ON REES AND FORM RINGS OF ALMOST COMPLETE INTERSECTIONS

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1. Introduction. The problem of describing the behaviour of a given variety X under blowing up a subvariety $Y \subset X$ can be phrased as follows: How does the blowing up morphism $X' \longrightarrow X$ depend on the properties of X and Y ? From the homological point of view it makes sense to study also arithmetical properties of X' and its exceptional divisor, since the cohomology of $X' = \operatorname{Proj} R$ is closely related to the local cohomology of the graded ring R with respect to the maximal homogeneous ideal of R . Following this aspect we consider mainly for complete intersection - and almost complete intersection - ideals in a local Noetherian ring A some relationships that hold between the Gorenstein and Cohen-Macaulay (CM) property of the Rees ring $R(I) = \bigoplus_{n\geq 0} I^n t^n$, the associated graded ring $gr_A(I) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ and the corresponding properties of A and in particular of A/I itself.

One starting point are investigations of Brodmann [1], Valla [18] and Goto-Shimoda [2], where in particular the Cohen-Macaulay - and the Gorenstein-property of the form ring $gr_A(I)$ of an almost complete intersection ideal I in a Cohen-Macaulay ring were studied. Our notion of an almost complete intersection is slightly more general as in [1] and [18], s. definition (2.1), since I is not necessarily unmixed. [i.e. - by abuse of usual language - all minimal primes have the same height]. Brodmann [1] , proposition (5.2) proved that for an almost complete intersection I in a Gorenstein ring A the Cohen-Macaulay-property of A/I is equivalent to the Cohen-Macaulay-property of $gr_A(I)$ if moreover $\operatorname{Ass}(A/I) = \operatorname{Assh}(A/I)$.

Valla [18], theorem 8 has shown that under the same assumptions as in Brodmann's result the Cohen-Macaulay property of A/I does imply even the Gorenstein property of $gr_A(I)$.

Goto and Shimoda [2] investigated almost complete intersections ideals P , which are prime. The main theorems in [2] can also be shown, if the prime ideal property is replaced by $\mathrm{Ass}(A/I) = \mathrm{Assh}(A/I)$. Moreover it follows from [2] that - under the same assumptions as in Brodmann's result - the Cohen-Macaulay property of A/I is equivalent to the Gorenstein property of $gr_A(I)$.

We will show in theorem (4.1), that the same statement is also true <u>without</u> the unmixed-condition on the a.c.i. ideal I .

More explicitely, we treat in the first part of this report the following problem: Let I be an a.c.i. ideal (def.(2.1)) in a local Noetherian ring. When is A/I Cohen-Macaulay? We consider essentially 3 situations:

1) The associated graded ring $gr_A(I)$ is Gorenstein.
2) The Rees ring $R(I)$ is Gorenstein, where I is any ideal with $\mu(I) \leq 3$. This problem is reduced to a corresponding problem for an a.c.i. ideal in $R(I)_M$, where M is the maximal homogeneous ideal $R(I)$.
3) A is a regular local ring.

If I is in particular a <u>prime</u> ideal of height 2 in the third situation, then A/I is always Cohen-Macaulay, provided that A contains a field. That follows from the syzygy-theorem of Evans-Griffith. Moreover if $\dim A \leq 5$ and $\mathrm{Ass}(A/I) = \mathrm{Assh}(A/I)$ in this situation, then A/I is Cohen-Macaulay. As far as we know there exists an explicit counter example for situation 3 by Huneke, where I is prime and A/I is <u>not</u> Cohen-Macaulay. This is based on the following idea: Take a non-Cohen-Macaulay UFD ring A/I and link the defining ideal generically with an a.c.i. prime ideal [13]. We will describe necessary conditions on the multiplicity $e(A/I)$ for I being a Buchsbaum, non-Cohen-Macaulay ideal. A series of examples is given.

In the next part of this report we ask for the Gorenstein property of Rees and form rings of powers of ideals. First we mention the fact (s. proposition (6.2)) that for any ideal I of $ht(I) \geq 2$ in a Noetherian ring A there exists at most one power I^n such that the Rees ring $R(I^n)$ is Gorenstein.

Moreover if A is Gorenstein and I an almost complete intersection ideal of $ht(I) \geq 2$ such that A/I is Cohen-Macaulay, then

$$R(I^m) \text{ is Gorenstein iff } m = ht(I) - 1 ,$$
$$gr(I^m) \text{ is Gorenstein iff } ht(I) \equiv 1(m) .$$

Finally for equimultiple ideals I in a Cohen-Macaulay ring A we get the following equivalent conditions:

(i) A is Gorenstein and I is a complete intersection

(ii) $R(I^{ht(I)-1})$ is Gorenstein.

As a consequence one has for a d-dimensional local Cohen-Macaulay ring $(A,\mathfrak{m})$

$$A \text{ is regular iff } R(\mathfrak{m}^{d-1}) \text{ is Gorenstein.}$$

This result is due to Goto-Shimoda for $d = 2$. Independently it was also shown by A. Ooishi, using quite other methods (s. Appendix). Furthermore, if A is regular, then $gr(\mathfrak{m}^n)$ is Gorenstein for all n such that $d-1 \in (n)$.

2. Preliminaries.

Part I: In this section we recall some results on almost complete intersections needed in the sequel.

$(A,\mathfrak{m},k)$ is a local Noetherian ring, I an ideal in A; $\mu(I)$ denotes the minimal number of generators of I.

Definition (2.1):

(i) I is a complete intersection (c.i.) if $ht(I) = \mu(I)$

(ii) I is locally a complete intersection (l.c.i.) if

$$\forall P \in Min(A/I) : IA_P \text{ is c.i. in } A_P$$

(iii) I is an almost complete intersection (a.c.i.) if I is l.c.i. and $\mu(I) = ht(I) + 1$.

Remark (2.2): In [1] and [18] there is a somewhat stronger notion: "I is an almost complete intersection and a generic complete intersection" if $\mu(I) = ht(I) + 1$ and $\mu(I_P) = ht(I)$ for all $P \in Min(A/I)$.
Note that the ideal $I = (X,Y) \cap (Z) \subset k[[X,Y,Z]]$ is an a.c.i. ideal by definition (2.1) but not in the strong sense of [1] and [18]. We come back to this example in section 4, where theorem (4.1) shows immediately that $gr_A(I)$ cannot be Gorenstein.

As a consequence of the "Primbasissatz" , we have the following result [5]:

Proposition (2.3):

(i) Let $|k| = \infty$. Then if I is a.c.i. we find an ideal $J \subseteq I$ and an element $a \in I$, not contained in J , such that

$$I = J + aA \text{ and } J \text{ is c.i.} \quad .$$

Moreover if J is height-unmixed then $(J : a) = (J : a^2)$.

(ii) Conversely, if $I = J + aA$ such that J is a complete intersection and $(J : a) = (J : a^2)$, then I is either c.i. or a.c.i.

An important point is that in Cohen-Macaulay rings a.c.i. ideals are of linear type. (s. also [2],[5],[19]):

Proposition (2.4): Let A be a Cohen-Macaulay ring, such that all localizations A_p, $P \in \mathrm{Spec}(A)$, have infinite residue field. Let I be an ideal of A such that $\mu(I) = \mathrm{ht}(I) + 1$. Then the following are equivalent:

(i) I is an almost complete intersection.

(ii) I is of linear type, i.e. $R(I) = \mathrm{Sym}(I)$.

Remark (2.5): In Cohen-Macaulay rings almost complete intersections cannot be equimultiple, s. [5], proposition (1.2).

But in non-Cohen-Macaulay rings we can have both properties for I as the following examples show:

Example (2.6), s.[6]: Consider the following situation:

$$R := k[[t^5,t^6,t^7,t^8,t^9]]\,[X,Y,Z] \quad ,$$

where k is a field, and t,X,Y,Z are indeterminates. Let:

$$a := t^5 \; ; \quad b := t^8 \; ; \quad c := t^9 \; ;$$

$$G := R/(f :_R a^2) \quad , \quad \text{where} \quad f = aX + bY + cZ \quad ;$$

$$\mathfrak{m} := \text{maximal homogeneous ideal of } G \quad ;$$

$$A := G_{\mathfrak{m}} \quad \text{and} \quad P := (X,Y,Z)G_{\mathfrak{m}} \subset A \; .$$

By $l(I)$ we denote the analytic spread of I.

Then one has:
(1) A is non-Cohen-Macaulay of $\dim A = 3$
(2) $\mathrm{ht}(P) = l(P) = 2 < \mu(P) = 3$
(3) A/P is Cohen-Macaulay
(4) A_P is regular.

This shows that P is an equimultiple, almost complete intersection ideal. We remark that $R(P)$ is Cohen-Macaulay, but not

Gorenstein. Note that almost all results of the next sections need at least the Cohen-Macaulay-property of A , i.e. example (2.6) had to be treated by quite other methods; see [6].

Example (2.7): $A := k[[s^2, s^3, st, t]]$ is a non-Cohen-Macaulay, Buchsbaum ring of dimension 2. Consider the prime ideal $P := (st,t)$.

Then we have: (1) $ht(P) = l(P) = 1 < \mu(P) = 2$

(2) A_P is regular.

Hence P is a.c.i. (and equimultiple), but not of linear type, since

$$\mathrm{Sym}(P) \cong A[X,Y]/\mathfrak{a} \quad \text{where}$$
$$\mathfrak{a} = (tX - stY,\ s^2X - s^3Y,\ s^3X - s^4Y,\ stX - s^2tY)$$

and

$$R(P) \cong A[X,Y]/(\mathfrak{a}, X^2 - s^2Y^2) \quad .$$

Remark (2.8): In the last example A/P is Cohen-Macaulay but $R(P)$ is not, since otherwise $\operatorname{depth} A \geq \dim(A/P) + 1 = 2$, which is not the case.

Proposition (2.9) (Depth-Formula): Let I be an almost complete intersection ideal (in the sense of (2.1)) in a Cohen-Macaulay ring A . Then the following hold for $G = gr_A(I)$ and $R = R(I)$:

(i) $\operatorname{depth}(G) = \min\{\dim A\ ;\ \operatorname{depth}(A/I) + ht(I) + 1\}$

(ii) $\operatorname{depth}(R) = \min\{\dim A\ ;\ \operatorname{depth}(A/I) + 2\}$ if $ht(I) = 0$

$= \min\{\dim A + 1\ ;\ \operatorname{depth}(A/I) + ht(I) + 2\}$ if $ht(I) \geq 1$.

These formulas were shown by Brodmann [1] for a.c.i. ideals in the strong sense. But one can check that the same proof works for a.c.i. ideals in the sense of (2.1), since the only thing needed in Brodmann's proof is the fact that $I^n \cap a_1 A = a_1 \cdot I^{n-1}$, where I is generated by a d-sequence $a_1, \ldots, a_h, a_{h+1}$ (and $a_1, \ldots, a_h$ form a regular sequence).

Part II. Brief survey of the theory of Approximation Complexes.

To study the relationships between the various graded rings associated to an ideal I in a (commutative) Noetherian ring A , one can also use the socalled Approximation Complexes of I ; see [8], [9], [10]. The situation is in particular good when these complexes are exact, a condition that is fulfilled for some important families of ideals. In this case it is possible to investigate the arithmetical properties of those graded rings as if they were Cohen-Macaulay or Gorenstein. One important point in this theory is that under the "good" situation I is of linear type. Assuming for simplicity that A is local with maximal ideal $\mathfrak{m}$ and $\dim A = d$, we consider the Koszul complex of R w.r.t. $\underline{a} := \{a_1, \ldots, a_n\}$, where $a_1, \ldots, a_n$ generate I . By S we denote the polynomial ring $A[X_1, \ldots, X_n]$. We then get two complexes:

$$\mathcal{Z}(\underline{a}) \; : \; 0 \longrightarrow \mathcal{Z}_n \longrightarrow \ldots \longrightarrow \mathcal{Z}_1 \longrightarrow \mathcal{Z}_0 \longrightarrow 0$$

and

$$\mathcal{M}(\underline{a}) \; : \; 0 \longrightarrow \mathcal{M}_n \longrightarrow \ldots \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_0 \longrightarrow 0 \quad ,$$

where $\mathcal{Z}_i = Z_i K(\underline{a}) \otimes_A S$ and $\mathcal{M}_i = H_i K(\underline{a}) \otimes_A S$. Here the $Z_i K(\underline{a})$ denotes the cycles of the Koszul complex $K(\underline{a})$ and the $H_i K(\underline{a})$ the Koszul homology.

Both complexes can be taken as <u>complexes of graded modules over S</u> with mappings of <u>degree -1</u> . We list the main properties of these complexes:

<u>(1)</u> The homology of $\mathcal{Z}(\underline{a})$ and $\mathcal{M}(\underline{a})$ is independent of the system of generators $\underline{a}$.

<u>(2)</u> $H_0(\mathcal{Z}(\underline{a})) = \mathrm{Sym}_A(I)$

<u>(3)</u> $H_0(\mathcal{M}(\underline{a})) = \mathrm{Sym}_A(I/I^2)$

<u>(4)</u> The following are equivalent:

(i) $\mathcal{M}(\underline{a})$ is acyclic.

(ii) $\mathcal{Z}(\underline{a})$ is acyclic and I is of linear type.

Now we assume that A is Cohen-Macaulay.

(5) Suppose (i) For any prime ideal $P \supseteq I$, $\mu(I_P) \leq ht(P)$

(ii) For any $r \geq 0$ and for any prime ideal $P \supseteq I$

$$\operatorname{depth}_{A_P}(H_r(K(\underline{a}))_P) \geq \inf(ht(P/I), r).$$

Then $\mathcal{M}(\underline{a})$ is acyclic.

It turns out that condition (ii) above is independent of the system of generators of I, and that it is fulfilled if I is strongly Cohen-Macaulay (sCM), that is, if for any $r \geq 0$, $H_r(K(\underline{a}))$ is zero or a maximal Cohen-Macaulay A/I-module.

(6) Assume: (i) For any prime ideal $P \supseteq I$, $\mu(I_P) \leq ht(P) + 1$.

(ii) For any $r \geq 0$ and for any prime ideal $P \supseteq I$,

$$\operatorname{depth}_{A_P}(H_r(K(\underline{a}))_P) \geq \inf(ht(P/I), r) - 1.$$

Then $\mathcal{Z}(\underline{a})$ is acyclic.

There is an important connection between the theory of Approximation Complexes and the theory of d-sequences. In fact:

(7) If I is generated by a d-sequence, then $\mathcal{M}(\underline{a})$ is acyclic.

(8) Assume that $|A/\mathfrak{m}| = \infty$. If $\mathcal{M}(\underline{a})$ is acyclic, then I can be generated by a d-sequence (but the given $\underline{a}$ is not necessarily a d-sequence!).

(10) Finally we give a list of some important families of sCM-ideals. For this we mainly follow [8]. A is always Cohen-Macaulay.

The following ideals are strongly Cohen-Macaulay:

a) Complete intersection ideals

b) Ideals $I \subseteq A$ such that A/I is Cohen-Macaulay and $\mu(I) = ht(I) + 1$.

c) If A is Gorenstein, A/I is Cohen-Macaulay and $\mu(I) = ht(I) + 2$, then I is sCM.

3. R(I) is Gorenstein and $\mu(I) = 2$ (domain-case).

To motivate the special case $\mu(I) = 2$ we mention the following result, see [3], proposition 1.1 and also proposition 2.4:

Proposition (3.1): Let I be a parameter-ideal in A . If $\dim(A) \geq 2$ and $\operatorname{depth}(A) \geq 2$, then the following conditions are equivalent:

(i) $R(I)$ is Gorenstein.

(ii) A is Gorenstein and $\dim(A) = 2$.

The proof of (i) $\Rightarrow$ (ii) is in [3]. The converse follows also from [3]: From A is Gorenstein (Cohen-Macaulay is enough) we conclude that $R(I)$ is Cohen-Macaulay. Moreover I is generated by a regular sequence, hence A/I and $gr_A(I)$ are Gorenstein. Therefore by [15], theorem (3.7) $R(I)$ is Gorenstein [see also §4 , where we show that in this case the a-invariant of $gr_A(I)$ is $-ht(I) = -2$].

Corollary (3.2): Let $I \subset A$ be a complete intersection ideal (i.e. $\mu(I) = ht(I)$, hence I is generated by a subsystem of parameters). Assume that $\operatorname{grade}(I) \geq 2$. If $R(I)$ is Gorenstein then $ht(I) = 2$.

These facts may convince us to start with an ideal generated by 2 elements with $ht(I) \geq 1$. In the following proposition due to J. Ribbe we do not assume that I is an a.c.i..

Proposition (3.3): Let $I \subset A$ and assume that $(A,\mathfrak{m})$ is a domain. Assume

(i) $R(I)$ is Gorenstein

(ii) $\mu(I) = 2$ and $ht(I) \geq 1$.

Then A is a Cohen-Macaulay ring.

Remark (3.4): Proposition (3.3) is also true if $(A,\mathfrak{m})$ is any local Noetherian ring. This general fact will be proven in section 4, theorem (4.1).

Corollary (3.3.1): Assume (i) and (ii) of proposition (3.3). If $ht(I) = 2$, then $gr_A(I)$ is Gorenstein.

Proof of (3.3.1): Since A is Cohen-Macaulay by (3.3) and $\mu(I) = ht(I) = 2$, I is generated by a regular sequence of two elements. Moreover $A \cong K_A$ (where K_A is the canonical module of A) by [15], theorem 3.1, hence A and A/I are Gorenstein rings. This implies that $gr_A(I) \cong A/I[X_1 , X_2]$ is Gorenstein.

Corollary (3.3.2): Assume (i) and (ii) of proposition (3.3). If I is an unmixed $(Ass(A/I) = Assh(A/I))$ almost complete intersection ideal of $ht(I) = 1$, then A is a non-Gorenstein, but Cohen-Macaulay ring.

Proof of (3.3.2): This follows from (3.3) and [2], corollary (4.6).

Example for (3.3.2): Let $A = k[[s^2 , s^3 , t , st, s^2u, tu]]$, where k is a field and s,t,u are indeterminates. Let $P = (s^2u , tu)$, which is an almost complete intersection ideal of $ht(P) = 1$. A is Cohen-Macaulay, but not Gorenstein and $R(P)$ is Gorenstein by [2].

Proof of proposition (3.3): We put $R = R(I)$ and

$R_+ = 0 \oplus It \oplus I^2t^2 \oplus \ldots$

$M := \mathfrak{m} + R_+$, the maximal homogeneous ideal of R

$B := R_M \supset P := R_+B$.

Then we have:

1) $B/P \cong A$ and $ht(P) = 1$. P is prime since A is a domain.

2) $\mu_B(P) = 2 = ht(P) + 1$

3) $gr_B(P) \cong R$, since $R = A[R_1]$.

4) B_P is regular.

That means, P is an almost complete intersection prime ideal such that $gr_B(P) \cong R$ is Gorenstein by assumption. This implies that $B/P \cong A$ is Cohen-Macaulay by [2], theorem (1.1).

Remark (3.5): To extend proposition (3.3) to any local ring $(A,\mathfrak{m})$ we first need to prove theorem (1.1) of [2] for any almost complete intersection ideal (which is by our definition in general not unmixed). This we do in section 4.

Remark (3.6): The method used in the proof of proposition (3.3) does not work in the same way for an ideal I with $\mu(I) \geq 3$, since $ht(R_+B) = 1$ and $\mu(R_+B) \geq 3$.

4. R(I) is Gorenstein and $\mu(I) = 2$ or 3 .

The main tool for this section is the following theorem generalizing theorem (1.1) in [2] . For simplicity we assume that $|A/\mathfrak{m}| = \infty$.

Theorem (4.1): Let I be an almost complete intersection in a local ring A . If $gr_A(I)$ is Gorenstein, then A/I is Cohen-Macaulay.

Remark (4.2): The ideal $I = (X,Y) \cap (Z) \subset A = k[[X,Y,Z]]$ is an a.c.i. ideal by definition (2.1). Since I is not unmixed, $gr_A(I)$ cannot be Gorenstein by this theorem. This can of course also be seen by the following well known argument:

$gr_A(I) = A[T_1,T_2]/J$, where J is generated by the homogeneous polynomials $F(T_1,T_2)$ satisfying $F(XZ,YZ) \in I^{\deg(F)+1}$. Hence

$$gr_A(I) = k[[X,Y,Z]]\,[T_1,T_2]/(XZ,YZ,YT_1 - XT_2) .$$

So we have $\mu(J) = 3 = ht(J) + 1$, and J is an a.c.i. in the strict sense of [1], [18], hence $gr_A(I)$ cannot be Gorenstein by an old result of Kunz.

Proof of (4.1): By (2.3) and the Primbasissatz [5], theorem 4.2 we may assume that

$$I = a \cdot A \ , \quad ht(I) = 0 \quad \text{and} \quad (0 : a) = (0 : a^2) \quad .$$

Then we have $I \cap (0 : I) = (0)$ and by the depth-formula (2.9) we have $\operatorname{depth}(A/I) \geq d - 1$, where $d = \dim(A/I)$. So it is enough to show

$$(*) \qquad \operatorname{Ext}_A^{d-1}(k,A/I) = 0 \quad .$$

Put $J = 0 : I$. Note that in our situation $(ht(I) = 0)$ we have $d = \dim(A/I) = \dim A$.

Claim 1: A/J is Cohen-Macaulay of dimension d

Proof: Consider the exact sequence

$$0 \longrightarrow A/J \xrightarrow{\cdot a} A \longrightarrow A/I \longrightarrow 0$$

and note that $\operatorname{depth}(A/I) \geq d - 1$ and $\operatorname{depth}(A) = d$.

<u>Claim 2</u>: J is a CM-A-module of dimension d .

<u>Proof</u>: This follows from the exact sequence

(i) $0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$

since A and A/J are d-dimensional Cohen-Macaulay rings.

To use the assumption that $gr_A(I)$ is Gorenstein, we recall (see [1], [2], [18]) that

(1) $$gr_A(I) \cong \frac{A[X]}{X \cdot JA[X] + IA[X]} ,$$

where X is an indeterminate. We denote the right side of (1) by G , and the irrelevant maximal ideal of G by $\mathfrak{m}$.

If t is a non-zero element of A then

(2) $$G_{\mathfrak{m}}/(X-t)G_{\mathfrak{m}} \cong A/(tJ + I) .$$

Now the next step is to choose some special t , such that $A/(tJ+I)A$ becomes Gorenstein as a ring:

<u>Claim 3</u>: There exists an element $t \in J$, such that $t \notin \bigcup_{P \in Assh(A/I)} P$ i.e. t is a parameter on A/I .

<u>Proof</u>: By assumption (I is a.c.i.) we have $\mu(IA_P) = ht(P)$ for all $P \in Min(A/I)$. Hence $\left(\frac{a}{1}\right)A_P = IA_P = 0$ for $P \in Assh(A/I)$. Therefore there exists an element $t \notin \bigcup_{P \in Assh(A/I)}$, such that $a \cdot t = 0$, hence $t \in J$, which proves the claim.

This element t we use in the isomorphism (2) above. Since also t^2 is a parameter on A/I we have:

$$\dim(G_{\mathfrak{m}}/(X-t)G_{\mathfrak{m}}) = \dim(A/(tJ+I)) \leq \dim(A/(t^2A+I)) \leq d-1 .$$

hence $(X-t)$ is a parameter on G, so $A/(tJ+I)$ is a Gorenstein ring of dimension $d-1$.

Now consider the canonical exact sequence

(3) $$0 \longrightarrow A \longrightarrow A/I \oplus A/tJ \longrightarrow A/I+tJ \longrightarrow 0 \quad .$$

The corresponding Ext-sequence induces

(4) $$0 \longrightarrow \mathrm{Ext}_A^{d-1}(k,A/I) \oplus \mathrm{Ext}_A^{d-1}(k,A/Jt) \longrightarrow \mathrm{Ext}_A^{d-1}(k,A/I+Jt) \quad .$$

Putting $I+tJ =: \mathfrak{a}$, we have

$$\begin{aligned} \mathrm{Ext}_A^{d-1}(k,A/\mathfrak{a}) &\cong \mathrm{Hom}_A(k,A/(\mathfrak{a};x_1,\dots,x_{d-1})) \\ &\cong \mathrm{Hom}_{A/\mathfrak{a}}(k,A/(\mathfrak{a},x_1,\dots,x_{d-1})) \\ &\cong \mathrm{Ext}_{A/\mathfrak{a}}^{d-1}(k,A/\mathfrak{a}) \quad , \end{aligned}$$

where $x_1,\dots,x_{d-1}$ form a regular sequence on $A/\mathfrak{a}$. Therefore $\mathrm{Ext}_A^{d-1}(k,A/\mathfrak{a}) \cong k$ in (4), because $A/\mathfrak{a}$ is a Gorenstein ring.

Thus it is enough to show that

(**) $$\mathrm{Ext}_A^{d-1}(k,A/Jt) \neq 0 \quad .$$

For that we consider the exact sequence

(5) $$0 \longrightarrow J/tJ \longrightarrow A/tJ \longrightarrow A/J \longrightarrow 0 \quad .$$

Note that $J/tJ = J/(a+t)J$.

<u>Claim 4</u>: $a+t$ is regular on A.

<u>Proof</u>: Suppose $a+t \in P$ for some $P \in \mathrm{Ass}(A)$. Since $I \cap J = (0) \subseteq P$, we have to consider two cases:

<u>Case 1</u>: $I \subseteq P$. This implies $t \in P$. Since A is Cohen-Macaulay (even Gorenstein), $\dim(A/P) = \dim A = \dim(A/I)$, i.e. $P \in \mathrm{Assh}(A/I)$. That is a

contradiction to the choice of t .

Case 2: $J \subseteq P$; then $t \in P$, which gives again a contradiction.

From claim 2 and claim 4 we conclude that J/tJ is a Cohen-Macaulay-A-module of dimension $d-1$.

Now consider the exact sequence induced by (5) :

$$(6) \qquad 0 \longrightarrow \operatorname{Ext}_A^{d-1}(k,J/tJ) \xrightarrow{\varphi} \operatorname{Ext}^{d-1}(k,A/tJ) \longrightarrow 0 \quad ,$$

i.e. φ is an isomorphism and $\operatorname{Ext}_A^{d-1}(k,J/tJ) \neq 0$, since J/tJ is Cohen-Macaulay of dimension $d-1$. This implies the desired relation (**), q.e.d. (Theorem (4.1)).

Now we can prove the following theorem, which generalizes proposition (3.3).

Theorem (4.3): Let I be an ideal of a local ring A satisfying the following condition:

(i) $R = R(I)$ is Gorenstein

(ii) $\mu(I) = 2$ and $ht(I) \geq 1$.

Then A is Cohen-Macaulay and $\operatorname{depth}(A/I) \geq \dim(A/I) - 1$.

Proof: We use the same ideas as in the proof of proposition (3.3).

Put $\qquad R = R(I)$, $M = \mathfrak{m} + R_+$, $B = R_M \supset J = R_+B$.

As in section 3 it can be seen that

1) $B/J \cong A$ and $ht(J) = 1$

2) $\mu(J) = \mu(I) = 2$

3) $gr_B(J) \cong R$ is Gorenstein by (i).

Note that $\mathrm{Min}_B(B/J) = \{J + qB/q \in \mathrm{Min}(A)\}$, since $B/J \cong R/R_+ \cong A$.

<u>Claim</u>: a) $\mu(JB_{J+qB}) \leq 1$ for all $q \in \mathrm{Min}(A)$

b) $\mathrm{ht}(J + qB) \geq 1$ for all $q \in \mathrm{Min}(A)$

c) $\mu(JB_Q) = \mathrm{ht}(Q)$ for all $Q \in \mathrm{Min}_B(B/J)$

<u>Proof</u>: a) Let $a_1,\ldots,a_n$ be any system of generators of I , and take any $q \in \mathrm{Min}(A)$. Since $\mathrm{ht}(I) \geq 1$, say $a_1 \notin q$. Then we get in the ring $B_{J+qB} \cong R_{R_+ +qR}$

$$\frac{a_i t}{1} = \frac{a_i}{a_1} \cdot \frac{a_1 t}{1} \quad \text{for } i = 2,\ldots,n \quad .$$

(Note that $a_1 \notin R_+ + qR$ in our situation).

This proves a), because $J = (a_1t/1,\ldots,a_nt/1)$.

b) It is enough to show that $\mathrm{ht}(R_+ + qR) \geq 1$. But this follows from $I \not\subseteq q$.

c) From a) and b) we obtain

$$1 \leq \mathrm{ht}(J + qB) = \mathrm{ht}(JB_{J+qB}) \leq \mu(JB_{J+qB}) \leq 1 \quad ;$$

this gives the claim.

Therefore J is an a.c.i. ideal in B and $\mathrm{gr}_B(J)$ is Gorenstein by assumption, hence $B/J \cong A$ is Cohen-Macaulay by (4.1).

Since $R(I)$ is Cohen-Macaulay, $\mathrm{depth}(A/I) \geq \dim A - l(I)$ by [16]. Moreover $1 \leq \mathrm{ht}(I) \leq l(I) \leq 2$, hence $\mathrm{depth}(A/I) \geq \dim A - (\mathrm{ht}(I) + 1)$. This proves the last claim of (4.3).

We can prove a somewhat similar statement for $\mu(I) = 3$ provided that we assume from the beginning that A is Cohen-Macaulay.

Theorem (4.4): Let $(A,\mathfrak{m})$ be a Cohen-Macaulay ring and I an ideal in A. Assume that

(i) $R(I)$ is Gorenstein

(ii) $\mu(I) = 3$ and $1 \leq ht(I) \leq 2$

(iii) I is an ideal of linear type.

Then $\operatorname{depth}(A/I) \geq \dim(A/I) - 1$.

Proof: We use the same notations as in the proof of (4.3).
Since B is Gorenstein, $\mu(J) = ht(J) + 2$ and $B/J = A$ is Cohen-Macaulay, we know by [8], prop. (5.5), that J is strongly Cohen-Macaulay. Moreover J is of linear type. Then we know by [14], theorem (2.4), that for all $k \geq 1$:

$$\operatorname{depth}(J^k/J^{k+1}) \geq \operatorname{depth}(B/J) - k .$$

For $k = 1$ we get $\operatorname{depth}(J/J^2) = \operatorname{depth}(I) \geq \operatorname{depth}(A) - 1$, hence (since A is CM):

$$\operatorname{depth}(I) \geq \dim A - 1 .$$

From the exact sequence $0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$ we get by the depth-lemma the inequality

$$\operatorname{depth}(A/I) \geq \dim A - 2 \geq \dim(A/I) - 1 .$$

Remarks: a) Using the same argument as in the proof of theorem (4.3), part d), which was based on the inequality $\operatorname{depth}(A/I) \geq \dim A - l(I)$ in [16], we would only get that $\operatorname{depth}(A/I) \geq \dim(A/I) - 2$.

b) The condition (iii) in theorem (4.4) can be replaced by a weaker condition(iii)': $\mu(IA_P) \leq 2$ for all associated height one prime ideals P of I.

c) If we assume $ht(I) = 2$ in (ii) of (4.4), then the statement of the theorem follows immediately from the depth-formula (2.9). The

interesting case in (4.4) is the height one case.
The following example (mentioned to us by P. Schenzel) shows that in case $ht(I) = 1$ in theorem (4.4) with (iii)' instead of (iii) we cannot expect A/I to be Cohen-Macaulay:

Consider the ring $\tilde{B} = k[[s^{a+b}, s^b \cdot t^a, s^a \cdot t^b, t^{a+b}]]$, where $a = 2n-1$, $b = 2n+1$; that describes a non-singular curve. The defining equations of $\tilde{B}$ in $C = k[[X,Y,Z,W]]$ are

$$Q = YZ - XW$$
$$F_0 = Y^{2n+1} - X^2Z^{2n-1}$$
$$F_1 = Y^{2n}W - XZ^{2n}$$
$$F_2 = Y^{2n-1}W^2 - Z^{2n+1} \quad .$$

Take $\mathfrak{P} := (Q,F_0,F_1,F_2)$; then $ht(\mathfrak{P}) = 2$ and

$$R(\mathfrak{P}) = k[[X,Y,Z,W,U,T_0,T_1,T_2]]/\mathfrak{h} \quad ,$$

where $\mathfrak{h}$ is generated by

$$WT_0 - YT_1 - XZ^{2n-1}U$$
$$ZT_0 - XT_1 - Y^{2n}U$$
$$WT_0 - YT_1 - Z^{2n}U$$
$$ZT_1 - XT_2 - Y^{2n-1}WU$$
$$T_0T_1 - T_1^2 + Y^{2n-1}Z^{2n-1}U^2 \quad .$$

Then by writing these elements as the Pfaffians of a suitable skew symmetric matrix, one can see that $R(\mathfrak{P})$ is a Gorenstein ring (of dimension 5), hence $gr_C(\mathfrak{P})$ is Gorenstein of dimension 4.

Now take $A := k[[X,Y,Z,W]]/(Q)$ and $P := \mathfrak{P}A/Q$. Since the initial form Q^* is a non-zero-divisor in $gr_C(\mathfrak{P})$, the ring $gr_C(\mathfrak{P})/(Q^*) \cong gr_A(P)$ is Gorenstein.

Moreover: $\mu(P) = 3$

$ht(P) = 1$

A_p is regular

A and $R(P)$ are Gorenstein ,

but $\widetilde{B} = A/P$ is not Cohen-Macaulay.

At the end of this section we describe two situations where $\mu(I) \leq ht(I) + 2$ and $R(I)$ is Gorenstein. The proof-strategy is somewhat different from corresponding strategies in [2], [11], since we use the computation of the a-invariant $a(gr_A(I))$.

Recall that for a positively graded Noetherian ring $R = \bigoplus_{i \geq 0} R_i$ defined over a local ring R_0 and a Noetherian graded R-module G the a-invariant of G is defined as

$$a(G) = \max\{j \in \mathbb{Z} : H_M^r(G)_j \neq 0 \} \quad ,$$

where $r = \dim(G)$ and $H_M^r(G)$ is the r-th local cohomology w.r.t. the maximal homogeneous ideal M of R . Then R is Gorenstein if and only if R is Cohen-Macaulay and the canonical module of R is $K_R \cong R(a(R))$.

We start with the following lemma. For completeness we sketch a proof.

<u>Lemma (4.5)</u>: Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian graded algebra defined over a local ring R_0 and $G = \bigoplus_{n \geq 0} G_n$ a Noetherian graded R-module. Let $x \in R_1$ be a regular element on G . Then we get for the a-invariants of G and G/xG

$$a(G) \leq a(G/xG) - 1 \quad .$$

Moreover, if G is Cohen-Macaulay, the equality holds.

<u>Proof</u>: Consider the exact sequence of graded modules

$$(1) \qquad 0 \longrightarrow G(-1) \xrightarrow{\cdot x} G \longrightarrow G/xG \longrightarrow 0 \quad .$$

Putting $r = \dim_R(G)$ and M = maximal homogeneous ideal of R, we get from (1) the long exact cohomology-sequence

$$(2) \qquad \ldots \longrightarrow H_M^{r-1}(G)_i \longrightarrow H_M^{r-1}(G/xG)_i \longrightarrow H_M^r(G)_{i-1} \longrightarrow H_M^r(G)_i \longrightarrow 0 \quad .$$

For $i = a(G)+1$ we have

$$(3) \qquad H_M^r(G)_{i-1} \neq 0 \quad \text{and} \quad H_M^r(G)_i = 0 \quad ;$$

then (2) implies $H_M^{r-1}(G/xG)_i \neq 0$, i.e. $a(G/xG) \geq i = a(G)+1$. This proves the first part of the claim. The second part follows immediately from (2) since now $H_M^{r-1}(G) = 0$.

Proposition (4.6): Let $(A,\mathfrak{m})$ be a Cohen-Macaulay ring of dimension d. Let I be a strongly Cohen-Macaulay ideal in A. Assume $\mu(I_p) \leq ht(P)$ for all prime ideals $P \supseteq I$. Then

$$a(gr_A(I)) = -ht(I) \quad .$$

Proof-idea: Put $s = ht(I) = grade(I)$ and $\mu(I) = s+t$. Note that the homology $H_i(K.)$ of the Koszul complex $K.$ of A with respect to some minimal system of generators of I is zero for all $i > t$.

Moreover $gr_A(I)$ is CM and the $\mathcal{M}$-complex is exact by [8], which implies in particular that I is of linear type. The $\mathcal{M}$-complex gives a resolution of $gr_A(I)$:

$$(4) \qquad 0 \longrightarrow \mathcal{M}_t \xrightarrow{\varphi_t} \ldots \longrightarrow \mathcal{M}_1 \xrightarrow{\varphi_1} \mathcal{M}_0 \longrightarrow gr_A(I) \longrightarrow 0 \quad ,$$

where $\mathcal{M}_i = H_i(K) \otimes A[X_1,\ldots,X_{s+t}]$, $\dim(\mathcal{M}_i) = d+t$ and $\mathcal{M}_i$ is CM over $A[X_1,\ldots,X_{s+t}]$. The idea is now to compute $a := a(gr_A(I))$ via the a-invariants $a(\mathcal{M}_i)$:

Applying lemma (4.5) exactly (s+t) times we see that

(5) $$a(M_i) = a(H_i) - \mu(I) = -\mu(I) \quad ,$$

since (4) is considered as a sequence of $A[X_1,\dots,X_{s+t}]$-modules. Note that by construction the morphisms φ_i are of degree 1. Therefore we get the following exact sequences for the cokernels D_i of φ_{i+1}

$$0 \longrightarrow D_{i+1}(-1) \longrightarrow M_i \longrightarrow D_i \longrightarrow 0$$

with morphisms of degree zero.

Since G and M_i are CM, we get for the local cohomology w.r.t. the maximal homogeneous ideal M of $A/I[X_1,\dots,X_{s+t}]$

(6) $$H^d_M(G)_j \cong H^{d+1}_M(D_1)_{j-1} \cong \dots \cong H^{d+t-1}_M(D_{t-1})_{j-t+1} \quad .$$

Moreover we have the exact sequence

(7) $$0 \longrightarrow H^{d+t-1}_M(D_{t-1})_{j-t+1} \longrightarrow H^{d+t}_M(D_t)_{j-t} \longrightarrow H^{d+t}_M(M_{t-1})_{j-t+1} \quad .$$

Note that

(8) $$H^{d+t}_M(D_t)_{j-t} \cong H^{d+t}_M(M_t)_{j-t} \quad , \quad \text{since}$$

$$0 \longrightarrow D_{t+1}(-1) \longrightarrow M_t \longrightarrow D \longrightarrow 0 \quad .$$

Case 1: $j > -h$, where $h = ht(I)$, i.e. $j - t > -\mu(I) := -n$. Therefore $H^{d+1}_M(M_t)_{j-t} = 0$, since $a(M_t) = -n$. But by (6), (7) and (8) we know that

$$H^d_M(G)_j \cong H^{d+t-1}_M(D_{t-1})_{j-t+1} \subset H^{d+1}_M(M_t)_{j-t} = 0 \quad ,$$

hence $a(G) \leq -h$.

Case 2: $j = -h$. Then (6) implies:

$$H^d_M(G)_{-h} \cong H^{d+t-1}_M(D_{t-1})_{-h-t+1} \cong H^{d+t}_M(M_t)_{-n} \neq 0 \quad ,$$

hence $a(G) = ht(I)$.

Proposition (4.7): Let $(A,\mathfrak{m})$ be a Gorenstein ring and let I be an ideal in A of height 2, satisfying the following properties

(i) A/I is Cohen-Macaulay
(ii) $\mu(I) \leq ht(I) + 2$
(iii) $\mu(I_P) \leq ht(P)$ for $P \supseteq I$.

Then $R(I)$ is Gorenstein.

The assumptions imply that I is sCM. Therefore $Sym(I/I^2) \cong gr_A(I)$ is Gorenstein by [10], theorem (6.1), and $a(gr_A(I)) = -2$ by proposition (4.6). Since $R(I) \cong Sym(I)$ is CM by [10], theorem (6.1), this implies that $R(I)$ is Gorenstein by [15], theorem (3.1).

For almost complete intersections we can formulate a natural generalization of theorem (1.2) in [2].

Proposition (4.8): Let A be a local Cohen-Macaulay ring and let I be an almost complete intersection ideal of $ht(I) \geq 2$. Then the following hold:

(i) $R(I)$ is not Gorenstein if $ht(I) > 2$

(ii) Assume $ht(I) = 2$. Then $R(I)$ is Gorenstein if and only if $gr_A(I)$ is Gorenstein.

Corollary (4.9): Let I be an almost complete intersection ideal in A of $ht(I) \geq 2$. Assume that $gr_A(I)$ is Gorenstein. Then

(i) $a(gr_A(I)) = -2$ if $ht(I) = 2$

(ii) $a(gr_A(I)) \neq -2$ if $ht(I) > 2$.

Proof of (4.8): To (i): Since $R(I) \cong Sym(I)$, this follows from a result of Rossi, s. [17] , theorem (6.7).

To (ii): Assume that R(I) is Gorenstein. Then $gr_A(I)$ is CM by the depth-formula (2.9). Therefore $gr_A(I)$ is even Gorenstein by [15], theorem (3.1).

Conversely, assume that $gr_A(I)$ is Gorenstein. Then R(I) is CM , A is Gorenstein and A/I is Cohen-Macaulay by theorem (4.1), hence I is sCM of linear type. [Note that $H_0(K.) = A/I$ and $H_1(K.) = K_{A/I}$ which is CM. Moreover $H_i(K.) = 0$ for $i > 1$. So I is indeed sCM]. This proves the claim because of (4.7).

Remark (4.10): The second part of the proof of (ii) is using the fact that $gr_A(I)$ Gorenstein implies A/I is CM by theorem (4.1). Another proof-idea is to show that the a-invariant of an a.c.i. ideal I with a Cohen-Macaulay graded ring $gr_A(I)$ is $-ht(I)$.[Note that here we do not assume that A/I is CM]. Then by [15], theorem (3.1) we get the claim.

5. A is regular.

The most interesting problem of this section is the following question of Valla [18]:

If $(A,\mathfrak{m})$ is regular and P an a.c.i. prime ideal in A . When is A/P a Cohen-Macaulay ring?

Proposition (5.1). Let $(A,\mathfrak{m})$ be a regular local ring containing a field and P an a.c.i. prime ideal in A. Then the following hold:

(i) If $ht(P) = 2$, then A/P is Cohen-Macaulay

(ii) If $\dim(A) \leq 5$, then A/P is Cohen-Macaulay.

Proof: (i) comes from [20], theorem (2.1).

For proving (ii) note that $ht(P) \geq 2$ since P is not principlal. If $\dim(A/P) \in \{1,2\}$, then A/P is Cohen-Macaulay by (2.9) and proposition 10 in [18]; if $\dim(A/P) = 3$, then $ht(P) = 2$ and A/P is Cohen-Macaulay by (i).

Without any restrictions on the regular ring A and the a.c.i. prime ideal P the statement " A/P is Cohen-Macaulay" is not true in general. There is an explicit counterexample of Huneke based on the following idea : Take a non Cohen-Macaulay UFD ring A/I and link the defining ideal generically with an a.c.i. prime ideal. This gives an example of an a.c.i. prime ideal which is not perfect.

The following theorem (5.2) which is essentially due to N.V. Trung describes situations where A/I is Buchsbaum for any ideal I in a regular local ring A with $ht(I) > 0$. As a consequence we get in Corollary (5.3) a necessary condition for A/I being Buchsbaum but not Cohen-Macaulay, where I is now an a.c.i. and $depth(A/I) \geq 2$.

This gives in Corollary (5.4) a sufficient condition for the Cohen-Macaulayness of those ideals in the case that $ht(I)$ is 2 or 3.

For a regular local ring $(A,\mathfrak{m})$ with infinite residue field and an ideal I of A with $(0) \neq I \subset \mathfrak{m}^2$ we put:

$n = ord_{\mathfrak{m}}(I)$; $h = ht_A(I)$; $d = \dim(A/I)$; $t = depth(A/I)$;

$B = A/I$; $\mathfrak{m}_B$ = maximal ideal of B ;

$e = e(B)$ = Samuel multiplicity of $\mathfrak{m}_B$.

Moreover we choose elements $x_1,\dots,x_d \in \mathfrak{m}$ so that $a_i = x_i \bmod I$ form a minimal reduction $\underline{a}$ of $\mathfrak{m}_B$. We also set $h_i = l_B(H^i_{\mathfrak{m}_B}(B))$, where H^i denotes the local cohomology.

<u>Theorem (5.2)</u>: Let $(A,\mathfrak{m})$ be a regular local ring and I an ideal in A. Assume that A/I is Buchsbaum with $d \geq 2$ and $ht(I) \geq 2$. Then:

(i) $$e(A/I) \geq \binom{n-1+h}{h} + \sum_{i=1}^{d-1} \binom{d-1}{i-1} h_i - d(\mu-h-1) ,$$

if $\mu = \mu(I) \geq h+1$

(ii) $e(A/I) \geq \binom{n+h}{h} - h$ if $\mu = h$ and hence A/I is Cohen-Macaulay.

<u>Proof</u>: I) Since A/I is Buchsbaum, we have

$$e(B) = 1 + \sum_{i=1}^{d-1} \binom{d-1}{i-1} h_i + l(\mathfrak{m}_B/K_B) ,$$

where $K_B = \sum_{i=1}^{d} (a_1,\dots,\hat{a}_i,\dots,a_d) : a_i$.

Put $K_A = \sum_{i=1}^{d} (I,x_1,\dots,\hat{x}_i,\dots,x_d) : x_i$. Clearly

$$l(A/K_A) = l(\mathfrak{m}/K_A) + 1 = l(\mathfrak{m}_B/K_B) + 1 ,$$

hence

$$e(B) = l(A/K_A) + \sum_{i=1}^{d-1} \binom{d-1}{i-1} h_i .$$

Note that $x_1,\dots,x_d$ are of order 1 w.r.t. $\mathfrak{m}$, forming a regular sequence in A and a s.o.p. for B.

II) For the next step in the proof we recall the following lemma.

<u>Lemma</u>: Under the assumptions made above we get:

$$\mu((I,x)/(x) = \mu(I) - l(I : x/\mathfrak{m}I : x)$$

for any element $x \in A$.

Proof of the lemma: The equality follows from the exact sequence:

$$0 \longrightarrow I : x/\mathfrak{m}I : x \xrightarrow{\cdot x} I/\mathfrak{m}I \longrightarrow (I,x)/\mathfrak{m}I+(x) \longrightarrow 0 \quad .$$

III) To continue in the proof of (5.2) we remark that

(1) $\mu(I) \geq \mu((I,\underline{x}) / (\underline{x})) \geq ht((I,\underline{x}) / (\underline{x})) = ht(I)$,

where $\underline{x} = \{x_1,\dots,x_d\}$.

We consider two cases:

Case 1: $\mu((I,\underline{x}) / (\underline{x})) = h$.

Then $J := (I,\underline{x})/(\underline{x})$ is an ideal of the principal class in the regular ring $A/(\underline{x})$, hence $A/(I,\underline{x})$ is Gorenstein, i.e.

(2) $l((I,\underline{x}) : \mathfrak{m}/(I,\underline{x})) = 1$.

Moreover

$$K_A = \sum_{i=1}^{d} (I,x_1,\dots,\hat{x}_i,\dots,x_d) : \mathfrak{m} \quad ,$$

since A/I is Buchsbaum.

This implies $K_A \subseteq (I,\underline{x}) : \mathfrak{m}$, i.e.

$$\begin{aligned} l(A/K_A) \geq l(A/(I,\underline{x}) : \mathfrak{m}) &= l(A/(I,\underline{x})) - l((I,\underline{x}) : \mathfrak{m}/(I,\underline{x})) \\ &= l(A/(I,\underline{x})) - 1 \quad , \text{ by } (2) \\ &= l(A/(\mathfrak{m}I,\underline{x}) - l((I,\underline{x})/(\mathfrak{m}I,\underline{x})) - 1 \\ &\geq l(A/(\mathfrak{m}^{n+1},\underline{x})) - h - 1 \\ &= \binom{n+h}{h} - (h+1) \quad . \end{aligned}$$

Case 2: $ht(I) + 1 \leq \mu((I,\underline{x}) / (\underline{x}))$.

Put $\delta := \mu(I) - \mu((I,\underline{x}) / (\underline{x}))$. Since

$$\mu((I,x_1,\ldots,\hat{x}_i,\ldots,x_d) / (mI,x_1,\ldots,\hat{x}_i,\ldots,x_d)) \leq \mu(I)$$

we get be the above lemma for each i

$$\delta_i := l((I,x_1,\ldots,\hat{x}_i,\ldots,x_d) : x_i / (mI,x_1,\ldots,\hat{x}_i,\ldots,x_d) : x_i)$$
$$\leq \delta \leq \mu(I) - ht(I) - 1 .$$

Hence, for each i there is an ideal J_i generated by at most $\mu(I) - ht(I) - 1$ elements, such that

$$(I,x_1,\ldots,\hat{x}_i,\ldots,x_d) : x_i = (mI,x_1,\ldots,\hat{x}_i,\ldots,x_d) : x_i + J_i .$$

Note that

$$mJ_i \subseteq (mI,x_1,\ldots,\hat{x}_i,\ldots,x_d) : x_i \subseteq (m^n,\underline{x}) ,$$

since $ord(x_j) = 1$ and $ord(I) = n$.

Since

$$K_A = \sum_{i=1}^{d}\left((mI,x_1,\ldots,\hat{x}_i,\ldots,x_d) : x_i + J_i\right) \subseteq \sum_{i=1}^{d} ((m^n,\underline{x}) + J_i$$

we get

$$l(A/K_A) \geq l(A/(m^n,\underline{x})) - \sum_{i=1}^{d} l\left(((m^n,\underline{x}) + \sum_{r=1}^{i} J_r) / ((m^n,\underline{x}) + \sum_{r=1}^{i-1} J_r))\right)$$
$$\geq \binom{n-1+h}{h} - d \cdot (\mu(I) - ht(I) - 1) .$$

Comparing the bounds in the cases 1 and 2 we obtain the claims of the theorem. Note that (ii) follows from the proof of case 1 using the fact that now $K_A = (I,\underline{x})$.

Moreover, the situation $\mu(I) = ht(I) + 1$ is covered as well by case 1 as by case 2. It is easy to check that under our assumptions ($ord(I) \geq 2$ and $ht(I) \geq 2$) both cases imply

$$e(A/I) \geq \binom{n-1+h}{h} + \sum_{i=1}^{d-1} \binom{d-1}{i-1} h_i \quad .$$

Example 1: $A = k[x_1,x_2,x_3]$

$$I = (x_1^2,x_2^2,x_1x_2x_3)$$
$$= (x_1^2,x_2^2,x_1x_2) \cap (x_1^2,x_2^2,x_3)$$

Then: A/I is Buchsbaum, non-Cohen-Macaulay, $depth(A/I) = 0$,

$n = 2$, $h = ht(I) = 2$, $\mu(I) = 3 = h + 1$,

$e(A/I) = l(k[x_1,x_2] \;/\; (x_1^2,x_2^2,x_1x_2)) = 3$.

Note that the bound in this example is $\binom{(n-1)+h}{h} = 3$.

Example 2: $A/I = k[[s^2,s^3,st,t]] \cong k[[x,y,z,w]]/I$ is Buchsbaum, non-Cohen-Macaulay.

Then

$ht(I) = 2$, $d = 2$, $t = 1$,

$I = (x^3-y^2; xw^2-z^2; yw^3-z^3; xz-yw)$, hence

$\mu(I) = h + 2$, $e(A/I) = 2$; $n = 2$.

Then the bound is: $h + 1 + 1 - 2 = 2$

Corollary (5.3): Let $(A,\mathfrak{m})$ be regular and assume

(1) A/I is Buchsbaum, non-Cohen-Macaulay

(2) I is a.c.i.

Then

$$e(A/I) \geq \binom{n-1+h}{h} + 2t - 1 \quad .$$

If in particular $n = 2$, $t \geq 2$ then $e(A/I) \geq h + 4$. If $t \geq 3$, then $e(A/I) \geq \binom{n-1+h}{h} + 5$.

Example 3: $A/I = k[[t^6;t^4s^2;t^3s^3;t^2s^4;s^6]]$ is Cohen-Macaulay;

$ht(I) = 3$, $\dim A = 5$, $d = n = t = 2$;

$e = 6 < h + 4 = 7$.

Note that $I = (x_4^2 - x_2x_5; x_2x_4 - x_1x_5; x_3^2 - x_1x_5; x_2^2 - x_1x_4)$ if $A = k[[x_1,x_2,x_3,x_4,x_5]]$, i.e. I is an a.c.i. prime ideal.

Example 4: $A/I = k[[t^3;t^2s;ts^2;s^3;ut^3;us^3]]$ is Cohen-Macaulay;

$ht(I) = 3$, $\dim A = 6$, $d = t = 3$, $n = 2$;

$e = 6 < \binom{(n-1)+h}{h} + 5 = 9$.

I is an a.c.i. prime ideal (twisted cubic).

Corollary (5.4): Let $(A,\mathfrak{m})$ be regular and assume

(1) A/I is Buchsbaum

(2) I is a.c.i.

(3) $t \geq 2$.

(4) $e < \binom{n-1+h}{h} + 2t - 1$.

Then A/I is Cohen-Macaulay.

Remark: N.V. Trung mentioned to us that the bound of theorem (5.2) is not sharp in general:

Example 5 (Trung):

$$A/I = k[[t^6; t^5 s; t^4 s^2; t^2 s^4; ts^5; s^6]]$$
$$= k[[x_1, x_2, x_3, x_4, x_5, x_6]]/I$$

is a Buchsbaum ring. We have:

$$ht(I) = 4 \; ; \quad n = 2 \; ; \quad \mu(I) = 8 \; ;$$
$$I(A/I) = 1 \quad \text{and} \quad e(A/I) = 6 \quad ,$$

where $I(A/I)$ is the I-invariant of the Buchsbaum ring. Using Theorem (5.2) we get

$$6 \geq 5 - 2(8 - 4 - 1) + 1 = 0 \quad .$$

Trung could generalize statement (i) of Theorem (5.2) for Buchsbaum rings as follows:

Proposition (5.5) (N.V. Trung): Let I be as above. Suppose that A/I is a Buchsbaum ring. Then

$$e(A/I) \geq \binom{n+h}{h} - \mu(I) - I(A/I) \quad .$$

This bound is sharp for the special example 5:

$$e \geqq 15 - 8 - 1 = 6 \quad .$$

We shall come back to these phenomenons in a joint preprint with N.V. Trung, where new bounds for $e(A/I)$ will be given.

6. On the Gorensteinness of Rees and form rings of powers of ideals.

Given an ideal I in a local ring $(A,\mathfrak{m})$ it is well-known that the Cohen-Macaulayness of the Rees algebra $R(I) = A[It]$ (where t is an indeterminate) implies the Cohen-Macaulayness of all Rees algebras $R(I^n)$, $n \in \mathbb{N}$. The same conclusion holds for the form rings $gr_A(I)$ and $gr_A(I^n)$.

In this section we ask for the corresponding Gorenstein property. If $gr_A(I)$ is Gorenstein and if $R(I)$ is Cohen-Macaulay, it comes out that the integers n for which $R(I^n)$ or $gr_A(I^n)$ are Gorenstein are closely related to the a-invariant of the form ring $gr_A(I)$. Under this aspect we formulate some results concerning the powers of strongly Cohen-Macaulay ideals which are of linear type (in particular a.c.i. ideals) and the powers of the maximal (or of an $\mathfrak{m}$-primary) ideal in a Gorenstein local ring.

For completeness we define the reduction exponent $\delta(I)$ of an ideal I by

$$\delta(I) := \min \left\{ n \,\middle|\, \begin{array}{l} \exists \text{ minimal reduction } q \text{ of } I \\ \text{s.t. } I^{n+1} = q I^n \end{array} \right\} .$$

First we place together some auxiliary results needed in the sequel, which are virtually known.

Proposition (6.1): Let I be a proper ideal in a local ring $(A,\mathfrak{m})$ of dimension d. Then the following hold:

a) If the Rees algebra $R(I)$ is Cohen-Macaulay of dimension $d+1$, then $a(R(I)) = -1$ and $a(gr_A(I)) \leq -1$.

b) If $R(I)$ is Cohen-Macaulay and $grade(I) \geq 2$, then $R(I)$ is Gorenstein iff $a(gr_A(I)) = -2$ and the rings A and $gr_A(I)$ are quasi-Gorenstein (i.e. the canonical modules of the specific rings are isomorphic to the suitably shifted rings).

c) If A is Cohen-Macaulay and I a strongly Cohen-Macaulay ideal satisfying $\mu(IA_P) \leq ht(P)$ for all prime ideals $P \supseteq I$, then $a(gr_A(I)) = -ht(I)$ and $R(I)$ is Cohen-Macaulay. Moreover, if A is even Gorenstein, then $gr_A(I)$ is so.

d) If $R(I)$ and A are Cohen-Macaulay rings, then $gr_A(I)$ is Gorenstein iff $(1,t)^{-a-2}(-1)$ is a canonical module of $R(I)$, where

$a := a(gr_A(I))$ and $(1,t)^m$ denotes the $R(I)$-submodule of the polynomial ring $A[t]$ which is generated by $1,t,\dots,t^m$ in case $m \geq 0$ or $(1,t)^{-1} = I\,R(I)$ in case $m = -1$.

e) Assume that A is Gorenstein and $R(I)$ is Cohen-Macaulay. If $(1,t)^m(-1)$ is a canonical module of $R(I)$ for some integer $m \geq -1$, then $R(I^{m+1})$ is Gorenstein.

f) If the reduction exponent of the maximal ideal $\mathfrak{m}$ of A is $0,1$ or 2, then $gr_A(\mathfrak{m})$ is Cohen-Macaulay (or Gorenstein) iff A is so.

g) If I is $\mathfrak{m}$-primary and $gr_A(I)$ is Cohen-Macaulay, then $a(gr_A(I)) = \delta(I) - \dim(A)$.

Proof: a) can be deduced from the proof of proposition (2.1) in [15]. b) is theorem (3.1) in [15]. c) is in the first part our proposition (4.6). The second part of c) follows from [10]; see also proof of proposition (4.7). d) comes from corollary (2.5) in [12], if one takes care of the correct gradings in the proof given there. f) is a well-known result of J. Sally. g) follows from (4.5) using that the initial forms of the generators of a minimal reduction of I are a regular sequence on $gr_A(I)$. To prove e) we note that for $m = -1$ there is nothing to prove. Hence we may assume $m \geq 0$. Then, denoting $(1,t)^m(-1)$ by K, we get:

$$K_j = (1,t)^m(-1)_j = \begin{cases} 0 & \text{if } j \leq 0 \\ A & \text{if } 1 \leq j \leq m+1 \\ I^{j-(m+1)} & \text{if } j \geq m+2 \end{cases}$$

Now recall that the Veronesean $K^{(m+1)}$ is a canonical module of the Veronesean $R(I)^{(m+1)} = R(I^{m+1})$.

We get

$$(K^{(m+1)})_j = K_{j(m+1)} = \begin{cases} 0 & \text{if } j \le 0 \\ A & \text{if } j = 1 \\ I^{j(m+1)-(m+1)} & \text{if } j \ge 2 \end{cases}$$

Hence $R(I^{m+1})(-1) = K^{(m+1)}$, which proves e).

For Rees algebras of powers of ideals we prove the following auxiliary result.

Proposition (6.2): Let I be an ideal of height ≥ 2 in a Noetherian local ring A. Then at most one power of I has a Gorenstein Rees algebra.

Proof: Assume that $R(I^s)$ and $R(I^t)$ are Gorenstein. Since $R(I^{st}) = R(I^s)^{(t)} = R(I^t)^{(s)}$ and since $R(I^s)(-1)$ and $R(I^t)(-1)$ are canonical modules of $R(I^s)$ and $R(I^t)$ (see (5.1), a) for the correct shifting degree -1) we know that both, $R(I^s)(-1)^{(t)}$ and $R(I^t)(-1)^{(s)}$ are canonical modules of $R(I^{st})$; hence they must be isomorphic. Comparing their homogeneous parts of degree $j \ge 1$ we see that the ideals $I^{s(tj-1)}$ and $I^{t(sj-1)}$ are isomorphic as A-modules. By the following lemma (6.3) we get $s = t$.

Lemma (6.3): Let I be an ideal of height ≥ 2 in a Noetherian ring A. If the two powers I^s and I^t are isomorphic, then $s = t$.

Proof: We can assume that A is a local ring with maximal ideal $\mathfrak{m}$. The isomorphism $I^s \cong I^t$ induces isomorphisms $I^{sj}/\mathfrak{m}I^{sj} \cong I^{tj}/\mathfrak{m}I^{tj}$ for all numbers j. Now, there is a polynomial $P = \sum_{i=0}^{l-1} a_i X^i \in Q[X]$ of degree $l-1$ (where l denotes the analytic spread of I) such that $P(i) = \lambda(I^i/\mathfrak{m}I^i)$ for $i \gg 0$ (λ denotes the length). From $P(sj) = P(tj)$ for $j \gg 0$ we get $a_{l-1}s^{l-1} = a_{l-1}t^{l-1}$. Since $l \ge ht(I) \ge 2$, we get $s = t$.

The following result shows the relation beween the a-invariants of $gr_A(I)$ and $gr_A(I^n)$.

Proposition (6.4): Let I be an ideal in a local ring A and assume that $gr_A(I)$ is Cohen-Macaulay. Then

$$a(gr_A(I^n)) = \left[\frac{a(gr_A(I))}{n} \right] ,$$

where $[\quad]$ denotes the smallest integral part.

Proof: Put $d = \dim(A)$, $a = a(gr_A(I))$, $l = \left[\frac{a}{n} \right]$ and write $a = ln + r$ with $r \in \{0,\ldots,n-1\}$. For every $i \in \{1,\ldots,n\}$ there is an exact sequence of $R(I^n)$-modules:

$$0 \longrightarrow I^{n-i+1} gr_A(I^n) \longrightarrow I^{n-i} gr_A (I^n) \longrightarrow gr_A(I)(n-i)^{(n)} \longrightarrow 0 .$$

Let N be the maximal homogeneous ideal of $R(I^n) \cong R(I)^{(n)}$ and M the maximal homogeneous ideal of $R(I)$. Then (see [7], proposition (47.5)):

$$H_N^i \left(gr_A(I)(n-i)^{(n)} \right) \cong H_M^i \left(gr_A(I)(n-i) \right)^{(n)} .$$

Hence for every $j \in \mathbf{Z}$ and $i \in \{1,\ldots,n\}$ there is an exact sequence

$$0 \longrightarrow H_N^d(I^{n-i+1} gr_A(I^n))_j \longrightarrow H_N^d(I^{n-i} gr_A(I^n))_j \longrightarrow H_M^d(gr_A(I))_{nj+n-i} \longrightarrow 0 .$$

First consider these sequences for $j \geq l+1$. Since $nj+n-i \geq n(l+1)+n-i > nl+r = a$, we have $H_M^d(gr_A(I))_{nj+n-i} = 0$ for each $i \in \{1,\ldots,n\}$. Using the above sequences it follows inductively that $H_N^d(I^{n-i} gr_A(I^n))_j = 0$ for $i = 0,1,\ldots,n$; in particular $H_N^d(gr_A(I^n))_j = 0$ for any $j \geq l+1$, i.e. $a(gr_A(I^n)) \leq l$.

To finish the proof, consider the cohomology sequences from above in degree $j = 1$. Since $H^d_M(gr_A(I))_{nl+r} \neq 0$ we get $H^d_N(I^r gr_A(I^n))_1 \neq 0$ and then successively $H^d_N(I^{r-k} gr_A(I^n))_1 \neq 0$ for $k = 1,\dots,r$. Hence $H^d_N(gr_A(I^n))_1 \neq 0$, q.e.d.

Theorem (6.5): Let I be an ideal of height ≥ 2 in the local ring A . Assume that $gr_A(I)$ is Gorenstein and $R(I)$ is Cohen-Macaulay. Then the following hold for $n \in \mathbb{N}$:

a) $R(I^n)$ is Gorenstein iff $n = -a(gr_A(I)) - 1$

b) $gr_A(I^n)$ is Gorenstein iff $-a(gr_A(I)) \equiv 1 \bmod(n)$.

Proof: To a) Put $a = a(gr_A(I))$. Since $gr_A(I)$ is Gorenstein and $R(I)$ is Cohen-Macaulay we know by (6.1) that $K := (1,t)^{-a-2}(-1)$ is a canonical module of $R(I)$. It follows that $R(I^{-a-1})$ is Gorenstein, by (6.1),e). This is by (6.2) the only power of I which has a Gorenstein Rees algebra. To b) Put $b = a(gr_A(I^n))$. We can assume that $n \geq 2$. Note that $R(I^n)$ and $gr_A(I^n)$ are Cohen-Macaulay rings and that $K^{(n)}$ is a canonical module of $R(I)^{(n)} \cong R(I^n)$. Hence by (6.1),d), $gr_A(I^n)$ is Gorenstein if and only if the $R(I^n)$-module $L := (1,t)^{-b-2}(-1)$ is isomorphic to $K^{(n)}$. To finish the proof, we point out that this last statement holds iff $-a \equiv 1 \bmod (n)$. First we note that $b = \left[\frac{a}{n}\right]$ by (6.4), hence $a = bn + r$ with $r \in \{0,\dots,n-1\}$.

Claim: $L \cong K^{(n)}$ iff $r = n-1$.

Assume that $r = n-1$. Then we get for each $j \in \mathbb{Z}$:

$$L_j = (1,t)^{-b-2}(-1)_j = \begin{cases} 0 & \text{if } j \leq 0 \\ A & \text{if } 1 \leq j \leq -b-1 \\ I^{n(j-1+b+2)} & \text{if } j \geq -b \end{cases}$$

and

$$(K^{(n)})_j = \left((1,t)^{-a-2}(-1)^{(n)}\right)_j = \left((1,t)^{-bn-n-1}\right)_{jn-1}$$

$$= \begin{cases} 0 & \text{if} \quad jn-1 \leq 0 \\ A & \text{if} \quad 0 \leq jn-1 \leq -bn-n-1 \\ I^{(jn-1)+(bn+n+1)} & \text{if} \quad jn-1 \geq -bn-n \end{cases}$$

$$= \begin{cases} 0 & \text{if} \quad j \leq 0 \\ A & \text{if} \quad 1 \leq j \leq -b-1 \\ I^{n(j+b+1)} & \text{if} \quad j \geq -b-1+\frac{1}{n} \quad \text{, i.e. if} \quad j \geq -b \end{cases} .$$

Hence $L \cong K^{(n)}$.

For the converse assume that $L \cong K^{(n)}$; in paticular $L_{-b} \cong K_{-bn} = K_{r-a}$, i.e. $I^n \cong I^{r+1}$. Since $ht(I) \geq 2$ is follows $n = r+1$ by (6.3) . This proves the claim and statement b) of the theorem.

As an immediate consequence of (6.5) we get the following proposition.

Proposition (6.6): Let A be a Gorenstein local ring and I a strongly Cohen-Macaulay ideal satisfying $\mu(IA_p) \leq ht(P)$ for all prime ideals $P \supseteq I$. Assume that $h := ht(I) \geq 2$. Then:

a) $R(I^n)$ is Gorenstein iff $n = h-1$

b) $gr_A(I^n)$ is Gorenstein iff $h \equiv 1 \bmod (n)$.

Proof: Use (6.1), c) together with (6.5).

Remark: For $\mathfrak{m}$-primary ideals I one can avoid the assumption that $R(I)$ is Cohen-Macaulay in the statements of theorem (6.5). For a proof we refer to the theorems 2 and 3 in the appendix by A. Ooishi, where he only considers $\mathfrak{m}$-primary ideals.

In the following we discuss the Rees algebra of powers of $\mathfrak{m}$-primary ideals in situations where $R(I)$ is Cohen-Macaulay.

Proposition (6.7): Let $(A,\mathfrak{m})$ be a d-dimensional Gorenstein local ring with reduction exponent $\delta(\mathfrak{m}) \leq 2$. Then the following hold for $n \in \mathbb{N}$:

(1) $gr_A(\mathfrak{m}^n)$ is Gorenstein iff $d - \delta(\mathfrak{m}) = 1 \bmod (n)$

(2) $R(\mathfrak{m}^n)$ is Gorenstein iff $n = d - \delta(\mathfrak{m}) - 1$.

In particular: If A is regular, then $R(\mathfrak{m}^{d-1})$ is Gorenstein and if A is a quadratic hypersurface, then $R(\mathfrak{m}^{d-2})$ is Gorenstein.

Proof: This follows as a direct consequence of (6.5) and (6.1),f) and g).

Next we show to which extent the Gorensteinness of $R(\mathfrak{m}^n)$ determines the structure of the local ring $(A,\mathfrak{m})$ via the reduction exponent $\delta(\mathfrak{m})$. First we need a lemma.

Lemma (6.8): Let I be an $\mathfrak{m}$-primary ideal in the d-dimensional local ring $(A,\mathfrak{m})$ and $q = (x_1,\ldots,x_d)$ a minimal reduction of I . Assume that $gr_A(I^n)$ is Cohen-Macaulay for some natural number n . Put $a := a(gr_A(I^n))$. Then

$$I^{na+n+d} \subseteq q \quad .$$

If moreover $R(I^n)$ is Gorenstein and $ht(I) \geq 2$, then $I^{d-n} \subseteq q$.

Corollary (6.8.1): If under the assumptions of (6.8) the Rees algebra $R(I^{d-1})$ is Gorenstein, then I is a complete intersection ideal.

Proof of lemma (6.8): Clearly, $J := (x_1^n,\dots,x_d^n)$ is a minimal reduction of I^n and the ideal J^* generated by the initial forms $(x_i^n)^*$ in $gr_A(I^n)$ is a complete intersection in the Cohen-Macaulay ring $gr_A(I^n)$. Since $G = \frac{gr_A(I^n)}{J^*}$ is artinian and $a(G) = a+d$, we get $G_{a+d+1} = 0$. It follows

$$I^{n(a+d+1)} = I^{n(a+d)} \cdot J \subseteq J$$

and from this

$$I^{n(a+d+1)-d(n-1)} \subseteq q \quad ,$$

i.e. $I^{n(a+n+d)} \subseteq q$.

If moreover $R(I^n)$ is Gorenstein, then $a = -2$ (by (6.1),b)), hence $I^{d-n} \subseteq q$.

As a consequence we obtain:

Proposition (6.9): An equimultiple ideal I of height h in a Gorenstein local ring A is a complete intersection iff $R(I^{h-1})$ is Gorenstein.

Proof: By (6.6) the "Only if"-part is already clear. For the converse let q be a minimal reduction of I. Since $IA_p = qA_p$ for all $p \in \mathrm{Assh}(A/I)$ (by corollary(6.8.1)), we get $I = q$ (since $\mathrm{Ass}_A\left(\frac{I}{q}\right) \subseteq \mathrm{Ass}_A\left(\frac{A}{q}\right) = \mathrm{Assh}\left(\frac{A}{q}\right) = \mathrm{Assh}\left(\frac{A}{I}\right)$ and hence $\mathrm{Ass}_A\left(\frac{I}{q}\right)$ is empty).

Theorem (6.10): Given a d-dimensional local Gorenstein ring $(A,\mathfrak{m})$ and an integer $i \in \{1,2,3\}$. Then

$$R(\mathfrak{m}^{d-i}) \text{ is Gorenstein iff } \delta(\mathfrak{m}) = i-1 \ .$$

Proof: If $\delta(\mathfrak{m}) = i-1$ then $R(\mathfrak{m}^n)$ is Gorenstein for $n = d - \delta(\mathfrak{m}) - 1 = d-i$ by (6.7). Conversely, assume that $R(\mathfrak{m}^{d-i})$ is Gorenstein and let q be a minimal reduction of $\mathfrak{m}$. By (6.8) we know that $\mathfrak{m}^i \subseteq q$. Hence, for $i \in \{1,2\}$ we get $\mathfrak{m}^i = q\mathfrak{m}^{i-1}$ and $\mathfrak{m}^{i-1} \neq q\mathfrak{m}^{i-2}$, i.e. $\delta(\mathfrak{m}) = i-1$ for these two cases.

For $i = 3$ assume that $R(\mathfrak{m}^{d-3})$ is Gorenstein. We know that $\mathfrak{m}^3 \subseteq q\mathfrak{m}$, where $q = (x_1,\dots,x_d)$ is a minimal reduction of $\mathfrak{m}$.

<u>Claim</u>: (i) $\mathfrak{m}^3 \cap q^2 = \mathfrak{m}q^2$

(ii) $(q :_A \mathfrak{m}) = q + \mathfrak{m}^2$

(iii) $\mathfrak{m}^3 = \mathfrak{m}^2 q$

Note that from (iii) it follows $\delta(\mathfrak{m}) = 2$, since $\delta(\mathfrak{m}) \notin \{0,1\}$ by (6.2) and (6.7).

<u>Proof of the Claim</u>: To (i): Let $u \in \mathfrak{m}^3 \cap q^2$. Write $u = \sum_i \alpha_i M_i$, where $\alpha_i \in A$ and M_i are monomials in $x_1,\dots,x_d$ of degree two. Put $F := x_1^{d-4} \cdot \dots \cdot x_d^{d-4}$. For fixed i there is a monomial V in $x_1,\dots,x_d$ of degree $d(d-4)-2$ such that $M_i \cdot V = F$ and $M_j \cdot V \in J := \left(x_1^{d-3},\dots,x_d^{d-3}\right)$ for all $j \neq i$. Hence

$$Vu = \alpha_i V M_i + \sum_{j\neq i} \alpha_j M_j V = \alpha_i F + W ,$$

where

$$Vu \in \mathfrak{m}^{d(d-4)-2} \cdot \mathfrak{m}^3 = \mathfrak{m}^{d^2-4d+1}$$

and $W \in J$. This implies

$$\alpha_i F \in J + \mathfrak{m}^{d^2-4d+1} .$$

Since (see proof of (6.8))

$$\mathfrak{m}^{d^2-4d+3} = \mathfrak{m}^{(d-3)(d-1)} \subseteq J$$

it follows

$$\alpha_i F\mathfrak{m}^2 \subseteq J = \left(x_1^{d-3}, \ldots, x_d^{d-3}\right)$$

hence

$$\alpha_i \mathfrak{m}^2 \subseteq q = (x_1, \ldots, x_d) \quad .$$

Since $\mathfrak{m}^2 \not\subseteq q$ (otherwise $R(\mathfrak{m}^{d-2})$ or $R(\mathfrak{m}^{d-1})$ would be also Gorenstein), we get $\alpha_i \in \mathfrak{m}$.

To (ii): Since A/q is Gorenstein, we have

$$k \cong \frac{(q :_A \mathfrak{m})}{q} \supseteq \frac{\mathfrak{m}^2 + q}{q} \qquad ,$$

where $k := A/\mathfrak{m}$ and $\frac{\mathfrak{m}^2+q}{q} \neq 0$.

Therefore $(q :_A \mathfrak{m}) = q + \mathfrak{m}^2$.

To (iii): Let $u \in \mathfrak{m}^3 \subseteq q\mathfrak{m}$. Write $u = \sum_{i=1}^{d} \alpha_i x_i$, where $\alpha_i \in \mathfrak{m}$. For fixed i there is a monomial V in $x_1, \ldots, x_d$ of degree $d(d-4) - 1$ such that

$$Vx_i = F = x_1^{d-4} \cdot \ldots \cdot x_d^{d-4}$$

and

$$Vx_j \in J = \left(x_1^{d-3}, \ldots, x_d^{d-3}\right) \quad \text{for all } j \neq i \ .$$

We get

$$Vu = \alpha_i V x_i + \sum_{j \neq i} \alpha_j x_j V = \alpha_i F + W ,$$

where $Vu \in \mathfrak{m}^{d(d-4)-1}\mathfrak{m}^3 = \mathfrak{m}^{d^2-4d+2}$ and $W \in J$. From $\mathfrak{m}^{d^2-4d+3} = \mathfrak{m}^{(d-1)(d-3)} \subseteq J$, we conclude

$$\alpha_i F \mathfrak{m} \subseteq J \quad \text{and} \quad \alpha_i \mathfrak{m} \subseteq q \quad ,$$

that means by (ii)

$$\alpha_i \in (q :_A \mathfrak{m}) = \mathfrak{m}^2 + q \qquad .$$

Now write $\alpha_i = \beta_i + \gamma_i$ with $\beta_i \in \mathfrak{m}^2$ and $\gamma_i \in q$. Then

$$(*) \qquad u = \sum_{i=1}^{d} \alpha_i x_i = \sum_{i=1}^{d} \beta_i x_i + \sum_{i=1}^{d} \gamma_i x_i$$

implies

$$\sum_{i=1}^{d} \gamma_i x_i \in q^2 \cap \mathfrak{m}^3 = \mathfrak{m} q^2$$

(by (i)), since $u \in \mathfrak{m}^3$ and $\sum_{i=1}^{d} \beta_i x_i \in \mathfrak{m}^2 q$.

Finally it follows by (*) that

$$u \in \mathfrak{m} q^2 + \mathfrak{m}^2 q \subseteq \mathfrak{m}^2 q \quad .$$

This proves (iii) of the claim and so (6.10).

Remark: A. Ooichi proves in his appendix the following result: Given a d-dimensional Gorenstein local ring $(A,\mathfrak{m})$ with multiplicity e , then $R(\mathfrak{m}^{d-e})$ is Gorenstein iff A is a hypersurface.

This is a generalization of our theorem (6.10), where only the cases $e \in \{1,2,3\}$ were treated.

Appendix

On the Gorenstein Property of the Associated Graded Rings and the Rees Algebras of an Ideal

Akira OOISHI

Let $(A,\mathfrak{m},k)$ be a noetherian local ring with $\dim A = d$ and I an $\mathfrak{m}$-primary ideal of A. For simplicity, we assume that k is an infinite field. Put

$$(1-t)^d F(G(I),t) = a_0 + a_1 t + \dots + a_s t^s \ , \ a_s \neq 0 \ ,$$

where $F(G(I),t) = \sum_{n=0}^{\infty} l(I^n/I^{n+1})t^n$ is the Hilbert series of $G(I)$. We say that $G(I)$ is <u>symmetric</u> if $a_i = a_{s-i}$ for any i, $0 \le i \le s$. The following theorem generalizes a result by J. Watanabe [4]:

<u>Theorem 1</u>. Assume that A is Gorenstein. Then $G(I)$ is Gorenstein if and only if $G(I)$ is Cohen-Macaulay and symmetric.

<u>Proof</u>. We may assume that $G(I)$ is Cohen-Macaulay. Take a minimal reduction $J = (x_1,\dots,x_d)$ of I and put $B = A/J$ and $L = I/J$. Then B is an artinian Gorenstein local ring; $x_1^*,\dots,x_d^*$ is a $G(I)$-regular sequence, $G(L) \cong G(I)/(x_1^*,\dots,x_d^*)$ and $(1-t)^d F(G(I),t) = F(G(L),t)$. Hence we may assume that $d = 0$.

Assume that $G(I)$ is Gorenstein. Then by duality,

$$\underline{\mathrm{Hom}}_{A/I}(G(I), E_{A/I}(k)) \cong K_{G(I)} \cong G(I)(s) \ ,$$

where $K_{G(I)}$ is the canonical module of $G(I)$ (cf. [1]). Hence

$$a_i = l(\mathrm{Hom}_{A/I}(G(I)_i, E_{A/I}(k))) = l(G(I)_{s-i}) = a_{s-i} \ .$$

Conversely, assume that $G(I)$ is symmetric, i.e.,

$$l(I^i/I^{i+1}) = l(I^{s-i}/I^{s-i+1}) \ , \quad 0 \leq i \leq s \ .$$

Then $l(I^i) + l(I^{s-i+1}) = l(I^{i+1}) + l(I^{s-i})$ and we get $l(I^i) + l(I^{s-i+1}) = l(I^{s+1}) + l(A) = l(A)$. Since A is Gorenstein,

$$\begin{aligned} l(I^i) &= l(A) - l(I^{s-i+1}) \\ &= l(A/I^{s-i+1}) = l((0 : I^{s-i+1})) \ . \end{aligned}$$

Combined with the inclusion $I^i \subset (0 : I^{s-i+1})$, we get $I^i = (0 : I^{s-i+1})$. Hence

$$(I^{i+1} : I) = ((0 : I^{s-i}) : I) = (0 : I^{s-i+1}) = I^i \ .$$

Therefore

$$\Big[\mathrm{Soc}\ G(I)\Big]_i \subseteq I^i \cap (I^{i+1} : \mathfrak{m}) \cap (I^{i+2} : I))/I^{i+1} = 0 \quad \text{for} \quad i < s$$

and

$$\Big[\mathrm{Soc}\ G(I)\Big]_s = I^s \cap (0 : \mathfrak{m}) = (0 : I) \cap (0 : \mathfrak{m}) = (0 : \mathfrak{m}) \cong k \ .$$

Hence $\mathrm{Soc}\, G(I) \cong k$ and $G(I)$ is Gorenstein. □

Henceforth we assume that $d \geq 1$. We denote by $\delta(I)$ the <u>reduction exponent</u> of I , namely, $\delta(I)$ is the smallest integer n satisfying the equality $JI^n = I^{n+1}$ for some parameter ideal J contained in I . Recall that if $G(I)$ is Cohen-Macaulay, then $\delta(I) = s = \deg F(G(I),t) + d = a(G(I)) + d$. Define the integers $e_i(I) = e_i$, $0 \leq i \leq d$, by the following condition:

$$l(A/I^{n-1}) = e_0 \binom{n+d}{d} - e_1 \binom{n+d-1}{d-1} + \cdots + (-1)^d e_d \quad \text{for} \quad n \gg 0 \ .$$

Corollary. If $G(I)$ is Gorenstein, then $\delta(I) = 2e_1(I)/e(I)$.

Proof. By [3, Proposition 1.1],

$$e_0 = a_0 + a_1 + \ldots + a_s$$

and $$e_1 = a_1 + 2a_2 + \ldots + sa_s \quad .$$

Since $a_i = a_{s-i}$ by the assumption, we get

$$\begin{aligned} 2e_1 &= (a_1 + \ldots + sa_s) + (sa_0 + \ldots + a_{s-1}) \\ &= s(a_0 + \ldots + a_s) = se_0 \ . \end{aligned}$$ □

It is easy to show the following

Lemma. $e_1(I^r) = r^{d-1}\{e_1(I) + e(I)(d-1)(r-1)/2\}$.

Theorem 2. (1) If $G(I^r)$ is Gorenstein, then

$$2e_1(I) \equiv e(I)(d-1) \pmod{e(I)r} \ .$$

(2) Assume that $G(I)$ is Gorenstein. Then $G(I^r)$ is Gorenstein if and only if $a(G(I)) \equiv -1 \pmod{r}$.

Proof. Put $e(I) = e$, $e_1(I) = e_1$ and $\delta(I) = s$.

(1) By Corollary and Lemma above,

$$\begin{aligned} er^d\,\delta(I^r) &= e(I^r)\delta(I^r) \\ &= 2e_1(I^r) = 2r^{d-1}\{e_1 + e(d-1)(r-1)/2\} \ . \end{aligned}$$

Hence $2e_1 \equiv -e(d-1)(r-1) \equiv e(d-1) \pmod{er}$.

(2) Assume that $G(I^r)$ is Gorenstein. Then

$$se = 2e_1 \equiv e(d-1) \pmod{er} .$$

Therefore $s \equiv d-1 \pmod r$ and $a(G(I)) = s-d \equiv -1 \pmod r$.

To show the converse, we may assume that A is complete. First we remark that the theory of local cohomology and canonical modules for graded rings defined over local rings developed by Ikeda (see [1], pp. 139-141) also holds for a noetherian graded ring $R = \bigoplus_{n\in\mathbf{Z}} R_n$ with $R_0 = A$ and with unique maximal homogeneous ideal. In particular, R is Gorenstein if and only if R is Cohen-Macaulay and $K_R \cong R(a)$ for some $a \in \mathbf{Z}$, where K_R is the canonical module of R . The integer a is uniquely determined by R and we denote it by $a(R)$. We apply this to $R = A[It,t^{-1}]$. Put $u = t^{-1}$. Since $R/(u) \cong G(I)$ is Gorenstein, R is Gorenstein, and it is easy to see that $a(R) = a(G(I))+1$. By the assumption, $a(R) \equiv 0 \pmod r$, hence $a := a(R) = rb$ for some $b \in \mathbf{Z}$. Then $R^{(r)}$ is Cohen-Macaulay and

$$K_{R^{(r)}} \cong K_R^{(r)} \cong R(a)^{(r)} \cong R^{(r)}(b) .$$

Hence $R^{(r)} \cong A[I^r t,t^{-1}]$ is Gorenstein by the above criterion. Therefore $G(I^r) \cong R^{(r)}/(u)$ is also Gorenstein. □

<u>Corollary</u>: $G(I^r)$ is Gorenstein for any r if and only if $G(I)$ is Gorenstein and $\delta(I) = d-1$.

<u>Theorem 3</u>: Assume that A is Cohen-Macaulay and $d \geq 2$.

(1) If $R(I^r)$ is Gorenstein, then

$$r = -2e_1(I)/e(I) + d - 1 .$$

In particular, $r \leq d-1$.

(2) Assume that $G(I)$ is Gorenstein. Then $R(I^r)$ is Gorenstein if and only if $r = -a(G(I)) - 1$ (in particular $\delta(I) \leq d-2$).

Proof. Put $e(I) = e$ and $e_1(I) = e_1$.

(1) By [1, Corollary (3.7)] , $\delta(I^r) = d-2$. Hence

$$\begin{aligned}(d-2)er^d &= (d-2)e(I^r) \\ &= 2e_1(I^r) = r^{d-1}\left\{2e_1 + e(d-1)(r-1)\right\} .\end{aligned}$$

Therefore we get $(d-1-r)e = 2e_1$, i.e. $r = -2e_1/e + d - 1$. Since $e_1 \geq 0$, we get $r \leq d-1$.

(2) Since $G(I)$ is Gorenstein, $a(G(I)) = \delta(I) - d = 2e_1/e - d$. Assume that $R(I^r)$ is Gorenstein. Then by (1),

$$r = -2e_1/e + d - 1 = -a(G(I)) - 1 .$$

Conversely, assume that $r = -a(G(I)) - 1$. Then $G(I^r)$ is Gorenstein by Theorem 2 and

$$a(G(I^r)) = \left[a(G(I))/r \right] = \left[(-r-1)/r \right] = -2 .$$

Therefore $R(I^r)$ is Gorenstein by [1, Corollary (3.7)]. □

By localization, we get the following

Corollary: Let J be an ideal of a Cohen-Macaulay local ring with $\operatorname{ht} J \geq 2$. Then

(1) $R(J^n)$ is not Gorenstein for any $n \geq \operatorname{ht} J$.

(2) If $R(J^n)$ is Gorenstein, then $R(J^n)$ is not Gorenstein for any $m \neq n$.

Corollary: Assume that A is Cohen-Macaulay and $d \geq 2$.

(1) $R(I^{d-1})$ if Gorenstein if and only if A is Gorenstein and I is

o

a parameter ideal. In particular, $R(\mathfrak{m}^{d-1})$ is Gorenstein if and only if A is a regular local ring.

(2) Assume that $d \geq 3$. Then $R(\mathfrak{m}^{d-2})$ is Gorenstein if and only if $e(R) = 2$.

(3) Assume that $d \geq 4$. Then $R(\mathfrak{m}^{d-3})$ is Gorenstein if and only if A is Gorenstein and $emb(A) = e(A) + d - 2$.

(4) Put $e(R) = e$ and assume that $d > e$. Then $R(\mathfrak{m}^{d-e})$ is Gorenstein if and only if $emb(A) = d + 1$, i.e., A is a hypersurface.

Proof: The "if" parts follow from Theorem 3, (2). To show the "only if" parts, we recall that

(1) $e_1(I) = 0$ if and only if I is a parameter ideal (cf. [2, Theorem 4.1,(2)]).

(2) Since $e_1(\mathfrak{m}) \geq e(\mathfrak{m}) - 1$, $2e_1(\mathfrak{m}) = e(\mathfrak{m})$ if and only if $e(\mathfrak{m}) = 2$ (cf. [2, Lemma 4.2]).

(3) If A is Gorenstein, then $e_1(\mathfrak{m}) = e(\mathfrak{m})$ if and only if $emb(A) = e(A) + d - 2$ (cf. [3, Theorem 3.6, (2)]).

(4) A is a hypersurface if and only if $e_1(\mathfrak{m}) = e(e-1)/2$ (cf. [3, Theorem 3.4, (6)]). □

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