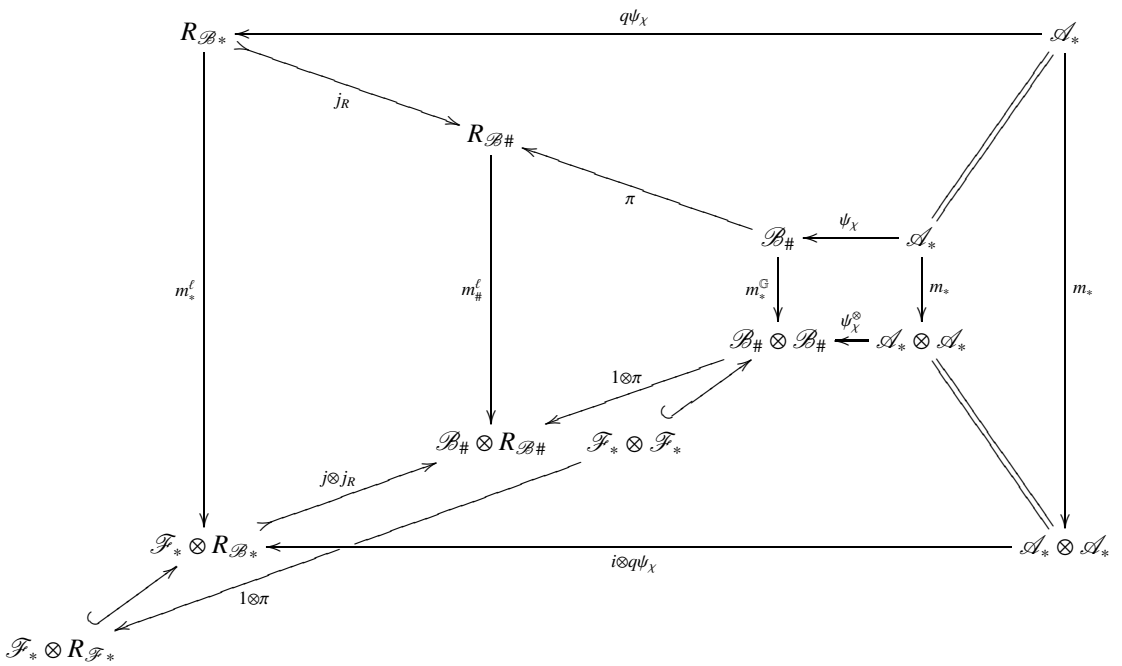
(15.17) Lemma. *The difference $m^{\mathbb{G}}_*\psi_{\chi} - \psi^{\otimes}_{\chi}m_*$ lifts to an $\mathbb{F}$-linear map $\nabla_{\chi} : \mathscr{A}_* \to \mathscr{F}_*\otimes\mathscr{F}_*$ such that one has*

$$(a_{\psi})_* = (1\otimes\pi)\nabla_{\chi}$$
$$(b_{\psi})_* = (\pi\otimes 1)\nabla_{\chi}.$$

Here $\pi : \mathscr{F}_* \twoheadrightarrow R_{\mathscr{F}*}$ is induced by the inclusion $R_{\mathscr{F}} \subset \mathscr{F}_0$.

Proof. We will only prove the first equality; the proof for the second one is completely similar.

The following diagram



commutes except for the innermost square, whose deviation from commutativity is ∇_{χ} and lies in the image of $\mathscr{F}_*\otimes\mathscr{F}_* \hookrightarrow \mathscr{B}_\#\otimes\mathscr{B}_\#$, and the outermost square, whose deviation from commutativity is a_{ψ_*} and lies in the image of $\mathscr{F}_*\otimes R_{\mathscr{F}*} \hookrightarrow \mathscr{F}_*\otimes R_{\mathscr{B}*}$. It follows that $(1\otimes\pi)\nabla_{\chi}$ and a_{ψ_*} have the same image under $j\otimes j_R$, and since the latter map is injective we are done. □

Using the splitting ψ we get the following analogue of (15.9).

(15.18) Proposition. *The comonoid $\mathscr{R}^{\mathbb{F}}$ in the category $\mathbf{Alg}^{pair}_{\mathbb{1}}$ described in* (5.4) *is completely determined by the multiplication maps a_{ψ} and b_{ψ}. In fact, the identification*

$$\mathscr{R}^{\mathbb{F}}_1 = \mathscr{A}\oplus R_{\mathscr{F}}$$

induced by the splitting $s = \psi$ identifies the diagonal of $\mathscr{R}^{\mathbb{F}}$ with $\Delta_{\mathscr{A}}\oplus\Delta_R$ (see (15.5)*,* (15.6)*), and the bimodule structure of $\mathscr{R}^{\mathbb{F}}_1$ with*

$$f(\alpha, r) = (f\alpha - a_{\psi}(f, r), fr)$$
$$(\alpha, r)f = (\alpha f - b_{\psi}(r, f), rf)$$

for $f\in\mathscr{F}_0$, $r\in R_{\mathscr{F}}$, $\alpha\in\mathscr{A}$.

16. C H $\mathscr{B}^{\mathbb{F}}$

The Hopf pair algebra $\mathscr{V} = \mathscr{B}^{\mathbb{F}}$ in (6.15), given by the algebra of secondary cohomology operations, satisfies the following crucial condition which we deduce from [1, 16.1.5].

(16.1) Theorem. *There exists a right $\mathscr{F}_0$-equivariant splitting*

$$u : \mathscr{R}^{\mathbb{F}}_1 = R_{\mathscr{B}} \otimes \mathbb{F} \to \mathscr{B}_1 \otimes \mathbb{F} = \mathscr{B}^{\mathbb{F}}_1$$

of the projection $\mathscr{B}^{\mathbb{F}}_1 \to \mathscr{R}^{\mathbb{F}}_1$, see (6.4)*, such that the following holds. The diagram*

$$\begin{array}{ccccccc}
\mathscr{A} \oplus_{\varkappa} \Sigma\mathscr{A} & \rightarrowtail & \mathscr{B}^{\mathbb{F}}_1 & \longrightarrow & \mathscr{B}^{\mathbb{F}}_0 & \twoheadrightarrow & \mathscr{A} \\
{\scriptstyle q}\downarrow\uparrow{\scriptstyle \bar{u}} & & {\scriptstyle q}\downarrow\uparrow{\scriptstyle u} & & \| & & \| \\
\mathscr{A} & \rightarrowtail & \mathscr{R}^{\mathbb{F}}_1 & \longrightarrow & \mathscr{R}^{\mathbb{F}}_0 & \twoheadrightarrow & \mathscr{A}
\end{array}$$

commutes, where $\bar{u}$ is the inclusion. Moreover in the diagram of diagonals, see (6.5)*,*

$$\begin{array}{ccccc}
\mathscr{B}^{\mathbb{F}}_1 & \xrightarrow{\Delta_{\mathscr{B}}} & (\mathscr{B}^{\mathbb{F}} \hat{\otimes} \mathscr{B}^{\mathbb{F}})_1 & \hookleftarrow & \Sigma\mathscr{A} \otimes \mathscr{A} \\
\uparrow{\scriptstyle u} & & \uparrow{\scriptstyle u\hat{\otimes}u} & & \\
\mathscr{R}^{\mathbb{F}}_1 & \xrightarrow{\Delta_R} & (\mathscr{R}^{\mathbb{F}} \hat{\otimes} \mathscr{R}^{\mathbb{F}})_1 & &
\end{array}$$

the difference $\Delta_{\mathscr{B}} u - (u\hat{\otimes}u)\Delta_R$ lifts to $\Sigma\mathscr{A} \otimes \mathscr{A}$ and satisfies

$$\xi\bar{\pi} = \Delta_{\mathscr{B}} u - (u\hat{\otimes}u)\Delta_R : \mathscr{R}^1_{\mathbb{F}} \xrightarrow{\bar{\pi}} \bar{R} \xrightarrow{\xi} \Sigma\mathscr{A} \otimes \mathscr{A}$$

where ξ is dual to ξ_* in (14.8)*. Here $\bar{\pi}$ is the projection $\mathscr{R}_{\mathbb{F}} \twoheadrightarrow R_{\mathscr{F}} \twoheadrightarrow \bar{R}$. The cocycle ξ is trivial if p is odd.*

(16.2) Definition. Using a splitting u of $\mathscr{B}^{\mathbb{F}}$ as in (16.1) we define a *multiplication operator*

$$A : \mathscr{A} \otimes R_{\mathscr{B}} \to \Sigma\mathscr{A}$$

by the equation

$$A(\bar{\alpha} \otimes x) = u(\alpha x) - \alpha u(x)$$

for $\alpha \in \mathscr{F}_0$, $x \in R_{\mathscr{B}}$. Thus $-A$ is a multiplication map as studied in [1, 16.3.1]. Fixing a splitting s of $\mathscr{R}^{\mathbb{F}}$ as in (15.2) we define an *s-multiplication operator A_s* to be the composite

$$A_s : \mathscr{A} \otimes R_{\mathscr{F}} \xrightarrow{1\otimes s} \mathscr{A} \otimes R_{\mathscr{B}} \xrightarrow{A} \Sigma\mathscr{A}\,.$$

Such operators have the properties of the following s-multiplication maps.

(16.3) Definition. Let s be a splitting of $\mathscr{R}^{\mathbb{F}}$ as in (15.2) and let U_s, a_s, b_s be defined as in section 15. An *s-multiplication map*

$$A_s : \mathscr{A} \otimes R_{\mathscr{F}} \to \mathscr{A}$$

is an $\mathbb{F}$-linear map of degree -1 satisfying the following conditions with $\alpha, \alpha', \beta, \beta' \in \mathscr{F}_0$, $x, y \in R_{\mathscr{F}}$

(1) $A_s(\alpha, x\beta) = A_s(\alpha, x)\beta + \varkappa(\alpha) b_s(x, \beta)$
(2) $A_s(\alpha\alpha', x) = A_s(\alpha, \alpha' x) + \varkappa(\alpha) a_s(\alpha', x) + (-1)^{\deg(\alpha)} \alpha A_s(\alpha', x)$
(3) $\delta A_s(\alpha, x) = (A_s)_{\#}(\alpha \otimes \Delta x) + L(\alpha, x) + \nabla_{\xi}(\alpha, x) + \delta\varkappa(\alpha) U_s(x)$.

Here $(A_s)_{\#} : \mathscr{A} \otimes R^{(2)}_{\mathscr{F}} \to \mathscr{A} \otimes \mathscr{A}$ is defined by the equalities

$$(A_s)_{\#}(\alpha \otimes x \otimes \beta') = \sum (-1)^{\deg(\alpha_r)\deg(x)} A_s(\alpha_\ell, x) \otimes \alpha_r \beta',$$
$$(A_s)_{\#}(\alpha \otimes \beta \otimes y) = \sum (-1)^{\deg(\alpha_r)\deg(\beta) + \deg(\alpha_\ell) + \deg(\beta)} \alpha_\ell \beta \otimes A_s(\alpha_r, y),$$

where as always

$$\delta(\alpha) = \sum \alpha_\ell \otimes \alpha_r \in \mathscr{A} \otimes \mathscr{A}.$$

Two s-multiplication maps A_s and A'_s are *equivalent* if there exists an $\mathbb{F}$-linear map

$$\gamma : R_{\mathscr{F}} \to \mathscr{A}$$

of degree -1 such that the equality

$$A_s(\alpha, x) - A'_s(\alpha, x) = \gamma(\alpha x) - (-1)^{\deg(\alpha)}\alpha\gamma(x)$$

holds for any $\alpha \in \mathscr{A}$, $x \in R_{\mathscr{F}}$ and moreover γ is right $\mathscr{F}_0$-equivariant and the diagram

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\delta} & \mathscr{A} \otimes \mathscr{A} \\ \uparrow{\scriptstyle \gamma} & & \uparrow{\scriptstyle \gamma_\#} \\ R_{\mathscr{F}} & \xrightarrow[\Delta]{} & R^{(2)}_{\mathscr{F}} \end{array}$$

commutes, with $\gamma_\#$ given by

$$\gamma_\#(x \otimes \beta) = \gamma(x) \otimes \beta,$$
$$\gamma_\#(\alpha \otimes y) = (-1)^{\deg(\alpha)}\alpha \otimes \gamma(y)$$

for $\alpha, \beta \in \mathscr{F}_0$, $x, y \in R_{\mathscr{F}}$.

(16.4) Theorem. *There exists an s-multiplication map A_s and any two such s-multiplication maps are equivalent. Moreover each s-multiplication map is an s-multiplication operator as in* (16.2) *and vice versa.*

Proof. We apply [1, 16.3.3]. In fact, we obtain by A_s the multiplication operator

$$A : \mathscr{A} \otimes R_{\mathscr{B}} = \mathscr{A} \otimes \mathscr{A} \oplus \mathscr{A} \otimes R_{\mathscr{F}} \to \Sigma\mathscr{A}$$

with $A(\alpha \otimes x) = A_s(\alpha \otimes \bar{x}) + \varkappa(\alpha)\xi$ where $(\bar{x}, \xi) \in R_{\mathscr{F}} \oplus \mathscr{A} = R_{\mathscr{B}} \otimes \mathbb{F}$ corresponds to x, that is $s(\bar{x}) + \iota(\xi) = x$ for $\iota : \mathscr{A} \subset R_{\mathscr{B}} \otimes \mathbb{F}$. □

(16.5) Remark. For the splitting $s = \phi$ of $\mathscr{R}^{\mathbb{F}}$ in (15.3) the maps

$$A_{n,m} : \mathscr{A} \to \mathscr{A}$$

are defined by $A_{n,m}(\alpha) = A_\phi(\alpha \otimes [n, m])$, with $[n, m]$ the Adem relations in $R_{\mathscr{F}}$. Using formulæ in (16.3) the maps $A_{n,m}$ determine the ϕ-multiplication map A_ϕ completely. The maps $A_{n,m}$ coincide with the corresponding maps $A_{n,m}$ in [1, 16.4.4]. In [1, 16.6] an algorithm for determination of $A_{n,m}$ is described, leading to a list of values of $A_{n,m}$ on the elements of the admissible basis of $\mathscr{A}$. The algorithm for the computation of $A_{n,m}$ can be deduced from theorem (16.4) above.

(16.6) Remark. Triple Massey products $\langle \alpha, \beta, \gamma \rangle$ with $\alpha, \beta, \gamma \in \mathscr{A}$, $\alpha\beta = 0 = \beta\gamma$, as in (6.16) can be computed by A_s as follows. Let $\bar{\beta}\bar{\gamma} \in R_{\mathscr{B}}$ be given as in (6.16). Then $\bar{\beta}\bar{\gamma} \otimes 1 \in R_{\mathscr{B}} \otimes \mathbb{F}$ satisfies

$$\bar{\beta}\bar{\gamma} \otimes 1 = s(\bar{x}) + \iota(\xi)$$

with $\bar{x} \in R_{\mathscr{F}}$, $\xi \in \mathscr{A}$ and $\langle \alpha, \beta, \gamma \rangle$ satisfies

$$A_s(\alpha \otimes \bar{x}) + \varkappa(\alpha)\xi \in \langle \alpha, \beta, \gamma \rangle .$$

Compare [1, 16.3.4].

Now it is clear how to introduce via a_s, b_s, U_s, ξ, $\varkappa$, and A_s a Hopf pair algebra structure on

(16.7)
$$\begin{array}{ccc} \mathscr{A} \oplus \Sigma\mathscr{A} \oplus R_{\mathscr{F}} & \xrightarrow{q} & \mathscr{A} \oplus R_{\mathscr{F}} \\ \| & & \| \\ \mathscr{B}^{\mathbb{F}}_1 & & \mathscr{R}^{\mathbb{F}}_1 \end{array}$$

which is isomorphic to $\mathscr{B}^{\mathbb{F}}$, compare (15.9).

In the next section we describe an algorithm for the computation of a ψ-multiplication map, where ψ is the comultiplicative splitting of $\mathscr{R}^{\mathbb{F}}$ in (15.12). For this we compute the dual map $(A_\psi)_*$ of A_ψ.

17. C H $\mathscr{B}_{\mathbb{F}}$

For the comultiplicative splitting $s = \psi$ of $\mathscr{R}^{\mathbb{F}}$ in (15.12) we introduce the following ψ-comultiplication maps which are dual to the ψ-multiplication maps in (16.3).

(17.1) Definition. Let a_{ψ_*} and b_{ψ_*} be given as in (15.17). A *ψ-comultiplication map*

$$A^{\psi} : \mathscr{A}_* \to \mathscr{A}_* \otimes R_{\mathscr{F}*}$$

is an $\mathbb{F}$-linear map of degree +1 satisfying the following conditions.

(1) The maps in the diagram

$$\begin{array}{ccc} \mathscr{A}_* \otimes R_{\mathscr{F}*} & \xleftarrow{A^{\psi}} & \mathscr{A}_* \\ \downarrow{\scriptstyle 1\otimes m_*^r} & & \downarrow{\scriptstyle m_*} \\ \mathscr{A}_* \otimes R_{\mathscr{F}*} \otimes \mathscr{F}_* & \xleftarrow[A^{\psi}\otimes i]{} & \mathscr{A}_* \otimes \mathscr{A}_* \end{array}$$

satisfy

$$(1 \otimes m_*^r)A^{\psi} - (A^{\psi} \otimes i)m_* = (\varkappa_* \otimes b_{\psi_*})m_*.$$

Here $\varkappa_*$ is computed in (9.7) and m_*^r is defined in (15.16).

(2) The maps in the diagram

$$\begin{array}{ccc} \mathscr{A}_* \otimes R_{\mathscr{F}*} & \xleftarrow{A^{\psi}} & \mathscr{A}_* \\ \downarrow{\scriptstyle m_*\otimes 1} & & \\ \mathscr{A}_* \otimes \mathscr{A}_* \otimes R_{\mathscr{F}*} & & \downarrow{\scriptstyle A^{\psi}} \\ \downarrow{\scriptstyle 1\otimes i\otimes 1} & & \\ \mathscr{A}_* \otimes \mathscr{F}_* \otimes R_{\mathscr{F}*} & \xleftarrow[1\otimes m_*^{\ell}]{} & \mathscr{A}_* \otimes R_{\mathscr{F}*} \end{array}$$

satisfy

$$(1 \otimes i \otimes 1)(m_* \otimes 1)A^{\psi} = (1 \otimes m_*^{\ell})A^{\psi} + (\tau \otimes i \otimes 1)(1 \otimes A^{\psi})m_* + (\varkappa_* \otimes a_{\psi_*})m_*.$$

Here m_*^{ℓ} is as in (15.15), and $\tau : \mathscr{A}_* \to \mathscr{A}_*$ is given by $\tau(\alpha) = (-1)^{\deg(\alpha)}\alpha$.

(3) For $x, y \in \mathscr{A}_*$ the product xy in the algebra $\mathscr{A}_*$ satisfies the formula

$$A^{\psi}(xy) = A^{\psi}(x)m_*(y) + (-1)^{\deg(x)}m_*(x)A^{\psi}(y) + L_*(x, y) + \nabla_{\xi_*}(x, y).$$

Here L_* and ∇_{ξ_*} are given in 12.12 and 14.16 respectively, with $L_* = \nabla_{\xi_*} = 0$ for p odd.

Two ψ-comultiplication maps $A^{\psi}, \bar{A}^{\psi}$ are *equivalent* if there is a derivation

$$\gamma_* : \mathscr{A}_* \to R_{\mathscr{F}*}$$

of degree +1 satisfying the equality

$$A^{\psi} - \bar{A}^{\psi} = m_*^{\ell}\gamma_* - (\tau \otimes \gamma_*)m_*.$$

As a dual statement to (16.4) we get

(17.2) Theorem. *There exists a ψ-comultiplication map A^{ψ} and any two such ψ-comultiplication maps are equivalent. Moreover each ψ-comultiplication map A^{ψ} is the dual of a ψ-multiplication map A_{ψ} in* (16.4) *with $A^{\psi} = A_{\psi_*}$.*

□

Now dually to (16.7), it is clear how to introduce via a_{ψ_*}, b_{ψ_*}, ξ_*, $\varkappa_*$, and A^{ψ} a Hopf pair coalgebra structure on

$$\begin{array}{ccc} \mathscr{A}_* \oplus \Sigma\mathscr{A}_* \oplus R_{\mathscr{F}*} & \xleftarrow{i} & \mathscr{A}_* \oplus R_{\mathscr{F}*} \\ \| & & \| \\ \mathscr{B}^1_{\mathbb{F}} & & \mathscr{R}^1_{\mathbb{F}} \end{array}$$

which is isomorphic to $\mathscr{B}_{\mathbb{F}}$, compare (15.18).

18. A

We finally briefly indicate how to use the obtained expressions to perform actual calculations on a computer. Notably, knowledge of either a multiplication map such as A_ϕ or of a comultiplication map such as A^ψ enables one to calculate all ordinary and matric triple Massey product in the Steenrod algebra, as well as the $d_{(2)}$ differential of the Adams spectral sequence, and hence its E_3 term — see [1, 2].

Using conditions in (17.1) one obtains an algorithm computing the comultiplication map A^ψ as follows. At nth step, $n = 1, 2, ...$, one picks the Milnor generator ζ_n and seeks the value $A^\psi(\zeta_n)$ as a linear combination

$$\sum \epsilon_{a,r} a \otimes r$$

where $a \otimes r$ runs through the basis of the 2^nth homogeneous component of $\mathscr{A}_* \otimes R_{\mathscr{F}_*}$ formed by tensors $a \otimes r$ with $\deg(a) + \deg(r) = 2^n$ where a is a monomial in Milnor generators ζ_k and r a monomial in the elementary Lyndon elements $M_{d_1,...,d_\ell}$ containing at least one element not in $\{M_1, M_{2,1}, M_{4,2,1}, ...\}$ (see description after (11.2)).

The conditions (1) and (2) in (17.1) impose linear equations on the numbers $\epsilon_{a,r}$; solution is not uniquely determined, so one fixes one possible solution and obtains the corresponding value $A^\psi(\zeta_n)$. After that, condition (3) in (17.1) gives all values of A^ψ on the subalgebra of $\mathscr{A}_*$ generated by the ζ_k for $k \leqslant n$. One then moves to step $n + 1$.

One sees easily that this inductive procedure indeed works, that is, at each step n the conditions (1) and (2) are expressible by values $A^\psi(\zeta_k)$ with $k \leqslant n$.

(18.1) Remark. In [1, Chapter 16], an algorithm for computation of the multiplication map A_ϕ is presented. This algorithm has been implemented and used in [2] to compute values of the differential $d_{(2)}$ in the Adams spectral sequence up to degree 40. The main computational strain of that algorithm lies in the fact that in the intermediate steps one has to compute auxiliary values of the maps such as ∇_ϕ which involve expressions much larger than the final answer.

The dual approach presented here gives an algorithm for A^ψ which is more economic in this respect, in the sense that the intermediate values to compute are closer in size to the final value. It is thus expected that with the same computational resources, values in much higher degrees can be reached.

R

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