COMPACT FLAT SPACETIMES

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Abstract

We give a precise description of compact flat spacetimes and use this to draw conclusions about the existence of closed timelike and null geodesics. Other related results and comments are also given.

1. Introduction

Recall that a spacetime (M, g) is a time-oriented Lorentz manifold with signature $-, +, \ldots, +$. A tangent vector X to M is said to be timelike, spacelike or null according to whether $g(X, X) < 0, > 0$ or $= 0$, respectively. Similarly, a smooth curve in M is said to be timelike, spacelike or null if its tangent vector field is always timelike, spacelike or null, respectively. We are interested in compact flat spacetimes which are, according to [5], quotients of the form $\mathbb{R}_1^n/\Gamma$, where $\mathbb{R}_1^n$ denotes the n-dimensional Minkowski space and Γ is a discrete cocompact subgroup of isometries which, by [13], must be virtually polycyclic (equivalently, contains a solvable subgroup of finite index).

As mentioned in [16], it was our great wish to complete our line of investigation of compact flat spacetimes by classifying them, but after having almost had finished the classification we read in [6] that this had been done already in [14], where it was shown that a compact flat n-dimensional spacetime is, up to a finite cover, either a k-step nilmanifold with $k \leq 3$, or a solvmanifold $T_A^n = \mathbb{R} \times T^{n-1}/\left(t, z\right) \sim \left(t+1, Az\right)$, where $A \in Sl_{n-1}\left(\mathbb{Z}\right)$ is a Lorentz matrix, that is, A is diagonalizable over $\mathbb{C}$ and its eigenvalues are $1, \lambda, \lambda^{-1}, a_1, \ldots, a_{n-3}$, where λ is positive real and each a_i satisfies $|a_i| = 1$.

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The aim of this paper is to give a precise classification of compact flat spacetimes using different techniques, and then use this to study the existence of the closed timelike and null geodesics. Our approach is purely geometric, given that the methods we employed can be regarded as an adaptation to the n-dimensional case of the ideas that had been used in [8] for the case $n = 4$. This relies on an exact description by means of exponential coordinates of the Lie groups which may act isometrically and simply transitively on Minkowski space, and on an extension of the classical Bieberbach's theorem to discrete subgroups of the affine group $Aff(\mathbb{R}^n)$ (cf. [6]). The paper is organized as follows. In Section 2, we classify the unipotent Lie groups which can act isometrically and simply transitively on Minkowski space. We show that they are very special k-step nilpotent Lie groups $U_{(a,b)}$ that we parametrize by pairs (a, b). In this case we see that the nilpotency class k is ≤ 3, and that $U_{(a,b)}$ is abelian (i.e., $k = 1$) if and only if $a = b = 0$; it is 2-step nilpotent if and only if $a = 0$ and $b \neq 0$; and it is 3-step nilpotent otherwise (i.e., $a \neq 0$). Since a geodesically complete and flat left invariant Lorentz metric on a simply connected Lie group N is the same as an isometric simply transitive action of N on Minkowski space $\mathbb{R}^n_1$, we deduce from this classification that a nilpotent Lie group can admit a flat left invariant Lorentz metric if and only if it is isomorphic to some $U_{(a,b)}$.

In Section 3, we classify the nonunipotent Lie groups that can act isometrically and simply transitively on Minkowski space. We show that they fall to two categories depending on whether the linear part of the unipotent radical of the algebraic hull of the group contains a hyperbolic automorphism. The fundamental result of this section is Theorem 3.2. It asserts that if G is a nonunipotent Lie group that arises as the crystallographic hull of some discrete cocompact subgroup, then G is isomorphic to a 2-step sovable Lie group which may be obtained by extending (in some precise sense) the three-dimensional 2-step solvable Lie group G_3 having X, Y, Z with $[X, Y] = Y$, $[X, Z] = -Z$, $[Y, Z] = 0$ as a basis for its Lie algebra. This result is obtained by using a result in [6] generalizing the first theorem of Bieberbach on crystallographic groups to discrete subgroups of $Aff(\mathbb{R}^n)$. Theorem 3.2 will be afterwards a crucial tool for the study of the existence of closed timelike and null geodesics in compact flat spacetimes.

In Section 4, we classify compact flat spacetimes in light of the results obtained in the previous sections. We prove in Theorem 4.1 that, up to finite covers, one such spacetime is either a k-step nilmanifold with $k \leq 3$, or a 2-step solvmanifold that is diffeomorphic to the product $T^m \times T^{2k+3}_A$, where $n = m + 2k + 3$ and A

is some Lorentz matrix in $Sl_{2k+3}(\mathbb{Z})$. This contains the results of [2] for $n = 3$ and results of [8] for $n = 4$ as special cases. We should also mention the paper [7] from that a more precise classification for the case $n = 3$ might be deduced.

In Section 5 we prove, using Theorem 4.1, a special case of an old conjecture of Auslander [3] stating that every nilpotent Lie group N which acts affinely and simply transitively on $\mathbb{R}^n$ has a nontrivial central translation. Namely, we prove that if N acts affinely and simply transitively on $\mathbb{R}^n$ and the action preserves a flat Lorentz metric on $\mathbb{R}^n$, then N contains a nontrivial translation in its center. More precisely, such a translation lies in the derived subgroup $[N, N]$ of N.

In Section 6, we discuss the problem of existence of closed nonspacelike geodesics in compact flat spacetimes. We prove that a compact flat spacetime contains no closed timelike geodesic if and only if it is a nilmanifold $U_{(a,b)}/\Gamma$ with $b \neq 0$. We also prove that any compact flat spacetime contains a closed null geodesic, and we deduce an alternate proof for a previous result [16] stating that a compact flat spacetime M contains a closed timelike geodesic if and only if the fundamental group $\pi_1(M)$ contains a nontrivial timelike translation.

2. Unipotent Lie groups acting isometrically and simply transitively on $\mathbb{R}^n_1$

The following result classifies unipotent Lie groups which can act isometrically and simply transitively on Minkowski space.

Theorem 2.1. *Let U be a unipotent Lie group acting isometrically and simply transitively on Minkowski space $(\mathbb{R}^n_1, g)$. Then, with respect to suitable linear coordinates (x, y, z) on $\mathbb{R}^n \cong \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}$ with $y = (y_1, \ldots, y_{n-2})$, the metric g has the form $2dxdz + dy_2^2 + \ldots + dy_{n-2}^2$ and U is conjugate to the unipotent group $U_{(a,b)}$ which consists of all pairs*

$$\left(\begin{pmatrix} 1 & -w & -\frac{1}{2}ww^t \\ 0 & Id_{n-2} & w^t \\ 0 & 0 & 1 \end{pmatrix}, \begin{bmatrix} x - \frac{1}{2}wy^t - \frac{1}{6}ww^t \\ y^t + \frac{1}{2}zw^t \\ z \end{bmatrix} \right),$$

with $w = (a_2 y_2, b_2 z, a_4 y_2, b_4 z, \ldots, a_{n-2} y_{n-2}, b_{n-2} z)$ if n is even, and $w = (b_1 z, a_2 y_2, b_3 z, a_4 y_4, \ldots, a_{n-3} y_{n-3}, b_{n-2} z)$ if n is odd; here $a = (a_2, \ldots, a_{n-2})$, $b = (b_2, \ldots, b_{n-2})$ if n is even, and $a = (a_2, \ldots, a_{n-3})$, $b = (b_1, \ldots, b_{n-2})$ if n is odd, *and the conjugacy class of $U_{(a,b)}$ is uniquely determined by (a,b) up to rescaling $(a,b) \to (\lambda a, \lambda^2 b)$, with $\lambda > 0$.*

Proof. Recall that $Isom(\mathbb{R}_1^n) = O(n-1,1) \ltimes \mathbb{R}^n$, where the factor $\mathbb{R}^n$ acts on $\mathbb{R}_1^n$ by translations. Let $\pi : Isom(\mathbb{R}_1^n) \to \mathbb{R}^n$ be the natural projection which associates each element of $Isom(\mathbb{R}_1^n)$ to its translation part. Since U acts simply transitively, we see that π is bijective from U onto $\mathbb{R}^n$. This allows us to define a map $\phi : \mathbb{R}^n \to O(n-1,1)$ so that $U = Graph\ \phi$. In this respect, write $(\phi(v), v)$ for elements of U. Since U is unipotent, it is algebraic and therefore coincides with its unipotent radical. Let $L : Isom(\mathbb{R}_1^n) \to O(n-1,1)$ denote the morphism which associates each element of $Isom(\mathbb{R}_1^n) = O(n-1,1) \ltimes \mathbb{R}^n$ to its linear part. Since U is unipotent, $L(U)$ is conjugate to a subgroup of the group $U(n,\mathbb{R})$ of upper triangular matrices with diagonal entries 1. On the other hand, we can easily see that $L(U)$ preserves a null vector (compare [8], p. 388). So, we choose linear coordinates $x_1, \ldots, x_n$ in $\mathbb{R}^n$ so that $L(U)$ fixes the vector $e_1 = (1,0,\ldots,0)^t$, where t denotes the transpose. Denote by J the matrix of the metric g relative to the pseudo-orthonormal basis $\{e_1, \ldots, e_n\}$ associated to the above system of coordinates, with $g(e_1,e_1) = g(e_n,e_n) = 0$ and $g(e_1,e_n) = 1$. Since $B = \phi(v)$ satisfies $B^tJB = J$ and $Be_1 = e_1$ we get $B^t(Je_1) = Je_1$, and since Je_1 is null and $B^t(e_n) = e_n$ we may assume that $Je_1 = e_n$, that is, g has the form $2dx_1dx_n + dx_2^2 + \ldots + dx_{n-1}^2$. Relative to the above basis, an element $(A(v), v)$ of the Lie algebra $\mathcal{U}$ of U takes the form

$$A(v) = \begin{pmatrix} 0 & \alpha(v) & \beta(v) \\ 0 & K(v) & \gamma(v) \\ 0 & 0 & 0 \end{pmatrix},$$

with $\alpha(v) = (\alpha_1(v), \ldots, \alpha_{n-2}(v))$, $\beta(v) \in \mathbb{R}$, $\gamma(v) = (\gamma_1(v), \ldots, \gamma_{n-2}(v))^t$ and $K(v)$ is a an upper triangular $(n-1)$-squared matrix with diagonal entries 0.

¿From the identity $A^tJ + JA = 0$ it follows that $K^t = -K$, $\beta = 0$, and $\gamma^t = -\alpha$. That is, $K \equiv 0$, $\beta = 0$, and $\gamma_2 = -\alpha_1, \ldots, \gamma_{n-1} = \alpha_{n-1}$. Set $v = (x,y,z)^t$ with $y = (y_1, \ldots, y_{n-2})$, and write $\alpha_i\left((x,y,z)^t\right) = c_i x + \sum_{j=1}^{n-2} \alpha_{i,j} y_j + b_i z$. For $v, w \in \mathbb{R}^n$, we have $[(A(v), v), (A(w), w)] = (A(v)A(w) - A(w)A(v), A(v)w - A(w)v)$.

We check easily that $A(v)A(w) - A(w)A(v) = 0$ and that $(\widetilde{x}, \widetilde{y}, \widetilde{z})^t = A(v)w - A(w)v = (-\alpha(v)w_2 + \alpha(w)v_2, \alpha^t(v)w_3 - \alpha^t(w)v_3, 0)^t$, where here $v = (v_1, v_2, v_3)^t$ and $w = (w_1, w_2, w_3)^t$. Since $[(A(v), v), (A(w), w)] \in \mathcal{U}$, we get $\alpha\left((\widetilde{x}, \widetilde{y}, \widetilde{z})^t\right) = 0$ for arbitrary $v, w \in \mathcal{U}$. On the one hand, for $v = (1,0,0)^t$

and $w=(0,k_i,0)^t$, with $k_i=(0,\ldots,0,1,0,\ldots,0)$ where the 1 is at the *ith* rank, we see that $\alpha(v)=(c_1,\ldots,c_{n-2})$ and $\alpha(w)=(\alpha_{1,i},\ldots,\alpha_{n-2,i})$. Thus $(\widetilde{x},\widetilde{y},\widetilde{z})^t=(-c_i,0,0)^t$, and hence $\alpha\left((\widetilde{x},\widetilde{y},\widetilde{z})^t\right)=-c_i(c_1,\ldots,c_{n-2})=0$. Since i is arbitrary, this yields $c_1=\ldots=c_{n-2}=0$. On the other hand, for $v=(0,0,1)^t$ and $w=(0,k_i,0)^t$ we see that $\alpha(v)=(b_1,\ldots,b_{n-2})^t$ and $\alpha(w)=(\alpha_{1,i},\ldots,\alpha_{n-2,i})$. Thus $(\widetilde{x},\widetilde{y},\widetilde{z})^t=(b_i,(\alpha_{1,i},\ldots,\alpha_{n-2,i}),0)^t$, and hence

$$\alpha\left((\widetilde{x},\widetilde{y},\widetilde{z})^t\right)=\left(\sum_{j=1}^{n-2}\alpha_{1,j}\alpha_{j,i},\ldots,\sum_{j=1}^{n-2}\alpha_{n-2,j}\alpha_{j,i}\right)^t=0 \tag{2.1}$$

Setting $[\alpha]=(\alpha_{i,j})_{1\leq i,j\leq n-2}$ it follows from (2.1) by letting i vary that $[\alpha]^2=0$.

We distinguish two cases. Assume first that n is even and remark that, in this case, there exists a basis relative to which $[\alpha]$ is a product of Jordan blocks of the form

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \quad a\in\mathbb{R}.$$

Notice that since $[\alpha]$ acts on a spacelike subspace, we may choose Euclidean orthonormal coordinates, say $y_1,\ldots,y_{n-2}$, relative to which $[\alpha]$ is a product of Jordan blocks of the above form. In that coordinates, we see that the α_i have the form $\alpha_{2k-1}\left((x,y,z)^t\right)=a_{2k}y_{2k}+b_{2k-1}z$ and $\alpha_{2k}\left((x,y,z)^t\right)=b_{2k}z$, $1\leq k\leq\frac{n-2}{2}$. Without loss of generality, by assuming $b_{n-2}\neq 0$ and $a_{2k}\neq 0$, $1\leq k\leq\frac{n-2}{2}$, we may change the coordinates $y_2,y_4,\ldots,y_{n-2}$ so that $b_{2k-1}=0$ for $1\leq k\leq\frac{n-2}{2}$. In matricial words, we may suppose that $(\alpha_1,\alpha_3,\ldots,\alpha_{n-3})^t=diag\{a_2,a_4,\ldots,a_{n-2}\}(y_2,y_4,\ldots,y_{n-2})^t$. This finally allows us to choose an orthonormal basis relative to which we have

$$(A(v),v)=\left(\begin{pmatrix} 0 & -w & 0 \\ 0 & 0 & w^t \\ 0 & 0 & 0 \end{pmatrix},\begin{bmatrix} x \\ y^t \\ z \end{bmatrix}\right),$$

with $w=(a_2y_2,b_2z,a_4y_4,b_4z,\ldots,a_{n-2}y_{n-2},b_{n-2}z)$ and $y=(y_1,\ldots,y_{n-2})$.

By computing

$$\exp\begin{pmatrix} A(v) & v \\ 0 & 0 \end{pmatrix},$$

we see that this is precisely an element of the unipotent group $U_{(a,b)}$ described by Theorem 2.1 when n is even. Now, the second case where n is odd can be handled similarly. This completes the proof of Theorem 2.1. ■

In light of Theorem 2.1, we easily see that the nilpotency class k of $U_{(a,b)}$ is ≤ 3. Precisely, $U_{(a,b)}$ is abelian (i.e., $k = 1$) if and only if $a = b = 0$; it is 2-step nilpotent if and only if $a = 0$ and $b \neq 0$; and it is 3-step nilpotent otherwise (i.e., $a \neq 0$). In the case $U_{(a,b)}$ is 2-step nilpotent, we see that $U_{(a,b)}$ is a trivial central extension of the three-dimensional Heisenberg group H_3. More generally, we proved in [17] that a 2-step nilpotent Lie group N can admit a flat left invariant Lorentz metric if and only if N is a trivial central extension of H_3.

Theorem 2.2. *A nilpotent Lie group N can admit a flat left invariant Lorentz metric if and only if it is isomorphic to some unipotent group $U_{(a,b)}$. In particular, the nilpotency class of N is ≤ 3.*

Proof. It is clear that a geodesically complete and flat left invariant Lorentz metric on a simply connected Lie group N is the same as a simply transitive action by isometries of N on Minkowski space $\mathbb{R}_1^n$. On the other hand, the following result is contained in [1] : a flat left invariant pseudo-Riemannian metric on a Lie group G is geodesically complete if and only if G is unimodular (that is, the linear transformation Ad_g has determinant ± 1 for every $g \in G$). Since any nilpotent Lie group is unimodular, Theorem 2.2 follows now as a consequence of Theorem 2.1 and the result above. ■

Corollary 2.3. *A k-step nilpotent Lie group with $k > 3$ cannot admit a flat left invariant Lorentz metric.*

3. Nonunipotent Lie groups acting isometrically and simply transitively on $\mathbb{R}_1^n$

Nonunipotent Lie groups which can act isometrically and simply transitively on Minkowski space are classified by the following result.

Theorem 3.1. *Let G be a nonunipotent Lie group that acts isometrically and simply transitively on Minkowski space $(\mathbb{R}_1^n, g)$, $n \geq 3$. Then, with respect to suitable linear coordinates (x, y, z) on $\mathbb{R}^n \cong \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}$ with $y = (y_1, \ldots, y_{n-2})$, the metric g has the form $2dxdz + dy_2^2 + \ldots + dy_{n-2}^2$ and, with the notation $\overline{y} = (y_{2k+1}, \ldots, y_{n-2})$ and $n = m + 2(k+1)$, the group G is conjugate to either one of the following solvable Lie groups*

(i) $G_{1,n} = \left\{ \left(\begin{array}{cccccc} e^{\mu(\overline{y})} & & & & & 0 \\ & e^{J_1(\overline{y})} & & & & \\ & & \ddots & & & \\ & & & e^{J_k(\overline{y})} & & \\ & & & & Id_m & \\ 0 & & & & & e^{-\mu(\overline{y})} \end{array} \right), \left[\begin{array}{c} x \\ y_1 \\ \vdots \\ y_{2k} \\ \overline{y} \\ z \end{array} \right] \right\},$

where μ *and the* θ_i *are linear functions depending only in* $\overline{y}$, *and*

$$e^{J_i(\overline{y})} = \begin{pmatrix} \cos\theta_i(\overline{y}) & \sin\theta_i(\overline{y}) \\ -\sin\theta_i(\overline{y}) & \cos\theta_i(\overline{y}) \end{pmatrix}, \quad 1 \leq i \leq k.$$

(ii) $G_{2,n} = \left\{ \left(\begin{array}{cccccc} 1 & & & & & 0 \\ & e^{J_1(x,\overline{y},z)} & & & & \\ & & \ddots & & & \\ & & & e^{J_k(x,\overline{y},z)} & & \\ & & & & Id_m & \\ 0 & & & & & 1 \end{array} \right), \left[\begin{array}{c} x \\ y_1 \\ \vdots \\ y_{2k} \\ \overline{y} \\ z \end{array} \right] \right\},$

where the θ_i *are linear functions depending in* $x, \overline{y}, z$, *and*

$$e^{J_i(x,\overline{y},z)} = \begin{pmatrix} \cos\theta_i(x,\overline{y},z) & \sin\theta_i(x,\overline{y},z) \\ -\sin\theta_i(x,\overline{y},z) & \cos\theta_i(x,\overline{y},z) \end{pmatrix}, \quad 1 \leq i \leq k.$$

Proof. By [13], G is solvable. Since G is nonunipotent, it follows (see [4]) that the algebraic hull $A(G)$ (that is, the Zariski closure) of G decomposes as a semidirect product $T \ltimes U$, where U denotes the unipotent radical of $A(G)$ and T is a nontrivial reductive abelian subgroup of $O(n-1,1)$. Let $L : Isom(\mathbb{R}^n_1) \to O(n-1,1)$ denote the morphism which associates each element of $Isom(\mathbb{R}^n_1) = O(n-1,1) \ltimes \mathbb{R}^n$ to its linear part. Since U is unipotent, $L(U)$ is conjugate to a subgroup of the group $U(n,\mathbb{R})$ of upper triangular matrices with diagonal entries 1. On the one hand, it is easy to see that the normalizer $N(U)$ of U in $Isom(\mathbb{R}^n_1)$ decomposes as $N(U) = R \ltimes U$, where R is a product of a maximal torus of $SO(n-1)$ with a reductive one-parameter subgroup of $O(n-1,1)$ acting by hyperbolic automorphisms. Notice that since U is normal in $A(G)$, we have $A(G) \subseteq N(U)$ by definition of $N(U)$. On the other hand, as we can check easily, $L(U)$ preserves a null vector (compare [8], p. 388). We choose linear coordinates $x_1, \ldots, x_n$ in $\mathbb{R}^n$ so that $L(U)$ fixes the vector $e_1 = (1, 0, \ldots, 0)^t$, where t denotes the transpose. Denote by J the matrix of the metric g relative to the peudo-orthonormal basis $\{e_1, \ldots, e_n\}$ associated to the above system of coordinates, with $g(e_1, e_1) = g(e_n, e_n) = 0$ and $g(e_1, e_n) = 1$. Let $\phi = (\phi(v), v)$ be an element of G, and write as usual $\phi(v) = \phi_u(v).\phi_s(v) = \phi_s(v).\phi_u(v)$ for unique semi-simple $\phi_s(v)$ and unipotent $\phi_u(v)$ (notice that $(\phi_u(v), v) \in U$ and $\phi_s(v) \in R$). Since $A = \phi_u(v)$ satisfies $A^t JA = J$ and $Ae_1 = e_1$, we get $A^t(Je_1) = Je_1$. Since Je_1 is null and $A^t(e_n) = e_n$, we may assume that $Je_1 = e_n$, that is, g has the form $2dx_1 dx_n + dx_2^2 + \ldots + dx_{n-1}^2$. Relative to the above basis, ϕ takes the form

$$\phi = (\phi(v), v) = \left(\begin{pmatrix} e^{\mu(v)} & a_{1,2} & & \cdots & & a_{1,n} \\ 0 & e^{J_1(v)} & a_{2,4} & \cdots & & \\ & & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & e^{J_k(v)} & \ddots & \\ & & & & Id_m & a_{n-1,n} \\ 0 & & \cdots & & 0 & e^{-\mu(v)} \end{pmatrix}, \begin{bmatrix} v_1 \\ \vdots \\ \vdots \\ v_{2k+1} \\ \vdots \\ v_n \end{bmatrix} \right),$$

where μ, the θ_i and the $a_{i,j}$ are functions in v, and

$$J_i(v) = \begin{pmatrix} 0 & \theta_i(v) \\ -\theta_i(v) & 0 \end{pmatrix}, \quad 1 \leq i \leq k.$$

In this case, $\phi_s(v) = diag\left(e^{\mu(v)}, e^{J_1(v)}, \ldots, e^{J_k(v)}, 1, \ldots, 1, e^{-\mu(v)}\right)$ and $\phi_u(v)$ equals the matrix obtained by taking $\mu(v) = \theta_1(v) = \cdots = \theta_k(v) = 0$ in the matrix representing $\phi(v)$. Since $\phi_u(v).\phi_s(v) = \phi_s(v).\phi_u(v)$ we see that $a_{1,2} = \ldots = a_{1,n} = a_{2,n} = \ldots = a_{n-1,n} = 0$. Moreover, we deduce from the equation $g = \phi(v)^* g$ that all the remaining $a_{i,j}$ are identically equal to zero.

Now we turn to the Lie algebra $\mathcal{G}$ of G. Clearly, $\mathcal{G}$ consists of elements of the form $(A(v), v) = (diag(\mu(v), J_1(v), \ldots, J_k(v), 0, \ldots, 0, -\mu(v)), v)$. Note that μ and the θ_i are linear functions in $v_1, \ldots, v_n$. In order to simplify computations, change the notation $v = (v_1, \ldots, v_n)$ with $v = (r, v_1, w_1, \ldots, v_k, w_k, u_1, \ldots, u_m, s)$, where $n = 2(k+1) + m$. With this notation, we see that for two elements v and v' we get $[(A(v), v), (A(v'), v')] = (0, v'')$, with v'' having $r'' = \mu(v) r' - \mu(v') r$, $v_i'' = \theta_i(v) w_i' - \theta_i(v') w_i$, $w_i'' = -\theta_i(v) v_i' + \theta_i(v') v_i$, and $s'' = -\mu(v) s' + \mu(v') s$

as components. Since $\mathcal{G}$ is a Lie algebra, it follows that

$$\mu\left(v''\right)=\theta_1\left(v''\right)=\cdots=\theta_k\left(v''\right)=0. \tag{3.1}$$

We distinguish two cases according to whether μ is identically zero or not. If $\mu\equiv 0$, set $\theta_i\left(v\right)=a_ir+b_is+\sum_{j=1}^{k}\left(a_{ij}v_j+b_{ij}w_j\right)+\sum_{j=1}^{m}\delta_{ij}u_j$. In this case, we have for v'' that $r''=s''=u''_j=0$, $1\leq j\leq m$. Thus, the equation $\theta_i\left(v''\right)=0$ reduces to

$$\sum_{j=1}^{k}\left\{\left(a_{ij}w'_j-b_{ij}v'_j\right)\theta_i\left(v\right)-\left(a_{ij}w_j-b_{ij}v_j\right)\theta_i\left(v'\right)\right\}=0.$$

Letting v and v' such that $r=r'=s=s'=u_j=u'_j=0$, $1\leq j\leq m$, we deduce from the above equation that $a_{ii}^2+b_{ii}^2=0$, that is, $a_{ii}=b_{ii}=0$. Now, if there exists j_0 such that $a_{ij_0}\neq 0$ or $b_{ij_0}\neq 0$, letting v with $v_j=w_j=u_l=0$, $1\leq j\leq k$, $1\leq l\leq m$, and v' with $r'=s'=0$, we deduce that $\theta_{j_0}\equiv 0$. It follows that $\left(A\left(v\right),v\right)=diag\left(0,J_1\left(r,u,s\right),\ldots,J_k\left(r,u,s\right),0,\ldots,0\right)$, where $J_i\left(r,u,s\right)=\begin{pmatrix}0 & \theta_i\left(r,u,s\right)\\ -\theta_i\left(r,u,s\right) & 0\end{pmatrix}$ with $\theta_i\not\equiv 0$, $1\leq i\leq k$, (the notation $\theta_i\left(r,u,s\right)$ means that θ_i depends only in r,u,s with $u=\left(u_1,\ldots,u_m\right)$). This covers case (ii).

Assume now that $\mu\not\equiv 0$. In this case, we deduce from equations (3.1) that μ does not depend in r,s and that each θ_i depends only in $u=\left(u_1,\ldots,u_m\right)$. Setting $\mu\left(v\right)=\sum_{i=1}^{k}\left(\alpha_iv_i+\beta_iw_i\right)+\sum_{i=1}^{m}\delta_iu_i$ and $\theta_i\left(v\right)=\sum_{j=1}^{m}\delta_{ij}u_j$, we see that the equation $\mu\left(v''\right)=0$ yields

$$\sum_{i=1}^{k}\left\{\left(\alpha_iw'_i-\beta_iv'_i\right)\sum_{j=1}^{m}\delta_{ij}u_j+\left(\beta_iv_i-\alpha_iw_i\right)\sum_{j=1}^{m}\delta_{ij}u'_j\right\}=0.$$

Note that we deduce nothing from the equations $\theta_1\left(v''\right)=\cdots=\theta_k\left(v''\right)=0$, because $u''=0$ and each θ_i depends only in u.

If there exists i_0 such that $\alpha_{i_0}\neq 0$ or $\beta_{i_0}\neq 0$, say $\alpha_{i_0}\neq 0$, then by fixing j_0 and letting $u_{j_0}=1$, $u_j=0$ for any $j\neq j_0$, and $u'=\left(u'_1,\ldots,u'_m\right)=\left(0,\ldots,0\right)$, we see that $\sum_{i=1}^{k}\left(\alpha_iw'_i-\beta_iv'_i\right)\delta_{ij_0}$. Letting in addition $w_{i_0}=1$, $w_i=0$ for $i\neq i_0$, and $v'=\left(v'_1,\ldots,v'_k\right)=\left(0,\ldots,0\right)$, we get finally that $\alpha_{i_0}\delta_{i_0j_0}=0$, that is, $\delta_{i_0j_0}=0$. This implies that $\theta_{i_0}\equiv 0$, given that j_0 has been chosen arbitrarily. It follows that either $\left(A\left(v\right),v\right)=\left(diag\left(\mu\left(v,w,u\right),0,\ldots,0,-\mu\left(v,w,u\right)\right),v\right)$ or $\left(A\left(v\right),v\right)=\left(diag\left(\mu\left(u\right),J_1\left(u\right),\ldots,J_k\left(u\right),0,\ldots,0,-\mu\left(u\right)\right),v\right)$, where $J_i\left(u\right)=$

$\begin{pmatrix} 0 & \theta_i(u) \\ -\theta_i(u) & 0 \end{pmatrix}$ with $\theta_i \not\equiv 0$, $1 \leq i \leq k$, (the notation $\mu(v,w,u)$ (resp. $\mu(u)$ and $\theta_i(u)$) means that μ (resp. μ and θ_i) depends in $v_1, w_1, \ldots, v_k, w_k, u_1, \ldots, u_m$ (resp. $u = (u_1, \ldots, u_m)$)). This covers case (i), and therefore achieves the proof of Theorem 3.1. ■

We refine Theorem 3.1 in the case that G is a nonunipotent Lie group arising as the crystallographic hull of some discrete cocompact subgroup.

Theorem 3.2. *Let Γ be a nonunipotent group that acts discontinuously and isometrically on Minkowski space $(\mathbb{R}_1^n, g)$ with compact quotient $\mathbb{R}_1^n/\Gamma$, $n \geq 3$. Then, with respect to suitable linear coordinates (x,t,y,z) on $\mathbb{R}^n \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-3} \times \mathbb{R}$ with $y = (y_1, \ldots, y_{n-3})$, the metric g has the form $2dxdz + dt^2 + dy_1^2 + \ldots + dy_{n-3}^2$ and Γ has a subgroup of finite index which is conjugate to a discrete subgroup of the 2-step solvable Lie group*

$$G_{c,k}^n = \left\{ \left(\begin{pmatrix} e^t & & & & & & 0 \\ & 1 & & & & & \\ & & e^{J_1(t)} & & & & \\ & & & \ddots & & & \\ & & & & e^{J_k(t)} & & \\ & & & & & Id_m & \\ 0 & & & & & & e^{-t} \end{pmatrix}, \begin{bmatrix} x \\ t \\ y_1 \\ \vdots \\ \vdots \\ y_{n-3} \\ z \end{bmatrix} \right) \right\},$$

with $c = (c_1, \ldots, c_k) \in \mathbb{R}^k$, k and m satisfying $n = m + 2k + 3$, and

$$e^{J_i(t)} = \begin{pmatrix} \cos(c_i t) & \sin(c_i t) \\ -\sin(c_i t) & \cos(c_i t) \end{pmatrix}, \quad 1 \leq i \leq k.$$

Proof. By [13] Γ is virtually solvable, that is, it contains a solvable subgroup of finite index. Assume without loss of generality that Γ is itself contained in a solvable Lie group G. Being nonunipotent, G is conjugate in $Isom(\mathbb{R}_1^n)$ to either $G_{1,n}$ or $G_{2,n}$ by Theorem 3.1. Let L be the morphism which associates each element of $Isom(\mathbb{R}_1^n) = O(n-1,1) \ltimes \mathbb{R}^n$ to its linear part, and let $\overline{L(\Gamma)}^0$ be the identity component of the topological closure of $L(\Gamma)$. Setting $L(\Gamma)_{nd} = L(\Gamma) \cap \overline{L(\Gamma)}^0$ and $\Gamma_{nd} = \Gamma \cap L^{-1}(L(\Gamma)_{nd})$, and recalling that Γ is cocompact we deduce by Theorem 1.2.1 of [6] that Γ_{nd} is unipotent. This implies that G is not conjugate to $G_{2,n}$, and thus G is conjugate to $G_{1,n}$. The same argument of Theorem 1.2.1

in [6] applied to Γ considered as a subgroup of $G_{1,n}$ shows that if $\mu(\overline{y}) = 0$ then $\theta_i(\overline{y}) = 0$ for all i. In other words, G is conjugate to $G_{1,n}$ with the additional condition that $\ker \mu \subseteq \bigcap_{i=1}^{k} \ker \theta_i$.

Let us now see to what extent can this last inclusion be simplified. First, with the notation of Theorem 3.1, note that by setting $\mu(\overline{y}) = \sum_{j=1}^{m} \delta_j y_{2k+j}$ and $\theta_i(\overline{y}) = \sum_{j=1}^{m} \alpha_{ij} y_{2k+j}$, where $\overline{y} = (y_{2k+1}, \ldots, y_{n-2})$, it follows from the above inclusion that there exists some j_0 such that $\delta_{j_0} \neq 0$ and $\alpha_{ij_0} \neq 0$ for all i, $1 \leq i \leq k$. Thus the implication

$$[\mu(\overline{y}) = 0] \Rightarrow [\theta_i(\overline{y}) = 0 \text{ for all } i]$$

yields the implication

$$\left[y_{2k+j_0} = -\frac{1}{\delta_{j_0}} \sum_{j \neq j_0} \delta_j y_{2k+j} \right] \Rightarrow \left[\sum_{j \neq j_0} \left(\alpha_{ij} - \frac{\alpha_{ij_0}}{\delta_{j_0}} \delta_j \right) y_{2k+j} = 0 \text{ for all } i \right].$$

Since $\overline{y}$ is arbitrary in the vector space $\ker \mu$, we deduce that $\alpha_{ij} = \frac{\alpha_{ij_0}}{\delta_{j_0}} \delta_j$ for all i, j. Thus, by setting $v' = \left(x, y_1, \ldots, y_{2k}, y'_{2k+1}, \ldots, y'_{n-2}, z\right)$ with $y'_{2k+j_0} = \sum_{j=1}^{m} \delta_j y_{2k+j}$ and $y'_{2k+j} = y_{2k+j}$ for $j \neq j_0$, $1 \leq j \leq m$, we see that $\mu(v') = y'_{2k+j_0}$ and $\theta_i(v') = \frac{\alpha_{ij_0}}{\delta_{j_0}} y'_{2k+j_0}$ for all i.

By setting $c_i = \frac{\alpha_{ij_0}}{\delta_{j_0}}$, $1 \leq i \leq k$, and choosing the new linear coordinates $x'_1, \ldots, x'_n$ in $\mathbb{R}^n$ such that $x'_{l_0} = \sum_{j=1}^{m} \delta_j x_j$ and $x'_l = x_l$ for $l \neq l_0$ where $l_0 = n - m + j_0 - 1$, we see that

$$(A(v), v) = (diag(y_{2k+j_0}, J_1(y_{2k+j_0}), \ldots, J_k(y_{2k+j_0}), 0, \ldots, 0, -y_{2k+j_0}), v),$$

with $J_i(y_{2k+j_0}) = \begin{pmatrix} 0 & c_i y_{2k+j_0} \\ -c_i y_{2k+j_0} & 0 \end{pmatrix}$, $1 \leq i \leq k$. The proof of Theorem 3.2 is complete. ■

Remark 1. *To see that $G^n_{c,k}$ is 2-step solvable, consider its Lie algebra $\mathcal{G}^n_{c,k}$. This consists of elements of the form*

$$(A(v), v) = (diag(t, 0, J_1(t), \ldots, J_k(t), 0, \ldots, 0, -t), v),$$

where $v = (x, t, y_1, \ldots, y_{n-3}, z)$ and $J_i(t) = \begin{pmatrix} 0 & c_i t \\ -c_i t & 0 \end{pmatrix}$, $1 \leq i \leq k$.

Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$, that is, e_i is the vector having all components equal to 0 except the i-th one which is equal to 1. Recalling that $n = m+2k+3$, and setting $X = (A(e_2), e_2)$, $Y = (A(e_1), e_1)$, $Z = (A(e_n), e_n)$, $V_i = \left(A\left(e_{2(i+1)-1}\right), e_{2(i+1)-1}\right)$, $W_i = \left(A\left(e_{2(i+1)}\right), e_{2(i+1)}\right)$ with $1 \leq i \leq k$, and $U_j = \left(A\left(e_{2(k+1)+j}\right), e_{2(k+1)+j}\right)$ with $1 \leq j \leq m$, we get a basis for $\mathcal{G}^n_{c,k}$ with $[X, Y] = Y$, $[X, Z] = -Z$, $[X, V_i] = -c_i W_i$, $[X, W_i] = c_i V_i$, $1 \leq i \leq k$, and all the other brackets are zeros. It follows that $\left[\left[\mathcal{G}^n_{c,k}, \mathcal{G}^n_{c,k}\right], \left[\mathcal{G}^n_{c,k}, \mathcal{G}^n_{c,k}\right]\right] = 0$, that is, $\mathcal{G}^n_{c,k}$ is a 2-step solvable Lie algebra.

Remark 2. *It should be noticed that the group $G^n_{c,k}$ is not algebraic and its unipotent radical $U\left(G^n_{c,k}\right)$ equals the n-dimensional group T_n of translations of $\mathbb{R}^n_1$. To see this, set $v = (0, t, 0, \ldots, 0)^t$ and consider the element*

$$(\phi(v), v) = \left(diag\left(e^t, 1, e^{J_1(t)}, \ldots, e^{J_k(t)}, 1, \ldots, 1, e^{-t}\right), v\right)$$

of $G^n_{c,k}$. This splits as a product $(\phi(v), v) = gh = hg$ with $g = (\phi_u(v) = Id, v)$ and $h = (\phi_s(t), 0)$, where $\phi_s(t) = diag\left(e^t, 1, e^{J_1(t)}, \ldots, e^{J_k(t)}, 1, \ldots, 1, e^{-t}\right)$. Since $U\left(G^n_{c,k}\right) = \left\{(\phi_u(v), v) : (\phi(v), v) \in G^n_{c,k}\right\}$ it follows that $U\left(G^n_{c,k}\right) = T_n$. This agrees with Theorem III.1 in [3] that if a connected, simply connected solvable Lie group G acts transitively by affine transformations on $\mathbb{R}^n$, then the unipotent radical U of the algebraic hull $A(G)$ also acts simply transitively by affine transformations on $\mathbb{R}^n$.

4. Classification of compact flat spacetimes

Compact flat spacetimes are classified by the following result.

Theorem 4.1. *Let M be an n-dimensional compact flat spacetime, $n \geq 3$. Then, up to a finite cover, M is congruent to either*

(i) *a k-step nilmanifold $\mathbb{R}^n_1/\Gamma$ with $k \leq 3$, where Γ is a lattice in some unipotent Lie group $U_{(a,b)}$ that is described by Theorem 2.1, or*

(ii) *a 2-step solvmanifold $\mathbb{R}^n_1/\Gamma$, where Γ is a lattice in some nonunipotent Lie group $G^n_{c,k}$ that is described by Theorem 3.2.*

Proof. By [5] the universal covering of M is Minkowski space $(\mathbb{R}_1^n, g)$. Therefore, without loss of generality $M \cong \mathbb{R}_1^n/\Gamma$ where , by [13], $\Gamma \cong \pi_1(M)$ is a lattice in a connected and simply connected n-dimensional solvable Lie group G that acts isometrically and simply transitively on $\mathbb{R}_1^n$. If G is unipotent, then it is isomorphic to some group $U_{(a,b)}$ that is described by Theorem 2.1. In this case, since $\Gamma \subset G$ is cocompact and G is nilpotent, it follows that G must be rational [20]. This covers case (i). If G is nonunipotent, then by Theorem 3.2 G is conjugate to some 2-step solvable Lie group $G_{c,k}^n$. In this case, it is easy to see that Γ coincides with the subgroup of $G_{c,k}^n$ consisting of elements $(r, t, q, s) \in \mathbb{Z}^n \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^{n-3} \times \mathbb{Z}$, with $q = (q_1, \ldots, q_{n-3})$. This covers case (ii), and therefore completes the proof of Theorem 4.1. ■

We shall now describe and classify the lattices of the 2-step solvable Lie groups $G_{c,k}^n$. Recall first that a matrix $A \in GL_n(\mathbb{Z})$ is called of Lorentz type in [14] if it is diagonalizable over $\mathbb{C}$ and its eigenvalues are $1, \lambda, \lambda^{-1}, a_1, \ldots, a_{n-3}$, where λ is positive real and each a_i satisfies $|a_i| = 1$. It is then clear that given a Lorentz matrix $A \in SL_{n-1}(\mathbb{Z})$ there exist $t \in \mathbb{R}$, an integer $k \geq 0$, and $c = (c_1, \ldots, c_k) \in \mathbb{R}^k$ such that A is conjugate in $SL_{n-1}(\mathbb{R})$ to the matrix $A_{c,k}^t = diag\left(e^t, e^{J_1(t)}, \ldots, e^{J_k(t)}, 1, \ldots, 1, e^{-t}\right)$, where $J_i(t)$ stands for the 2×2-matrix appearing in the notation of Theorem 3.2. This defines for A a mapping

$$\begin{array}{ccc} \mathbb{R} & \rightarrow & SL_{n-1}(\mathbb{R}) \\ t & \mapsto & A_{c,k}^t. \end{array}$$

Let G_A be the Lie group diffeomorphic to $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ with the group law

$$(t, w)(t', w') = \left(t + t', A_{c,k}^t w + w'\right), \quad t, t' \in \mathbb{R} \text{ and } w, w' \in \mathbb{R}^{n-1}.$$

Clearly G_A is isomorphic to $G_{c,k}^n$. Let $\Gamma_A \cong \mathbb{Z} \ltimes \mathbb{Z}^{n-1}$ be the discrete subgroup of G_A consisting of elements with integral coordinates. It is then easy to see that Γ_A is a lattice in $G_A \cong G_{c,k}^n$.

With this notation, we deduce from Theorem 3.2 the following result which generalizes in higher dimensions results in [7] and [8]. This also generalizes Proposition II.2.2 in [12].

Theorem 4.2. *Every n-dimensional compact flat spacetime of type $G_{c,k}^n/\Gamma_A$ is finitely covered by a T^{n-1}-bundle over the circle. More precisely, up to a finite* *cover,* $G^n_{c,k}/\Gamma_A$ *is diffeomorphic to* $T^m \times T^{2k+3}_A$ *with :*

$$T^{2k+3}_A = \mathbb{R} \times T^{2(k+1)} / (t, w) \sim \left(t+1, \widetilde{A}(w)\right),$$

where $n = m + 2k + 3$*, and* $\widetilde{A}$ *denotes the automorphism of the torus* $T^{2(k+1)}$ *induced by* A*.*

Remark 3. *It is interesting to note that* G_A *depends in general on the matrix* A*. However, if* A *is conjugate to a matrix of type* $A^t_{c,k}$ *with* $c = (c_1, \ldots, c_k) = (0, \ldots, 0)$*, that is,* A *is now purely a hyperbolic automorphism of the torus, then* G_A *becomes isomorphic to the solvable group* $E(1,1) = O(1,1) \ltimes \mathbb{R}^2$ *of rigid motions of Minkowski plane* $\mathbb{R}^2_1$*. In this particular case, if* Γ_A *is a lattice in* G_A *then the* 2*-step solvmanifold* $M = \mathbb{R}^n_1/\Gamma_A$ *becomes congruent to a product of a Euclidean torus* T^{n-3} *with the three-dimensional Lorentz* 2*-step solvmanifold* $\mathbb{R}^3_1/\Gamma$*, where* Γ *is a lattice in the* 2*-step solvable Lie group* $G_3 = \mathbb{R} \ltimes \mathbb{R}^2$ *with the factor* $\mathbb{R}$ *acting on* $\mathbb{R}^2_1$ *by hyperbolic automorphisms and the normal factor* $\mathbb{R}^2$ *consisting purely of translations. Notice that lattices in* G_3 *exist (see [10], p. 511, or [12], p. 291-292).*

In [12], Proposition II.2.3, it is shown that if Γ is lattice in the 2-step solvable Lie group G_3 described in the remark above then there exists a matrix A in $SL_2(\mathbb{Z})$ with trace $tr(A) > 2$ such that Γ is conjugate in G_3 to Γ_A. The following proposition which appears as a simple consequence of Theorem 3.2 generalizes this result. This shows that the spacetimes $G^n_{c,k}/\Gamma_A$ described above are the only ones to consider.

Proposition 4.3. *Let* Γ *be a lattice in some* 2*-step solvable Lie group* $G^n_{c,k}$*,* $n \geq 3$*. Then, there exists a Lorentz matrix* A *in* $SL_{n-1}(\mathbb{Z})$ *such that* Γ *is conjugate in* $G^n_{c,k}$ *to* Γ_A*.*

5. Around a conjecture of L. Auslander

In [3], Auslander has been conjectured that every nilpotent Lie group N which acts affinely and simply transitively on $\mathbb{R}^n$ has a nontrivial central translation. This is true if $n \leq 3$ [7] or if N has a degree of nilpotence $k = 1$, that is, N is abelian [9]. This has also been proved in the case of filiform Lie groups (i.e., the index of the Lie algebra is maximal) with odd dimension [19]. However, this conjecture does not hold in general (see [9] for a counter-example in dimension 4 which is 3-step nilpotent). We believe that Auslander's conjecture holds for 2-step nilpotent Lie groups, and we are actively working toward this result. In light of Theorem 2.1, we prove here that Auslander's conjecture holds if N is a nilpotent subgroup of the group $E\,(1, n-1)$ of affine Lorentz motions (see also [1] for an alternate proof of this fact).

Theorem 5.1. *Let N be a nilpotent Lie group acting affinely and simply transitively on $\mathbb{R}^n$. If the action preserves a flat Lorentz metric on $\mathbb{R}^n$ then N contains a nontrivial translation in its center. More precisely, such a translation lies in the derived subgroup $[N, N]$ of N.*

Proof. Being a nilpotent group and acting simply transitively by affine transformations, N is unipotent [21]. By Theorem 2.1, N is conjugate to $U_{(a,b)}$ for some pair (a, b). Now, the element τ_x of $U_{(a,b)}$ defined by $w = 0$, $y = 0$, and $z = 0$ is clearly a nontrivial translation which lies in the derived subgroup of $U_{(a,b)}$. ■

6. Closed geodesics in compact flat spacetimes

Establishing the existence of closed timelike geodesics in compact spacetimes is one of the important problems in Lorentzian geometry. Based on an old result in [22], it is natural to conjecture that any compact spacetime which admits a globally hyperbolic regular covering contains a closed timelike geodesic. In a recent work [15] we provided, among other results, examples of compact flat Lorentz space forms without closed timelike geodesics. Since each of such spacetimes admits Minkowski space as its universal covering, this provides a negative answer to the above conjecture. We proved, however, that any compact flat spacetime contains a closed timelike or null geodesic. In a second paper [17], we proved that in N is a simply connected 2-step nilpotent Lie group with a flat left invariant Lorentz metric g, then for any lattice Γ in N the compact flat Lorentz 2-step nilmanifold $(N/\Gamma, g)$ contains no closed timelike geodesic. A natural question that arises in this context is : *Does any compact flat Lorentz nilmanifold contain no closed timelike geodesic ?*

The following result which characterizes completely the closed timelike and null geodesics in compact flat spacetimes also answers negatively the above question.

Theorem 6.1. *Let M be a compact, flat spacetime of dimension $n \geq 3$. Then M necessarily contains a closed null geodesic, and it contains a closed timelike geodesic if and only if it is congruent to either :*

(i) *a nilmanifold $\mathbb{R}_1^n/\Gamma$, where Γ is a lattice in some $U_{(a,b)}$ with $b = 0$. More precisely, M is a torus if $a = 0$, or a 3-step nilmanifold if $a \neq 0$.*

(ii) *a 2-step solvmanifold $\mathbb{R}_1^n/\Gamma$, where Γ is a lattice in some $G_{c,k}^n$.*

Proof. Recall that the universal covering of M is Minkowski space $\mathbb{R}_1^n$, and that the geodesics of M are projections under the natural projection $\pi : \mathbb{R}_1^n \to M$ of geodesics (i.e., straight lines) of $\mathbb{R}_1^n$. In light of Theorem 4.1, we distinguish two cases.

Case 1. M is a nilmanifold $\mathbb{R}_1^n/\Gamma$, where Γ is a lattice in some $U_{(a,b)}$. In this case, Γ is precisely the subgroup of $U_{(a,b)}$ consisting of elements with integral entries. Let $c(t) = (\alpha t + x_0, \beta_1 t + y_{1,0}, \ldots, \beta_{n-2} t + y_{n-2,0}, \delta t + z_0)$ be a geodesic in $\mathbb{R}_1^n$ that is translated an amount ω by an element $\phi \in \Gamma$, that is, $\phi \cdot c(t) = c(t + \omega)$ for all $t \in \mathbb{R}$. According to the form can ϕ take (Theorem 2.1), this is equivalent to the following system of equations

$$\begin{cases} -w(\beta t + y_0) - 1/2ww^t(\delta t + z_0) + x - 1/2wy^t - 1/6ww^t = \alpha\omega \\ (\delta t + z_0)w^t + y^t + 1/2zwt = \omega\beta^t \\ z = \delta\omega, \end{cases} \tag{6.1}$$

where $\beta = (\beta_1, \ldots, \beta_{n-2})^t$, $y_0 = (y_{1,0}, \ldots, y_{n-2,0})^t$, and w is as in Theorem 2.1. Since t is arbitrary, we necessarily get $w\beta + 1/2\delta ww^t = 0$ and $\delta w^t = 0$. If $b \neq 0$, there is some $b_{i_0} \neq 0$, and so it follows from $\delta w^t = 0$ that $\delta z = 0$. Substituting this into the third equation of (6.1), we deduce that $\delta = 0$. Thus $g(\dot{c}, \dot{c}) = 2\alpha\delta + \beta^2 = \beta^2 \geq 0$, and hence c cannot be timelike. This however shows that M contains closed null geodesics, as it suffices to choose $\beta = 0$ in $c(t)$.

If $b = 0$, letting $\phi(x, y, z)$ be an element of Γ with $y = 0$ we see that the geodesic $c(t) = (\alpha t + x_0, 0, \ldots, 0, \delta t + z_0)$ in $\mathbb{R}_1^n$ with $x = \alpha\omega$, and $z = \delta\omega$ may be translated an amount ω by $\phi(x, 0, z)$. Since x, z in $\phi(x, 0, z)$ may be chosen so that $xz < 0$ or $= 0$, we deduce that M contains both closed timelike and null geodesics.

Case 2. M is a 2-step solvmanifold $\mathbb{R}_1^3/\Gamma$, where Γ is a lattice in some $G_{c,k}^n$. Let $c(t) = (\alpha t + x_0, \eta t + t_0, \beta_1 t + y_{1,0}, \ldots, \beta_{n-3} t + y_{n-3,0}, \delta t + z_0)$ be a geodesic in $\mathbb{R}_1^n$ that is translated an amount ω by an element $\phi(x,s,y,z) \in \Gamma$, where $y = (y_1, \ldots, y_{n-3})$. In this case, with the notation of Theorem 3.2, the equation $\phi \cdot c(t) = c(t+\omega)$ is equivalent to the following system of equations

$$\begin{cases} e^s(\alpha t + x_0) + x = \alpha(t+\omega) + x_0 \\ s = \eta\omega \\ e^{J_i(s)}(\beta_i t + y_{i,0}) + y_i = \beta_i(t+\omega) + y_{i,0}, \quad 1 \le i \le n-3 \\ e^{-s}(\delta t + z_0) + z = \delta(t+\omega) + z_0, \end{cases}$$

Since t is arbitrary, we get from the first and the last equations that $s = 0$ (that is, ϕ is a pure translation) or $\alpha = \delta = 0$ (that is, c is spacelike). Now, by letting for instance $\phi(a,0,0,b)$ be an element of Γ with either $ab = -1$, or else $a = 0$ and $b \neq 0$, we see that the geodesic $c(t) = (at, 0, \ldots, 0, bt)$ of $\mathbb{R}_1^n$ is translated an amount 1 by $\phi(a,0,0,b)$. This shows that M contains closed timelike and null geodesics. The proof of Theorem 6.1 is complete. ■

Theorem 6.1 and its proof can also provide an alternate proof to the following result established already in [16].

Theorem 6.2. *A compact flat spacetime M contains a closed timelike geodesic if and only if the fundamental group $\pi_1(M)$ contains a nontrivial timelike translation.*

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