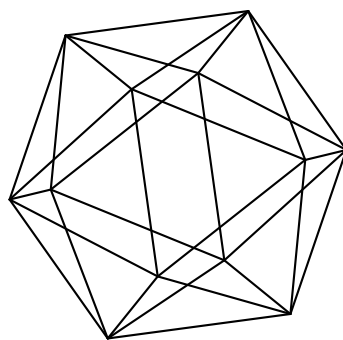


Max-Planck-Institut für Mathematik Bonn

The ABCD of topological recursion

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The ABCD of topological recursion

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Abstract

Kontsevich and Soibelman reformulated and slightly generalised the topological recursion of [24], seeing it as a quantization of certain quadratic Lagrangians in T^*V for some vector space V . KS topological recursion is a procedure which takes as initial data a quantum Airy structure – a family of at most quadratic differential operators on V satisfying some axioms – and gives as outcome a formal series of functions on V (the partition function) simultaneously annihilated by these operators. Finding and classifying quantum Airy structures modulo the gauge group action, is by itself an interesting problem which we study here. We provide some elementary, Lie-algebraic tools to address this problem, and give some elements of the classification for $\dim V = 2$. We also describe four more interesting classes of quantum Airy structures, coming from respectively Frobenius algebras (here we retrieve the 2d TQFT partition function as a special case), non-commutative Frobenius algebras, loop spaces of Frobenius algebras and a \mathbb{Z}_2 -invariant version of the latter. This \mathbb{Z}_2 -invariant version in the case of a semi-simple Frobenius algebra corresponds to the topological recursion of [24].

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1 Introduction

The topological recursion (TR) is a formalism developed by Eynard, Orantin [24, 27] and Chekhov [13] which has in recent years found many applications in random matrices [20, 10], enumerative geometry [12, 23, 22, 2], intersection theory on the moduli space of curves [38, 25, 21], integrable systems [7, 40, 6], topological strings [11, 29, 30], quantum field theories [8, 9, 4], see [19] for a recent overview. In its simplest version, it takes as input a spectral curve Σ embedded as a Lagrangian in $(\mathbb{C} \times \mathbb{C}, dx \wedge dy)$, and returns a collection of meromorphic forms $\omega_{g,n}$ defined on $\text{Sym}^n \Sigma$, indexed by integers $g \geq 0$ and $n \geq 1$. It also returns scalars $F_g = \omega_{g,0}$ for $g \geq 0$, which enjoy a property of symplectic invariance [26, 28]. The proof of the symplectic invariance assumes Σ is compact, the embedding algebraic, and it is a computational tour de force: it does not explain why this property is true and does not allow an easy generalization to weaker conditions. These applications hint at interpreting the topological recursion as a quantization procedure, but a thorough understanding of its underlying (symplectic) geometric nature is still incomplete.

Kontsevich and Soibelman [36] recently proposed a new point of view and setting for TR which generalizes the TR of [24]. We refer to it as KS-TR. Their starting point is the notion of classical Airy structure, *i.e.* a Lagrangian defined by quadratic equations in a symplectic vector space T^*V . The initial data for KS-TR is a lift of the former to a sub-Lie algebra of the Weyl algebra of V , which they call a *quantum Airy structure* (Definition 2.1). Quantum Airy structures are equivalently determined by their coefficients, collected in four tensors (A, B, C, D) , which must satisfy relations (9)-(13) coming from the “Lie subalgebra” condition. The outcome of KS-TR is a formal function on V of the form

$$Z = \exp\left(\sum_{g \geq 0} \hbar^{g-1} S_g\right)$$

annihilated by the differential operators determining the quantum Airy structure (Proposition 2.1). The n -th order Taylor coefficients $F_{g,n}$ of S_g are computed as in TR by induction on $2g - 2 + n$ using (A, B, C, D) , and encode the same information as the $\omega_{g,n}$ did in TR. More precisely, the data of a spectral curve can be used to produce a quantum Airy structure, such that the $F_{g,n}$ computed by KS-TR are the coefficients of the decomposition of $\omega_{g,n}$ computed by TR in a suitable basis of meromorphic differentials (Section 3.5 in [36], and Section 6.1 here).

Kontsevich and Soibelman emphasize in [36] the geometry of Lagrangians in T^*V and the relations between KS-TR and deformation quantization. The present work is complementary to [36]. It focuses on the study of the relations defining quantum Airy structures, with the aim to exhibiting initial data for KS-TR.

Outline

Let us summarize the content of the article.

In Section 2 we concisely present the KS-TR formalism. We write down explicitly in Section 2.2 the relations satisfied by (A, B, C, D) , for which we give a graphical interpretation as three coupled IHX-like relations (Figure 3). The existence of the partition function (Proposition 2.1) is proved in [36] by general holonomicity arguments. We prove it in Section 2.4 by direct computations. Section 5 shows that the partition function can be explicitly computed when some of the tensors (A, B, C) are zero.

In Section 3, we give an equivalent characterization of classical and quantum Airy structures in

terms of “torsion free” symplectic representations of Lie algebras $\rho_1; V \rightarrow \mathfrak{sp}(T^*V)$, together with the data of a Lagrangian linear embedding $J; V \rightarrow T^*V$. As a result, we describe in Section 4 an action of the group of at most quadratic differential operators on quantum Airy structures (the analog of a gauge group) and their partition function. Therefore, we are especially interested in quantum Airy structure modulo the action of this group. One can define in this way the moduli space of quantum Airy structures (Section 4.2), and deformation theory of quantum Airy structures is governed by twisted Lie algebra cohomology. We also define (Section 4.4) an action of commuting flows corresponding to translations in V . It means that, from a given quantum Airy structure $(L_i)_{i \in I}$, we can obtain a deformed Airy structure $(L_i^{(t)})_{i \in I}$ parametrized by t in a formal neighborhood of 0 in V . The action of translation is non-linear even at the infinitesimal level, thus non-trivial modulo the gauge group action.

The remaining of the paper is devoted to exhibiting examples of quantum Airy structures. In Section 6, we study general properties of the above symplectic representations, and apply them to prove some results aiming towards a of classification finite-dimensional quantum Airy structures forming semi-simple Lie algebras. In particular, representation theory allows us to construct a quantum Airy structure forming the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. In Section 7, we give a classification of abelian quantum Airy structures in dimension two and three, a partial classification of two-dimensional quantum Airy structures, and an example of a non-trivial quantum Airy structure for a non semi-simple three-dimensional Lie algebra.

We then progress towards more geometric examples. In Section 8, we describe four classes of quantum Airy structures, associated respectively to Frobenius algebras, non-commutative Frobenius algebras, the loop space of Frobenius algebras, and a \mathbb{Z}_2 -invariant version of the latter. Our proposal for the Frobenius algebra class (Section 8.1) satisfies rather trivially the axioms of a quantum Airy structure, and contains as a special case the enumeration of the trivalent graphs underlying TR and the partition functions of 2d TQFT’s (Lemma 8.6). For the three other classes, checking that our proposal is a quantum Airy structure is a computation; it is perhaps not enlightening, but we have not found more elegant proofs. For the Frobenius and non-commutative Frobenius algebra class, we are able to give an explicit formula – in the form of a finite-dimensional path integral, well-defined at the level of formal power series – for the partition function in full generality (Section 8.2). For the loop space of Frobenius algebras class (Section 8.3-8.4), the partition function necessarily has $S_0 = 0$, but $(S_g)_{g \geq 1}$ can be non-trivial. For the \mathbb{Z}_2 -invariant version, a priori all S_g can be non-trivial.

The class of quantum Airy structures we describe in Proposition 8.14 for \mathbb{Z}_2 -invariant loop spaces of Frobenius algebras are in correspondence with local spectral curves, and KS-TR gets identified with TR in this case. Section 9.1 explains this correspondence to TR in more detail. In Section 9.2, we explain how the gauge group action on quantum Airy structures relates to Givental group action on Lagrangian cones. This puts our understanding the correspondence between TR and correlation functions of semi-simple cohomological field theories established in [18] within the original spirit of Givental quantization procedure [31]. Independently, Section 11 interpretes the recursion for quantum Airy structure on loop spaces as a dynamic on Young diagrams.

We conclude in Section 12 with a list of open problems raised throughout the article.

Comment

We stress that KS-TR is not only a reformulation of TR. It comes with new non-trivial examples of initial data, *e.g.* having a finite-dimensional V (Sections 8.1-8.2), and the case where V is infinite-dimensional and attached to a curve without reference to a local involution (Proposition 8.13). The latter may be used to propose a TR for spectral curves without ramification points, see Section 10. Although the motivations mainly come from geometry, KS-TR can be presented only resorting to multilinear algebra and combinatorics, without complex analysis. The beginners or non-geometers interested in the theory of topological recursion – *e.g.* for the enumeration of maps [22] – may find the simplicity of this new framework (concentrated in Section 2) appealing.

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2 Kontsevich-Soibelman approach to topological recursion

2.1 Setting

Let V be a vector space over \mathbb{C} . It could be finite or infinite-dimensional. We will mostly work in a basis $(e_i)_{i \in I}$ of V , and with its dual basis $(x_i)_{i \in I}$ which form a set of linear coordinates on V . In the cases where $\dim V = \infty$, convergence issues will not play a role in this article. For general discussions it will be implicitly assumed that all seemingly infinite sums are actually finite or make sense after introducing if necessary suitable filtrations or completions. For specific examples where $\dim V = \infty$ we will justify that the sums contain only finitely many non-zero terms. We equip T^*V with its canonical symplectic structure, and consider its Weyl algebra

$$\mathcal{W}_V^{\hbar} = \mathbb{C}[\hbar] \langle (x_i, \partial_i)_{i \in I} \rangle / \langle [\partial_i, x_i] = \hbar \rangle.$$

Kontsevich and Soibelman [43, 36] proposed the following setting, motivated by the problem of quantization of Lagrangians in T^*V defined by quadratic equations.

Let $(x_i)_{i \in I}$ be the linear coordinates on V . By convention, i, j, k, \dots are fixed indices, while indices a, b, c, \dots should be summed over I . For instance, $A_{a,b}^i x_a x_b := \sum_{a,b \in I} A_{a,b}^i x_a x_b$.

Definition 2.1 *A quantum Airy structure on V is a sequence $(L_i)_{i \in I}$ of elements of \mathcal{W}_V^{\hbar} of the form*

$$L_i = \hbar \partial_i - \frac{1}{2} A_{a,b}^i x_a x_b - \hbar B_{a,b}^i x_a \partial_b - \frac{\hbar^2}{2} C_{a,b}^i \partial_a \partial_b - \hbar D^i, \quad (1)$$

where \hbar is a formal parameter and $A_{j,k}^i, B_{j,k}^i, C_{j,k}^i$ and D^i are scalars, which form a Lie subalgebra of \mathcal{W}_V^\hbar , i.e.

$$[L_i, L_j] = \hbar f_{i,j}^a L_a \quad (2)$$

for some scalars $f_{i,j}^k$.

In this definition, we can always assume that $A_{j,k}^i = A_{k,j}^i$ and $C_{j,k}^i = C_{k,j}^i$. The coefficients defining a quantum Airy structure can be rearranged in a basis-free way

$$A \in \text{Hom}(V^{\otimes 3}, \mathbb{C}), \quad B \in \text{Hom}(V^{\otimes 2}, V), \quad C \in \text{Hom}(V, V^{\otimes 2}), \quad D \in \text{Hom}(V, \mathbb{C}), \quad (3)$$

by the assignments

$$A(e_i \otimes e_j \otimes e_k) = A_{j,k}^i, \quad B(e_i \otimes e_j) = B_{j,a}^i e_a, \quad C(e_i) = C_{a,b}^i e_a \otimes e_b, \quad D(e_i) = D^i.$$

Equation (2) puts strong constraints on A, B, C, D . They will be studied in Section 2.2 in a pedestrian way, and in Section ?? in a more abstract way. We remark that for any choice of A and D , $(A, B = C = 0, D)$ defines a (rather trivial) quantum Airy structure. The justification for the name ‘‘Airy structures’’ will appear in the examples provided in Section 8.1. The notion of classical Airy structure will only be presented in Section 3.2, as it does not play a central role here although it served as motivation in [36].

The Weyl algebra \mathcal{W}_V^\hbar naturally acts by differential operators on functions on V . Equation (2) is a sufficient condition for the existence of a function Z on V which is a common solution to $L_i \cdot Z = 0$ for all $i \in I$. More precisely, we have

Proposition 2.1 *There exists a unique formal series*

$$Z = \exp \left(\sum_{\substack{g \geq 0 \\ n \geq 1}} \frac{\hbar^{g-1}}{n!} \sum_{i_1, \dots, i_n \in I} F_{g,n}(i_1, \dots, i_n) x_{i_1} \cdots x_{i_n} \right), \quad (4)$$

where $F_{g,n}(i_1, \dots, i_n)$ are scalars, invariant under permutation of the $(i_m)_{m=1}^n$, such that $F_{0,1}(i) = F_{0,2}(i, j) = 0$ for all i, j , and

$$\forall i \quad L_i \cdot Z = 0.$$

More precisely,

$$F_{0,3}(i, j, k) = A_{j,k}^i, \quad F_{1,1}(i) = D^i, \quad (5)$$

and for $2g - 2 + n \geq 2$

$$\begin{aligned} F_{g,n}(i_1, J) &= \sum_{m=2}^n B_{i_m, a}^{i_1} F_{g, n-1}(a, J \setminus \{i_m\}) \\ &+ \frac{1}{2} C_{a,b}^{i_1} \left(F_{g-1, n+1}(a, b, J) + \sum_{\substack{J' \sqcup J'' = J \\ h' + h'' = g}} F_{h', 1+|J'|}(a, J') F_{h'', 1+|J''|}(b, J'') \right), \end{aligned} \quad (6)$$

where $J = \{i_2, \dots, i_n\}$ is a $(n-1)$ -uple of indices in I .

Proof. The uniqueness is obvious. We take $i_1 \in I$, insert (4) into the equation $L_{i_1} \cdot Z = 0$, and for each $g \geq 0$, $n \geq 1$ and $(i_2, \dots, i_n) \in I^{n-1}$ we collect the coefficient of $\hbar^{g-1} x_{i_2} \cdots x_{i_n}$. The equations for

$(g, n) = (0, 1), (0, 2), (0, 3)$ are

$$\begin{aligned}
0 &= -F_{0,1}(i_1) + C_{ab}^i F_{0,1}(a) F_{0,1}(b), \\
0 &= -F_{0,2}(i_1, i_2) + B_{i_2, a}^{i_1} F_{0,1}(a) + C_{a, b}^{i_1} F_{0,1}(a) F_{0,2}(b, i_2), \\
0 &= -F_{0,3}(i_1, i_2, i_3) + A_{i_2, i_3}^{i_1} + (B_{i_2, a}^{i_1} F_{0,2}(a, i_3) + B_{k, a}^{i_1} F_{0,2}(a, i_2)) \\
&\quad + \frac{1}{2} C_{a, b}^{i_1} (F_{0,1}(a) F_{0,3}(b, i_2, k) + F_{0,2}(a, i_2) F_{0,2}(b, i_3)).
\end{aligned} \tag{7}$$

As we take $F_{0,1}(i) = F_{0,2}(i, j) = 0$, we indeed solve the two first equations. The third equation yields $F_{0,3}(i_1, i_2, i_3) = A_{i_2, i_3}^{i_1}$. For $(g, n) = (1, 1)$ we find $F_{1,1}(i_1) = D^{i_1}$. In general, isolating the term $F_{g,n}(i_1, \dots, i_n)$ readily gives (6). In this equation, the order of the indices in the set J' and J'' do not matter as the $F_{g',n'}(j_1, \dots, j_n)$ were assumed symmetric under permutation of $(j_m)_{m=1}^n$.

Conversely, the existence is guaranteed by the constraints (2). Alternatively, and effectively, we can define $F_{g,n}$ by formula (6) inductively on $2g - 2 + n$, provided we justify that the result is symmetric when i_1 is permuted with the other i_m s. We will show this is true later in Proposition 2.4, by direct computations involving the relations between (A, B, C, D) following from (2). We see for instance that the symmetry of $F_{0,3}$ imposes that $A_{j,k}^i = A_{i,k}^j$, hence A must be fully symmetric in its three indices. It will indeed be a consequence (see Section 4.1) of (2) for operators of the form (1). \square

Formula (6) has a graphical interpretation (Figure 1), which contains the same kind of terms as the topological recursion introduced in [24]. One can therefore propose, following Kontsevich and Soibelman, an elementary definition of the topological recursion.

- the initial data is a quantum Airy structure, *i.e.* (A, B, C, D) as in (3) satisfying the relations we will write in Section 2.
- the outcome are symmetric tensors $F_{g,n} \in \text{Hom}(V^{\otimes n}, \mathbb{C})$ indexed by $2g - 2 + n > 0$, which we consider as the Taylor expansion coefficients of a formal series/function on V , whose exponential is denoted Z and called the partition function.

The topological recursion of [24] rather takes as initial data a spectral curve, *i.e.* a simple branched covering $x : \Sigma \rightarrow \Sigma_0$ between two Riemann surfaces, together with a meromorphic 1-form $\omega_{0,1}$ on Σ , and a fundamental meromorphic bidifferential of the second kind $\omega_{0,2}$ on Σ^2 . We will show in Section 9 that this data determines a quantum Airy structure based on $V = H^0(U, K_U) / x^* H^0(U_0, K_{U_0})$ where U_0 is a (small enough) neighborhood of the ramification points (zeroes of dx) in Σ_0 , and $U := x^{-1}(U_0)$. The $F_{g,n}$ computed by (6) are then coefficients of the decompositions of the meromorphic n -differentials $\omega_{g,n}$ defined by [24] in a suitable basis of meromorphic forms (Proposition 9.2). Therefore, Kontsevich-Soibelman topological recursion can be seen as a generalisation of [24].

2.2 The relations between (A, B, C, D)

Let $L := (L_i)_{i \in I}$ be differential operators of the form (1). We now describe the necessary and sufficient conditions on (A, B, C, D) for L to be a quantum Airy structure, *i.e.* satisfy (2). Evaluating the commutator between the first terms with a pure single derivative and the B -terms we again obtain

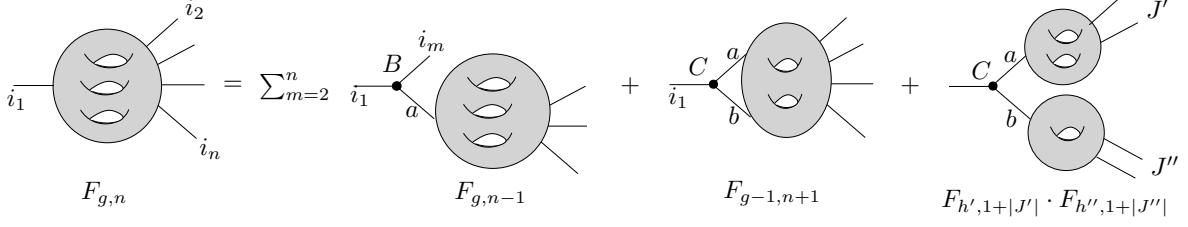


Figure 1: $F_{g,n}(i_1, \dots, i_n)$ is represented as a surface of genus g with n boundaries, carrying the labels i_1, \dots, i_n . In this pictorial language, the terms appearing in the recursion (6) are all the topologies resulting from removal of a pair of pant P bounding the first boundary. The weight of P is a B or a C depending on whether it has one or two external boundary components.

terms with pure single derivatives. Because this commutator is the only source of such terms in the right-hand side, we immediately conclude that the structure constants $f_{i,j}^k$ are determined by the B -terms alone

$$\forall i, j, k, \quad f_{i,j}^k = B_{j,k}^i - B_{i,k}^j. \quad (8)$$

Evaluating now the commutator between L_i and L_j and comparing with the right-hand side in Equation (2) we obtain further constraints on (A, B, C, D) . First, the absence of a linear term in x_i immediately implies the full symmetry of the coefficients A

$$\forall i, j, k, \quad A_{j,k}^i = A_{i,k}^j, \quad (9)$$

as anticipated for the symmetry of $F_{0,3}$ in (5). We obtain three more relations matching the coefficients of the terms $x_k x_\ell$, $\partial_k \partial_\ell$, $x_k \partial_\ell$, for any i, j, k, ℓ

$$B_{j,a}^i B_{k,\ell}^a + B_{k,a}^i B_{a,\ell}^j + C_{\ell,a}^i A_{a,k}^j = (i \leftrightarrow j), \quad (10)$$

$$B_{j,a}^i C_{k,\ell}^a + C_{k,a}^i B_{a,\ell}^j + C_{\ell,a}^i B_{a,k}^j = (i \leftrightarrow j), \quad (11)$$

$$B_{j,a}^i A_{k,\ell}^a + B_{k,a}^i A_{a,\ell}^j + B_{\ell,a}^i A_{a,k}^j = (i \leftrightarrow j). \quad (12)$$

And matching the coefficient of $\hbar \cdot 1$ we find, for all i, j

$$B_{j,a}^i D^a + \frac{1}{2} C_{a,b}^i A_{a,b}^j = (i \leftrightarrow j). \quad (13)$$

Consequently we have the lemma

Lemma 2.2 $(L_i)_{i \in I}$ is a quantum Airy structure if and only if (A, B, C, D) satisfy the zero torsion conditions (8)-(9), the **BB-AC** relation (10), the **BC** relation (11), the **BA** relation (12), and the **D** relation (13). \square

Equation (8) can be taken as a definition of the structure constants f , and one can check by direct computation that the above relations imply the Jacobi identity for f . The full symmetry of A could be added to the axioms of quantum Airy structures. We call (8)-(9) “zero torsion condition” for a reason explained in Section 3.1. The three relations for (A, B, C) are rather non-trivial. If $d = \dim V$, let us count the number of unknowns and *a priori* independent equations determining quantum Airy structures. A has $\frac{d(d+1)(d+2)}{6}$ independent coefficients, B has d^3 coefficients, and C has $\frac{d^2(d+1)}{2}$ coefficients. The **BB-AC** relation is antisymmetric in i, j , so gives $\frac{d(d-1)}{2} \cdot d^2$ constraints. The **BC** and the **AC** relations are antisymmetric in i, j , symmetric in k, ℓ , so gives $\frac{d(d-1)}{2} \cdot \frac{d(d+1)}{2}$ constraints.

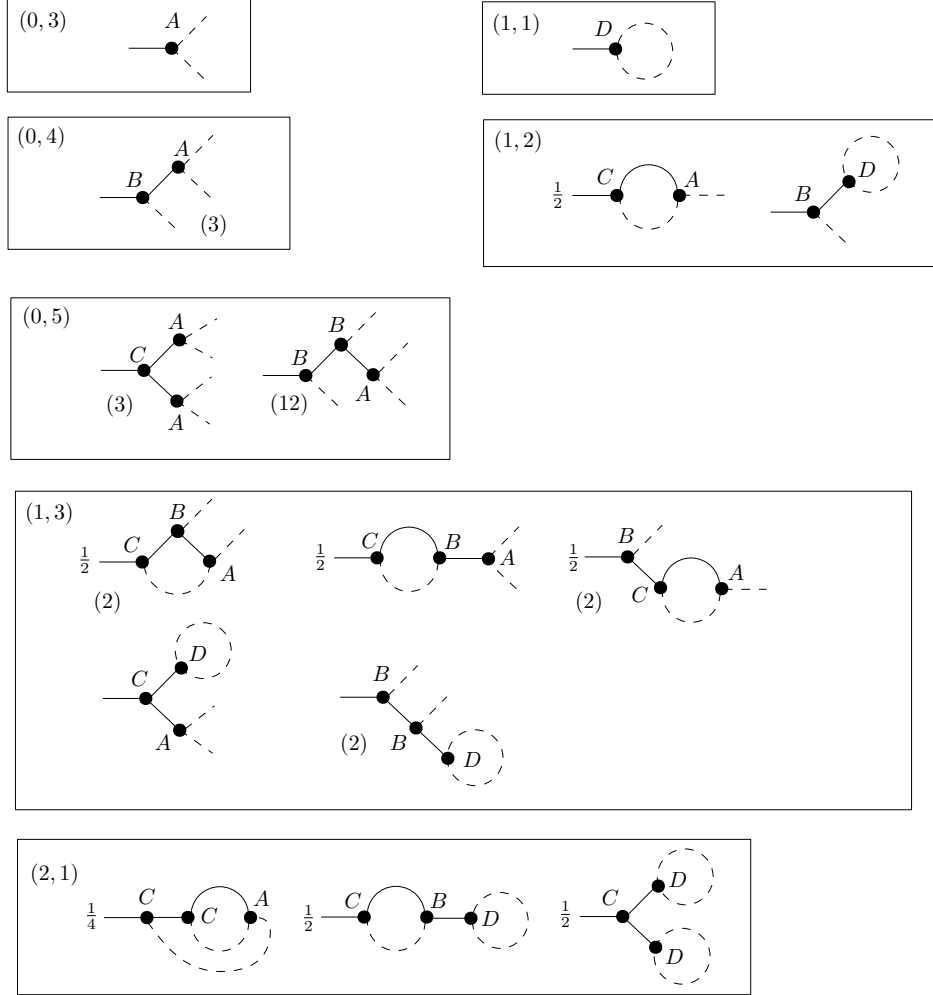


Figure 2: Unfolding (6) gives [27, Section 3] a set $\mathfrak{G}_{g,n}(1)$ of pairs (G, T) where G is a trivalent graph with first Betti number g and n leaves, with cyclic order at each vertex, and T is a spanning tree rooted at the first leaf, having the extra property that edges which are not in T must connect vertices v and v' which are *parent*. This means that the geodesic in T from the root to v contains (or is contained) in the geodesic from the root to v' . Vertices incident to a loop are assigned a D , vertices incident to one external leg are B s, vertices incident to two external legs are A s. Internal vertices can be A, B, C as prescribed by the recursive construction of the graph – which is remembered by the spanning tree rooted at the first leg. We have listed these graphs for low values of (g, n) . The $\frac{1}{2^p}$ is the symmetry factor which arises from the repetition of factor of $\frac{1}{2}$ in the C -term of (6). A (k) indicates that there are k such graphs, which differ by the labeling $2, \dots, n$ of the legs. When two such graphs are related by an exchange of the two legs outgoing from a C , the two graphs give the same contribution to $F_{g,n}$, and we listed it as a single graph with a factor of $\frac{1}{2}$ less.

So, as far as (A, B, C) are concerned, we have $\frac{d}{3}(5d^2 + 3d + 1)$ unknowns, and $\frac{d^2(d-1)(2d+1)}{2}$ constraints. The first values are

d	1	2	3	4	5	6
#unknowns	3	18	55	124	235	398
#constraints	0	10	63	216	550	1170

For $d \geq 3$, we find that the three relations form an overdetermined system. Therefore, it is *a priori*

not obvious that non-zero solutions for (A, B, C) can be found at all. If (A, B, C) is a solution, the set of allowed D satisfying (13) is an affine space, hence easier to describe. We will however show in Sections 8 and 7 that many non-trivial solutions can be found.

Lemma 2.3 *Assume (A, B, C) solves the **BB-AC**, **BC** and **BA** relations, as well as (8) and (9).*

- (i) *If V is an abelian Lie algebra (namely $f_{i,j}^k = 0$), any choice of D gives a quantum Airy structure.*
- (ii) *If $\text{tr } B^i := \sum_a B_{a,a}^i$ exists (for instance, it is always the case when $\dim V < \infty$), then*

$$D_{\text{ref}}^i = \frac{1}{2} \text{tr } B^i$$

completes (A, B, C) into a quantum Airy structure. Further, (A, B, C, D) completes (A, B, C) into a quantum Airy structure if and only if for any $v_1, v_2 \in V$, $(D - D_{\text{ref}})([v_1, v_2]) = 0$ where we consider $D \in V^$.*

In particular, when V is finite-dimensional and semi-simple, $[V, V] = V$ thus, for given (A, B, C) satisfying the relations, there is a unique way to complete into a quantum Airy structure (A, B, C, D) .

Proof. The claim follows from the observation that, summing (10) over $k = \ell \in I$, we find

$$f_{j,a}^i \text{Tr } B^a + C_{a,b}^i A_{a,b}^j - C_{a,b}^j A_{a,b}^i = 0.$$

□

Note that

$$\hbar B_{a,b}^i x_a \partial_b = \frac{\hbar}{2} B_{a,b}^i (x_a \partial_b + \partial_b x_a) - \frac{\hbar}{2} \text{tr } B^i$$

when it makes sense. So, the solution exhibited in (ii) corresponds to choosing the symmetric ordering of x_a and ∂_b , rather than the normal order $x_a \partial_b$. When V is infinite-dimensional, we will in Section 8.3 see cases where $\text{tr } B^i$ is not well-defined, but we can nevertheless find other solutions for D .

2.3 Graphical interpretation of the relations

The structure of the indices (i, j, k, l) and the summation index a is the same in the three relations **BB-AC**, **BC** and **BA**. They can in fact be presented as a system of three IHX relations (Figure 3). Another graphical form is given in Figure 4.

This IHX form is in fact not surprising. The original IHX relation is the graphical interpretation of the Jacobi relation for a Lie algebra \mathfrak{l} , and the Jacobi relation itself expresses that the adjoint representation $\mathfrak{l} \rightarrow \text{GL}(\mathfrak{l})$ is a homomorphism of Lie algebras. It is a special case, for the adjoint representation, of the STU relation expressing that one has a representation $\mathfrak{l} \rightarrow \text{End}(\mathfrak{m})$, where \mathfrak{m} is a module for \mathfrak{l} . We will see in Section 3.1 that the three relations for (A, B, C) are equivalent to requiring that the adjoint action $\text{ad}_L : V \rightarrow \text{End}(T_{\hbar}^* V)$ is a representation of the Lie algebra $(V, (f_{i,j}^k)_{i,j,k})$. So, these 3 relations come from specializing the STU relation to this representation which has the special form $[L_i, \cdot]$ with L_i of the form (1).

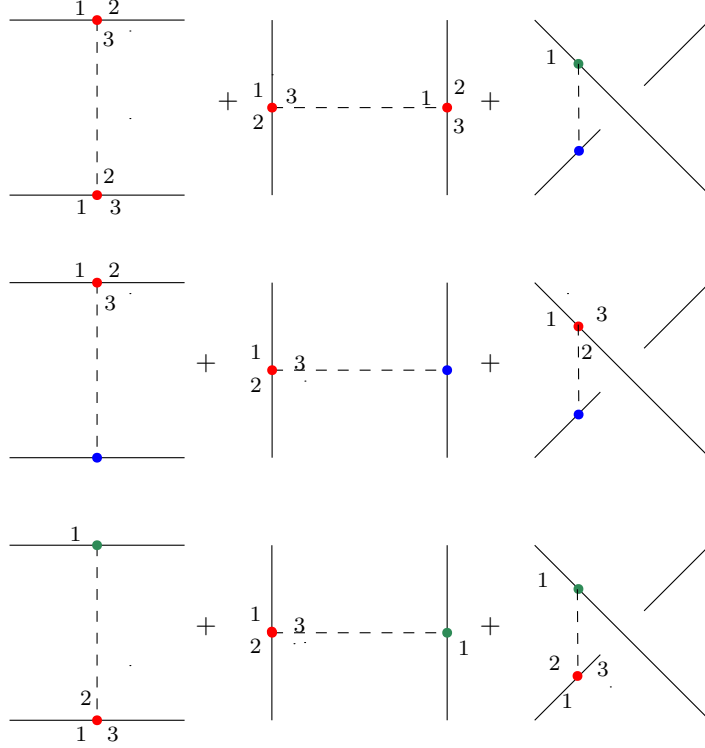


Figure 3: A red dot indicates a $B_{i_2, i_3}^{i_1}$ with incident edges labeling as indicated by the numbers. A blue dot represents an $A_{\bullet\bullet}$. A green dot represents a $C_{\bullet\bullet}^{i_1}$, with first incident edge carrying the upper index. The three relations are that these combinations are symmetric with respect to permutation of the two left legs.

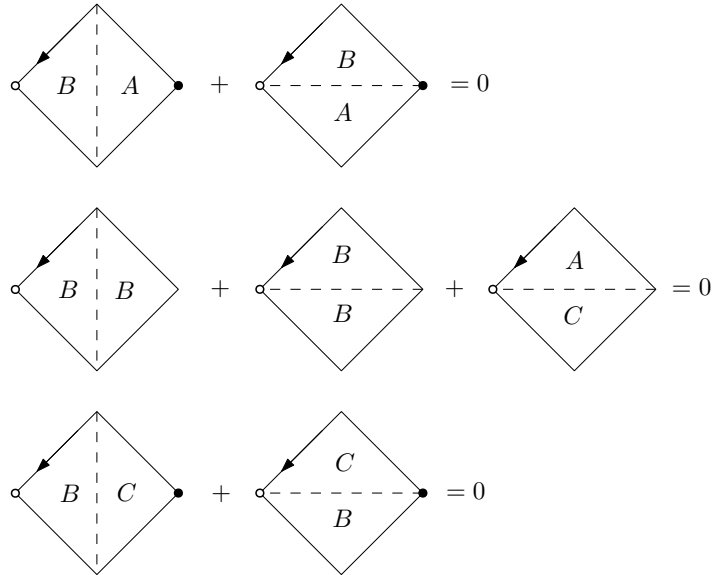


Figure 4: Each edge carries an index. Indices carried by dashed edge are summed over. \bullet means symmetrisation of the indices of the edge incident at that vertex, while \circ means antisymmetrization. The arrow indicates which index is placed first, and thus defines the order of composition of the two operators.

2.4 Proof of symmetry

Proposition 2.4 *If $(L_i)_{i \in I}$ is a quantum Airy structure, $F_{g,n}(i_1, \dots, i_n)$ defined recursively by (6) for $2g - 2 + n > 0$ is symmetric under permutation of $(i_m)_{m=1}^n$.*

Remark 2.2 *The converse of Proposition 2.4 is not true. Indeed, if $A = 0$ and $D = 0$, one finds that $F_{g,n} = 0$ for all g, n , hence it is symmetric, whether or not B and C satisfy the – still non-trivial – relations **BB-AC** and **BC**.*

Proof. We already saw in Section 2.1 that $F_{0,3}(i, j, k) = A_{j,k}^i$, which is fully symmetric due to (9), and for $F_{1,1}$ there is nothing to check. We compute from (6)

$$F_{0,4}(i, j, k, \ell) = B_{j,a}^i A_{k,\ell}^a + \text{cycl. } (j, k, \ell)$$

which is fully symmetric thanks to (12), and

$$F_{1,2}(i, j) = B_{j,a}^i D^a + \frac{1}{2} C_{a,b}^i A_{a,b}^j$$

which is fully symmetric thanks to (13). So the result holds for $2g - 2 + n \leq 2$. Take (g, n) such that $2g - 2 + n > 2$, and assume we have proved fully symmetry of $F_{g',n'}$ for all $2g' - 2 + n' \leq 2g - 2 + n$. Let $K = \{k_1, \dots, k_n\}$. Let us define $F_{g,n+2}(i, j, K)$ by applying (6) with first index i . The resulting terms are in the range of the induction hypothesis, thus completely symmetric under permutation of (j, k_1, \dots, k_n) . So, we only need to prove it is symmetric under permutation of i and j . For this purpose, we use again (6) with first index j , except for the term involving $B_{j,a}^i F_{g,n+1}(a, K)$, for which we use (6) with first index a . Denote $K^{[k]} := K \setminus \{k\}$ and $K^{[k,\ell]} := K \setminus \{k, \ell\}$. We also implicitly use

the full symmetry of A and the symmetry of C in its two lower indices. We find that

$$\begin{aligned}
& F_{g,n+2}(i, j, K) \\
= & B_{j,a}^i F_{g,n+1}(a, K) + \sum_{k \in K} B_{k,a}^i F_{g,n+1}(a, j, K^{[k]}) + \frac{1}{2} C_{a,b}^i F_{g-1,n+3}(a, b, j, K) \\
& + \sum_{\substack{h'+h''=g \\ K' \sqcup K''=K}} C_{a,b}^i F_{h',2+|K'|}(a, j, K') F_{h'',1+|K''|}(b, K'') + \delta_{g,0} \delta_{n,1} A_{j,k_1}^i \\
= & B_{j,a}^i \left\{ \sum_{k \in K} B_{k,b}^a F_{g,n}(b, K^{[k]}) + \frac{1}{2} C_{b,c}^a F_{g-1,n+2}(b, c, K) \right. \\
& + \sum_{\substack{h'+h''=g \\ K' \sqcup K''=K}} \left. \frac{1}{2} C_{b,c}^a F_{h',1+|K'|}(b, K') F_{h'',1+|K''|}(c, K'') + \delta_{g,0} \delta_{n,2} A_{k_1,k_2}^a + \delta_{g,1} \delta_{n,0} D^a \right\} \\
& + \sum_{k \in K} B_{k,a}^i \left\{ B_{a,b}^j F_{g,n}(b, K^{[k]}) + \sum_{\ell \in K^{[k]}} B_{\ell,b}^j F_{g,n}(b, a, K^{[k,\ell]}) + \frac{1}{2} C_{b,c}^j F_{g-1,n+2}(b, c, a, K^{[k]}) \right. \\
& + \sum_{\substack{h'+h''=g \\ K' \sqcup K''=K^{[k]}}} \left. \frac{1}{2} F_{h',2+|K'|}(b, a, K') F_{h'',1+|K''|}(c, K'') + \delta_{g,0} \delta_{n,2} A_{a,k}^j \right\} \\
& + \frac{1}{2} C_{a,b}^i \left\{ B_{a,c}^j F_{g-1,n+2}(b, c, K) + B_{b,c}^j F_{g-1,n+2}(a, c, K) \right. \\
& + \sum_{k \in K} B_{k,c}^j F_{g-1,n+2}(b, c, a, K^{[k]}) + \frac{1}{2} C_{c,d}^j F_{g-2,n+4}(c, d, a, b, K) \\
& + \sum_{\substack{h'+h''=g-1 \\ K' \sqcup K''=K}} C_{c,d}^j F_{h',3+|K'|}(a, b, c, K') F_{h'',1+|K''|}(d, K'') + C_{c,d}^j F_{h',2+|K'|}(a, c, K') F_{h'',2+|K''|}(b, d, K'') \\
& + \delta_{g,1} \delta_{n,0} A_{a,b}^j \left. \right\} \\
& + \sum_{\substack{h'+h''=g \\ K' \sqcup K''=K}} C_{a,b}^i F_{h'',1+|K''|}(b, K'') \left\{ B_{a,c}^j F_{h',1+|K'|}(c, K') + \sum_{k \in K'} B_{k,c}^j F_{h',1+|K'|}(c, a, K^{[k]}) \right. \\
& + \frac{1}{2} C_{c,d}^j F_{h'-1,3+|K'|}(c, d, a, K') + \sum_{\substack{s+s'=h' \\ L \sqcup L'=K'}} C_{c,d}^j F_{s,2+|L|}(a, c, L) F_{s',1+|L'|}(d, L') + \delta_{h',0} \delta_{|K'|,1} A_{a,k'}^j \left. \right\}.
\end{aligned}$$

We now collect the various terms

$$\begin{aligned}
& F_{g,n+2}(i, j, K) \\
= & \delta_{g,0}\delta_{n,2}F_{0,4}(i, j, k_1, k_2) + \delta_{g,1}\delta_{n,0}F_{1,2}(i, j) \\
& + \sum_{k \in K} F_{g,n}(b, K^{[k]}) \left(B_{j,a}^i B_{k,b}^a + B_{a,b}^j B_{k,a}^i + A_{a,k}^j C_{a,b}^i \right) + \sum_{k \neq \ell \in K} F_{g,n}(a, b, K^{[k,\ell]}) B_{k,a}^i B_{\ell,b}^j \\
& + \frac{1}{2} F_{g-1,n+2}(b, c, K) \left(B_{j,a}^i C_{b,c}^a + B_{a,c}^j C_{a,b}^i + B_{b,c}^j C_{a,b}^i \right) + \sum_{k \in K} \frac{1}{2} F_{g-1,n+2}(b, c, a, K^{[k]}) \left(B_{k,a}^i C_{b,c}^j + B_{k,c}^j C_{a,b}^i \right) \\
& + \frac{1}{4} F_{g-2,n+4}(a, b, c, d, K) C_{a,b}^i C_{c,d}^j + \sum_{\substack{h'+h''=g \\ K' \sqcup K''=K}} \frac{1}{2} F_{h',1+|K'|}(b, K') F_{h'',1+|K''|}(c, K'') \left(C_{b,c}^a B_{j,a}^i + 2C_{a,b}^i B_{a,c}^j \right) \\
& + \sum_{k \in K} \sum_{\substack{h'+h''=g \\ K' \sqcup K''=K^{[k]}}} \frac{1}{2} F_{h',2+|K'|}(b, a, K') F_{h'',1+|K''|}(c, K'') \left(B_{k,a}^i C_{b,c}^j + B_{k,c}^j C_{a,b}^i \right) \\
& + \sum_{\substack{h'+h''=g-1 \\ K' \sqcup K''=K}} \frac{1}{2} F_{h',3+|K'|}(a, b, c, K') F_{h'',1+|K''|}(d, K'') \left(C_{c,d}^j C_{a,b}^i + C_{a,d}^i C_{c,b}^j \right) \\
& + \sum_{\substack{h'+h''=g-1 \\ K' \sqcup K''=K}} \frac{1}{2} F_{h',2+|K'|}(a, c, K') F_{h'',2+|K''|}(b, d, K'') C_{a,b}^i C_{c,d}^j \\
& + \sum_{\substack{h''+s+s'=g \\ K'' \sqcup L \sqcup L'=K}} F_{h'',1+|K''|}(b, K'') F_{s,2+|L|}(a, c, L) F_{s,1+|L'|}(d, L') C_{c,d}^j C_{a,b}^i.
\end{aligned}$$

We already proved that $F_{0,4}$ and $F_{1,2}$ are symmetric, and all the non-red terms are obviously symmetric by permutation of i and j . The three relations (10)-(11)-(12) allow the conclusion that the red terms are also symmetric. So, $F_{g,n}(i, j, K)$ is fully symmetric, and by induction this entails the claim. \square

3 Lie algebraic approach to Airy structures

3.1 Reformulation *via* the adjoint representation

In \mathcal{W}_V^h , we have two notions of degree: the \hbar -degree, and the variable degree assigning degree 1 to x_i and ∂_{x_i} . We denote $T_{\hbar}^*V = V^* \oplus V \cdot \hbar$ and consider the subspace $\mathcal{D}_V \subset \mathcal{W}_V^h$

$$\mathcal{D}_V = \bigoplus_{d=0}^2 \mathcal{D}_{V,d}, \quad \mathcal{D}_{V,0} = \mathbb{C} \cdot \hbar, \quad \mathcal{D}_{V,1} = T_{\hbar}^*V, \quad \mathcal{D}_{V,2} = \text{Sym}^2 T_{\hbar}^*V.$$

Note that the copy of V^* in \mathcal{D}_V contains the linear functions x_i , while the copy of V correspond to differential operators ∂_i . \mathcal{D}_V is naturally a sub-Lie algebra of \mathcal{W}_V^h . Let π_1 and π_1^* be the linear projections to the subspaces V^* and $V \cdot \hbar$ of $\mathcal{D}_{V,1}$. We are interested in linear maps $V \rightarrow \mathcal{D}_V$, which we parametrize – after a choice of basis $(e_i)_{i \in I}$ of V – in the form

$$L_i = M_a^i \hbar \partial_a + N_a^i x_a - \left(\frac{1}{2} A_{a,b}^i x_a x_b + \hbar B_{a,b}^i x_a \partial_b + \frac{\hbar^2}{2} C_{a,b}^i \partial_a \partial_b + \hbar D^i \right). \quad (14)$$

Definition 3.1 *We say that a linear map $L : V \rightarrow \mathcal{D}_V$ has normal form – with respect to a choice of basis $(e_i)_{i \in I}$ of V – if $M_j^i = \delta_{i,j}$ and $N_j^i = 0$. If the basis is not specified, we mean that there exists a basis for which L has normal form.*

Definition 3.2 *A quasi-Airy structure on V is the data of a Lie algebra structure on V , together with a homomorphism of Lie algebras $L : V \rightarrow \mathcal{D}_V$. In this case, we denote \mathcal{Z}_L the space of solutions*

of

$$\forall v \in V, \quad L(v) \cdot Z = 0$$

of the form

$$Z = \exp\left(\sum_{g \geq 0, n \geq 1} \frac{\hbar^{g-1}}{n!} \mathcal{F}_{g,n}\right), \quad \mathcal{F}_{g,n} \in \text{Sym}^n V^*.$$

Note that the relations imposed by $[L_i, L_j] = f_{i,j}^a L_a$ are a priori different than those described in Section 2.2 due to the presence of M and $N \neq 0$.

Tautologically, quasi-Airy structures having normal form are quantum Airy structures.

Let $L : V \rightarrow \mathcal{D}_V$ be a quasi-Airy structure. In particular, V has the structure of a Lie algebra, and we can consider the adjoint representation $\text{ad}_L : V \rightarrow \text{End}(\mathcal{D}_V)$ defined by $\text{ad}_L(v) = \hbar^{-1}[L(v), \cdot]$. Computation with (14) shows that ad_L has a block decomposition with respect to $\mathcal{D}_V = \bigoplus_{d=0}^2 \mathcal{D}_{V,d}$

$$\text{ad}_L = \begin{pmatrix} 0 & \varrho_{0,1} & \varrho_{0,2} \\ 0 & \varrho_1 & \varrho_{1,2} \\ 0 & 0 & \varrho_2 \end{pmatrix}.$$

If we choose a basis of V , since L has the form (14) the central block has a further decomposition with respect to $\mathcal{D}_{V,1} = V^* \oplus V \cdot \hbar$ of the form

$$\varrho_1(e_i) = \begin{pmatrix} -B^i & A^i \\ -C^i & (B^i)^T \end{pmatrix}, \quad (15)$$

where $X^i = (X_{j,k}^i)_{j,k}$ is considered as a matrix for $X \in \{A, B, C\}$. Note that (15) is the general form of a matrix Y such that YS is symmetric, S being the symplectic transformation

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We note that ϱ_1 is a representation of V on T_{\hbar}^*V if and only if $[\varrho_1(e^i), \varrho_1(e^j)] = f_{i,j}^a \varrho_1(e^a)$. Further, with respect to the same decomposition of $\mathcal{D}_{V,1}$

$$\varrho_{0,1} = (M^i \quad N^i). \quad (16)$$

We also define $\tilde{\varrho}_1 : V \rightarrow \text{End}(\mathbb{C} \cdot \hbar \oplus T_{\hbar}^*V)$ by

$$\tilde{\varrho}_1 = \begin{pmatrix} 0 & \varrho_{0,1} \\ 0 & \varrho_1 \end{pmatrix}. \quad (17)$$

An easy computation shows

Lemma 3.1 *Let $L : V \rightarrow \mathcal{D}_V$ be a linear map of normal form (1). Then ϱ_1 defined by (15) is a representation of V on T_{\hbar}^*V if and only if the **BB-AC**, **BC** and **BA** relations are satisfied.*

Lemma 3.2 *Let $L : V \rightarrow \mathcal{D}_V$ be a linear map of normal form, such that ϱ_1 is a representation of V on T_{\hbar}^*V . The three properties below are equivalent.*

- (i) $\tilde{\varrho}_1$ is a representation of V on $\mathbb{C} \cdot \hbar \oplus T_{\hbar}^*V$.
- (ii) the two extra relations (8) and (9) are satisfied.

(iii) if $\mathcal{J} : V \rightarrow T^*_\hbar V$ is the natural inclusion, the representation ϱ_1 satisfies

$$\forall v_1, v_2 \in V, \quad \varrho_1(v_1).\mathcal{J}(v_2) - \varrho_1(v_2).\mathcal{J}(v_1) = \mathcal{J}([v_1, v_2]). \quad (18)$$

The only property which may be missing in Lemma 3.2 for L to be a quantum Airy structure is the **D** relation (13) – compare however with Lemma 2.3, (ii). Indeed the D^i contribute to the constant part of the operators, and thus are not seen at the level of the adjoint action. Condition (iii) is very similar to a *zero torsion condition* for connections in vector bundles, which is the reason why we adopted this name to refer to the linear relation (18).

Remark that, if $(L_i)_{i \in I}$ are operators of normal form with $A = C = 0$, ϱ_1 is a representation if and only if ϱ_1 acting on V is a representation – given by the matrices $(B^i)^T$. In particular, if $(f^k_{i,j})$ are the structure constants of a Lie algebra structure on V , the choice $B^i_{j,k} = \frac{1}{2}f^k_{i,j}$ makes (V, ϱ_1^V) (where ϱ_1^V means ϱ_1 acting on V) the adjoint representation, and (V, ϱ_1^{V*}) its dual. One then sees that the **BB-AC** relation is equivalent to the Jacobi relation that $f^k_{i,j}$ indeed satisfy. In this case, $(L_i)_{i \in I}$ is the well-known expression of the adjoint representation by differential operators. It is however not a quantum Airy structure because the first – and very important – term $\hbar \partial_i$ is missing. Or, equivalently, this representation ϱ_1 in general violates the zero torsion condition (18).

3.2 Classical Airy structures from symplectic representations

This paragraph is a digression to relate Section 3.1 to the notion of *classical Airy structure* introduced in [36], which is the Poisson analog of Definition 2.1. Let $\mathcal{P}_{V,d}$ be the space of polynomial functions on T^*V of degree d , and

$$\mathcal{P}_V = \bigoplus_{d=0}^2 \mathcal{P}_{V,d}$$

equipped with its canonical Poisson bracket. We denote $p_{V^*} : \mathcal{P}_V \rightarrow V^* \subset \mathcal{P}_{V,1}$ the natural projection.

Definition 3.3 *A classical Airy structure is the data of a linear map $\Lambda : V \rightarrow \mathcal{P}_{V,1} \oplus \mathcal{P}_{V,2}$ such that $I := p_{V^*} \circ \Lambda$ is an isomorphism, and such that $\text{Im } \Lambda$ is closed under Poisson bracket.*

If $(e_i)_{i \in I}$ is a basis of V , we denote $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ the corresponding linear coordinates on T^*V . By defining $\Lambda_i := I_{i,a}^{-1} \Lambda(e_a)$, a classical Airy structure is uniquely determined by a family $(\Lambda_i)_{i \in I}$ of elements of \mathcal{P}_V of the form

$$\Lambda_i = y_i - \frac{1}{2}A^i_{a,b}x_ax_b - B^i_{a,b}x_ay_b - \frac{1}{2}C^i_{a,b}y_ay_b,$$

such that

$$\{\Lambda_i, \Lambda_j\} = f^a_{i,j} \Lambda_a \quad (19)$$

for some scalars $f^k_{i,j}$. Note that (19) implies that $\{(x, y) \in T^*V : \forall i \Lambda_i(x, y) = 0\}$ is a Lagrangian subvariety in T^*V . This Lagrangian for non-zero (A, B, C) is a perturbation of the canonical Lagrangian given by the zero section $\{(x, y) \in T^*V : \forall i y_i = 0\} \subset T^*V$.

The data of such $(\Lambda_i)_{i \in I}$ is the definition adopted by [36] for classical Airy structure. Here, we prefer to adopt the basis-independent Definition 3.3. The analysis we did for quantum Airy structures in the realm of the Weyl algebra can be transposed directly for classical Airy structures in the Poisson realm.

Proposition 3.3 *The data (A, B, C) defines a classical Airy structure if and only if it satisfies **BB-AC**, **BC**, **BA** relations, as well as the zero torsion conditions (8)-(9). This is also equivalent (Lemma 3.2) to requiring that $\tilde{\varrho}_1$ is a representation of the Lie algebra V defined by the structure constants $f_{i,j}^k$ in the basis $(e_i)_{i \in I}$. \square*

In other words, quantum Airy structures are just classical Airy structures together with the data D satisfying the affine relation (13).

This allows a more geometric perspective on Airy structures, which we are going to explain again but in a slightly different way. The keypoint, which is also put forward and exploited in [36], is that suitable family of infinitesimal symplectomorphisms in T^*V give rise to classical Airy structures.

Let us fix a Lie algebra structure on V . We then consider the vector space $W = V \oplus V^*$ with the canonical symplectic structure ω . Let $\pi_V : W \rightarrow V$ be the projection along V^* . We denote $\mathfrak{sp}(W)$ the Lie algebra of infinitesimal symplectomorphisms of W . Combining Proposition 3.3 with Lemma 3.2 we find

Lemma 3.4 *There is a one to one correspondence between classical Airy structures, and Lie algebra homomorphisms $\mathcal{L} : V \rightarrow \mathfrak{sp}(T^*V)$ together with a Lagrangian embedding $\mathcal{J} : V \rightarrow T^*V$ such that*

$$\forall v_1, v_2 \in V, \quad \mathcal{L}_{v_1} \mathcal{J}(v_2) - \mathcal{L}_{v_2} \mathcal{J}(v_1) = \mathcal{J}([v_1, v_2]). \quad (20)$$

The coefficients of the classical Airy structure are given in terms of the symplectic form and the action on W of the linear symplectomorphisms determined by \mathcal{L} in (21) below.

The advantage of this formulation is that given a Lie algebra homomorphism of V into $\mathfrak{sp}(T^*V)$, for instance coming from geometry, then one only need to construct \mathcal{J} such that the *linear* condition (20) is satisfied. We also remark that the Lie algebra structure on V is completely specified by the \mathcal{L} via (20).

Proof. First we will assume that we are given a Lie algebra homomorphism $\mathcal{L} : V \rightarrow \mathfrak{sp}(T^*V)$ as above, and from this we will construct a classical Airy structure. We pick a basis $(e_i)_{i \in I}$ of V . We let e_i^* be the dual basis of e_i , then $((0, e_i^*), (e_i, 0))_{i \in I}$ is a symplectic basis of $T^*V = W$, *i.e.*

$$\forall i, j, \quad \omega(e_i, e_j) = \omega(e_i^*, e_j^*) = 0, \quad \omega(e_i, e_j^*) = \delta_{i,j}.$$

We let $f_{i,j}^k$ be the structure constants of V in the basis e_i of V . One then has the decomposition

$$\forall w \in W, \quad w = -x_a e_a + y_a e_a^*, \quad x_i = \omega(e_i^*, w), \quad y_i = \omega(e_i, w).$$

Denoting $\mathcal{L}_i := \mathcal{L}_{e_i}$, we have by definition of a symplectic representation

$$\forall (w_1, w_2) \in W^2, \quad \omega(\mathcal{L}_i w_1, w_2) + \omega(w_1, \mathcal{L}_i w_2) = 0.$$

Hence \mathcal{L}_i is represented in the symplectic basis of $T^*V \cong V^* \oplus V$ by the matrix

$$H_i = \begin{pmatrix} -B^i & A^i \\ C^i & (B^i)^T \end{pmatrix}$$

where A^i and C^i are symmetric matrices with

$$A_{j,k}^i = -\omega(\mathcal{L}_i e_j, e_k), \quad B_{j,k}^i = \omega(\mathcal{L}_i e_j, e_k^*), \quad C_{j,k}^i = -\omega(\mathcal{L}_i e_j^*, e_k^*). \quad (21)$$

With this choice of (A, B, C) , one defines the hamiltonians

$$h_i(w) := \frac{1}{2}\omega(\mathcal{L}_i w, w) = -\left(\frac{1}{2}A_{a,b}^i x_a x_b + B_{a,b}^i x_a y_b + \frac{1}{2}C_{a,b}^i y_a y_b\right)$$

for $w = -x_a e_a + y_a e_a^*$. Then, we can define $\Lambda : e_i \rightarrow (w \mapsto \omega(e_i, w) + h_i(w))$, and we claim this is a classical Airy structure. Indeed, one can check using the commutation relations for \mathcal{L} that $\{w \in W : \forall i \ \omega(e_i, w) + h_i(w) = 0\}$ is a Lagrangian subvariety of T^*V .

Conversely, a classical Airy structure gives the desired representation by just using (21) to define \mathcal{L}_i in terms of (A, B, C) . \square

4 Moduli spaces of Airy structures

4.1 Group action

The affine extended symplectic group $\mathcal{G}_V := \exp(\mathcal{D}_V/\hbar)$ acts by conjugation on its Lie algebra \mathcal{D}_V , hence inducing an action on the set of quasi-Airy structures, as well as on the partition function Z

$$\mathcal{U} \in \mathcal{G}_V, \quad \tilde{L}_i = \mathcal{U} L_i \mathcal{U}^{-1}, \quad \tilde{Z} = \mathcal{U} \cdot Z.$$

It contains – and is generated by – the Heisenberg subgroup $\exp((\mathcal{D}_{V,0} + \mathcal{D}_{V,1})/\hbar)$, the metaplectic group $\exp(\mathcal{D}_{V,2}/\hbar)$. This perspective makes it clear that we ought to study (quasi-)Airy structures only up to the action of \mathcal{G}_V .

Computations show that the subgroup of \mathcal{G}_V which preserves the normal form (1) of quantum Airy structures only consists of multiplication by scalars (which are central and do not change the L_i), renormalization $(A, B, C, D) \rightarrow (\lambda^3 A, \lambda B, \lambda^{-1} C, \lambda D)$ for $\lambda \in \mathbb{C}^*$ (which can be realized via $\mathcal{U} = \lambda^{x_a \partial_a}$) and differential operators of order two

$$\mathcal{U} = \exp\left(\frac{\hbar}{2} u_{a,b} \partial_a \partial_b\right), \quad u_{i,j} = u_{j,i} \in \mathbb{C}. \quad (22)$$

If (A, B, C, D) are the coefficients of a quantum Airy structure, the new quantum Airy structure obtained by the action of (22) has coefficients $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ given by

$$\begin{aligned} \tilde{A}_{j,k}^i &= A_{j,k}^i. \\ \tilde{B}_{j,k}^i &= B_{j,k}^i + u_{j,a} A_{a,k}^i. \\ \tilde{C}_{j,k}^i &= C_{j,k}^i + u_{j,a} B_{a,k}^i + u_{k,a} B_{j,a}^i + u_{j,a} u_{k,b} A_{a,b}^i. \\ \tilde{D}^i &= D^i + \frac{1}{2} u_{a,b} A_{a,b}^i. \end{aligned} \quad (23)$$

We have just proved

Corollary 4.1 *If $(u_{i,j})_{i,j \in I}$ is a symmetric matrix, (23) is a symmetry of the relations of Section 2.2.*

This can also be checked directly by inserting (23) in the relations. At the level of the partition functions, if $u = (u_{i,j})_{i,j}$ is invertible, Wick's theorem shows that the action of (22) can be realized by a formal Gaussian convolution

$$\exp\left(\frac{\hbar}{2} u_{a,b} \partial_a \partial_b\right) \cdot Z(x) = \int_V \frac{d\xi}{\det(2\pi\hbar \text{id})^{1/2}} \exp\left(-\frac{(u^{-1})_{a,b} \xi_a \xi_b}{2\hbar}\right) Z(x + \xi),$$

where $d\xi = \prod_{i \in I} d\xi_i$ is the Lebesgue measure on V .

Remark 4.1 Another easy transformation of the L_i is the rescaling of \hbar acting as $L_i \rightarrow z^{-1} \exp(z\hbar\partial_\hbar)L_i$. It transforms (A, B, C, D) into $(z^{-1}A, B, zC, D)$. We prefer not to include it in \mathcal{G}_V .

Lemma 4.2 If L is a quasi-Airy structure, its \mathcal{G}_V -orbit contains a quantum Airy structure if and only if $(\pi_1 \oplus \pi_1^*) \circ L : V \rightarrow T_\hbar^*V$ is a (linear) Lagrangian embedding of V .

Proof. If $L = (L_i)_{i \in I}$ is a quantum Airy structure, then $\pi_1^* \circ L$ is the isomorphism between V and V^* induced by the choice of a basis $(e_i)_{i \in I}$. In particular, $(\pi_1 \oplus \pi_1^*) \circ L$ is a Lagrangian embedding of V into T_\hbar^*V . These properties remain true for \tilde{L} in the \mathcal{G}_V -orbit of L . Conversely, let L be a quasi-Airy structure such that $(\pi_1 \oplus \pi_1^*) \circ L : V \rightarrow T_\hbar^*V$ is a Lagrangian embedding. We can always compose it with a linear symplectomorphism of T_\hbar^*V bringing this Lagrangian to V^* . It is well-known that \mathcal{G}_V contains elements which can realize as automorphisms of the Weyl algebra the linear symplectomorphisms

$$\tilde{x}_i = \alpha_{i,a}x_a + \beta_{i,a}\hbar\partial_a, \quad \hbar\tilde{\partial}_i = \gamma_{i,a}x_a + \varepsilon_{i,a}\hbar\partial_a$$

for arbitrary matrices $(\alpha, \beta, \gamma, \varepsilon)$ satisfying the symplectic conditions

$$\begin{cases} \alpha\beta^T \text{ symmetric,} \\ (\gamma\varepsilon^T) \text{ symmetric,} \\ \alpha\varepsilon^T - \beta\gamma^T = \text{id}_V. \end{cases}$$

Therefore, we can find \tilde{L} in the \mathcal{G}_V -orbit of L such that $\pi_1 \circ \tilde{L} = 0$ and $\pi_1^* \circ \tilde{L}$ is an isomorphism. In other words, in a given basis, \tilde{L}_i has the form (14) with $\tilde{N}_j^i = 0$ and $M = (\tilde{M}_j^i)_{i,j}$ invertible. So, putting $\check{L}_i = (M^{-1})_{i,a}\tilde{L}_a$ gives an operator in normal form, *i.e.* a quantum Airy structure. \square

Corollary 4.3 A quasi-Airy structure $(L_i)_{i \in I}$ has a quantum Airy structure in its \mathcal{G}_V -orbit if and only if ϱ_1 is a representation of the Lie algebra V into T_\hbar^*V , and $\tilde{\varrho}_1$ is a one-dimensional extension of this representation such that $v \mapsto (v, \varrho_{0,1}(v))$ is a Lagrangian embedding of V into T_\hbar^*V .

Proof. Comparing Lemmas 3.1-3.2 with the relations found in Section 2.2 shows that $(L_i)_{i \in I}$ of normal form is a quantum Airy structure if and only if ϱ_1 is a representation of V in T_\hbar^*V and $\tilde{\varrho}_1$ is a one-dimensional extension of this representation such that $\varrho_{0,1} = (\psi_{V^*}, 0_V)$ where $\psi_{V^*} : V \rightarrow V^*$ is the isomorphism determined by the choice of basis in which $(L_i)_{i \in I}$ is defined. In this case, $\tilde{\varrho}_1$ determines an exact sequence of V -modules:

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}\cdot\hbar \oplus T_\hbar^*V \longrightarrow (\mathbb{C}\cdot\hbar \oplus T_\hbar^*V)/\mathbb{C} \longrightarrow 0,$$

If $L_i \in \mathcal{D}$ does not have normal form, the block $\varrho_{0,1}$ nevertheless gives the map $(\pi_1^* \oplus \pi_1) \circ L$, and the general claim is a consequence of Lemma 4.2. \square

4.2 Definition of moduli spaces

We shall now introduce various moduli spaces associated to Airy structures.

Let V be a (finite-dimensional) Lie algebra. We denote $\mathfrak{A}_V^{\text{cl}}$ (resp. \mathfrak{A}_V , $\mathfrak{A}_V^{\text{q}}$) the set of classical (resp quasi-, quantum) Airy structures based on the Lie algebra V . Of course, each of them is a subset of the set of all Airy structures where we also vary the Lie algebra structure on the vector space V . However, we restrict here to study the set of Airy structures based on a fixed Lie algebra.

As a subset of $\mathbb{A}_V := (\text{Sym}^3 V^*) \times (V^* \otimes V^* \otimes V) \times (\text{Sym}^2 V^* \otimes V)$ cut out by the (finitely many) quadratic **BB-AC**, **BC** and **BA** relations and the linear relation (8) (we have included the relation (9) in the definition of \mathbb{A}_V), $\mathfrak{A}_V^{\text{cl}}$ naturally has the structure of an affine algebraic variety. Likewise, \mathfrak{A}_V is an affine algebraic variety. $\mathfrak{A}_V^{\text{q}}$ is obviously a subvariety of \mathfrak{A}_V . It can also be seen as a subvariety of $V^* \times \mathbb{A}_V$ cut out by the extra **D** relation, where D^i are the coordinates in the first factor. In fact, as the **D** relation is affine, it arises as the total space of an affine subbundle $\pi : \mathfrak{A}_V^{\text{q}} \rightarrow \mathfrak{A}_V^{\text{cl}}$ of the trivial vector bundle $V^* \times \mathbb{A}_V \rightarrow \mathbb{A}_V$ restricted to $\mathfrak{A}_V^{\text{cl}}$. According to Lemma 2.3, π has a section given by

$$D_{\text{ref}}^i := \frac{1}{2} \text{Tr } B^i.$$

Mapping D to $(D - D_{\text{ref}})$ turns $\mathfrak{A}_V^{\text{q}}$ into a trivial vector bundle over $\mathfrak{A}_V^{\text{cl}}$, with fiber

$$(V')^\perp = \{\varphi \in V^* : \forall x, y \in V, \varphi([x, y]) = 0\}.$$

As we saw in Section 4.1, \mathcal{G}_V acts algebraically on $\mathfrak{A}_V^{\text{cl}}$ and \mathfrak{A}_V , and its algebraic subgroup $\tilde{\mathcal{G}}_V$ preserving Airy structures acts on $\mathfrak{A}_V^{\text{q}}$. Since we are interested in Airy structures up to the action of \mathcal{G}_V , the appropriate algebraic way to proceed would be to consider the quotient stack or the GIT quotients $\mathfrak{A}_V^{\text{cl}}//\mathcal{G}_V$ and $\mathfrak{A}_V//\mathcal{G}_V$. In this paper we shall just consider the set-theoretic quotients $\mathfrak{M}_V^{\text{cl}} = \mathfrak{A}_V^{\text{cl}}/\mathcal{G}_V$ and $\mathfrak{M}_V = \mathfrak{A}_V/\mathcal{G}_V$ with the induced topology from $\mathfrak{A}_V^{\text{cl}}$ and \mathfrak{A}_V respectively, and present some preliminary remarks about this quotient space, which we will call the moduli space of quasi-Airy structures. We caution the reader that this quotient space will in general not even be Hausdorff. The same comment applies to the moduli space of quantum Airy structures $\mathfrak{M}_V^{\text{q}} = \mathfrak{A}_V^{\text{q}}/\tilde{\mathcal{G}}_V$. The fibration $\pi : \mathfrak{A}_V^{\text{q}} \rightarrow \mathfrak{A}_V^{\text{cl}}$ is $\tilde{\mathcal{G}}_V$ -equivariant, therefore we have a natural fibration $\mathfrak{M}_V^{\text{q}} \rightarrow \mathfrak{M}_V$. The description of its fibers can in principle be obtained by looking at the action of \mathcal{G}_V on D via (23).

4.3 Deformation theory

As the moduli space of classical Airy structures just consists of Lie algebra homomorphisms modulo inner automorphisms of the target Lie algebra \mathcal{D}_V , one can use the theory of deformation of algebraic structures to study their moduli space. The deformations of the Lie algebra homomorphism $L : V \rightarrow \mathcal{D}_V$ are governed by the differential graded algebra $E_L^\bullet := \text{Hom}(\Lambda^\bullet V, \mathcal{D}_V)$, which is equipped with the Cartan-Eilenberg differential

$$(d_{E_L^\bullet} \varphi)(v_1 \wedge \cdots \wedge v_{n+1}) := \sum_{\ell=1}^{n+1} (-1)^{\ell+1} \varphi(v_1 \wedge \cdots \wedge \widehat{v}_\ell \wedge \cdots \wedge v_{n+1}) L(v_\ell) + \sum_{\ell < m} (-1)^{\ell+m} \varphi([v_\ell, v_m]_V \wedge v_1 \wedge \cdots \wedge \widehat{v}_\ell \wedge \cdots \wedge \widehat{v}_m \wedge \cdots \wedge v_{n+1})$$

for $\varphi \in E_L^n$ and the bracket

$$[\varphi, \psi]_{E_L^\bullet}(v_1 \wedge \cdots \wedge v_{n+m}) := \sum_{\substack{\sigma \in \mathfrak{S}_{n+m} \\ \sigma(1) < \cdots < \sigma(m) \\ \sigma(m+1) < \cdots < \sigma(m+n)}} \varepsilon(\sigma) [\varphi(v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(m)}), \psi(v_{\sigma(m+1)} \wedge \cdots \wedge v_{\sigma(m+n)})]_{\mathcal{D}}$$

for $\varphi \in E_L^m$ and $\psi \in E_L^n$ where $v_1 \wedge \cdots \wedge \widehat{v}_\ell \wedge \cdots \wedge v_n$ stands for $v_1 \wedge \cdots \wedge v_n$ with v_ℓ removed and $[\cdot, \cdot]_X$ stands for the bracket in X . We denote $H_L^i(V, \mathcal{D}_V)$ the cohomology of this complex, and $Z_L^n(V, \mathcal{D}_V)$ the space of n -cocycles. In particular we get a quadratic map

$$\Omega : H_L^1(V, \mathcal{D}_V) \rightarrow H_L^2(V, \mathcal{D}_V),$$

given by

$$\Omega(\varphi) = [\varphi, \varphi]_{E_L^\bullet}.$$

We get the following proposition as a direct consequence of the results of [41].

Proposition 4.4 *Let $L : V \rightarrow \mathcal{D}_V$ be a classical Airy structure. If $H_L^1(V, \mathcal{D}_V) = 0$, then L is rigid, i.e. all continuous deformations of L remain in its \mathcal{G}_V -orbit. In general there exists an open neighbourhood U of zero in $H_L^1(V, \mathcal{D}_V)$ and an open neighbourhood \mathfrak{U} of $[L] \in \mathfrak{M}_V^{\text{cl}}$ such that*

$$\mathfrak{U} \cong (U \cap \Omega^{-1}(0)) / \mathcal{G}_V(L)$$

where $\mathcal{G}_V(L)$ is the stabiliser of L in \mathcal{G}_V .

Corollary 4.5 *If we have that $H_L^2(V, \mathcal{D}) = 0$ and that the action of $\mathcal{G}_V(L)$ on $H_L^1(V, \mathcal{D})$ factors through a finite group, then M will have the structure of an orbifold near $[L] \in \mathfrak{M}_V^{\text{cl}}$ and*

$$T_{[L]} \mathfrak{M}_V^{\text{cl}} \cong H_L^1(V, \mathcal{D}_V).$$

We have seen by explicit computations that $\tilde{\rho}_1 : V \rightarrow \text{End}(\mathbb{C}\hbar \oplus T_{\hbar}^*V)$ is a Lie algebra homomorphism if and only if $\rho : V \rightarrow \mathcal{D}_V$ is a Lie algebra homomorphism. So, we can in fact replace in Proposition 4.4 and Corollary 4.5 the module \mathcal{D}_V by the module $\mathbb{C}\hbar \oplus T_{\hbar}^*V$. However, if we only used the module T_{\hbar}^*V , we would miss the constraints imposed the zero torsion condition (18).

4.4 Translations

So far, the symmetries we have described act linearly on the coefficients of quantum Airy structures. Among them, translations $x_i \rightarrow x_i + t_i$ transform a quasi-Airy structure into operators $(\tilde{L}_i)_{i \in I}$ such that $(\tilde{L}^i - P^i)_{i \in I}$ is a quasi-Airy structure, for some constants P^i . The solution of $\tilde{L}_i \cdot \tilde{Z} = 0$ for all i is $Z(t+x)$, which is the Taylor expansion of $Z(\xi)$ around the point $\xi = t$. If we write $Z(x) = \exp(\sum_{g \geq 0} \hbar^{g-1} S_g(x))$, and assume momentarily that S_g has a non-zero radius of convergence uniformly in g ,

$$Z(t+x) = \exp\left(\sum_{g \geq 0} \sum_{n \geq 1} \sum_{i_1, \dots, i_n} \frac{\hbar^{g-1}}{n!} F_{g,n}^{(t)}(i_1, \dots, i_n) x_{i_1} \cdots x_{i_n}\right),$$

where now $F_{0,1}^{(t)}(i) = \partial_i S_0(t)$ and $F_{0,2}^{(t)}(i, j) = \partial_i \partial_j S_0(t)$ are a priori now zero. It is natural to suspect that

$$\tilde{Z}(x) := \exp\left(-\frac{F_{0,1}^{(t)}(a)x_a}{\hbar} - \frac{F_{0,2}^{(t)}x_a x_b}{2\hbar}\right) Z(x+t)$$

is the partition function of a new, t -dependent quantum Airy structure. The next theorem will confirm and make sense of this, using formal series in t .

We introduce the graded vector space $\hat{K} = \mathbb{K}[[t_i]_{i \in I}]$, by assigning degree 1 to each t_i . We get the decomposition into homogeneous pieces

$$\hat{K} = \prod_{n \geq 0} \hat{K}^{[n]}.$$

Let (A, B, C, D) be a quantum Airy structure. We first describe a formal replacement for $\partial_i S_0(t)$ and $\partial_i \partial_j S_0(t)$. These are elements of \hat{K} whose homogeneous components are inductively defined

$$G_{0,1}(i) = \sum_{n \geq 2} G_{0,1}^{[n]}(i), \quad G_{0,2}(i, j) = \sum_{n \geq 1} G_{0,2}^{[n]}(i, j),$$

with

$$G_{0,1}^{[2]}(i) = \frac{1}{2} A_{a,b}^i t_a t_b, \quad (24)$$

$$\forall n \geq 3, \quad G_{0,1}^{[n]}(i) = B_{a,b}^i t_a G_{0,1}^{[n-1]}(b) + \frac{1}{2} C_{a,b}^i \sum_{\substack{n_1+n_2=n \\ n_1, n_2 \geq 2}} G_{0,1}^{[n_1]}(a) G_{0,1}^{[n_2]}(b), \quad (25)$$

and

$$G_{0,2}^{[1]}(i, j) = A_{j,a}^i t_a, \quad (26)$$

$$\forall n \geq 2, \quad G_{0,2}^{[n]}(i, j) = B_{a,b}^i t_a G_{0,2}^{[n-1]}(b, j) + B_{j,a}^i G_{0,1}^{[n]}(a) + C_{a,b}^i \sum_{\substack{n_1+n_2=n \\ n_1 \geq 2, n_2 \geq 1}} G_{0,1}^{[n_1]}(a) G_{0,2}^{[n_2]}(b, j). \quad (27)$$

Then, we define $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with coefficients in \hat{K} , again inductively by their homogeneous components. The initial conditions are

$$\tilde{X}^{[0]} = X, \quad X \in \{A, B, C, D\} \quad (28)$$

and the recursions read for $n \geq 1$

$$(\tilde{A}^{[n]})_{j,k}^i = B_{a,b}^i (A^{[n-1]})_{j,k}^b t_a + B_{j,a}^i G_{0,2}^{[n]}(a, k) + B_{k,a}^i G_{0,2}^{[n]}(a, j) \quad (29)$$

$$+ C_{a,b}^i \left(\sum_{\substack{n_1+n_2=n \\ n_1 \geq 2, n_2 \geq 0}} G_{0,1}^{[n_1]}(a) (A^{[n_2]})_{j,k}^b + \sum_{\substack{n_1+n_2=n \\ n_1, n_2 \geq 1}} G_{0,2}^{[n_1]}(a, j) G_{0,2}^{[n_2]}(b, k) \right). \quad (30)$$

$$(\tilde{B}^{[n]})_{j,k}^i = B_{a,b}^i (B^{[n-1]})_{j,k}^b t_a + C_{k,a}^i G_{0,2}^{[n]}(a, j) + \sum_{\substack{n_1+n_2=n \\ n_1 \geq 2, n_2 \geq 0}} C_{a,b}^i G_{0,1}^{[n_1]}(a) (B^{[n_2]})_{j,k}^b. \quad (31)$$

$$(\tilde{C}^{[n]})_{j,k}^i = B_{a,b}^i (C^{[n-1]})_{j,k}^b t_a + \sum_{\substack{n_1+n_2=n \\ n_1 \geq 2, n_2 \geq 0}} C_{a,b}^i G_{0,1}^{[n_1]}(a) (C^{[n_2]})_{j,k}^b. \quad (32)$$

$$(\tilde{D}^{[n]})^i = B_{a,b}^i (D^{[n-1]})^b t_a + C_{a,b}^i \sum_{\substack{n_1+n_2=n \\ n_1 \geq 2, n_2 \geq 0}} G_{0,1}^{[n_1]}(a) (D^{[n_2]})^b. \quad (33)$$

Proposition 4.6 *If (A, B, C, D) is a quantum Airy structure whose partition function has Taylor coefficients $F_{g,n}(i_1, \dots, i_n)$, then $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is a quantum Airy structure with coefficients in \hat{K} whose partition function has Taylor coefficients*

$$\tilde{F}_{g,n}(i_1, \dots, i_n) = \sum_{m \geq 0} \frac{1}{m!} \sum_{j_1, \dots, j_m} F_{g,n}(i_1, \dots, i_n, j_1, \dots, j_m) t_{j_1, \dots, j_m} \in \hat{K}. \quad (34)$$

The above formulas form a non-linear⁵, infinitesimal symmetry of quantum Airy structures. In the convergent case, this symmetry is the one expected.

Lemma 4.7 *Assume V finite-dimensional. Let $Z = \exp(\sum_{g \geq 0} \hbar^{g-1} S_g(x))$ be the partition function of a quantum Airy structure, where $S_g \in \mathbb{K}[[x_i]_{i \in I}]$. If S_0 has positive radius of convergence, then for any $g \geq 0$, S_g has a radius of convergence bounded from below by a positive constant independent of g . We denote \underline{S}_g the analytic function defined by those series at least in a neighborhood of 0 in V . The formal series defining $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, G_{0,1}, G_{0,2})$ have positive radius of convergence, and we also use underlined letters to denote the analytic functions of t they define. Then*

$$\underline{G}_{0,1}(t; i) = \partial_i \underline{S}_0(t), \quad \underline{G}_{0,2}(t; i, j) = \partial_i \partial_j \underline{S}_0(t), \quad \underline{A}_{j,k}^i(t) = \partial_i \partial_j \partial_k \underline{S}_0(t), \quad \underline{D}_{j,k}^i(t) = \partial_i \underline{S}_1(t).$$

Further, for x and t in a neighborhood of 0,

$$\tilde{\underline{S}}_g(t; x) = \underline{S}_g(x + t).$$

⁵We mean that \tilde{X} does not depend linearly of X .

Proof. A computation shows that the translation $x \rightarrow x + t$, followed by conjugation by

$$\mathcal{U} = \exp\left(\hbar^{-1}\left(G_{0,1}(a)x_a + \frac{1}{2}G_{0,2}(a,b)x_ax_b\right)\right)$$

transforms the quantum Airy structure (A, B, C, D) into the quasi-Airy structure $(\hat{L}_i - \hat{P}^i)_{i \in I}$ with coefficients (see (14) for notations, we replaced here M_j^i by $M_{i,j}$ and N_j^i by $N_{i,j}$ for convenience)

$$\begin{aligned} \hat{M}_{i,j} &= \delta_{i,j} - B_{j,a}^i t_a - C_{j,a}^i G_{0,1}(a). \\ \hat{N}_{i,j} &= -A_{j,a}^i t_a + (\delta_{i,b} - B_{a,b}^i t_a - C_{a,b}^i G_{0,1}(a))G_{0,2}(b,j) - B_{j,a}^i G_{0,1}(a). \\ \hat{P}^i &= -\frac{1}{2}A_{a,b}^i t_a t_b - \frac{1}{2}C_{a,b}^i G_{0,1}(a)G_{0,1}(b) + (\delta_{i,b} - B_{a,b}^i t_a)G_{0,1}(b). \\ \hat{A}_{j,k}^i &= A_{j,k}^i + B_{j,a}^i G_{0,2}(a,k) + B_{k,a}^i G_{0,2}(a,j) + C_{a,b}^i G_{0,2}(a,j)G_{0,2}(b,k). \\ \hat{B}_{j,k}^i &= B_{j,k}^i + C_{k,a}^i G_{0,2}(a,j). \\ \hat{C}_{j,k}^i &= C_{j,k}^i. \\ \hat{D}^i &= D^i. \end{aligned}$$

We indeed remark that the operations of translation and conjugation by the exponential of a quadratic form preserve the Lie commutation relations, so $(\hat{L}_i - \hat{P}^i)_{i \in I}$ is indeed a quasi-Airy structure, with same structure constants. This determines a quantum Airy structure provided \hat{M} is an invertible matrix, and provided one can choose $\hat{N}_{i,j} = \hat{P}^i = 0$ for all i, j . In this case, the quantum Airy structure is $\tilde{L}_i = (\hat{M}^{-1})_{i,a} \hat{L}_a$, *i.e.* its coefficients are $\tilde{X}^i = (\hat{M}^{-1})_{i,a} \hat{X}^a$ for $X \in \{A_{j,k}, B_{j,k}, C_{j,k}, D\}$.

We can indeed solve the equation $\hat{P}^i = 0$ by choosing $G_{0,1}(i)$ as in (24)-(25). Then, $G_{0,2}(i,j)$ is obtained by solving perturbatively $\hat{N}_{i,j} = 0$, leading to (26)-(27). Inserting these series in the expression of the coefficients of \tilde{L}_i

$$\tilde{X}^i = \hat{M}_{i,a}^{-1} \hat{X}^a, \quad X \in \{A_{j,k}, B_{j,k}, C_{j,k}, D\}$$

leads to formulas (28) and (29)-(33). The partition function for this new quantum Airy structure is

$$\tilde{Z}(x) = \exp\left(-\frac{G_{0,1}(a)x_a}{\hbar} - \frac{G_{0,2}(a,b)x_ax_b}{2\hbar}\right)Z(x+t)$$

and by consistency, we deduce that

$$G_{0,1}(i) = \partial_i S_0(t), \quad G_{0,2}(i,j) = \partial_i \partial_j S_0(t), \quad \tilde{A}_{j,k}^i = \partial_i \partial_j \partial_k S_0(t), \quad \tilde{D}^i = \partial_i S_1(t) \quad (35)$$

and the Taylor coefficients of \tilde{Z} are given by (34), both in the sense of formal series in $(t_i)_{i \in I}$.

Now assume that V is finite-dimensional and $S_0(t)$ has a non-zero radius of convergence. The equation $L_i \cdot Z = 0$ implies for $g \geq 1$

$$\hat{M}_{i,a}(t) \partial_a S_g(t) = \delta_{g,1} D^i + \frac{1}{2} C_{a,b}^i \left(\partial_a \partial_b S_{g-1}(t) + \sum_{\substack{g_1+g_2=g \\ g_1, g_2 \geq 1}} \partial_a S_{g_1}(t) \partial_b S_{g_2}(t) \right).$$

We recall that $\hat{M}_{i,a}(t) = \delta_{i,a} - B_{a,b}^i t_b - C_{a,b}^i \partial_b S_0(t)$. As $S_0(t) = O(t)$, we have $\hat{M}(t) = \text{Id} + O(t)$. Hence, the (finite-dimensional) matrix $\hat{M}(t)$ is invertible for t small enough. So, we can prove by induction on $g \geq 1$ that $S_g(t)$, as a solution of the (compatible) system of linear ODEs with analytic coefficients in a neighborhood of $t = 0$,

$$\partial_i S_g(t) = [\hat{M}^{-1}(t)]_{i,c} \left\{ \delta_{g,1} D^c + \frac{1}{2} C_{a,b}^c \left(\partial_a \partial_b S_{g-1}(t) + \sum_{\substack{g_1+g_2=g \\ g_1, g_2 \geq 1}} \partial_a S_{g_1}(t) \partial_b S_{g_2}(t) \right) \right\}$$

is the formal Taylor series at 0 of an analytic function $\underline{S}_g(t)$ in the neighborhood Ω of 0 on which $\hat{M}(t)$ is invertible and $S_0(t)$ is analytic. This contains a neighborhood of 0 independent of g .

Independently, as $S_0(t)$ is analytic in a neighborhood of 0, the equality

$$G_{0,1}(i) = \partial_i S_0(t) \quad (36)$$

provides a definition of $\underline{G}_{0,1}$ as the analytic function $\partial_i \underline{S}_0(t)$, whose Taylor series at 0 is $G_{0,1}(i)$, hence such that (36) holds at the level of analytic functions. Then, the expression for the formal series $G_{0,2}(i, j)$ obtained by enforcing $N_{i,j} = 0$ above shows that it is the formal Taylor series at 0 of an analytic function $\underline{G}_{0,2}(i, j)$ for t in Ω . And the expression of $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ in terms of $S_0(t)$ and its first and second order derivatives shows they upgrade in the same way to analytic functions of $t \in \Omega$, in such a way that the equality between formal series at 0 continue to hold at the level of analytic functions of $t \in \Omega$. \square

5 Formulas for the partition function in two simple cases

5.1 The case $C = 0$

Quantum Airy structures with $C = 0$ give rise to a compatible system of linear ODEs for the partition function

$$\forall i \quad \hbar(\delta_{i,b} - B_{a,b}^i x_a) \partial_{x_b} \ln Z = \frac{1}{2} A_{a,b}^i x_a x_b + \hbar D^i.$$

The partition function in this case can be computed in exact form. Let us introduce matrices $\mathbf{B}_j = (B_{j,b}^a)_{a,b}$ and column vectors $\mathbf{A}_{j,k} = (A_{j,k}^a)_a$ and $\mathbf{D} = (D^a)_a$, as well as the formal power series

$$\begin{aligned} \psi_0(z) &:= \frac{1}{2z^3} \left(-\ln(1-z) - z - \frac{z^2}{2} \right) = \frac{1}{6} + O(z), \\ \psi_1(z) &:= -\frac{1}{z} \ln(1-z) = 1 + O(z), \end{aligned}$$

which we will apply to matrices.

Proposition 5.1 *The partition function of a quantum Airy structure having $C = 0$ reads*

$$Z = \exp \left(\hbar^{-1} S_0(x) + S_1(x) \right),$$

where

$$S_0(x) = \left[\psi_0 \left(\sum_j x_j \mathbf{B}_j \right) \cdot \mathbf{A}_{a,b} \right]_c x_a x_b x_c, \quad S_1(x) = \left[\psi_1 \left(\sum_j x_j \mathbf{B}_j \right) \cdot \mathbf{D} \right]_a x_a,$$

and \cdot is the multiplication between a matrix and a column vector.

Proof. We use the expression of the Taylor coefficients $F_{g,n}(i_1, \dots, i_n)$ of the partition function as sums over trivalent graphs. Since $C = 0$, all trivalent vertices should be incident to one leaf (a B), two leaves (an A), or a loop (a D). This drastically simplifies the structure of the graphs which can contribute to the sum, in particular their genus is 0 and 1. Thus $F_{g,n} = 0$ for $g \geq 2$.

In genus 0 (Figure 5), the graphs are characterized by the sequence $\sigma(1), \dots, \sigma(n-3)$ of leaves successively attached when moving away from the root, and the pair $\{m, m'\}$ of leaves terminating the graph. Thus:

$$\begin{aligned} F_{0,n}(i_1, \dots, i_n) &= \sum_{\sigma} \sum_{m < m'} B_{i_{\sigma(1)}, a_1}^{i_1} B_{i_{\sigma(2)}, a_2}^{a_2} \cdots B_{i_{\sigma(n-3)}, a_{n-3}}^{a_{n-3}} A_{i_m, i_{m'}}^{a_{n-3}} \\ &= \sum_{\sigma} \sum_{m < m'} \left[\mathbf{B}_{i_{\sigma(1)}} \mathbf{B}_{i_{\sigma(2)}} \cdots \mathbf{B}_{i_{\sigma(n-3)}} \cdot \mathbf{A}_{i_m, i_{m'}} \right]_{i_1} \end{aligned}$$

where the first sum ranges over bijections $\sigma : \{1, \dots, n-3\} \rightarrow \{2, \dots, n\} \setminus \{m, m'\}$ and we recall that the indices $a_1, a_2, \dots, a, b, c, \dots$ are implicitly summed over. Then

$$\sum_{i_1, \dots, i_n} F_{0,n}(i_1, \dots, i_n) \frac{x_{i_1} \cdots x_{i_n}}{n!} = \frac{1}{n!} \sum_{\sigma} \sum_{m < m'} \left[(x_d \mathbf{B}_d)^{n-3} \cdot \mathbf{A}_{a,b} \right]_c x_a x_b x_c.$$

The set of $(\sigma, \{m, m'\})$ appearing in this sum is in bijection with permutations of $\{2, \dots, n\}$, thus

$$\sum_{i_1, \dots, i_n} F_{0,n}(i_1, \dots, i_n) \frac{x_{i_1} \cdots x_{i_n}}{n!} = \frac{1}{n} \left[(x_d \mathbf{B}_d)^{n-3} \cdot \mathbf{A}_{a,b} \right]_c x_a x_b x_c,$$

and summing over $n \geq 3$ gives the announced expression.

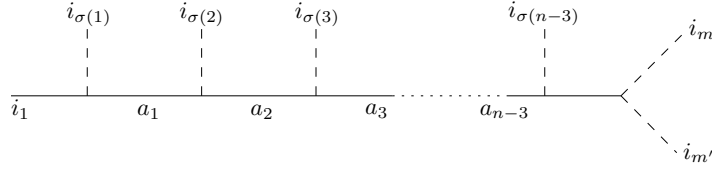


Figure 5: Genus 0 graphs without inner trivalent vertex.

In genus 1 (Figure 6), the same graphs appear except that the terminal vertex is a loop instead of a pair of leaves. Thus for $n \geq 1$

$$\begin{aligned} F_{1,n}(i_1, \dots, i_n) &= \sum_{\sigma} B_{i_{\sigma(1)}, a_1}^{i_1} B_{i_{\sigma(2)}, a_2}^{a_2} \cdots B_{i_{\sigma(n-1)}, a_{n-1}}^{a_{n-2}} D^{a_{n-1}} \\ &= \sum_{\sigma} \left[\mathbf{B}_{i_{\sigma(1)}} \mathbf{B}_{i_{\sigma(2)}} \cdots \mathbf{B}_{i_{\sigma(n-1)}} \cdot \mathbf{D} \right]_{i_1}, \end{aligned}$$

where σ are bijections from $\{1, \dots, n-1\}$ to $\{2, \dots, n\}$. Therefore

$$\sum_{i_1, \dots, i_n} F_{1,n}(i_1, \dots, i_n) \frac{x_{i_1} \cdots x_{i_n}}{n!} = \frac{1}{n} \left[(x_b \mathbf{B}_b)^{n-1} \cdot \mathbf{D} \right]_a x_a,$$

and summing over $n \geq 1$ gives the announced expression. \square

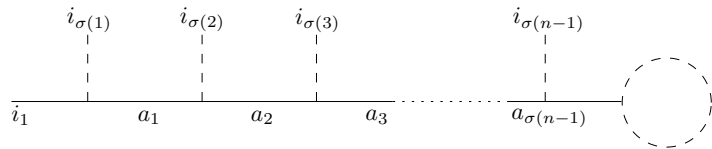


Figure 6: Genus 1 graphs without inner trivalent vertex.

We remark that, if $(L_i)_{i \in I}$ forms an abelian quantum Airy structure, all matrices $\varrho_1(e^i)$ in (15) commute, therefore they can simultaneously be brought into an upper-triangular form. So, abelian quantum Airy structures are always, up to the action of \mathcal{G}_V , equivalent to an abelian quantum Airy structure with $C = 0$, to which Proposition 5.1 can be applied.

5.2 The case $A = 0$

Quantum Airy structures with $A = 0$ are related in principle to quantum Airy structures with $C = 0$ by an element of \mathfrak{G}_V . Indeed, the automorphism of the Weyl algebra $(x_i, \partial_{x_i}) \rightarrow (\hbar \partial_{x_i}, -x_i)$ exchanges C -terms and A -terms. This transformation yields operators which are not anymore in normal form, but it can be pre- and post-composed with elements in $\exp(\mathcal{D}_{V,2}/\hbar)$ so as to be brought back to quantum Airy structures. So the partition function of a quantum Airy structures with $A = 0$ is computable in principle from Proposition 5.1.

Here we take a direct route, by examining which graphs may give non-zero contributions.

Lemma 5.2 *For a quantum Airy structure with $A = 0$, we have $F_{0,n} = 0$ for all $n \geq 1$.*

Proof. The recursion for $n \geq 4$

$$F_{0,n}(i_1, \dots, i_n) = \sum_{m=2}^n B_{i_m, a}^{i_1} F_{0,n-1}(a, i_2, \dots, \widehat{i_m}, \dots, i_n)$$

together with the initial data $F_{0,3}(i_1, i_2, i_3) = A_{i_2, i_3}^{i_1}$ has $F_{0,n} = 0$ as the unique solution. \square

In higher genera, simplification occurs when furthermore $B = 0$.

Proposition 5.3 *Let \mathfrak{T}_g be the set of rooted trivalent trees with g leaves. The root edge is denoted r . If v is a vertex, we denote $e_0(v)$ the edge closer to the root, and $e_{\pm}(v)$ the two other edges – in arbitrary order. If $T \in \mathfrak{T}_g$, let $E(T)$ be the set of unoriented edges (including the root edge, denoted r), $V(T)$ the set of trivalent vertices, and $L(T)$ the set of leaves (univalent vertices distinct from the root). If ℓ is a leaf, we denote $e(\ell)$ its incident edge. Note that the cardinality of $\text{Aut } T$ is a power of 2, as this group consists of the permutations of $\{e_-(v), e_+(v)\}$ for each v , which preserve T .*

The partition function of a quantum Airy structure with $A = B = 0$ is computed by

$$F_{g,n}(i_1, \dots, i_n) = \delta_{n,1} f_g(i_1),$$

with $f_0 = 0$ and for $g \geq 1$

$$f_g(i) := \sum_{T \in \mathfrak{T}_g} \frac{1}{|\text{Aut } T|} \sum_{\substack{a: E(T) \rightarrow I \\ v \in V(T) \\ a(r)=i}} \prod_{a(e_-(v)), a(e_+(v))} C_{a(e_-(v)), a(e_+(v))}^{a(e_0(v))} D^{a(e(\ell))}. \quad (37)$$

Proof. Let us write $Z = \exp(\sum_{g \geq 0} \hbar^{g-1} S_g)$. $L_i \cdot Z = 0$ turns into the differential recursion

$$\forall g \geq 0, \quad \partial_{x_i} S_g = \frac{1}{2} C_{a,b}^i (\partial_{x_a} \partial_{x_b} S_{g-1}(x) + \sum_{h_1+h_2=g} \partial_{x_a} S_{h_1} \partial_{x_b} S_{h_2}) + \delta_{g,1} D^i. \quad (38)$$

We know from Lemma 5.2 that $S_0 = 0$. Therefore, the equation above for $g = 0$ implies $S_1 = \sum_i D^i x_i$, and by induction that S_g must be a linear function. As $A = B = 0$, in the graphs in $\mathfrak{G}_{g,n}(1)$ with non-zero contribution the Betti number g can only arise from loops receiving a D -weight. They should thus form g leaves of the graph attached to the spanning tree. \square

6 Finite-dimensional Airy structures and representation theory

6.1 General properties

Lemma 6.1 *Assume V is a finite-dimensional Lie algebra, $L : V \rightarrow \mathcal{D}_V$ a homomorphism of normal form and that there exists a V -submodule $\varrho_M : V \rightarrow \text{End}(M)$ of $\mathbb{C}\cdot\hbar \oplus T_\hbar^*V$ and an isomorphism*

$$\Phi : \mathbb{C}\cdot\hbar \oplus T_\hbar^*V \rightarrow \mathbb{C}\cdot\hbar \oplus M, \quad \Phi \tilde{\varrho}_1 \Phi^{-1} = \varrho_t \oplus \varrho_M, \quad (39)$$

where ϱ_t is the one-dimensional trivial representation and $\Phi|_{\mathbb{C}\cdot\hbar} = \text{Id}$. Then there exists $\alpha \in M^*$ such that

$$\forall v \in V, \quad \varrho_{0,1}(v) \circ \Phi^{-1} = \alpha \circ \varrho_M(v) \circ \pi_M,$$

where $\pi_M : \mathbb{C}\cdot\hbar \oplus M \rightarrow M$ is the projection onto M with kernel $\mathbb{C}\cdot\hbar$. Further,

$$V_\alpha := \{v \in V : \text{Im } \varrho_M(v) \subseteq \text{Ker } \alpha\}$$

must vanish and

$$\begin{aligned} \pi_M \circ \Phi \circ \mathcal{J}(V) &= \{m \in M : \forall v \in V \quad \alpha(\varrho_M(v)m) = 0\}, \\ \pi_M \circ \Phi \circ \mathcal{J}([V, V]) &\subset \text{Ker } \alpha. \end{aligned}$$

where \mathcal{J} is the Lagrangian embedding from Lemma 3.2.

We note that if V is semi-simple then the trivial submodule $\mathbb{C}\cdot\hbar$ in $\mathbb{C}\cdot\hbar \oplus T_\hbar^*V$ will have a complementary submodule M as assumed in Lemma 6.1, thus the conclusion of this Lemma applies in this case. We will then use the notation

$$\tilde{\mathcal{J}} = \pi_M \circ \Phi \circ \mathcal{J}.$$

Lemma 6.2 *Suppose that V is a finite-dimensional semi-simple Lie-algebra and $L : V \rightarrow \mathcal{D}_V$ a Lie algebra homomorphism of normal form. Let $\varrho_M : V \rightarrow \text{End}(M)$ be the symplectic V -submodule of $\mathbb{C}\cdot\hbar \oplus T_\hbar^*V$ with the properties stated in Lemma 6.1. If ω_M is the V -invariant symplectic structure on M , then there exists an $m_\alpha \in \tilde{\mathcal{J}}(V) \subset M$ such that*

$$\alpha = \omega_M(m_\alpha, \cdot).$$

Moreover $\tilde{\mathcal{J}}(V)$ is a Lagrangian subspace of M on which α vanishes and

$$\varrho_M(V)m_\alpha = \tilde{\mathcal{J}}(V).$$

There exists a unique $v_\alpha \in V$ such that

$$\varrho_M(v_\alpha)m_\alpha = m_\alpha.$$

Proof of Lemma 6.1. We consider some V -module structure $\tilde{\varrho}_1 : V \rightarrow \text{End}(\mathbb{C}\cdot\hbar \oplus T_\hbar^*V)$ of the form (17). Then $\mathbb{C}\cdot\hbar$ is a trivial one-dimensional submodule. By projection from $\mathbb{C}\cdot\hbar \oplus T_\hbar^*V$ to $\mathbb{C}\cdot\hbar$ along T_\hbar^*V , we get a linear functional $\tilde{\alpha} \in (\mathbb{C}\cdot\hbar \oplus T_\hbar^*V)^*$. If we now let β be the projection onto T_\hbar^*V along $\mathbb{C}\cdot\hbar$, then $\tilde{\alpha} + \beta = \text{Id}_{\mathbb{C}\cdot\hbar \oplus T_\hbar^*V}$ and

$$\tilde{\varrho}_1 = (\tilde{\alpha} + \beta) \circ \Phi^{-1} \circ \varrho_M \circ \Phi = \tilde{\alpha} \circ \Phi^{-1} \circ \varrho_M \circ \Phi + \beta \circ \Phi^{-1} \circ \varrho_M \circ \Phi,$$

where the first factor on the right hand side gives $\varrho_{0,1}$ by definition. We now let $\alpha = \tilde{\alpha} \circ \Phi^{-1}|_M$ and get the claimed formula for $\varrho_{0,1}$. If $(e_i)_{i \in I}$ is a basis of V , the linear forms $\alpha \circ \varrho_M(e_i)$ on V must be linearly independent. Therefore, if $v = \sum_{i \in I} v_i e_i$ is solution to $\varrho_{0,1}(v) = \sum_{i \in I} v_i \alpha \circ \Phi^{-1} \circ \varrho_M(e_i) = 0$, we must have $v = 0$. This implies $V_\alpha = 0$. From the normal form of L , we see that

$$V = \{m \in M : \forall v \in V \quad \varrho_{0,1}(v)(\Phi^{-1}(m)) = 0\},$$

from which it follows that

$$\pi_M \circ \Phi \circ \mathcal{J}(V) = \{m \in M : \forall v \in V \quad \alpha(\varrho_M(v)m) = 0\}.$$

But then (18) implies that

$$\pi_M \circ \Phi \circ \mathcal{J}([V, V]) \subset \text{Ker } \alpha.$$

□

Proof of Lemma 6.2. There exists a unique $m_\alpha \in M$ such that $\alpha = \omega_M(m_\alpha, \cdot)$, where ω_M is the symplectic form, by the non-degenerateness of ω_M . Since

$$\alpha \circ \varrho_M(v) \pi_M \circ \Phi \circ \mathcal{J}(u) = 0$$

for $u, v \in V$ by the normal form assumption, we see that

$$\varrho_M(V)m_\alpha \subset \pi_M \circ \Phi \circ \mathcal{J}(V).$$

As $V_\alpha = 0$, $\alpha \circ \varrho_M : V \rightarrow M^*$ is injective. Thus

$$\varrho_M(V)m_\alpha = \tilde{\mathcal{J}}(V).$$

Since V is semi-simple, we get that $[V, V] = V$. Lemma 6.1 then implies that $\alpha(\tilde{\mathcal{J}}(V)) = 0$. Thus m_α must be in the Lagrangian subspace $\tilde{\mathcal{J}}(V)$, and we see that there must exist $v_\alpha \in V$ such that

$$\varrho_M(v_\alpha)m_\alpha = m_\alpha.$$

If $\tilde{v}_\alpha \in V$ was another such element, then $\varrho_M(v_\alpha - \tilde{v}_\alpha)m_\alpha = 0$. Thus $v_\alpha - \tilde{v}_\alpha \in V_\alpha$. By Lemma 6.1, we get that $v_\alpha = \tilde{v}_\alpha$, hence the claimed uniqueness. □

We shall now establish a kind of converse to Lemma 6.2.

Theorem 6.3 *Suppose that V is a finite-dimensional Lie algebra and $\varrho_M : V \rightarrow \text{End}(M)$ is a symplectic representation of dimension twice the dimension of V and further assume that there exists $m_\alpha \in M$ such that $\varrho_M(V)m_\alpha \subset M$ is a Lagrangian subspace.*

Then for any choice of a basis $(w_i)_{i \in I}$ of a Lagrangian complement W to $\varrho_M(V)m_\alpha$, one can construct a quantum Airy structure $L : V \rightarrow \mathcal{D}_V$ as follows. Let v_i be the unique basis of V such that $(w_i, (\varrho_M(v_i)m_\alpha))_{i \in I}$ is a symplectic basis of M . Now let (A^i, B^i, C^i) be defined by the matrix of $\varrho_M(v_i)$ in this basis as in Equation 15, and let $D^i = \frac{1}{2} \text{Tr}(B^i)$. Then L_i is determined by (A^i, B^i, C^i, D^i) as in Equation (1).

Proof. We define the map $\tilde{\mathcal{J}} : V \rightarrow \varrho_M(V)m_\alpha$ by the formula

$$\tilde{\mathcal{J}}(v) = \varrho_M(v)m_\alpha.$$

We then immediately see that

$$\forall u, v \in V, \quad \varrho_M(u)\tilde{\mathcal{J}}(v) - \varrho_M(v)\tilde{\mathcal{J}}(u) = \tilde{\mathcal{J}}([u, v]).$$

Further, since $\varrho_M(V)m_\alpha$ is Lagrangian, we see that $\tilde{\mathcal{J}}$ is an isomorphism. For any choice of a basis $(w_i)_i$ of a Lagrangian complement W to $\varrho_M(V)m_\alpha$, we pick the basis v_i of V as specified in the Theorem. By using the basis $(w_i, (\varrho_M(v_i)m_\alpha))_{i \in I}$ of M and the basis $(v_i^*, v_i)_{i \in I}$ of T_h^*V , we induce a symplectic representation ϱ_1 of V on T_h^*V and by the above property of $\tilde{\mathcal{J}}$, we see that

$$\forall u, v \in V, \quad \varrho_1(u)\mathcal{J}(v) - \varrho_1(v)\mathcal{J}(u) = \mathcal{J}([u, v]).$$

But then we can simply define L_i as specified in the Theorem above and because $\varrho_{0,1} = \alpha \circ \varrho_M$, where $\alpha = \omega_M(m_\alpha, \cdot)$, we see by the way the basis of V is chosen that L_i are of normal form. It then follows from Lemmas 3.1-3.2 that (A, B, C) satisfies the **BB-AC**, **BC** and **BA** relations and the zero torsion condition (18) with respect to the Lie algebra structure on V . Then letting D be given as in the Theorem, we see by Lemma 2.3 that (A, B, C, D) induces a quantum Airy structure. \square

6.2 The example of $\mathfrak{sl}_2(\mathbb{C})$

The previous considerations allow the construction a quantum Airy structure based on $\mathfrak{sl}_2(\mathbb{C})$.

Theorem 6.4 *The simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ supports the following quantum Airy structure*

$$\begin{aligned} L_1 &= \hbar\partial_1 - \hbar(3x_1\partial_1 + 5x_2\partial_2 + x_3\partial_3) - \frac{9\hbar}{2}, \\ L_2 &= \hbar\partial_2 - \hbar\left(\frac{8}{3}x_3\partial_1 + 3x_1\partial_2\right) - \frac{3\hbar^2}{80}\partial_3^2, \\ L_3 &= \hbar\partial_3 - \hbar\left(\frac{5}{3}x_2\partial_1 + 3x_1\partial_3\right) + 60x_3^2. \end{aligned}$$

Its partition function is $Z(x) = \frac{\mathfrak{Z}(x)}{\mathfrak{Z}(0)}$ with

$$\begin{aligned} \mathfrak{Z}(x) &= x_2^{-\frac{3}{5}}(1-3x_1)^{-\frac{3}{2}}\left((1-3x_1)^2 - 10x_2x_3\right)^{\frac{1}{2}} \exp\left\{-\frac{8\left((1-3x_1)^2 - 10x_2x_3\right)^{\frac{5}{2}}}{125\hbar x_2^3}\right\} \\ &\quad \times \tilde{\mathbb{H}}_{\frac{1}{5}}^{(2)}\left(\frac{8\left((1-3x_1)^2 - 10x_2x_3\right)^{\frac{5}{2}}}{125i\hbar x_2^3}\right), \end{aligned}$$

where $\tilde{\mathbb{H}}^{(2)}(z) := \mathbb{H}^{(2)}(-iz)$ and $\mathbb{H}^{(2)}$ is the Hankel function of the second kind. This function satisfies the differential equation

$$(z^2\partial_z^2 + z\partial_z - (\nu^2 + z^2))\tilde{\mathbb{H}}_\nu^{(2)}(z) = 0$$

Proof. $V = \mathfrak{sl}_2(\mathbb{C})$ has three generators (H, E, F) satisfying the commutation relations

$$[H, E] = 2E, \quad [H, F] = -2F \quad \text{and} \quad [E, F] = H.$$

Let M be the 6-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. Recall that it has a basis (e_0, \dots, e_5) such that

$$\begin{aligned} \varrho_M(H)e_i &= (6-2i-1)e_i, \\ \varrho_M(E)e_i &= (6-i)e_{i-1}, \\ \varrho_M(F)e_i &= (i+1)e_{i+1}, \end{aligned}$$

for $i \in \{0, \dots, 5\}$ where by convention $e_{-1} = e_6 = 0$. Further M is symplectic with the following symplectic structure

$$\omega_M(e_i, e_{5-i}) = \begin{cases} 1 & i = 0 \\ -5 & i = 1 \\ 10 & i = 2 \end{cases}$$

and $\text{span}\{e_0, e_1, e_2\}$, $\text{span}\{e_3, e_4, e_5\}$ are Lagrangian. We are now going to pick⁶ $m_\alpha = e_1$. We get that

$$\mathcal{J}(H) = 3e_1, \quad \mathcal{J}(E) = 5e_0, \quad \mathcal{J}(F) = 2e_2.$$

It is then easy to check that

$$\forall u, v \in V, \quad \varrho_M(u)\mathcal{J}(v) - \varrho_M(v)\mathcal{J}(u) = \mathcal{J}([u, v])$$

as claimed in the proof of Theorem 6.3. One also finds

$$\alpha \circ \varrho_M(H)(e_4) = 15, \quad \alpha \circ \varrho_M(E)(e_5) = -5, \quad \alpha \circ \varrho_M(F)(e_3) = -20$$

and that all other evaluations on the basis e_0, \dots, e_5 are zero. Let now (H^*, E^*, F^*) be the basis of V^* which is dual to the basis (H, E, F) of V . We get a linear symplectomorphism $\varphi : T_h^*V \rightarrow M$ by mapping

$$\varphi(H^*) = \frac{1}{15}e_4, \quad \varphi(E^*) = -\frac{1}{5}e_5, \quad \varphi(F^*) = -\frac{1}{20}e_3,$$

and letting $\varphi|_V = \mathcal{J}$. The representation $\varrho_1 = \varphi^{-1}\varrho_M\varphi$ reads in the basis $(e_i)_{i=0}^5$

$$\varrho_1(H) = \begin{pmatrix} -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \varrho_1(E) = \begin{pmatrix} 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{8}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{8}{3} \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & -\frac{3}{40} & 0 & 0 & 0 \end{pmatrix}$$

$$\varrho_1(F) = \begin{pmatrix} 0 & 0 & -3 & 0 & 0 & 0 \\ -\frac{5}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -120 \\ 0 & 0 & 0 & 0 & \frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \end{pmatrix}$$

We now let $\alpha' = \alpha \circ \varphi$. Then

$$\alpha' \circ \varrho_1(H) = H, \quad \alpha' \circ \varrho_1(E) = E, \quad \alpha' \circ \varrho_1(F) = F,$$

where we here think of H, E, F as elements of $(T_h^*V)^*$. If we now define $\varrho_{0,1} = \alpha' \circ \varrho_1$, then it is of normal form. Comparing with (15), we get the three stated operators. Alternatively, one can check directly that they satisfy the desired $\mathfrak{sl}_2(\mathbb{C})$ commutation relations.

The general solution of $L_1 \cdot Z = 0$ takes the form

$$Z(x) = (1 - 3x_1)^{-\frac{3}{2}} \tilde{Z} \left((1 - 3x_1)^{-\frac{5}{3}} x_2, (1 - 3x_1)^{-\frac{1}{3}} x_3 \right)$$

⁶By Lemma 6.2, we know that m_α has to be the eigenvector of some v_α with eigenvalue 1, which we can assume is a multiple of H , but then we see that m_α must be proportional to one of the e_i 's. However $\varrho_M(V)m_\alpha$ must be Lagrangian. This means that m_α has to be proportional to either e_1 or e_4 .

In terms of the variables $y_2 = (1 - 3x_1)^{-\frac{5}{3}}$ and $y_3 = (1 - 3x_1)^{-\frac{1}{3}}x_3$, the equations $L_i \cdot Z = 0$ for $i \in \{2, 3\}$ become

$$\begin{aligned} \left\{ \hbar \partial_{y_2} - \hbar \left(\frac{40}{3} y_2 y_3 \partial_{y_2} + \frac{8}{3} y_3^2 \partial_{y_3} + 12 y_3 \right) - \frac{3\hbar^2}{80} \partial_{y_3}^2 \right\} \tilde{Z}(y_2, y_3) &= 0 \\ \left\{ \hbar \partial_{y_3} + 60 y_3^2 - \hbar \left(\frac{25}{3} y_2^2 \partial_{y_2} + \frac{5}{3} y_2 y_3 \partial_{y_3} + \frac{15}{2} y_2 \right) \right\} \tilde{Z}(y_2, y_3) &= 0 \end{aligned}$$

The two equations can be decoupled by a gauge transformation and a change of variables, and then reduce to a Bessel differential equation. Taking into account the initial conditions for the sought partition function picks up the solution $H^{(2)}$ of this differential equation. The final outcome is

$$\tilde{Z}(y_2, y_3) = c_\hbar \tilde{H}_{\frac{1}{5}}^{(2)} \left(\frac{8}{125\hbar} \frac{(1 - 10y_2 y_3)^{\frac{5}{2}}}{y_2^3} \right) \exp \left\{ \hbar^{-1} \left(-\frac{8}{125} y_2^{-3} + \frac{8}{5} y_2^{-2} y_3 - 12 y_2^{-1} y_3^2 \right) \right\}$$

for some c_\hbar independent of (y_2, y_3) , which is the announced result. \square

6.3 Towards a classification for simple Lie algebras

If V is a semi-simple Lie algebra and $L : V \rightarrow \mathcal{D}_V$ is a homomorphism of normal form, ϱ_1 defines a structure of symplectic V -module on $W = T^*V$, which must split in a direct sum of irreducible V -modules. Let \mathcal{R} be the set of equivalence class of irreducible V -modules which occurs in the decomposition of W and $n : \mathcal{R} \rightarrow \mathbb{Z}_{>0}$ the multiplicities. The Frobenius-Schur indicator distinguishes the following properties of an irreducible V -module M

$$\text{FS}(M) = \begin{cases} -1 & \text{if } M \cong M^* \text{ is symplectic} \\ 0 & \text{if } M \not\cong M^* \\ 1 & \text{if } M \cong M^* \text{ is symmetric} \end{cases}$$

We denote $\mathcal{R}_s \subset \mathcal{R}$ the subset consisting of the symplectic V -modules.

Lemma 6.5 *If $[M] \in \mathcal{R} \setminus \mathcal{R}_s$, then $[M^*] \in \mathcal{R} \setminus \mathcal{R}_s$. If $\text{FS}(M) = 0$, then $n([M]) = n([M^*])$. If $\text{FS}(M) = 1$, then $n([M])$ is even.*

Proof. Suppose we have an $[M] \in \mathcal{R} \setminus \mathcal{R}_s$ such that $M \not\cong M^*$. Let $m = n(M)$ and $m^* = n(M^*)$. The symplectic form of W restricted to $M^m \oplus (M^*)^{m^*}$ is then non-degenerate. Indeed, if it would be degenerate, it would induce a non-zero map from one of the M 's or M^* 's in this sum to some other module in the decomposition of W , which would then make M or M^* isomorphic to one of the other modules in the decomposition, thus we would have a contradiction.

Consider one copy of M inside $M^{\oplus m}$ and let

$$M^\perp = \{v \in M^m \oplus (M^*)^{m^*} \mid \omega(v, M) = 0\}.$$

We see that $M^m \subset M^\perp$. Let M' be a V -submodule of $M^m \oplus (M^*)^{m^*}$, which is a complement of M^\perp . But then the projection from M' to one of the M^* factors must be an isomorphism. Then we see that $(M \oplus M')^\perp$ must be isomorphic to $M^{m-1} \oplus (M^*)^{m^*-1}$ by projection onto the remaining factors of M 's and M^* 's and we can by induction conclude that $m - 1 = m^* - 1$.

We now turn to the case of an $[M] \in \mathcal{R} \setminus \mathcal{R}_s$ which is symmetric. Likewise, consider one copy of M inside M^m and restrict the symplectic form of W to M . If the restriction was non-zero, it must mean that M is a symplectic module and the restriction of the symplectic form of W is the

symplectic invariant form M admits, which is unique up to non-zero rescaling. Since M is assumed to be symmetric, it must then be isotropic. Consider now

$$M^\perp = \{v \in M^m \mid \omega(v, M) = 0\}.$$

Since M^\perp is a V -submodule, we can find a V -submodule $M' \subset M^m$ such that

$$M^m = M^\perp \oplus M'.$$

But then the symplectic form of W will induce an isomorphism between M^* and M' and will make $M \oplus M'$ a symplectic vector subspace of M^m , but since $M' \subset M^m$, it must also be isomorphic to M . To prove that m is now even, we proceed inductively by considering $(M \oplus M')^\perp \subset M^m$. We see that as a V -module $(M \oplus M')^\perp \cong M^{m-2}$. Thus, just as before, by induction, this module must be isomorphic to $M^{\oplus 2m'}$, thus $m = 2(m' + 1)$. \square

Let us denote \mathcal{R}_2 the quotient of $\mathcal{R} \setminus \mathcal{R}_s$ by the relation identifying the objects M and M^* , and $\mathcal{R}' = \mathcal{R}_s \cup \mathcal{R}_2$. Lemma 6.5 tells us that the multiplicity drops to a function $n : \mathcal{R}' \rightarrow \mathbb{Z}_{>0}$.

Corollary 6.6 *The dimension of the V -module W is*

$$2 \dim V = \sum_{M \in \mathcal{R}_s} n(M) \dim M + \sum_{M \in \mathcal{R}_2} 2n(M) \dim M. \quad (40)$$

In particular: if $M \in \mathcal{R}_s$, then $\dim M \leq 2 \dim V$; if $M \in \mathcal{R}_2$, then $\dim M \leq \dim V$. \square

Lemma 6.7 *W cannot be isomorphic as a V -module to the direct sum of the adjoint module and its dual.*

Proof. Assume that W is isomorphic as a V -module to $V \oplus V^*$. We use the Killing form $\langle \cdot, \cdot \rangle$ to identify V with V^* as a V -module, so we obtain an isomorphism of V -modules $\Psi : W \rightarrow V \oplus V$. Let $m_\alpha = (m'_\alpha, m''_\alpha)$ be the element of $V \oplus V$ and $\tilde{J} : V \rightarrow V \oplus V$ be the Lagrangian embedding provided by Lemma 6.2. Then, for all $v_1, v_2 \in V$ we have that

$$\begin{aligned} 0 &= \omega(\tilde{J}(v_1), \tilde{J}(v_2)) \\ &= \omega(\varrho(v_1)m_\alpha, \varrho(v_2)m_\alpha) \\ &= \langle \varrho(v_1)m'_\alpha, \varrho(v_2)m''_\alpha \rangle - \langle \varrho(v_1)m''_\alpha, \varrho(v_2)m'_\alpha \rangle \\ &= -\langle \varrho(v_2)\varrho(v_1)m'_\alpha, m''_\alpha \rangle + \langle \varrho(v_1)\varrho(v_2)m'_\alpha, m''_\alpha \rangle \\ &= \langle \varrho([v_1, v_2])m'_\alpha, m''_\alpha \rangle. \end{aligned}$$

The third equality exploits the fact that W is isomorphic to the direct sum of adjoint modules. Since V is semi-simple, we must have $[V, V] = V$, therefore $m''_\alpha \in (\varrho(V)m'_\alpha)^\perp$. We can always fix the Cartan subalgebra $\mathfrak{h} \subset V$, such that m'_α is a generic element of \mathfrak{h} . Then, $\varrho(V)m'_\alpha$ will span all the root spaces, therefore m''_α must belong to \mathfrak{h} as well. But then v_α defined in Lemma 6.2 could not be unique, thus contradicting the result of Lemma 6.2. \square

Lemma 6.8 *\mathcal{R} cannot contain the trivial V -module.*

Proof. Assume that the trivial representation is a V -submodule of W which we denote T . According to Lemma 6.5, it must have even multiplicity such that $T^{2n(T)}$ forms a symplectic V -submodule of

W , and there exists a symplectic V -submodule M'' such that W is isomorphic to $T^{2n(T)} \oplus M''$ as a V -module. Take $m_\alpha \in W$ as in Lemma 6.2, which we decompose as (m_α, m''_α) in $T^{2n(T)} \oplus M''$. Then $\varrho(V)m_\alpha = \varrho(V)m''_\alpha \in M''$. As M'' is a symplectic vector space, for dimensional reasons $\varrho(V)m_\alpha$ cannot be Lagrangian, thus contradicting the result of Lemma 6.2. \square

This dimension formula together with Lemmas 6.7-6.8 already puts strong constraints on V and the sets of V -modules \mathcal{R}_2 and \mathcal{R}_s . Since simple Lie algebras and their finite-dimensional irreducible representations are completely classified, we can use this classification to identify which V and M satisfy the naive dimension bounds of Corollary 6.6 and which could belong to \mathcal{R}' . For each simple V there are finitely many possible M 's. Then, one determines what are the possible \mathcal{R}' 's made out of these M 's and the possible multiplicity functions $n : \mathcal{R}' \rightarrow \mathbb{Z}_{>0}$, such that the dimension formula (40) holds. The outcome of this process is summarized in the following proposition.

Proposition 6.9 *The following simple Lie algebras do not admit classical Airy structure (and therefore, do not admit quantum Airy structures either): A_n for $n \notin \{1, 5\}$, C_n for $n \geq 6$, D_n for any $n \geq 4$, E_6 , E_7 , E_8 . For the remaining simple Lie algebras, the dimension bound (40) is respected for the following decompositions in irreducible modules⁷.*

- $\boxed{A_1}$ the only candidate is $\mathbf{6}(\mathfrak{s})$ – and it was realized in Theorem 6.4.
- $\boxed{A_5}$ the only candidate is $\mathbf{20}(\mathfrak{s})^2 \oplus \mathbf{15}^2$. Here, if we denote F the fundamental module, $\mathbf{20}(\mathfrak{s}) = \Lambda^3 F$ and $\mathbf{15} = S^2 F$.
- $\boxed{B_n}$ For $n \geq 3$, the only candidate is $(\mathbf{2n} + \mathbf{1})^{2n}$, where $(\mathbf{2n} + \mathbf{1})$ is the fundamental module. For $n = 2$, we have $\mathbf{5}^4$ as well as $\mathbf{4}(\mathfrak{s}) \oplus \mathbf{16}(\mathfrak{s})$. Here, $\mathbf{4}(\mathfrak{s})$ is the fundamental module for $\mathfrak{sp}(4)$ – using the isomorphism of this Lie algebra with $\mathfrak{so}(5) = B_2$.
- $\boxed{C_3}$ there are three candidates: $\mathbf{6}(\mathfrak{s})^7$, $\mathbf{14}'(\mathfrak{s})^3$ and $\mathbf{14}^2 \oplus \mathbf{14}'(\mathfrak{s})$. Here, $\mathbf{6}(\mathfrak{s})$ is the fundamental module.
- $\boxed{C_4}$ there are three candidates: $\mathbf{8}(\mathfrak{s})^8$, $\mathbf{8}(\mathfrak{s})^3 \oplus \mathbf{48}$ and $\mathbf{8}(\mathfrak{s})^2 \oplus \mathbf{27}^2$.
- $\boxed{C_5}$ the only candidate is $\mathbf{110}(\mathfrak{s})$.
- $\boxed{F_4}$ the only candidate is $\mathbf{26}^4$.
- $\boxed{G_2}$ the only candidate is $\mathbf{7}^4$.

We leave to future work the question of concluding on the existence of quantum Airy structures with those module contents, and of completing the classification of quantum Airy structure based on simple Lie algebras, and its extension to semi-simple Lie algebras. It is nevertheless remarkable that many simple Lie algebras are already excluded.

⁷We indicate an irreducible module by its dimension in bold characters. If there are several non isomorphic irreducible modules of the same dimension d , we use the notation \mathbf{d} , \mathbf{d}' , \mathbf{d}'' to distinguish them. We write (\mathfrak{s}) if and only if the module is symplectic.

7 Low dimensional examples

7.1 Twisted cohomology of Lie algebras in low dimensions

Let \mathfrak{g} be a complex Lie algebra and M be a \mathfrak{g} -module. We will consider a rather special case, which however suffices for our purposes to compute the needed Lie algebra cohomologies, namely that \mathfrak{g} has a codimension one ideal \mathfrak{i} such that

$$\mathfrak{g} = \mathfrak{i} \oplus \mathfrak{q}$$

as vector spaces. Pick $y \in \mathfrak{q} - \{0\}$. If v is an endomorphism of a vector space \mathcal{V} , we denote the v -invariant subspace $\mathcal{V}^{(v)} \subseteq \mathcal{V}$.

Lemma 7.1 *We have that*

$$H^q(\mathfrak{g}, M) \cong H^q(\mathfrak{i}, M)^{(y)} \oplus \frac{H^{q-1}(\mathfrak{i}, M)}{yH^{q-1}(\mathfrak{i}, M)}.$$

Proof. We consider the Hochschild-Serre spectral sequence [33] for the pair $(\mathfrak{i}, \mathfrak{g})$ with E_2 -page

$$E_2^{p,q} = H^p(\mathfrak{q}, H^q(\mathfrak{i}, M)).$$

But since \mathfrak{q} is one-dimensional, we see that this E_2 -page is concentrated in the columns $p = 0, 1$, thus the differential d_2 is trivial and the spectral sequence collapses at the E_2 -page. Since in general this spectral sequence converges to $H^q(\mathfrak{g}, M)$, we get that

$$H^q(\mathfrak{g}, M) \simeq H^0(\mathfrak{q}, H^q(\mathfrak{i}, M)) \oplus H^1(\mathfrak{q}, H^{q-1}(\mathfrak{i}, M)).$$

But

$$H^0(\mathfrak{q}, H^q(\mathfrak{i}, M)) \simeq H^q(\mathfrak{i}, M)^{(y)}, \quad H^1(\mathfrak{q}, H^{q-1}(\mathfrak{i}, M)) \simeq \frac{H^{q-1}(\mathfrak{i}, M)}{yH^{q-1}(\mathfrak{i}, M)}.$$

□

We recall that $y \in \mathfrak{q}$ acts on $u \in Z^q(\mathfrak{i}, M)$ by the formula

$$y(u)(x_1, \dots, x_q) = yu(x_1, \dots, x_q) - \sum_{i=1}^q u(x_1, \dots, [y, x_i], \dots, x_q). \quad (41)$$

The case where also \mathfrak{i} is one-dimensional is particularly simple. Let $x \in \mathfrak{i} - \{0\}$. Then $[y, x] = ax$ for some $a \in \mathbb{C}$. We then have that

$$H^0(\mathfrak{i}, M) \simeq M^{(x)}, \quad H^1(\mathfrak{i}, M) \simeq M/xM.$$

Here the last isomorphism is induced by mapping $u \in Z^1(\mathfrak{i}, M)$ to $u(x) \in M$. We then see that

$$y(u)(x) = yu(x) - u([y, x]) = (y - a \text{Id})u(x).$$

Thus we get that

$$H^1(\mathfrak{i}, M)^{(y)} = (M/xM)^{(y-a)}.$$

Lemma 7.2 *In the case where \mathfrak{g} is two-dimensional, with the notation as above, we have that*

$$H^0(\mathfrak{g}, M) \simeq M^{(x,y)}, \quad H^1(\mathfrak{g}, M) \simeq (M/xM)^{(y-a)} \oplus (M^{(x)}/yM^{(x)}),$$

and

$$H^2(\mathfrak{g}, M) \simeq \frac{M}{xM + (y-a)M}.$$

Proof. Since we have already computed H^0 and H^1 , we just need to compute H^2 . From the Lemma 7.1, we get that

$$H^2(\mathfrak{g}, M) \simeq H^1(\mathfrak{q}, H^1(\mathfrak{i}, M)) \simeq \frac{M/xM}{(y-a)M/xM}$$

from which the result follows. \square

Let us now consider the case where \mathfrak{i} has dimension two. Say $\mathfrak{i} = \text{span}\{x_1, x_2\}$, where $[x_2, x_1] = ax_1$. We then have a description of the cohomology of \mathfrak{i} from Proposition 7.2. However, in order to track the action of y , it is easier to work with the following model for H^1 of \mathfrak{i}

$$H^1(\mathfrak{i}, M) \simeq K/N,$$

where

$$\begin{aligned} K &= \ker(d_2 : M \times M \rightarrow M), & d_2(m_1, m_2) &= x_1 m_2 - (x_2 - a)m_1 \\ N &= \text{Im}(d_1 : M \rightarrow M \times M), & d_1(m) &= (x_1 m, x_2 m). \end{aligned}$$

If we now define $(a_{ij})_{i,j}$ by

$$[y, x_1] = a_{11}x_1 + a_{12}x_2, \quad [y, x_2] = a_{21}x_1 + a_{22}x_2 \quad (42)$$

then the Jacobi identity demands that

$$aa_{12} = 0, \quad aa_{22} = 0.$$

It is easy to check that the action of y on $H^1(\mathfrak{i}, M)$ is induced from the following action of y on K

$$y(m_1, m_2) = ((y - a_{11})m_1 - a_{12}m_2, (y - a_{22})m_2 - a_{21}m_1),$$

and using the above condition on a and $(a_{ij})_{i,j}$ one can easily verify that this action of y indeed does preserve K and that it maps N to itself.

In order to compute the action of y on $H^2(\mathfrak{i}, M)$, we simply use (41) to get that the action is induced by the following action of y on M

$$y(m) = (y - a_{11} - a_{22})m.$$

Putting the above together, we get the following result.

Lemma 7.3 *In the case where \mathfrak{g} is three-dimensional and has an ideal \mathfrak{i} of codimension one, with the notation as above, we have that*

$$H^0(\mathfrak{g}, M) \simeq M^{(x_1, x_2, y)}, \quad H^1(\mathfrak{g}, M) \simeq (K/N)^{(y)} \oplus (M^{(x_1, x_2)}/yM^{(x_1, x_2)}),$$

and

$$H^2(\mathfrak{g}, M) \simeq \left(\frac{M}{x_1 M + x_2 M + (y-a)M} \right)^{(y-a_{11}-a_{22})} \oplus \left(\frac{K}{N+yK} \right),$$

where $(a_{i,j})_{i,j}$ encode the commutation relations (42).

We can give an alternative description of the first cohomology group in the case where \mathfrak{g} has dimension three with generators x_1, x_2, y with structure constants as above, which is adapted to

impose the extra linear constraints necessary to compute the tangent space to the moduli space of quantum (rather than quasi-) Airy structures. Let us introduce

$$\begin{aligned} K^{(3)} &:= \ker(d_2^{(3)} : M \times M \times M \longrightarrow M \times M \times M), \\ d_2^{(3)}(m_1, m_2, m_3) &= \left(-(x_2 - a)m_1 + x_1m_2, -(y - a_{11})m_1 + a_{12}m_2 + x_1m_3, a_{21}m_1 - (y - a_{22})m_2 + x_2m_3 \right), \\ N^{(3)} &:= \operatorname{Im}(d_1^{(3)} : M \longrightarrow M \times M \times M), \\ d_1^{(3)}(m) &= (x_1m, x_2m, ym). \end{aligned}$$

Then we have that

$$H^1(\mathfrak{g}, M) \simeq K^{(3)}/N^{(3)}.$$

In this description a cocycle $u : \mathfrak{g} \rightarrow M$ is identified with $(m_1, m_2, m_3) \in K^{(3)}$ by

$$u(x_1) = m_1, \quad u(x_2) = m_2, \quad u(y) = m_3.$$

7.2 Abelian cases

From Lemma 2.3 and Section 3.1 we know that when V is a finite-dimensional abelian Lie algebra, the data of an abelian Lie subalgebra of $\mathfrak{sp}(T^*V)$ spanned by

$$\varrho_1(e_i) = \begin{pmatrix} -B^i & A^i \\ -C^i & (B^i)^T \end{pmatrix}$$

and of an arbitrary D is equivalent to the data of a quantum Airy structure. To describe all abelian quantum Airy structure, we should take our matrices $\varrho_1(e_i)$ belonging to a given maximal abelian Lie subalgebra of $\mathfrak{sp}(2n)$, and impose the constraint that A is fully symmetric. If we wish to do so up to \mathcal{G}_V -equivalence, one should first list all isomorphism classes of maximal abelian Lie subalgebras of $\mathfrak{sp}(2n)$. The reference [42] provides tools to do so, and achieves a complete classification in dimension two and three. Imposing further full symmetry of A , we describe below the five cases. The order in this list respects the order of the list of maximal abelian Lie subalgebras of [42]. In dimension $n \geq 4$ and higher, there exists continuous families of isomorphism classes of maximal abelian Lie subalgebras of $\mathfrak{sp}(2n)$, and the classification becomes more intricate. Therefore, in general finite dimension, the classification of abelian quantum Airy structures – and *a fortiori* of quantum Airy structures – modulo \mathcal{G}_V -action seems out of reach.

As commuting matrices can always be simultaneously trigonalized, we can always achieve $C^i = 0$ for all i via \mathcal{G}_V -transformations. It is then implicit in the lists below that $C = 0$. Any two different cases in this list represent two non-equivalent quantum Airy structures for generic values of the parameters. Within each case, we have not indicated the fundamental domain of parameters for the residual \mathcal{G}_V -action.

7.2.1 Dimension two

- 1] $A = 0$ and B^i diagonal.
- 2] $A^1 = A^2 = 0$ and $B^i = \begin{pmatrix} \alpha_i & 0 \\ 0 & 0 \end{pmatrix}$.
- 3] $A^1 = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$, $A^2 = \begin{pmatrix} \beta & \gamma \\ \gamma & \varepsilon \end{pmatrix}$, and $B^1 = B^2 = 0$.

$$\boxed{4} \quad A^1 = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}, A^2 = \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix}, B^1 = \begin{pmatrix} 0 & 0 \\ -\beta & 0 \end{pmatrix}, B^2 = 0.$$

$$\boxed{5} \quad A^1 = A^2 = 0 \text{ and } B^i = \begin{pmatrix} \alpha_i & \beta_i \\ 0 & \alpha_i \end{pmatrix}.$$

7.2.2 Dimension three

$$\boxed{1} \quad A = 0 \text{ and } B^i \text{ diagonal.}$$

$$\boxed{2} \quad A^1 = A^3 = 0, A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } B^i = \begin{pmatrix} \beta_i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma_i \end{pmatrix}.$$

$$\boxed{3} \quad A^1 = 0, A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \beta & \gamma \end{pmatrix}, A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & \gamma \\ 0 & \gamma & \varepsilon \end{pmatrix}, \text{ and } B^i = \begin{pmatrix} \kappa_i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\boxed{4} \quad A^1 = 0, A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \text{ and } B^i = \begin{pmatrix} -\beta_i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\alpha_i & 0 \end{pmatrix} \text{ with } \alpha_1 = \alpha_3 = 0.$$

$$\boxed{5} \quad A^i = 0 \text{ and } B^i = \begin{pmatrix} \alpha_i & 0 & 0 \\ 0 & \beta_i & \gamma_i \\ 0 & 0 & \beta_i \end{pmatrix}.$$

$$\boxed{6} \quad A^1 = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A^2 = A^3 = 0 \text{ and } B^i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta_i & \gamma_i \\ 0 & 0 & \beta_i \end{pmatrix}.$$

$$\boxed{7} \quad A \text{ fully symmetric and } B^i = 0.$$

$$\boxed{8} \quad A^1 = \begin{pmatrix} \alpha & \beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}, A^2 = \begin{pmatrix} \beta & \gamma & 0 \\ \gamma & \varepsilon & 0 \\ 0 & 0 & 0 \end{pmatrix}, A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \kappa \end{pmatrix},$$

$$\text{and } B^1 = B^2 = 0, B^3 = \begin{pmatrix} 0 & 0 & -\kappa \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\boxed{9} \quad A^1 = \begin{pmatrix} \alpha & \beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}, A^2 = \begin{pmatrix} \beta & \gamma & 0 \\ \gamma & \varepsilon & \kappa \\ 0 & \kappa & 0 \end{pmatrix}, A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\text{and } B^1 = B^3 = 0, B^2 = \begin{pmatrix} 0 & 0 & -\kappa \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\boxed{10} \quad A^1 = 0, A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \beta & 0 \end{pmatrix}, A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\text{and } B^1 = 0, B^2 = \begin{pmatrix} 0 & -\beta & -\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B^3 = \begin{pmatrix} 0 & 0 & -\beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\boxed{11} \quad A^1 = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & \beta + \gamma \\ \beta + \gamma & \gamma & \gamma \end{pmatrix}, A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta + \gamma & \gamma \\ 0 & \gamma & -\beta \end{pmatrix},$$

$$\text{and } B^1 = 0, B^2 = \begin{pmatrix} 0 & -\gamma & -\beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B^3 = \begin{pmatrix} 0 & \beta & -\beta - \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\boxed{12} \quad A = 0 \text{ and } B^i = \begin{pmatrix} \alpha_i & 0 & \beta_i \\ 0 & \alpha_i & \gamma_i \\ 0 & 0 & \alpha_i \end{pmatrix}.$$

$$\boxed{13} \quad A = 0 \text{ and } B^i = \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ 0 & \alpha_i & 0 \\ 0 & 0 & \alpha_i \end{pmatrix}.$$

$$\boxed{14} \quad A = 0 \text{ and } B^i = \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ 0 & \alpha_i & \beta_i \\ 0 & 0 & \alpha_i \end{pmatrix}.$$

7.3 Non-abelian two-dimensional Lie algebra

In dimension two, there is a unique non-abelian Lie algebra up to isomorphism. It is the Lie algebra of affine transformations of \mathbb{C} , generated by L_0 and L_1 with the commutation relation

$$[L_0, L_1] = -L_1. \quad (43)$$

We introduce a \mathbb{Z}_2 -grading by declaring t_i and ∂_{t_i} to have degree i for $i \in \{0, 1\}$. We remark that, using the conjugation by an operator of the form $\exp(\frac{\hbar}{2}u_{a,b}\partial_{x_a}\partial_{x_b})$, one can generically bring L_0 to a form where $C^0 = 0$.

Proposition 7.4 *The complete list of classical Airy structures such that*

- (1) L_i has degree i for $i \in \{0, 1\}$,
- (2) L_0 has no differential operators of order two, i.e. $C^0 = 0$,

reads as follows

$$\begin{array}{ll} \text{Ia} & L_0 = \hbar\partial_{x_0} - \alpha x_0^2 + \frac{\beta\alpha}{4}x_1^2 + 2\hbar x_0\partial_{x_0} + \hbar x_1\partial_{x_1} & L_1 = \hbar\partial_{x_1} + \frac{\alpha\beta}{2}x_0x_1 - \hbar\beta x_1\partial_{x_0} \\ \text{Ib} & L_0 = \hbar\partial_{x_0} - \alpha x_0^2 - \beta x_1^2 - 2\hbar x_0\partial_{x_0} - \hbar x_1\partial_{x_1} & L_1 = \hbar\partial_{x_1} - 2\beta x_0x_1 - 2\hbar x_0\partial_{x_1} \\ \text{Ic} & L_0 = \hbar\partial_{x_0} - \alpha x_0^2 - \hbar(\beta + 1)x_0\partial_{x_0} - \hbar\beta x_1\partial_{x_1} & L_1 = \hbar\partial_{x_1} - \hbar(\beta + 1)x_0\partial_{x_1} \\ \text{IIa} & L_0 = \hbar\partial_{x_0} + \frac{\alpha}{\beta}x_0^2 - \beta x_1^2 + 2\hbar x_0\partial_{x_0} - \hbar x_1\partial_{x_1} & L_1 = \hbar\partial_{x_1} - 2\beta x_0x_1 - \hbar\alpha\beta x_1\partial_{x_0} - 2\hbar x_0\partial_{x_1} - \hbar^2\alpha\partial_{x_0}\partial_{x_1} \\ \text{IIb} & L_0 = \hbar\partial_{x_0} + \frac{\alpha^2}{\beta}x_0^2 + \hbar\alpha x_0\partial_{x_0} + \hbar(1 - \alpha)x_1\partial_{x_1} & L_1 = \hbar\partial_{x_1} - \hbar\alpha x_0\partial_{x_1} - \hbar^2\gamma\partial_{x_0}\partial_{x_1} \end{array}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ (and non-zero when they appear in the denominator). Quantum Airy structures are obtained from these by adding an arbitrary constant $-\hbar D^0$ to L_1 , while $D^1 = 0$. \square

This result is obtained by inserting the form of the most general quantum Airy structure with properties (1) and (2) into the **BB-AC**, **BC** and **BA** relations, and solving the resulting equations. The **D** relation can be analyzed separately, and we indeed find that quantum Airy structures based on (43) must have $D^1 = 0$, while D^0 is unconstrained. We omit the details of this proof. The groups **I** and **II** correspond to $C = 0$ or $C^1 \neq 0$.

Now we want to consider the existence of deformations modulo \mathcal{G}_V of these structures. We remark that this may not preserve the homogeneity condition (1). In particular, with the following transformations – which belong to \mathcal{G}_V

$$t_0 \rightarrow e^{\xi_1}t_0, \quad \partial_0 \rightarrow e^{-\xi_1}\partial_0, \quad t_1 \rightarrow e^{\xi_2}t_1, \quad \partial_1 \rightarrow e^{-\xi_2}\partial_1, \quad \hbar \rightarrow e^{-\xi_1}\hbar, \quad L_1 \rightarrow e^{\xi_2 - \xi_1}L_1, \quad \xi_i \in \mathbb{C} \quad (44)$$

and assuming the parameters α, β, γ non-zero, and in each case one can use them to set the parameters to 1, except in case **Ic** where one can only set $\alpha = 1$ while β remains a free parameter.

Now let us assume that α, β, γ are generic. We compute the Lie algebra cohomologies governing the deformations of quantum Airy structures modulo \mathfrak{G}_V using Lemma 7.2, which in our case applies with $x = L_1, y = L_0$ and $a = -1$. The result is

	Ia	Ib	Ic	IIa	IIb
$\dim H^0$	1	1	2	1	2
$\dim H^1$	1	1	2	1	2
$\dim H^2$	0	0	0	0	0

We observe that H^2 always vanishes, *i.e.* there are no obstructions to deformations. As the conjugation by scalars acts trivially on \mathcal{D} , H^0 is at least one-dimensional. In the two cases where H^0 has dimension two, the extra generator in fact has a trivial action on H^1 , so $\dim H^1$ gives the number of independent deformations of the corresponding quantum Airy structure. In all cases, the value of D^0 gives a one-parameter deformation. In **Ic**, β gives the second deformation parameter – one sees for instance that it cannot be gauged out by the scaling transformations (44). In case **IIb**, one finds that the second one-parameter deformation (called t) is

$$\mathbf{IIb} : L_i^{(t)} = L_0 + tL'_i, \quad L'_0 = \hbar(1 - 2\alpha)x_0\partial_{x_1}, \quad L'_1 = \hbar^2\beta\partial_{x_1}^2 \quad (45)$$

Although the cohomology groups we computed only give the number of independent deformations within quasi-Airy structures, we actually find in these five examples that they are always realized within Airy structures.

The partition functions for each of these quantum Airy structures – with an arbitrary D^0 – can be computed by hand. As the group **I** has $C = 0$, we can also use Lemma 5.1.

$$\begin{aligned} Z_{\mathbf{Ia}} &= (1 + 2x_0 + \beta x_1^2)^{\frac{\alpha}{8\hbar} + \frac{D^0}{2}} \exp\left\{\hbar^{-1}\left(\frac{\alpha x_0(x_0-1)}{4} - \frac{\alpha\beta x_1^2}{8}\right)\right\} \\ Z_{\mathbf{Ib}} &= (1 - 2x_0)^{-\frac{\alpha}{8\hbar} - \frac{D^0}{2}} \exp\left\{\hbar^{-1}\left(-\frac{\alpha x_0(x_0+1)}{4} - \frac{\beta x_0 x_1^2}{1-2x_0}\right)\right\} \\ Z_{\mathbf{Ic}} &= (1 - (\beta + 1)x_0)^{-\frac{\alpha}{\hbar(\beta+1)^3} - \frac{D^0}{\beta+1}} \exp\left(-\frac{\alpha x_0(2+(\beta+1)x_0)}{2\hbar(\beta+1)^2}\right) \\ Z_{\mathbf{IIa}} &= (1 + 2x_0)^{-\frac{1}{4\hbar\alpha} + \frac{D^0}{4}} x_1^{\frac{1}{2\alpha\hbar} - \frac{D^0}{2}} \exp\left\{\hbar^{-1}\left(\frac{x_0(1-x_0)}{\alpha} - \frac{\beta x_1^2}{2}\right)\right\} J_{-\frac{1}{2\alpha\hbar} + \frac{D^0}{2}}\left[\hbar^{-1}x_1\left(\frac{\beta(1+2x_0)}{\alpha}\right)^{\frac{1}{2}}\right] \\ Z_{\mathbf{IIb}} &= (1 + \alpha x_0)^{-\frac{1}{\hbar\alpha\beta} + \frac{D^0}{\alpha}} \exp\left(\frac{x_0(2-\alpha x_0)}{\beta\hbar}\right) \end{aligned}$$

The most interesting case is $Z_{\mathbf{IIa}}$, where we see an appearance of the Bessel function $J_\nu(z)$. It is characterized by

$$(z^2\partial_z^2 + z\partial_z + (z^2 - \nu^2))J_\nu(z) = 0, \quad J_\nu(z) := \sum_{m \geq 0} \frac{\Gamma(\nu + 1)}{\Gamma(m + \nu + 1)} \frac{(-1)^m}{m!} \left(\frac{z}{2}\right)^{2\nu+m}.$$

Our normalization by a constant prefactor of $J_\nu(z)$ is not the conventional one, but we made it so that $J_\nu(z) = z^{2\nu}(1 + O(z))$ when $z \rightarrow 0$. In the case **IIb**, the partition function for the one-parameter deformation (45) is in fact independent of the parameter t , and independent of x_1 .

7.4 The case $n = 3$.

When considering isomorphism classes of three-dimensional non-trivial Lie algebra over \mathbb{C} , one finds [15] four rigid cases and a one-parameter family.

- \mathfrak{sl}_2 , which was treated in Theorem 6.4.
- The direct sum of the non-abelian two-dimensional algebra and the abelian Lie algebra of dimension 1. We do not study this direct sum case.
- The Heisenberg Lie algebra $[y_1, y_2] = y_3$, $[y_1, y_3] = [y_2, y_3] = 0$, which we do not treat.
- The Lie algebra $[y_1, y_2] = y_2$ and $[y_1, y_3] = y_2 + y_3$, which we also do not treat.
- The Lie algebra \mathfrak{l}_q defined by $[y_1, y_2] = y_2$, $[y_1, y_3] = qy_3$, $[y_2, y_3] = 0$, for $q \in \mathbb{C}^*$. We have $\mathfrak{l}_q \simeq \mathfrak{l}_r$ if and only if $q = r$ or $qr = 1$. For $q = -1$, this is the Lie algebra of the group of isometries of euclidean \mathbb{R}^2 .

We will not attempt here at the classification of Airy structures supported by each of these Lie algebras. We rather look for a non-trivial example of quantum Airy structure based on \mathfrak{l}_q , keeping as many non-zero elements as possible, which illustrates the non-triviality of the problem of exhibiting finite-dimensional quantum Airy structures. We rename the variables (x_0, x'_0, x_1) , and assign a \mathbb{Z}_2 -degree 0 to x_0, x'_0 , and 1 to x_1 . We postulate the commutations relation of \mathfrak{l}_q in the following form

$$[L_0, L_1] = L_1, \quad [L_0, L'_0] = qL'_0 \quad [L'_0, L_1] = 0, \quad \text{that is } L : (y_1, y_2, y_3) \rightarrow (L_0, L_1, L'_0)$$

and look for a quantum Airy structure such that L_0, L'_0 have degree 0 and L_1 has degree 1. For the same Lie algebra, computations shows that the other choice of grading $(y_1, y_2, y_3) \mapsto (L_0, L_1, L'_1)$ does not support non-trivial Airy structures.

Lemma 7.5 *Let ζ be a root of $P(\zeta) := 2\zeta^3 - 2\zeta^2 + 3\zeta - 1$ and α a complex parameter. The matrices, which we write with respect to the basis (e_0, e'_0, e_1) , $C^0 = 0$ and*

$$\begin{aligned} A^0 &= \begin{pmatrix} \frac{2}{3}(-5\zeta + 3) & \frac{4}{3}(-2\zeta^2 + \zeta - 1) & 0 \\ \frac{4}{3}(-2\zeta^2 + \zeta - 1) & \frac{2}{3}(-\zeta + 1) & 0 \\ 0 & 0 & 2\alpha \end{pmatrix} & B^0 &= \begin{pmatrix} 1 - 3\zeta & 3(-2\zeta^2 + \zeta - 1) & 0 \\ 3(-2\zeta^2 + \zeta - 1) & 3\zeta + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ A'^0 &= \begin{pmatrix} \frac{4}{3}(-2\zeta^2 + \zeta - 1) & \frac{2}{3}(-\zeta + 1) & 0 \\ \frac{2}{3}(-\zeta + 1) & 0 & 0 \\ 0 & 0 & -\alpha(2\zeta^2 + 1) \end{pmatrix} & B'^0 &= \begin{pmatrix} 3(-2\zeta^2 + \zeta - 1) & 3\zeta - 1 & 0 \\ -\zeta + 1 & -2\zeta^2 + \zeta - 1 & 0 \\ 0 & 0 & -2\zeta^2 - 1 \end{pmatrix} \\ A^1 &= \begin{pmatrix} 0 & 0 & 2\alpha \\ 0 & 0 & -\alpha(2\zeta^2 + 1) \\ 2\alpha & -\alpha(2\zeta^2 + 1) & 0 \end{pmatrix} & B^1 &= \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2\zeta^2 - 1 \\ 6\alpha & 3\alpha(2\zeta^2 + 1) & 0 \end{pmatrix} \\ C'^0 &= \begin{pmatrix} 6(-2\zeta^2 + \zeta - 1) & 6\zeta & 0 \\ 6(2\zeta^2 - \zeta + 1) & 0 & 0 \\ 0 & 0 & -\alpha^{-1}(2\zeta^2 + 1) \end{pmatrix} & C^1 &= \begin{pmatrix} 0 & 0 & 6 \\ 0 & 0 & 3(2\zeta^2 + 1) \\ 6 & 3(2\zeta^2 + 1) & 0 \end{pmatrix} \end{aligned}$$

together with

$$D^0 \text{ arbitrary}, \quad D'^0 = \frac{1}{2} \text{Tr } B'^0 = \frac{1}{2}(-5\zeta^2 + 4\zeta - 5), \quad D^1 = 0$$

define a quantum Airy structure based on the Lie algebra \mathfrak{l}_{-2} . It has

$$\dim H_L^i(V, \mathcal{D}_V) = \begin{cases} 1 & i = 0 \\ 2 & i = 1 \\ 1 & i = 2 \end{cases}.$$

The stabilizer $\mathcal{G}_V(L)$ is trivial. D^0 is the only deformation of this quantum Airy structure.

Proof. One can check by direct computation that the assignments in this Lemma indeed satisfies the desired commutation relations. We have however opted to provide some details explaining how we were led to this quantum Airy structure, in the hope that it may lead to the construction of other quantum Airy structures by similar techniques.

The main task is to find (A, B, C) defining a classical Airy structure, *i.e.* such that

$$[\tilde{\varrho}_1(e_0), \tilde{\varrho}_1(e_1)] = \tilde{\varrho}_1(e_1), \quad [\tilde{\varrho}_1(e_0), \tilde{\varrho}_1(e'_0)] = q\tilde{\varrho}_1(e'_0), \quad [\tilde{\varrho}_1(e'_0), \tilde{\varrho}_1(e_1)] = 0,$$

with $\tilde{\varrho}_1$ of the form (17). As our goal is not to find all possible solutions, we are going to postulate certain properties, which eventually lead to a solution.

1st postulate. L_0, L'_0 have degree 0, L_1 has degree 1.

2nd postulate. $C^0 = 0$ — for generic operators this can be achieved by a \mathcal{G}_V transformation.

The matrices then take the form

$$\begin{aligned} B^0 &= \left(\begin{array}{c|c} \tilde{B} & 0 \\ \hline 0 & b_{33} \end{array} \right), & B'^0 &= \left(\begin{array}{c|c} \tilde{B}' & 0 \\ \hline 0 & b'_{33} \end{array} \right), & C'^0 &= \left(\begin{array}{c|c} 2\tilde{C}' & 0 \\ \hline 0 & 2c'_{33} \end{array} \right), & A^0 &= \left(\begin{array}{c|c} 2\tilde{A} & 0 \\ \hline 0 & 2a_{33} \end{array} \right), \\ A'^0 &= \left(\begin{array}{c|c} 2\tilde{A}' & 0 \\ \hline 0 & 2a'_{33} \end{array} \right), & B_1 &= \left(\begin{array}{c|c} 0 & \mathbf{v}^T \\ \hline \mathbf{u} & 0 \end{array} \right), & C^1 &= \left(\begin{array}{c|c} 0 & 2\mathbf{c}^T \\ \hline 2\mathbf{c} & 0 \end{array} \right), & A^1 &= \left(\begin{array}{c|c} 0 & 2\mathbf{a}^T \\ \hline 2\mathbf{a} & 0 \end{array} \right), \\ \tilde{A}^T &= \tilde{A}, & (\tilde{A}')^T &= \tilde{A}', & (\tilde{C}')^T &= \tilde{C}' & \text{and} & a_{12} &= a'_{11}, & a_{22} &= a'_{12}, & a_{33} &= \mathbf{a}_1, & a'_{33} &= \mathbf{a}_2. \end{aligned} \quad (46)$$

where $\tilde{A}, \tilde{A}', \tilde{C}, \tilde{C}'$ are 2×2 matrices, and $\mathbf{a}, \mathbf{c}, \mathbf{u}, \mathbf{v}$ are row vectors of size two. The relation (8) between the structure constants of the Lie algebra and antisymmetric part of B results in

$$b'_{1,i} - b_{2,i} = q\delta_{i,2}, \quad i = 1, 2; \quad \mathbf{v}_2 = b'_{33}, \quad \mathbf{v}_1 = b_{33} + 1. \quad (47)$$

We first address the relation $[\tilde{\varrho}_1(0), \tilde{\varrho}_1(e'_0)] = q\tilde{\varrho}_1(e'_0)$. It results in the following relations

$$-\tilde{C}'\tilde{B} - \tilde{B}^T\tilde{C}' = q\tilde{C}', \quad -2c'_{33}b_{33} = qc'_{33}, \quad (48)$$

$$[\tilde{B}, \tilde{B}'] - 4\tilde{A}\tilde{C}' = q\tilde{B}', \quad -4c'_{33}a_{33} = qb'_{33}, \quad (49)$$

$$\tilde{B}\tilde{A}' + \tilde{A}'\tilde{B}^T - \tilde{B}'\tilde{A} - \tilde{A}(\tilde{B}')^T = q\tilde{A}', \quad 2b_{33}a'_{33} - 2b'_{33}a_{33} = qa'_{33}. \quad (50)$$

3rd postulate. We assume C'^0 is nondegenerate, *i.e.* $\det \tilde{C}' \neq 0$ and $c'_{33} \neq 0$.

Then, from the second equation in (48), we have that $b_{33} = -\frac{q}{2}$. From (50) we then obtain

$$b'_{33}a_{33} = -qa'_{33}.$$

The first equation in (48) implies that $\tilde{B} = S - \frac{q}{2}\text{Id}$, where the matrix S satisfies the equation

$$\tilde{C}'S = -S^T\tilde{C}', \quad (51)$$

which admits a one-parameter family of solutions. So, we express \tilde{B} in terms of \tilde{C}' and the parameter t

$$\tilde{B} = -\frac{q}{2}\text{Id} + t \begin{pmatrix} -c'_{12} & c'_{11} \\ -c'_{22} & c'_{12} \end{pmatrix}. \quad (52)$$

From (47) we then have $b'_{11} = -tc'_{22}$, $b'_{12} = tc'_{12} + \frac{q}{2}$, $\mathbf{v}_1 = 1 - \frac{q}{2}$, and $\mathbf{v}_1 = b'_{33}$. Due to general scaling transformations in \mathcal{G}_V for even and odd variables, we have two *free parameters*: t and a_{33} .

4th postulate $t = 1$ — we however keep $a_{33} = 2\alpha$ arbitrary.

We now solve (49) with respect to \tilde{A} . We first rewrite it in terms of S

$$[S, \tilde{B}'] - 4\tilde{A}\tilde{C}' = q\tilde{B}'.$$

Because $-S(\tilde{C}')^{-1} = (\tilde{C}')^{-1}\tilde{S}^T$ from (51), multiplying by $(\tilde{C}')^{-1}$ from the right, we obtain that

$$4\tilde{A} = S\tilde{B}'(\tilde{C}')^{-1} + \tilde{B}'(\tilde{C}')^{-1}S^T - q\tilde{B}'(\tilde{C}')^{-1}. \quad (53)$$

Introducing $R := \tilde{B}'(\tilde{C}')^{-1}$, we obtain that $4\tilde{A} = SR + RS^T - qR$. Another useful information can be extracted from the fact that \tilde{A} is symmetric: splitting R into symmetric and skew-symmetric parts, $R = R_{\text{sym}} + R_{\text{asym}}$, we observe that

$$SR_{\text{asym}} + R_{\text{asym}}S^T - qR_{\text{asym}} = 0,$$

wherefrom, accounting for an explicit form of S , the first two terms are mutually canceled, and we obtain that for $q \neq 0$,

$$R_{\text{asym}} = 0,$$

that is,

$$\tilde{B}'(\tilde{C}')^{-1} = (\tilde{C}')^{-1}(\tilde{B}')^T, \quad \text{or} \quad \tilde{C}'\tilde{B}' = (\tilde{B}')^T\tilde{C}', \quad (54)$$

which in the component form results in the condition

$$0 = c'_{12}(c'_{11} + c'_{22}) - c'_{11} + c'_{12}b'_{22} - c'_{22}b'_{21}. \quad (55)$$

Because $\tilde{B} = S - \frac{q}{2}\text{Id}$, equation (50) in terms of the matrices S , \tilde{A} , \tilde{A}' , and \tilde{B}' reads

$$S\tilde{A}' + \tilde{A}'S^T - \tilde{B}'\tilde{A} - \tilde{A}(\tilde{B}')^T = 2q\tilde{A}'.$$

We solve this equation as follows. It follows from this equation that

$$\tilde{A}'[S^T - q\text{Id}] = \tilde{B}'\tilde{A} + S_A, \quad S_A = \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix} \quad (56)$$

for some $z \in \mathbb{C}$.

Consider now the equation $[\tilde{\varrho}_1(e_0), \tilde{\varrho}_1(e_1)] = \tilde{\varrho}_1(e_1)$. Substituting the form (46) for the respective matrices, we obtain

$$(\tilde{B} - (b_{33} + 1)\text{Id})\mathbf{v}^T - 4\tilde{A}\mathbf{c}^T = 0, \quad (57)$$

$$\mathbf{u}(\tilde{B} - (b_{33} + 1)\text{Id}) - 4a_{33}\mathbf{c} = 0, \quad (58)$$

$$\mathbf{c}[\tilde{B} + (b_{33} + 1)\text{Id}] = 0, \quad (59)$$

$$[\tilde{B} + (b_{33} - 1)\text{Id}]\mathbf{a}^T - \tilde{A}\mathbf{u}^T - a_{33}\mathbf{v}^T = 0. \quad (60)$$

5th postulate. $\mathbf{c} \neq 0$.

Then, \mathbf{c} has to be in the kernel of the matrix $[S + (1 - q)\text{Id}]$. From the form of the matrix S we have that

$$\det(S + \lambda\text{Id}) = \lambda^2 - [c'_{12}]^2 + c'_{11}c'_{22}. \quad (61)$$

6th postulate. $A_{33} \neq 0$, and ± 1 are not eigenvalues of S . From (61), this implies $q \notin \{0, 2\}$.

The above system of equations become

$$[S - \text{Id}]\mathbf{v}^T = 4\tilde{A}\mathbf{c}^T, \quad (62)$$

$$-\mathbf{u}[S + \text{Id}] = 4a_{33}\mathbf{c}, \quad (63)$$

$$\mathbf{c}[S + (1 - q)\text{Id}] = 0, \quad (64)$$

$$[S - (1 + q)\text{Id}]\mathbf{a}^T = \tilde{A}\mathbf{u}^T + a_{33}\mathbf{v}^T. \quad (65)$$

Acting by $[S + (1 - q)\text{Id}]$ on (63) from the left, we obtain zero on the both sides; because $S + \text{Id}$ is non-degenerate by our 6th postulate, and commutes with $[S + (1 - q)\text{Id}]$, the vector \mathbf{u} is also a zero vector of $[S + (1 - q)\text{Id}]$ and it is therefore nonzero itself and proportional to \mathbf{c} , so

$$\mathbf{u} = -\frac{4a_{33}}{q}\mathbf{c}. \quad (66)$$

We next transform (65) using (66) and (62) we have that

$$\tilde{A}\mathbf{u}^T = -\frac{4a_{33}}{q}\tilde{A}\mathbf{c}^T = -\frac{a_{33}}{q}[S - \text{Id}]\mathbf{v}^T$$

and (65) becomes

$$[S - (q + 1)\text{Id}]\left[\mathbf{a}^T + \frac{a_{33}}{q}\mathbf{v}^T\right] = 0. \quad (67)$$

7th postulate. $\det(S - (q + 1)\text{Id}) \neq 0$.

Then

$$\mathbf{a} = -\frac{a_{33}}{q}\mathbf{v} = \left(-\frac{a_{33}}{q}\mathbf{v}_1, -\frac{a_{33}}{q}\mathbf{v}_2\right) = \left(-\frac{a_{33}}{q}\left(1 - \frac{q}{2}\right), +a'_{33}\right). \quad (68)$$

The condition $a'_{33} = \mathbf{a}_2$ is satisfied automatically, whereas the condition $a_{33} = \mathbf{a}_1$ implies

$$\boxed{q = -2}. \quad (69)$$

From now on, we set $q = -2$ in all further calculations. Note that thus the chosen q implies the non-degeneracy of the above determinants $\det(S + \lambda\text{Id})$ with $\lambda = -1, 1$, and with $\lambda = 1 + q = -1$.

Because $S + 3\text{Id}$ has a nonzero kernel, we have that

$$\det(S + 3\text{Id}) = 9 - [c'_{12}]^2 + c'_{11}c'_{22} = 0. \quad (70)$$

We now express a_{ij} from the formula (53). After a few computations we obtain

$$4a_{11} = -\frac{1}{9}\left[2(1 - c'_{12})(-[c'_{22}]^2 - [c'_{12}]^2 + c'_{12}) + 2c'_{11}(c'_{12}(c'_{11} + c'_{22}) - c'_{11})\right], \quad (71)$$

$$4a_{12} = -\frac{1}{9}\left[c'_{11}(-c'_{12}b'_{21} + c'_{11}b'_{22}) - c'_{22}(-[c'_{22}]^2 - [c'_{12}]^2 + c'_{12}) + 2(c'_{12}(c'_{11} + c'_{22}) - c'_{11})\right], \quad (72)$$

$$4a_{22} = -\frac{1}{9}\left[2(c'_{12} + 1)(-c'_{12}b'_{21} + c'_{11}b'_{22}) - 2c'_{22}(c'_{12}(c'_{11} + c'_{22}) - c'_{11})\right]. \quad (73)$$

Finally, let us consider the last remaining condition $[\tilde{\varrho}_1(e'_0), \tilde{\varrho}_1(e_1)] = 0$. We obtain the system

$$[\tilde{B}' - b'_{33}\text{Id}]\mathbf{v}^T = 4\tilde{A}'\mathbf{c}^T - 4c'_{33}\mathbf{a}^T,$$

$$\mathbf{u}[\tilde{B}' - b'_{33}\text{Id}] = 4\mathbf{a}\tilde{C}' - 4a'_{33}\mathbf{c},$$

$$\mathbf{c}[\tilde{B}' + b'_{33}\text{Id}] = \mathbf{v}\tilde{C}' + \mathbf{u}c'_{33},$$

$$[\tilde{B}' + b'_{33}\text{Id}]\mathbf{a}^T = \tilde{A}'\mathbf{u}^T + a'_{33}\mathbf{v}^T.$$

Because $\mathbf{a} + \frac{a_{33}}{q}\mathbf{v} = 0$, $2\mathbf{u} = 4a_{33}\mathbf{c}$, $q = -2$, and $-4a_{33}c'_{33} = qb'_{33}$, the above system simplifies dramatically. Two out of four equations become redundant, and the only nontrivial equations that survive are

$$\mathbf{u}\tilde{B}' = 4\mathbf{a}\tilde{C}', \quad (74)$$

$$\tilde{B}'\mathbf{a}^T = \tilde{A}'\mathbf{u}^T. \quad (75)$$

We add to this system equation (65), which, upon implementing (68) becomes

$$\mathbf{a}[S^T - \text{Id}] = \mathbf{u}\tilde{A}. \quad (76)$$

8th postulate. $\det B' \neq 0$.

Recall that $\mathbf{u}[S + 3\text{Id}] = \mathbf{c}[S + 3\text{Id}] = 0$. From equations (74) and (76), we obtain that

$$\mathbf{a}[S^T - \text{Id} - 4\tilde{C}'(\tilde{B}')^{-1}\tilde{A}] = 0.$$

Substituting (53), we obtain

$$\begin{aligned} & \mathbf{a}\left(S^T - \text{Id} - \tilde{C}'(\tilde{B}')^{-1}[S\tilde{B}'(\tilde{C}')^{-1} + \tilde{B}'(\tilde{C}')^{-1}S^T + 2\tilde{B}'(\tilde{C}')^{-1}]\right) \\ &= \mathbf{a}[-3\text{Id} - \tilde{C}'(\tilde{B}')^{-1}S\tilde{B}'(\tilde{C}')^{-1}] = \mathbf{a}\tilde{C}'(\tilde{B}')^{-1}[-3\text{Id} - S]\tilde{B}'(\tilde{C}')^{-1} \\ &= -\frac{1}{4}\mathbf{u}[S + 3\text{Id}]\tilde{B}'(\tilde{C}')^{-1} = 0, \end{aligned}$$

by (59), so the condition (74) is satisfied automatically.

From (74) and (75), we obtain that

$$\mathbf{a}[(\tilde{B}')^T - 4\tilde{C}'(\tilde{B}')^{-1}\tilde{A}] = 0. \quad (77)$$

Because $\tilde{A}'[2\text{Id} + S^T] = \tilde{B}'\tilde{A} + S_A$, see (56), multiplying from the right by $[2\text{Id} + S^T]$, we can perform the chain of transformations – where we have that $\tilde{B}'(\tilde{C}')^{-1}$ is symmetric –

$$\begin{aligned} 0 &= \mathbf{a}\left((\tilde{B}')^T[2\text{Id} + S^T] - 4\tilde{C}'\tilde{A} - 4\tilde{C}'(\tilde{B}')^{-1}S_A\right) \\ &= \mathbf{a}\tilde{C}'(\tilde{B}')^{-1}\left(\tilde{B}'[(\tilde{C}')^{-1}(\tilde{B}')^T][2\text{Id} + S^T] - 4\tilde{B}'\tilde{A} - 4S_A\right) \\ &= \mathbf{a}\tilde{C}'(\tilde{B}')^{-1}\left((\tilde{B}')^2(\tilde{C}')^{-1}[2\text{Id} + S^T] - \tilde{B}'[S\tilde{B}'(\tilde{C}')^{-1} + \tilde{B}'(\tilde{C}')^{-1}S^T + 2\tilde{B}'(\tilde{C}')^{-1}] - 4S_A\right) \\ &= \frac{1}{4}\mathbf{u}\left[-\tilde{B}'S\tilde{B}'(\tilde{C}')^{-1} - 4S_A\right] \\ &= \frac{1}{4}\mathbf{u}\left[-\tilde{B}'S - 4S_A\tilde{C}'(\tilde{B}')^{-1}\right]\tilde{B}'(\tilde{C}')^{-1}, \end{aligned}$$

that is, both $S + 3\text{Id}$ and $\tilde{B}'S + 4S_A\tilde{C}'(\tilde{B}')^{-1}$ share the same null vector \mathbf{u} . This imposes two more constraints on the matrix elements of $\tilde{B}'S\tilde{B}' + 4S_A\tilde{C}'$. Note that $S_A\tilde{C}' = -zS^T$, so both columns of the matrix $\tilde{B}'S\tilde{B}' - 4zS^T$ must be proportional to $(3 - c'_{12}, -c'_{22})^T$.

9th postulate. The row vector $(3 - c'_{12}, c'_{22})$ is non-zero.

Hence we obtain

$$-c'_{22}(-c'_{22})^2 + (9 - c'_{12})b'_{21} + 4zc'_{12} = (b'_{21}c'_{12}c'_{22} + b'_{22}(c'_{22})^2 + [b'_{21}]^2c'_{11} + b'_{21}b'_{22}c'_{12} - 4zc'_{11})(3 - c'_{12}) \quad (78)$$

and

$$\begin{aligned} & -c'_{22}(c'_{22}(c'_{12} - 1) + (9 - c'_{12})b'_{22} + 4zc'_{22}) \\ &= (-b'_{21}c'_{12}(c'_{12} - 1) - b'_{22}c'_{22}(c'_{12} - 1) + b'_{21}b'_{22}c'_{11} + [b'_{22}]^2c'_{12} - 4zc'_{12})(3 - c'_{12}). \quad (79) \end{aligned}$$

The last set of relations comes from (56). Here, we use that the matrix elements of \tilde{A} and \tilde{A}' are not independent. We obtain four equations:

$$(3 - 2c'_{12})a_{12} + c'_{11}a_{22} + c'_{22}a_{11} = 0, \quad (80)$$

$$(-c'_{12} + 5)a_{22} + c'_{11}a'_{22} - b'_{21}a_{11} - b'_{22}a_{12} = 0, \quad (81)$$

$$(c'_{12} + 2)a'_{22} - c'_{22}a_{22} - b'_{21}a_{12} - b'_{22}a_{22} = 0, \quad (82)$$

$$z = (2c'_{12} - 3)a_{22} + c'_{11}a'_{22} - c'_{12}a_{12}. \quad (83)$$

We now have eleven (inhomogeneous and, in principle, nonlinear) equations on the ten variables a_{11} , a_{12} , a_{22} , a'_{22} , b'_{21} , b'_{22} , z , c'_{11} , c'_{22} , and c'_{12} . Equations (71)–(73) and (83) are substitutions for a_{11} , a_{12} , a_{22} , and z . Equations (55) and (80) constitute a linear system determining b'_{21} and b'_{22} . We then determine a'_{22} from (81) and c'_{22} from the determinant equation (70) and substitute all the obtained expressions into the remaining three nonlinear equations (82), (78) and (79) on two variables c'_{11} and c'_{12} using MAPLE⁸. It turns out that (82) then factorizes into two factors, one rather large factor and the another is $9 - (c'_{12})^2 - (c'_{11})^2$. For the huge factor, its resultant with equations (78) and (79) is empty, whereas for the second factor we obtain a nonempty set of solutions. We now describe these solutions.

Let $9 - (c'_{12})^2 - (c'_{11})^2 = 0$. Then, using (70), either $c'_{11} = 0$, which results in inconsistencies in all the above calculations, or

$$c'_{11} + c'_{22} = 0,$$

which we assume in what follows. The substitutions then give

$$a_{11} = -\frac{1}{9}(5c'_{12} - 9), \quad a_{12} = \frac{2}{9}c'_{11}, \quad a_{22} = -\frac{1}{9}(c'_{12} - 3) \quad (84)$$

and

$$b'_{21} = 1 - \frac{c'_{12}}{3}, \quad b'_{22} = \frac{c'_{11}}{3}, \quad a'_{22} = 0. \quad (85)$$

Equation (82) is satisfied identically, expressing z out of (83), we obtain

$$z = -\frac{1}{9}(2c'_{12} - 3)(c'_{12} - 3) - \frac{2}{9}c'_{12}c'_{11},$$

and substituting all the above quantities into the matrix $\tilde{B}'S\tilde{B}' - 4zS^T$, we obtain that all matrix elements are multiplied by the same factor:

$$\tilde{B}'S\tilde{B}' - 4zS^T = \beta \begin{pmatrix} -c'_{12} & c'_{11} \\ c'_{11} & c'_{12} \end{pmatrix}, \quad \beta = 2(c'_{12} - 3)^2 + 2c'_{12}c'_{11}.$$

So, the only possibility for this matrix to be degenerate is to set $\beta = 0$. Then this matrix vanishes and all vectors are its null vectors. We therefore have that

$$c'_{11} = -\frac{(c'_{12} - 1)^2}{c'_{12}}, \quad (86)$$

and since we require $c'_{12} - 3 \neq 0$, using (70) we obtain a cubic equation determining c'_{12}

$$(3 - c'_{12})^3 = (c'_{12} + 3)[c'_{12}]^2 \quad \text{or} \quad 2[c'_{12}]^3 - 6[c'_{12}]^2 + 27c'_{12} - 27 = 0. \quad (87)$$

This equation admits one real root $c'_{12} = 1 + \left[1 + \frac{3}{4}\sqrt{78}\right]^{1/3} + \left[1 - \frac{3}{4}\sqrt{78}\right]^{1/3} \approx 1.1898$ and two complex conjugate roots $c'_{12} \approx 0.9051 \pm 3.2445i$. If we set $c'_{12} = 3\zeta$, we find that ζ are the roots of

$$P(\zeta) = 2\zeta^3 - 2\zeta^2 + 3\zeta - 1 \quad (88)$$

⁸We acknowledge a valuable help of Misha Shapiro at this part of the work.

Collecting all previous results and calling $\alpha = a_{33}$, we find that all coefficients of the matrices (A, B, C) as explicit elements in $\mathbb{Q}(\zeta) + \alpha\mathbb{Q}(\zeta)$. We can use (88) to express elements of $\mathbb{Q}(\zeta)$ as degree two polynomials in ζ . The result is the one announced in Lemma 7.5. Note that since the matrix $\tilde{B}'S\tilde{B}' - 4zS^T$ is null, from (77) we have another nice relation

$$\tilde{A}' = \frac{1}{4}\tilde{B}'(\tilde{C}')^{-1}(\tilde{B}')^T.$$

We have thus found a classical Airy structure – as can be directly checked. Lemma 2.3 tells us the quantum Airy structures which projects to this classical Airy structure are

$$D^i = \frac{1}{2}\mathrm{Tr} B^i + \Delta^i,$$

where Δ is an arbitrary elements of V^* such that Δ is zero on the space of commutators in the Lie algebra. As the latter is spanned by L'_0 and L_1 , we deduce that $\Delta^0 = \Delta^1 = 0$ while Δ^0 remains arbitrary.

These quantum Airy structures determine an (adjoint) module structure of V on \mathcal{D}_V , and the cohomology spaces $H^p(V, \mathcal{D}_V)$ for $p \in \{0, 1, 2\}$ can be computed with Lemma 7.3. We are indeed in a three-dimensional situation with a codimension 1 ideal \mathfrak{i} spanned by $(x_1, x_2) = (L_1, L'_0)$, the extra generator being $y = L_0 \in \mathfrak{q}$, and commutation relations (42) determined by

$$a = 0, \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \mathrm{diag}(-1, 2)$$

The result is that $\dim H_L^0(V, \mathcal{D}_V) = 1$ corresponding to the constant in \mathcal{D}_V , $\dim H^1 = 2$ with one generator corresponding to deformation by the constant Δ^0 , and $\dim H^2 = 1$. \square

8 Four classes of examples from geometry

In the remaining of the paper, we consider examples of quantum Airy structures which have a stronger geometric flavor – in relation with topological quantum field theories and enumerative geometry.

8.1 From Frobenius algebras (2d TQFTs)

8.1.1 Quantum Airy structures

Let \mathbb{A} be a Frobenius algebra (not necessarily unital), *i.e.* a finite-dimensional vector space together with a commutative, associative product $\mathbb{A} \otimes \mathbb{A} \rightarrow \mathbb{A}$ and a linear form $\varphi : \mathbb{A} \rightarrow \mathbb{C}$ such that the pairing $\mathbb{A} \otimes \mathbb{A} \rightarrow \mathbb{C}$ defined by $\langle v_1, v_2 \rangle = \varphi(v_1 v_2)$ is non-degenerate. We recall that

Lemma 8.1 [1] *If the product is given, all the other Frobenius algebra structures on \mathbb{A} are of the form $\tilde{\varphi}(v) := \varphi(vu)$ for some invertible element $u \in \mathbb{A}$.*

Let $(e_i)_{i \in I}$ be a basis, and $(e_i^*)_{i \in I}$ the basis such that

$$\langle e_i, e_j^* \rangle = \delta_{i,j}.$$

Proposition 8.2 *For any $\theta_A, \theta_B, \theta_C \in \mathbb{A}$*

$$\begin{aligned} A_{j,k}^i &= \varphi(e_i^* e_j^* e_k^* \theta_A). \\ B_{j,k}^i &= \varphi(e_i^* e_j^* e_k \theta_B). \\ C_{j,k}^i &= \varphi(e_i^* e_j e_k \theta_C). \end{aligned}$$

and arbitrary D^i defines a quantum Airy structure on $V = \mathbb{A}$. It has $f_{j,k}^i = 0$, i.e. the underlying Lie algebra is abelian.

Proof. Commutativity (resp. invariance) of the product shows that $B_{j,k}^i = B_{i,k}^j$ (resp. A is fully symmetric). Therefore, we just need to check the **BB-AC**, **BC** and **BA** relations of Section 2.2. By definition of the dual basis and the pairing we remark that

$$\forall v_1, v_2 \in \mathbb{A}, \quad v_1 v_2 = \varphi(v_1 v_2 e_a) e_a^*. \quad (89)$$

We recall the matrix notations $X^i = (X_{j,k}^i)_{j,k}$. Using invariance, associativity and commutativity of the product, we find

$$(B^i B^j)_{k,\ell} = \varphi(e_i^* e_k^* e_a \theta_B) \varphi(e_j^* e_\ell^* e_a \theta_B) = \varphi(e_i^* e_j^* e_k^* e_\ell^* \theta_B^2), \quad (90)$$

and likewise

$$(B^i A^j)_{k,\ell} = \varphi(e_i^* e_k^* e_j^* e_\ell \theta_B \theta_A), \quad (C^i B^j)_{k,\ell} = \varphi(e_i^* e_k e_j^* e_\ell \theta_C \theta_B).$$

These expressions are completely symmetric under permutation of i and j . So, pairs of terms in the **BB-AC**, **BC** and **BA** relations cancel each other. \square

As we see from the proof, this example provides a rather trivial solution of the three relations in the sense that the three terms in each of them are already symmetric in i and j .

Up to a change of basis in \mathbb{A} , we can and will assume in the remaining of this section that $(e_i)_{i \in I}$ is an orthonormal basis. We will sometimes need the element

$$H := e_a^2.$$

Let us make a last observation, in case θ_A is invertible. Using the symmetry (23) with $u_{i,j} = \varphi(e_i e_j \nu)$ and choosing $\nu = -\theta_B / \theta_A$, we find an equivalent quantum Airy structure with $\tilde{B} = 0$ and

$$\begin{aligned} \tilde{A}_{j,k}^i &= \varphi(e_i e_j e_k \theta_A). \\ \tilde{C}_{j,k}^i &= \varphi(e_i e_j e_k (\theta_C - \theta_A^{-1} \theta_B^2)). \\ \tilde{D}^i &= D^i - \frac{1}{2} \varphi(e_i H \theta_B). \end{aligned}$$

In the special case $\theta_B^2 = \theta_A \theta_C$, we have $\tilde{C} = 0$. On the contrary, if $\theta_B^2 \neq \theta_A \theta_C$, we cannot *a priori* get rid of \tilde{C} with the group action while keeping $\tilde{B} = 0$.

We can illustrate this phenomenon in a simple way when \mathbb{A} has dimension 1. In this case, we can perform a rescaling of x (this transformation is in \mathcal{G}_V) to achieve $\tilde{\theta}_A = 1$, and conjugation by an exponential of the Laplacian as above to achieve $\tilde{\theta}_B = 0$. However, when $\theta_B^2 - \theta_A \theta_C \neq 0$, we do not have further freedom to modify $\tilde{\theta}_C$. Thus, $\theta_B^2 - \theta_A \theta_C$ is a continuous parameter of deformation for \mathcal{G}_V -orbits. If we allowed \hbar -rescalings – this is not in the group \mathcal{G}_V – we could in fact achieve $\tilde{\theta}_C = 0$ or 1, so this deformation parameter coincides with \hbar -rescalings.

The action by the group of invertible elements of \mathbb{A} deforming the Frobenius structure according to Lemma 8.1 also provides a family of commuting flows on the moduli space of quantum Airy structures on the abelian Lie algebra \mathbb{A} , although we do not make statements about the independence of these flows.

The classical Airy structure is in this case simply given by the infinitesimal symplectomorphisms $(\mathcal{L}_i)_{i \in I}$, where

$$\begin{aligned} \mathcal{L}_i e_j^* &= -\langle \theta_B \cdot e_i^* \cdot e_j, e_a^* \rangle e_a^* - \langle \theta_C \cdot e_i^* \cdot e_j, e_a \rangle e_a \\ \mathcal{L}_i e_j &= -\langle \theta_A \cdot e_i^* \cdot e_j^*, e_a^* \rangle e_a^* + \langle \theta_B \cdot e_i^* \cdot e_j^*, e_a \rangle e_a \end{aligned}$$

The commutativity of the product directly implies that the commutation of the \mathcal{L}'_i 's together with the full symmetry of A and symmetry of B with respect to the two first indices.

8.1.2 Partition function for $\dim \mathbb{A} = 1$

Quantum Airy structures on a vector space of dimension 1 are just differential operators of the form

$$L = \hbar \partial_x - \frac{\theta_A x^2}{2} - \hbar \theta_B x \partial_x - \frac{\theta_C}{2} \partial_x^2 - \hbar D,$$

where $\theta_A, \theta_B, \theta_C, D$ are scalars. The differential equation $L \cdot Z = 0$ can be solved by elementary means to obtain the partition function. We have to distinguish several cases, all related by limiting procedures, which we will not discuss.

$$\boxed{\theta_C = 0}$$

We have $Z(x) = \exp\left(\frac{S_0(x)}{\hbar} + S_1(x)\right)$ with

$$S_0(x) = \frac{\theta_A}{2\theta_B^2} \left(-\ln(1 - \theta_B x) - \theta_B x - \frac{\theta_B^2 x^2}{2} \right), \quad S_1(x) = -\frac{D}{\theta_B} \ln(1 - \theta_B x).$$

The answer can be obtained combinatorially, as a corollary to Proposition 5.1. This formula still makes sense when $\theta_B = 0$, it becomes $Z(x) = \exp\left(\frac{\theta_A x^3}{6\hbar} + Dx\right)$.

$$\boxed{\theta_C \neq 0 \text{ and } \theta_B^2 = \theta_A \theta_C}$$

We find $Z(x) = \mathfrak{Z}(x)/\mathfrak{Z}(0)$ with

$$\begin{aligned} \mathfrak{Z}(x) &= \exp\left\{\frac{1}{\hbar}\left(\frac{x}{\theta_C} - \frac{\theta_B x^2}{2\theta_C}\right)\right\} \text{Bi}\left(\frac{1-2\theta_B x + \hbar\theta_C(\theta_B-2D)}{(2\theta_B\theta_C\hbar)^{2/3}}\right) \\ &= c_\hbar \exp\left\{\frac{1}{\hbar}\left(\frac{x}{\theta_C} - \frac{\theta_B x^2}{2\theta_C}\right)\right\} \int dt \exp\left(\frac{1}{\hbar}\left(\frac{1-2\theta_B x + \hbar\theta_C(\theta_B-2D)}{(2\theta_B\theta_C)^{2/3}} t - \frac{t^3}{3}\right)\right). \end{aligned}$$

where c_\hbar only depends on \hbar . Here $\text{Bi}(z)$ is the Bairy function, which is the solution of the differential equation $\text{Bi}''(z) = z\text{Bi}(z)$, whose asymptotics at $z \rightarrow \infty$ is

$$\text{Bi}(z) = \frac{e^{\frac{2}{3}z^{\frac{3}{2}}}}{\pi^{\frac{1}{2}}z^{\frac{1}{4}}}(1 + o(1)).$$

In particular, $Z(x) = \exp\left(\frac{S_0(x)}{\hbar} + S_1(x) + O(\hbar)\right)$ with

$$\begin{aligned} S_0(x) &= \frac{(1 - 2\theta_B x)^{\frac{3}{2}} - 1 + \frac{x}{\theta_C} - \frac{\theta_B x^2}{2\theta_C}}{3\theta_B\theta_C} \\ S_1(x) &= \frac{\theta_B - 2D}{2\theta_B} \left((1 - 2\theta_B x)^{\frac{1}{2}} - 1 \right) - \frac{1}{4} \ln(1 - 2\theta_B x). \end{aligned} \quad (91)$$

The integral above is a formal integral, *i.e.* it is to be evaluated by expansion around the (unique) saddle point which realizes $F_{0,3} = \theta_A$. However, choosing the corresponding steepest descent contour offers the possibility to define $Z(x)$ here as an entire function of x . This is also what the Bairy function does. We retrieve the Taylor coefficients $F_{g,n}$ by expanding $\text{Bi}(z)$ at $z \rightarrow \infty$. Elementary properties of the full asymptotic expansion of the Bairy function are collected in Appendix A.2.

$$\boxed{\theta_C \neq 0 \text{ and } \sigma^2 := \theta_B^2 - \theta_A \theta_C \neq 0}$$

We find that $Z(x) = \mathfrak{Z}(x)/\mathfrak{Z}(0)$ with

$$\begin{aligned} \mathfrak{Z}(x) &= \exp\left\{\frac{1}{\hbar}\left(\frac{x}{\theta_C} - \frac{\theta_B x^2}{2\theta_C}\right) - \frac{1}{2}\ln\left(1 - \frac{\sigma^2 x}{\theta_B}\right)\right\} \mathcal{W}_M\left(\frac{1}{4}\left(\frac{\theta_A}{\hbar\sigma^3} + \frac{2D - \theta_B}{\sigma}\right); \frac{1}{4}; \frac{\theta_B^2}{\hbar\sigma^3\theta_C}\left(1 - \frac{\sigma^2 x}{\theta_B}\right)^2\right) \\ &= c_\hbar \exp\left\{\frac{1}{\hbar}\left(\frac{x}{\theta_C} - \frac{\theta_B x^2}{2\theta_C}\right) + \ln\left(1 - \frac{\sigma^2 x}{\theta_B}\right)\right\} \int_{-\frac{1}{2}}^{\frac{1}{2}} dt \exp\left\{\frac{\theta_B^2}{\hbar\sigma^3\theta_C}\left(1 - \frac{\sigma^2 x}{\theta_B}\right)^2 t\right\} \left(\frac{1}{4} - t\right)^{-\frac{1}{4}} \left(\frac{1-t}{\frac{1}{2}+t}\right)^{\frac{1}{4}} \left(\frac{\theta_A}{\hbar\sigma^3} + \frac{2D - \theta_B}{\sigma}\right). \end{aligned} \quad (92)$$

Here $\mathcal{W}_M(\mu; \nu; z)$ is the Whittaker-M function, solving the differential equation

$$\left\{ \partial_z^2 + \left(-\frac{1}{4} + \frac{\mu}{z} + \frac{\frac{1}{4} - \nu^2}{z^2} \right) \right\} \mathcal{W}_M(\mu; \nu; z) = 0.$$

It is an entire function of z , but only its asymptotics near $z \rightarrow \infty$ matter to obtain the $F_{g,n}$ s. These asymptotics can be found by saddle point analysis in the integral formula, which leads to $Z(x) = \exp\left(\frac{S_0(x)}{\hbar} + S_1(x) + O(\hbar)\right)$ and

$$\begin{aligned} S_0(x) &= \frac{\theta_B}{2\sigma^2\theta_C} (1 - \frac{\sigma^2 x}{\theta_B}) (1 - 2\theta_B x + \sigma^2 x^2)^{\frac{1}{2}} + \frac{\theta_A}{2\sigma^3} \operatorname{argth} \left(\frac{\sigma}{\theta_B} \frac{(1 - 2\theta_B x + \sigma^2 x^2)^{\frac{1}{2}}}{1 - \frac{\sigma^2 x}{\theta_B}} \right) + \frac{x}{\theta_C} - \frac{\theta_B x^2}{2\theta_C} + c_0. \\ S_1(x) &= -\frac{1}{4} \ln(1 - 2\theta_B x + \sigma^2 x^2) + \frac{2D - \theta_B}{2\sigma} \operatorname{argth} \left(\frac{\sigma}{\theta_B} \frac{(1 - 2\theta_B x + \sigma^2 x^2)^{\frac{1}{2}}}{1 - \frac{\sigma^2 x}{\theta_B}} \right) + c_1. \end{aligned}$$

This formula still makes sense for $\theta_B = 0$, the main simplification being that the variable of the Whittaker function becomes $z \leftarrow \frac{\sigma x^2}{\hbar\theta_C}$. In fact, as a result of Proposition 2.1, $F_{g,n}$ is a polynomial in $\theta_A, \theta_B, \theta_C$ and D . The first values are given in Appendix A.3.

8.1.3 Partition function in general

In this paragraph, \mathbb{A} is assumed unital. When $\dim \mathbb{A} > 1$, we are going to show that the partition function for the quantum Airy structure of Proposition 8.2 is computed by a higher-dimensional version of the integral formulas of the previous paragraph. We heavily rely on the commutativity of the product in \mathbb{A} . We denote

$$D := D^a e_a \in \mathbb{A}.$$

An easy rewriting of Proposition 5.1 in terms of the linear form φ leads to

Lemma 8.3 *If $\theta_C = 0$, then we have that $Z = \exp\left(\frac{S_0(x)}{\hbar} + S_1(x)\right)$ with*

$$\begin{aligned} S_0(x) &= \varphi \left(\frac{\theta_A}{2\theta_B^3} \left(-\ln(1 - \theta_B x) - \theta_B x - \frac{\theta_B^2 x^2}{2} \right) \right). \\ S_1(x) &= \varphi \left(-\frac{D}{\theta_B} \ln(1 - \theta_B x) \right). \end{aligned}$$

This formula also makes sense when θ_B is not invertible. □

When $\theta_C \neq 0$, we will check the formulas below by direct computation with help of Dyson-Schwinger equations. Other proofs could be obtained exploiting the action of \mathfrak{G}_V on the partition function of Lemma 8.3.

Proposition 8.4 *If θ_C is invertible, and $\theta_B^2 = \theta_A \theta_C$, we have $Z(x) = \mathfrak{Z}(x)/\mathfrak{Z}(0)$ with*

$$\mathfrak{Z}(x) = \operatorname{Cste}_{\hbar} \exp \left\{ \frac{1}{\hbar} \varphi \left(\frac{x}{\theta_C} - \frac{\theta_B x^2}{2\theta_C} \right) \right\} \int \exp \left\{ \frac{1}{\hbar} \varphi \left(\frac{1 - 2\theta_B x + \hbar\theta_C(\theta_B - 2D)}{(2\theta_B\theta_C)^{2/3}} t - \frac{t^3}{3} \right) \right\} dt, \quad (93)$$

where dt is the Lebesgue measure on the linear coordinates on \mathbb{A} with respect to an orthonormal basis.

Proposition 8.5 *If $\theta_C \neq 0$ and $\sigma = \theta_B^2 - \theta_A \theta_C$ is invertible, we have $Z(x) = \mathfrak{Z}(x)/\mathfrak{Z}(0)$ with*

$$\begin{aligned} \mathfrak{Z}(x) &= \exp \left\{ \frac{1}{\hbar} \varphi \left(\frac{x}{\theta_C} - \frac{\theta_B x^2}{2\theta_C} \right) t + H \ln(1 - \frac{\sigma^2 x}{\theta_B}) \right\} \int \exp \left\{ \varphi \left(\frac{\theta_B^2}{\hbar\sigma^3\theta_C} (1 - \frac{\sigma^2 x}{\theta_B}) \right. \right. \\ &\quad \left. \left. + \frac{1}{4} \left(\frac{\theta_A}{\hbar\sigma^3} + \frac{2D - H\theta_B}{\sigma} - H \right) \ln\left(\frac{1}{2} - t\right) + \frac{1}{4} \left(-\frac{\theta_A}{\hbar\sigma^3} - \frac{2D - H\theta_B}{\sigma} - H \right) \ln\left(\frac{1}{2} + t\right) \right\} dt, \end{aligned}$$

and we recall that $H = e_a^2$. This formula also makes sense when θ_B is not invertible.

The intermediate cases in Proposition 8.4, when $\theta_C \neq 0$, but not invertible, or in Proposition 8.5 when $\sigma \neq 0$, but not invertible, will not be discussed, since they can be obtained by limiting procedures. For the same reason, we will only write a proof of Proposition 8.5. It is in fact a good exercise for the reader to repeat the scheme of this proof in the specific (and simpler) case of Proposition 8.4.

Proof. We consider a formal integral of the form

$$\mathbb{I}(x) = \exp \left\{ \varphi \left(\alpha_1 x + \frac{\alpha_2 x^2}{2} + \nu \ln(1 - \beta x) \right) \right\} \int \exp \left\{ \varphi \left(\gamma(1 - \beta x)^2 t + \tau_1 \ln\left(\frac{1}{2} - t\right) + \tau_2 \ln\left(\frac{1}{2} + t\right) \right) \right\} dt,$$

where $\alpha_1, \alpha_2, \nu, \gamma, \beta, \tau_1, \tau_2 \in \mathbb{A}$ are considered as parameters. If $f : \mathbb{A} \rightarrow \mathbb{K}$, we denote

$$\langle f(t) \rangle := \frac{\int f(t) \exp \left\{ \varphi \left(\gamma(1 - \beta x)^2 t + \tau_1 \ln\left(\frac{1}{2} - t\right) + \tau_2 \ln\left(\frac{1}{2} + t\right) \right) \right\} dt}{\int \exp \left\{ \varphi \left(\gamma(1 - \beta x)^2 t + \tau_1 \ln\left(\frac{1}{2} - t\right) + \tau_2 \ln\left(\frac{1}{2} + t\right) \right) \right\} dt},$$

which implicitly depends on x . This definition can be extended by linearity to a function $f : \mathbb{A} \rightarrow \mathbb{A}$. We also introduce $\varphi_i(v) := \varphi(e_i v)$ for $v \in \mathbb{A}$.

Let us compute the action of the differential operators of the quantum Airy structure on $\mathbb{I}(x)$.

$$\begin{aligned} -\mathbb{I}^{-1} \hbar \partial_{x_i} \mathbb{I} &= \hbar \varphi_i \left(-\alpha_1 - \alpha_2 x + \frac{\nu \beta}{1 - \beta x} + 2\gamma \beta(t) \right). \\ \mathbb{I}^{-1} \frac{1}{2} \varphi(e_i e_a e_b \theta_A) x_a x_b \mathbb{I} &= \varphi_i \left(\frac{x^2}{2} \theta_A \right). \\ \mathbb{I}^{-1} \hbar \varphi(e_i e_a e_b \theta_B) x_a \partial_{x_b} \mathbb{I} &= \hbar \varphi(e_i x e_b \theta_B) \varphi \left(e_b \left(\alpha_1 + \alpha_2 x - \frac{\nu \beta}{1 - \beta x} - 2\gamma \beta(1 - \beta x)x(t) \right) \right) \\ &= \hbar \varphi_i \left(\theta_B x \left(\alpha_1 + \alpha_2 x - \frac{\nu \beta}{1 - \beta x} - 2\gamma \beta(1 - \beta x)x(t) \right) \right). \end{aligned}$$

In the last line, we exploited the property (89) of the pairing in an orthonormal basis, *i.e.* such that $e_i^* = e_i$. In a similar fashion

$$\begin{aligned} &\mathbb{I}^{-1} \frac{\hbar^2}{2} \varphi(e_i e_a e_b \theta_C) \partial_{x_a} \partial_{x_b} \mathbb{I} \\ &= \frac{\hbar^2}{2} \varphi(e_i e_a e_b \theta_C) \left\{ \varphi \left(e_a \left(\alpha_1 + \alpha_2 x - \frac{\nu \beta}{1 - \beta x} \right) \right) \varphi \left(e_b \left(\alpha_1 + \alpha_2 x - \frac{\nu \beta}{1 - \beta x} x \right) \right) + \varphi \left(\alpha_2 e_a e_b - \frac{\nu \beta^2}{(1 - \beta x)^2} e_a e_b \right) \right. \\ &\quad - 4\varphi \left(e_a \left(\alpha_1 + \alpha_2 x - \frac{\nu \beta}{1 - \beta x} \right) \right) \varphi \left(2\gamma \beta(1 - \beta x) e_b(t) \right) \\ &\quad \left. + \left(4\varphi \left(\gamma \beta(1 - \beta x) e_a t \right) \varphi \left(\gamma \beta(1 - \beta x) e_b t \right) \right) + 2\varphi \left(2\gamma \beta^2 e_a e_b(t) \right) \right\} \\ &= \frac{\hbar^2}{2} \varphi_i \left(\theta_C \left\{ \left(\alpha_1 + \alpha_2 x - \frac{\nu \beta}{1 - \beta x} \right)^2 + \left(\alpha_2 - \frac{\nu \beta^2}{(1 - \beta x)^2} \right) H - 4 \left(\alpha_1 + \alpha_2 x - \frac{\nu \beta}{1 - \beta x} \right) \gamma \beta(1 - \beta x) \langle t \rangle \right. \right. \\ &\quad \left. \left. + 4\gamma^2 \beta^2 (1 - \beta x)^2 \langle t^2 \rangle + 2\gamma \beta^2 H \langle t \rangle \right\} \right), \end{aligned}$$

and we observe that $\mathbb{I}^{-1} \hbar D^i \mathbb{I} = \varphi_i(D)$. Summing all the terms yields an expression for $\mathbb{I}^{-1} L_i \mathbb{I}$, which involves in particular $\langle t^2 \rangle$.

We now write the Dyson-Schwinger relations satisfied by the averages $\langle \cdot \rangle$. They express the fact that (formal) integrals are invariant under change of variable. Here we use the infinitesimal change of variable $t \rightarrow t + \varepsilon \left(\frac{1}{4} - t^2 \right) e_j$ for a given j . We remark that $dt \rightarrow dt \left(1 - \varepsilon \varphi(2t H e_j) + O(\varepsilon^2) \right)$, and invariance of the integral at order ε gives the exact formula

$$\left\langle \varphi \left(e_j \left\{ -2tH + \gamma(1 - \beta x)^2 \left(\frac{1}{4} - t^2 \right) - \tau_1 \left(\frac{1}{2} + t \right) + \tau_2 \left(\frac{1}{2} - t \right) \right\} \right) \right\rangle = 0.$$

This formula can equivalently be derived by integration by parts. As it holds for any j , we also have the identity in \mathbb{A}

$$-2H\langle t \rangle + \gamma(1 - \beta x)^2 \left(\frac{1}{4} - \langle t^2 \rangle \right) - \tau_1 \left(\frac{1}{2} + \langle t \rangle \right) + \tau_2 \left(\frac{1}{2} - \langle t \rangle \right) = 0.$$

We use this identity to eliminate $\langle t^2 \rangle$ from $-\mathbb{I}^{-1}L_i\mathbb{I}$. After tedious but straightforward algebra, the terms can be collected as follows

$$\begin{aligned} -\mathbb{I}^{-1}L_i\mathbb{I} &= \varphi_i \left(\frac{\hbar^2\theta_C}{2}(\alpha_1^2 + \alpha_2 H + 2\nu\alpha_2) - \hbar\alpha_1 + \hbar\nu\theta_B + D + \frac{\hbar^2\theta_C}{2} \left(\frac{\gamma^2\beta^2}{2} + \gamma\beta^2(\tau_2 - \tau_1) \right) \right. \\ &\quad + x \left\{ -\hbar\alpha_2 + \hbar\alpha_1\theta_B + \hbar^2\theta_C\alpha_1\alpha_2 - \hbar^2\theta_C\gamma^2\beta^3 \right\} + x^2 \left\{ \frac{\theta_A}{2} + \hbar\theta_B\alpha_2 + \frac{\hbar^2\theta_C}{2}(\alpha_2^2 + \gamma^2\beta^4) \right\} \\ &\quad + \frac{\nu\beta}{1-\beta x} \left\{ \hbar\beta - \hbar\theta_B - \hbar^2\theta_C(\alpha_1\beta + \alpha_2) \right\} + \frac{\hbar^2\theta_C\nu\beta^2}{2(1-\beta x)^2}(\nu - H) \\ &\quad + 2\langle t \rangle \left\{ \hbar\gamma\beta + \hbar^2\theta_C\gamma\beta(-\alpha_1 + \nu\beta - \beta(\tau_1 + \tau_2 + 2H)) \right\} \\ &\quad \left. + x\gamma\beta \left\{ -\beta - \hbar\theta_B + \hbar^2\theta_C(\beta\alpha_1 - \alpha_2) \right\} + x^2\hbar\gamma\beta^2 \left\{ \hbar\theta_C\alpha_2 + \theta_B \right\} \right). \end{aligned}$$

Imposing that the coefficients – inside φ_i – of 1 , x , x^2 , $(1 - \beta x)^{-1}$, $(1 - \beta x)^{-2}$, $\langle t \rangle$, $\langle t \rangle x$ and $\langle t \rangle x^2$ separately vanish uniquely fixes

$$\alpha_1 = \frac{1}{\hbar\theta_C}, \quad \alpha_2 = -\frac{\theta_B}{\hbar\theta_C}, \quad \beta = \frac{\sigma^2}{\theta_B}, \quad \gamma = \frac{\theta_B^2}{\hbar\sigma^3\theta_C},$$

and

$$\nu = H, \quad \tau_1 = \frac{1}{4} \left(\frac{\theta_A}{\hbar\sigma^3} + \frac{2D - H\theta_B}{\sigma} - H \right), \quad \tau_2 = -\frac{1}{4} \left(\frac{\theta_A}{\hbar\sigma^3} + \frac{2D - H\theta_B}{\sigma} + H \right).$$

We then have found a common solution of $L_i Z = 0$ for all $i \in I$, and one can check by saddle point analysis that, for the above choice of parameters, it is indeed of the form

$$\mathbb{I}(x) = \exp \left(\sum_{n \geq 1} \sum_{i_1, \dots, i_n} \frac{\hbar^{g-1}}{n!} F_{g,n}(i_1, \dots, i_n) x_{i_1} \cdots x_{i_n} \right), \quad F_{0,1}(i) = F_{0,2}(i, j) = 0.$$

Invoking the uniqueness of such a solution (Proposition 2.1) concludes the proof. \square

8.1.4 Example: TQFT partition function

It is well known that unital Frobenius algebras are in one-to-one correspondence with two-dimensional topological quantum field theories [1]. In this context, the product on \mathbb{A} comes from the map $\mathbb{A}^{\otimes 2} \rightarrow \mathbb{A}$ that the TQFT attaches to a pair of pants. Let us choose $\theta_A = \theta_B = \theta_C = 1$ in Proposition 8.2. We notice that, up to dualization $F_{0,3}(i, j, k) = A_{j,k}^i$ represents the product. This is the manifestation of a more general, easy fact. Let $\Sigma_{g,n}$ is a topological surface with n boundaries oriented inward, and denote $\mathcal{F}(\Sigma_{g,n}) \in \text{Hom}(\mathbb{A}^{\otimes n}, \mathbb{C})$ the amplitude assigned to $\Sigma_{g,n}$ by the TQFT \mathcal{F} . Using the TQFT glueing rules, it can be computed (or defined) as follows. Take a pair of pants decomposition of $\Sigma_{g,n}$. Take the tensor product over all pairs of pants of the maps $\mathcal{F}(\Sigma_{0,3}) : \mathbb{A}^{\otimes 3} \rightarrow \mathbb{C}$, and apply the pairing $\mathbb{A} \otimes \mathbb{A} \rightarrow \mathbb{C}$ on the factors of \mathbb{A} corresponding to coinciding boundary components of the pair of pants. The result is a multilinear map $\mathbb{A}^{\otimes n} \rightarrow \mathbb{C}$. Owing to the associativity, commutativity and invariance of the product, the result does not depend on the choice of pair of pants decomposition.

Lemma 8.6 *Assume \mathbb{A} unital. Then the partition function corresponding to the quantum Airy structure of Proposition 8.2 with the choice of $D = \frac{H}{2}$ and $\theta_A = \theta_B = \theta_C = 1$, gives the amplitudes of the*

two-dimensional TQFT corresponding to \mathbb{A} , e.g. for $2g - 2 + n > 0$

$$F_{g,n} = |\mathfrak{G}_{g,n}| \mathcal{F}(\Sigma_{g,n}). \quad (94)$$

Here $\mathfrak{G}_{g,n}$ is the set of graphs described in the legend of Figure 1, and its weighted cardinality

$$|\mathfrak{G}_{g,n}(1)| := \sum_{G \in \mathfrak{G}_{g,n}(1)} \frac{1}{|\text{Aut } G|}$$

is computed by the generating series

$$\exp\left(\sum_{g \geq 0} \sum_{n \geq 1} \frac{\hbar^{g-1}}{n!} |\mathfrak{G}_{g,n}| \right) = \exp\left\{\frac{1}{\hbar}\left(x - \frac{x^2}{2}\right)\right\} \text{Bi}\left((2\hbar)^{-\frac{2}{3}}(1-x)\right).$$

Proof. We obtain $\Sigma_{1,1}$ by glueing two boundaries of the same pair of pants. The TQFT assigns to this $\mathcal{F}(\Sigma_{1,1})(e_i) = \varphi(e_i e_a e_a) = \varphi(e_i H) = 2D^i = 2F_{1,1}(i)$. This coincides with (94) as the unique graph in $\mathfrak{G}_{g,n}(1)$ has a symmetry factor $\frac{1}{2}$. Unfolding (6) using the fact that B and C here represent the product (maybe up to identification $\mathbb{A} \simeq \mathbb{A}^*$ with the pairing), we deduce that $F_{g,n}(i_1, \dots, i_n)$ is the sum over $\mathfrak{G}_{g,n}(1)$ of the TQFT amplitude computed with a pair of pants decomposition canonically defined by the graph, and evaluated on $e_{i_1} \otimes \dots \otimes e_{i_n}$. The fact that this amplitude is independent of the pair of pants decomposition leads to the factorization of the (automorphism-weighted) number of graphs in $\mathfrak{G}_{g,n}(1)$. \square

It is clear from the proof, that if one chooses for $\theta_A, \theta_B, \theta_C$ arbitrary scalars, one will rather obtain $F_{g,n} = G_{g,n}(\theta_A, \theta_B, \theta_C) \mathcal{F}(\Sigma_{g,n})$ where $G_{g,n}(\theta_A, \theta_B, \theta_C)$ are the Taylor coefficients of the partition function of the 1-dimensional quantum Airy structure $\hbar \partial_x - \frac{\theta_A x^2}{2} - \hbar \theta_B x \partial_x - \frac{\hbar^2 \theta_C}{2} \partial_x^2 - \frac{\hbar}{2}$, see Section 8.1.2.

8.2 From non-commutative Frobenius algebras

Consider now a non-commutative Frobenius algebra \mathbb{A} . It is defined as a Frobenius algebra except that we drop the commutativity axiom, and impose that $\varphi([v_1, v_2]) = 0$ for any $v_1, v_2 \in \mathbb{A}$. Here we denote $[\cdot, \cdot]$ the commutator, and $\{\cdot, \cdot\}$ the anticommutator. We choose to work directly with an orthonormal basis $(e_i)_{i \in I}$.

Proposition 8.7 *Let $\lambda_A, \lambda_B, \lambda_C$ be central elements in \mathbb{A} , such that $\lambda_B^2 + \lambda_A \lambda_C = 0$. The assignments*

$$\begin{aligned} A_{j,k}^i &= \varphi(\lambda_A \{e_j, e_k\} e_i), \\ B_{j,k}^i &= \varphi(\lambda_B [e_i, e_j] e_k), \\ C_{j,k}^i &= \varphi(\lambda_C \{e_i, e_j\} e_k), \end{aligned}$$

and $D = D^a e_a \in \mathbb{A}$ such that $\lambda_B D$ lies in the orthogonal complement of $[\mathbb{A}, \mathbb{A}]$, define a quantum Airy structure.

As λ_X for $X \in \{A, B, C\}$ are central, we can move them freely around inside $\varphi(\dots)$. Then, due to invariance of the product, A and C are fully symmetric, and $B_{j,k}^i$ is fully antisymmetric under permutations of (i, j, k) and equal to $\frac{1}{2} f_{i,j}^k$. We remark that, if $\lambda_B = \frac{1}{2}$, the Lie algebra structure on $V = \mathbb{A}$ determined by the quantum Airy structure coincides with the Lie algebra structure of \mathbb{A} given by the commutator.

Proof. Using the property $\varphi(x e_a) e_a = x$ of the pairing and the orthonormality of the basis $(e_i)_{i \in I}$, we compute

$$B_{j,a}^i B_{k,\ell}^a = \varphi(\lambda_B [e_i, e_j] e_a) \varphi(\lambda_B [e_a, e_k] e_\ell) = \varphi(\lambda_B^2 [[e_i, e_j], e_k] e_\ell).$$

With similar manipulations, expanding commutators/anticommutators and using cyclic invariance of φ to move e_ℓ to the last position, we find that

$$\begin{aligned}
& B_{j,a}^i B_{k,\ell}^a + B_{k,a}^i B_{a,\ell}^j + C_{\ell,a}^i A_{a,k}^j - (i \leftrightarrow j) \\
= & \varphi\left(2\lambda_B^2(e_i e_j e_k - e_j e_i e_k - e_k e_i e_j + e_k e_j e_i)e_\ell\right) \\
& + \varphi\left(\lambda_B^2(e_j e_i e_k - e_i e_k e_j - e_j e_k e_i + e_k e_i e_j - e_i e_j e_k + e_j e_k e_i + e_i e_k e_j - e_k e_j e_i)e_\ell\right) \\
& + \varphi\left(\lambda_A \lambda_C(e_j e_k e_i + e_k e_j e_i + e_i e_j e_k + e_i e_k e_j - e_i e_k e_j - e_k e_i e_j - e_j e_i e_k - e_j e_k e_i)e_\ell\right) \\
= & \varphi\left((\lambda_B^2 + \lambda_A \lambda_C)(e_i e_j e_k - e_j e_i e_k - e_k e_i e_j + e_k e_j e_i)e_\ell\right).
\end{aligned}$$

Likewise, we compute that

$$\begin{aligned}
& B_{j,a}^i C_{k,\ell}^a + C_{k,a}^i B_{a,\ell}^j + C_{\ell,a}^i B_{a,k}^j - (i \leftrightarrow j) \\
= & \varphi\left(2\lambda_B \lambda_C(e_i e_j e_k - e_j e_i e_k + e_k e_i e_j - e_k e_j e_i)e_\ell\right) \\
& + \varphi\left(\lambda_B \lambda_C(e_j e_i e_k - e_i e_k e_j + e_j e_k e_i - e_k e_i e_j - e_i e_j e_k + e_j e_k e_i - e_i e_k e_j + e_k e_j e_i)e_\ell\right) \\
& + \varphi\left(\lambda_B \lambda_C(e_k e_j e_i - e_j e_k e_i + e_i e_k e_j - e_i e_j e_k - e_k e_i e_j + e_i e_k e_j - e_j e_k e_i + e_j e_i e_k)e_\ell\right) \\
= & 0
\end{aligned} \tag{95}$$

and

$$\begin{aligned}
& B_{j,a}^i A_{k,\ell}^a + B_{k,a}^i A_{a,\ell}^j + B_{\ell,a}^i A_{a,k}^j - (i \leftrightarrow j) \\
= & \varphi\left(2\lambda_B \lambda_A(e_i e_j e_k - e_j e_i e_k + e_k e_i e_j - e_k e_j e_i)e_\ell\right) \\
& + \varphi\left(\lambda_B \lambda_C(e_j e_i e_k - e_j e_k e_i + e_i e_k e_j - e_k e_i e_j - e_i e_j e_k + e_i e_k e_j - e_j e_k e_i + e_k e_j e_i)e_\ell\right) \\
& + \varphi\left(\lambda_B \lambda_C(e_k e_j e_i - e_i e_k e_j + e_j e_k e_i - e_i e_j e_k - e_k e_i e_j + e_j e_k e_i - e_i e_k e_j + e_j e_i e_k)e_\ell\right) \\
= & 0.
\end{aligned}$$

Therefore, when $\lambda_B^2 + \lambda_A \lambda_C = 0$, we have a classical Airy structure. The last statement about D is a consequence of Lemma 2.3, noticing that $D^{\text{ref}} = \frac{1}{2} \text{tr} B^i = 0$ since $B_{j,k}^i = -B_{k,j}^i$. \square

If λ_A is invertible, the quantum Airy structures of Proposition 8.7 are transformed by the symmetries (23) with $u_{a,b} = \varphi\left(\frac{\lambda_B}{2\lambda_A} e_a e_b\right)$ into

$$\begin{aligned}
\tilde{A}_{j,k}^i &= \varphi(\mu_A \{e_i, e_j\} e_k), \\
\tilde{B}_{j,k}^i &= \varphi(\mu_B e_i e_j e_k), \\
\tilde{C} &= 0,
\end{aligned} \tag{96}$$

where the new parameters are related to the old ones by

$$\mu_A = \lambda_A, \quad \mu_B = 2\lambda_B.$$

If \mathbb{A} happens to be commutative, we retrieve particular cases of the quantum Airy structure of Section 8.1, namely the one with $(\theta_A, \theta_B, \theta_C) = (2\lambda_A, 0, 2\lambda_C)$ for the quantum Airy structure of Proposition 8.7, and the one with $(\theta_A, \theta_B, \theta_C) = (2\mu_A, \mu_B, 0)$ for (96). Generically they fit in the case $\theta_B^2 - \theta_A \theta_C \neq 0$.

If we assume $-\lambda_A = \lambda_B = \lambda_C = 1$, the associated infinitesimal symplectomorphisms $(\mathcal{L}_i)_{i \in I}$ have a particularly nice form

$$\mathcal{L}_i(e_j, 0) = ([e_i, e_j], 0) + (0, \{e_i, e_j\}),$$

and

$$\mathcal{L}_i(0, e_j) = (\{e_i, e_j\}, 0) - (0, [e_i, e_j]),$$

relative to the orthonormal basis (e_i) . By direct computation, one can check that $[\mathcal{L}_i, \mathcal{L}_j] = 2\varphi([e_i, e_j]e_a)\mathcal{L}_a$.

We already computed in Proposition 5.1 the partition function for quantum Airy structures having $C = 0$, and we get the following expression using the pairing φ .

Lemma 8.8 *The partition function \tilde{Z} of the quantum Airy structure (96) is $\tilde{Z}(x) = \exp\left(\frac{\tilde{S}_0(x)}{\hbar} + \tilde{S}_1(x)\right)$ with*

$$\begin{aligned}\tilde{S}_0(x) &= \varphi\left(\frac{\mu_A}{\mu_B^3}\left(-\ln(1 - \mu_B x) - \mu_B x - \frac{\mu_B^2 x^2}{2}\right)\right), \\ \tilde{S}_1(x) &= \varphi\left(-\frac{\tilde{D}}{\mu_B}\ln(1 - \mu_B x)\right).\end{aligned}$$

□

Transforming back to the initial quantum Airy structure, we deduce an integral formula for the expression for its partition function.

Corollary 8.9 *The partition function of the quantum Airy structure of Proposition 8.7 is $Z(x) = \mathfrak{Z}(x)/\mathfrak{Z}(0)$ with*

$$\mathfrak{Z}(x) = \int \exp\left\{\varphi\left(\frac{\lambda_A}{\hbar\lambda_B}\left(-\frac{x^2}{2} - \frac{x}{2\lambda_B} + \frac{1}{4}\left(t - 2x - \frac{1}{2\lambda_B}\right)^2 - \frac{\ln(1-2\lambda_B t)}{8\lambda_B^2}\right) - \frac{D}{2\lambda_B}\ln(1 - 2\lambda_B t)\right)\right\} dt.$$

□

8.3 From loop spaces

We consider $V = \mathbb{C}[[z]]$. We index a basis of V by non-negative integers k . If $f \in \mathbb{C}[z^{-1}, z]] \cdot dz$ (germ of meromorphic 1-forms), we denote

$$\varphi(f) = \operatorname{Res}_{z \rightarrow 0} f(z).$$

Let $(\xi_k)_{k \geq 0}$ be a linearly independent family of germs of meromorphic 1-forms, and $(\xi_k^*)_{k \geq 0}$ be a linearly independent family of germs of functions (elements of V), such that

$$\varphi(\xi_k \xi_\ell^*) = 0.$$

Let $\theta \in \mathbb{C}[z^{-1}, z]] \cdot (dz)^{-1}$. Inspired by the Frobenius algebra example, we declare

$$\begin{aligned}A_{j,k}^i &:= \varphi(\xi_i^* d\xi_j^* d\xi_k^* \theta), \\ B_{j,k}^i &:= \varphi(\xi_i^* d\xi_j^* \xi_k \theta), \\ C_{j,k}^i &:= \varphi(\xi_i^* \xi_j \xi_k \theta),\end{aligned}\tag{97}$$

for some θ yet to be fixed.

Proposition 8.10 *Let $u_{k,\ell} = u_{\ell,k}$ be scalars indexed by integers $k, \ell \geq 0$, and choose*

$$\xi_k = \left(\frac{k+1}{z^{k+2}} + \sum_{\ell \geq 0} u_{k,\ell} z^\ell\right) dz, \quad \xi_k^* = \frac{z^{k+1}}{k+1}, \quad \theta = \sum_{r \in \mathbb{Z}} t_r z^r (dz)^{-1},\tag{98}$$

where t_r are formal parameters. Then, A is fully symmetric. Further, the triple (A, B, C) given by (97) defines a classical Airy structure if and only if $\theta \in z^{-1}\mathbb{C}[[z]].(dz)^{-1}$. In this case, (A, B, C, D) defines a quantum Airy structure iff

$$\forall i \geq 1, \quad \sum_{r \geq -1} t_r D^{i+r} = 0, \quad (99)$$

The restriction on θ in this proposition implies that $A = 0$. Therefore, the partition function has $F_{0,n} = 0$ for all $n \geq 1$. The possibility to have a non-trivial partition function stems from the possible non-zero choices of D satisfying (99). Lemma B.1 in Appendix B gives vanishing rules for the $F_{g,n}$ of this quantum Airy structure. In particular, the case $t_{-1} \neq 0$ is not really interesting, as the partition function in this case is $Z = 1$, *i.e.* all $F_{g,n}$ vanish.

We further find

$$B_{j,k}^i = \sum_{r \geq -1} \frac{k+1}{i+1} t_r \delta_{i+j+r,k}.$$

So, we get the commutation relations

$$[\tilde{L}_i, \tilde{L}_j] = \sum_{r \geq -1} (j-i) t_r \tilde{L}_{i+j+r}, \quad \tilde{L}_i := (i+1)L_i. \quad (100)$$

This is just another form of a subalgebra of the Virasoro Lie algebra \mathfrak{Vir} . Recall that \mathfrak{Vir} is the Lie algebra defined by generators $(\mathcal{L}_i)_{i \in \mathbb{Z}}$ satisfying the commutation relations

$$[\mathcal{L}_i, \mathcal{L}_j] = (i-j)\mathcal{L}_{i+j}.$$

Lemma 8.11 *If $\theta(z) = t_{r_0} z^{r_0} + O(z^{r_0+1})$ with $t_{r_0} \neq 0$, define $\frac{z^{r_0}}{\theta(z)} = \sum_{k \geq 0} \tau_k z^k$, and for $n \geq r_0$*

$$\hat{L}_n = - \sum_{k \geq 0} \tau_k \tilde{L}_{n+k-r_0}.$$

Then, for all $m, n \geq r_0$, we have that $[\hat{L}_m, \hat{L}_n] = (m-n)\hat{L}_{m+n}$.

The proof is a straightforward computation and is omitted.

We can formulate a \mathbb{Z}_2 -symmetric version of Proposition 8.10.

Proposition 8.12 *Let $u_{k,\ell} = u_{\ell,k}$ be scalars indexed by integers $k, \ell \geq 0$, and choose*

$$\xi_k = \left(\frac{(2k+1)}{z^{2k+2}} + \sum_{\ell \geq 0} u_{k,\ell} z^{2\ell} \right) dz, \quad \xi_k^* = \frac{z^{2k+1}}{2k+1}, \quad \theta = \sum_{s \in \mathbb{Z}} t_s z^{2s} (dz)^{-1}.$$

and assume $\theta \in \mathbb{C}[z^{-2}, z^2]].(dz)^{-1}$. Then, A is fully symmetric. Further, the triple (A, B, C) given by (97) defines a classical Airy structure if and only if $\theta \in z^{-2}\mathbb{C}[[z^2]].(dz)^{-1}$. In this case, (A, B, C, D) defines a quantum Airy structure iff

$$\forall k \geq 0, \quad \sum_{s \geq 0} t_{s-1} D^{s+k} = \frac{t_{-1}^2}{8} \delta_{k,0} + \frac{t_{-1} t_0 + u_{0,0} t_{-1}^2}{4} \delta_{k,1}. \quad (101)$$

The restriction on θ in this proposition implies that

$$A_{j,k}^i = t_{-1} \delta_{i,j,k,0}. \quad (102)$$

$$B_{j,k}^i = \sum_{s \geq -1} \frac{2k+1}{2i+1} t_s \delta_{i+j+s,k} + t_{-1} u_{0,0} \delta_{i,j,k,0}. \quad (103)$$

In particular, if $t_{-1} = 0$, then $A = 0$ hence $F_{0,n} = 0$ for all n . Lemma B.2 in Appendix B gives more general vanishing rules for the $F_{g,n}$'s of this quantum Airy structure.

The commutation relations deduced from (103) are

$$[\tilde{L}_i, \tilde{L}_j] = \sum_{s \geq -1} (j-i)t_s \tilde{L}_{i+j+s}, \quad \tilde{L}_i := \frac{2i+1}{2} L_i.$$

They are the same as (100), so we obtain again a sub-Lie algebra of $\mathfrak{A}ir$.

Specializing from formal parameters t_r to complex-valued parameters, we obtain that $\theta(z) = \sum_{r \geq -1} t_r z^r (dz)^{-1}$ in Proposition 8.10 – and $\theta(z) = \sum_{s \geq -1} t_s z^{2s} (dz)^{-1}$ in Proposition 8.12 define quantum Airy structures. Conversely, if θ contains higher negative powers, (A, B, C, D) cannot be a quantum Airy structure for generic parameters $(t_r)_{r \in \mathbb{Z}}$ – but we do not rule out neither confirm the existence of non-generic θ for which these formulas define quantum Airy structures.

In particular, if we assume $D^k = 0$ for $k \geq 2$, (101) is satisfied iff

$$D^0 = \frac{t_0 + u_{0,0} t_{-1}}{8}, \quad D^1 = \frac{t_{-1}}{24}.$$

Note that, for fixed $a \geq 0$

$$B_{a,a}^i = (2a+1) \left(\delta_{i,0} t_0 + \delta_{i,1} \frac{t_{-1}}{3} \right) + \delta_{i,a,0} t_{-1} u_{0,0}.$$

Therefore, $\text{Tr } B^i$ is not well-defined, and even after zeta regularisation of the sum $\sum_{a \geq 0} (2a+1)$, the expression $\frac{1}{2} \sum_a B_{a,a}^i$ does not reproduce (8.3).

Proof of Proposition (8.10). First, we observe that the result for

$$\xi_k = \xi_k^{(0)} := \frac{k+1}{z^{k+2}} dz \tag{104}$$

implies the general result, since the (A, B, C) for general

$$\xi_k = \xi_k^{(0)} + \sum_{\ell \geq 0} u_{k,\ell} d\xi_\ell^*$$

in (98) is obtained from the (A, B, C) for (104) by the symmetries (23). For the choice (104), unfortunately, we do not know an elegant proof of the result, thus we will proceed by direct computation.

We start with the warm-up case $\theta(z) = z^r (dz)^{-1}$ for some integer $r \in \mathbb{Z}$, which yields

$$\begin{aligned} A_{j,k}^i &:= \frac{1}{i+1} \delta_{i+j+k+r+2,0}. \\ B_{j,k}^i &:= \frac{k+1}{i+1} \delta_{i+j+r,k}. \\ C_{j,k}^i &:= \frac{(j+1)(k+1)}{i+1} \delta_{i+r,j+k+2}. \end{aligned}$$

In the following computations, it is implicit that the index a is summed over, and it is important to keep in mind that indices are always ≥ 0 . We compute

$$\begin{aligned} B_{j,a}^i B_{k,\ell}^a &= \frac{\ell+1}{(i+1)(j+1)} (j+1) \delta_{i+j+r \geq 0} \delta_{i+j+k+2r,\ell}. \\ B_{k,a}^i B_{a,\ell}^j &= \frac{\ell+1}{(i+1)(j+1)} (i+k+r+1) \delta_{i+k+r \geq 0} \delta_{i+j+k+2r,\ell}. \\ C_{\ell,a}^i A_{a,k}^j &= \frac{\ell+1}{(i+1)(j+1)} (i+r-\ell-1) \delta_{i+r \geq \ell+2} \delta_{i+j+k+2r,\ell}. \end{aligned}$$

Therefore, the left-hand side of the **BA** relation reads

$$\begin{aligned}
& \frac{\ell+1}{(i+1)(j+1)} \delta_{i+j+k+2r,\ell} \left\{ (j-i)\delta_{i+j+r \geq 0} + (i+k+r+1)\delta_{i+k+r \geq 0} - (j+k+r+1)\delta_{j+k+r \geq 0} \right. \\
& \quad \left. + (i+r-\ell-1)\delta_{i+r \geq \ell+2} - (j+r-\ell-1)\delta_{i+r \geq k+2} \right\} \\
= & \frac{\ell+1}{(i+1)(j+1)} \delta_{i+j+k+2r,\ell} \left\{ (i+r)(2\delta_{i+j+r < 0} - \delta_{i+k+r+1,0} - \delta_{\ell+1,i+r}) \right. \\
& \quad \left. + k(\delta_{i+j+r < 0} - \delta_{i+k+r+1,0}) + \ell(-\delta_{i+j+r < 0} + \delta_{\ell+1,i+r}) \right\}, \tag{105}
\end{aligned}$$

where we exploited the constraint in the delta function prefactor to get rid of all j 's, and we used $\delta_A = 1 - \delta_{\bar{A}}$ where \bar{A} is the negation of A . The coefficients of $\delta_{i+k+r+1,0}$ cancel each other, and likewise for $\delta_{\ell+1,i+r}$. We are left with a non-zero multiple of

$$\text{L.H.S. of } \mathbf{BA} = (i-j)\delta_{i+j+r < 0} \delta_{i+j+k+2r,\ell}. \tag{106}$$

For non-negative indices i, j , the set $\{i+j+r < 0\}$ is empty if and only if $r \geq 0$. For $r = -1$, it consists only of $(i, j) = (0, 0)$, but the prefactor $(i-j)$ vanishes in this case, so (106) also vanishes identically if $r = -1$. When $r \leq -2$, one can find non-negative (i, j, k, ℓ) for which (106) is non-zero. Therefore, the **BB-AC** relation holds if and only if $r \geq -1$.

Next, we compute

$$\begin{aligned}
B_{j,a}^i A_{k,\ell}^a &= \frac{\delta_{i+j+k+\ell+2r+2,0}}{(i+1)(j+1)} (j+1)\delta_{i+j+r \geq 0}, \\
B_{k,a}^i A_{a,\ell}^j &= \frac{\delta_{i+j+k+\ell+2r+2,0}}{(i+1)(j+1)} (i+k+r+1)\delta_{i+k+r \geq 0}, \\
B_{\ell,a}^i A_{a,k}^j &= \frac{\delta_{i+j+k+\ell+2r+2,0}}{(i+1)(j+1)} (i+\ell+r+1)\delta_{i+\ell+r \geq 0}.
\end{aligned}$$

Applying the same principles used in the previous computation, the left-hand side of the **BA** relation reads

$$\text{L.H.S. of } \mathbf{BA} = \frac{\delta_{i+j+k+\ell+2r+2,0}}{(i+1)(j+1)} (i-j)\delta_{i+j+r+1 \leq 0}. \tag{107}$$

A similar analysis shows that (107) is identically zero (*i.e.* the **BA** relation is satisfied) if and only if $r \geq -1$.

We also compute

$$\begin{aligned}
B_{j,a}^i C_{k,\ell}^a &= \frac{(k+1)(\ell+1)}{(i+1)(j+1)} (j+1)\delta_{i+j+2r,k+\ell+2} \delta_{i+j+r \geq 0}, \\
C_{k,a}^i B_{a,\ell}^j &= \frac{(k+1)(\ell+1)}{(i+1)(j+1)} \delta_{i+j+2r,k+\ell+2} \delta_{i+r \geq k+2}, \\
C_{\ell,a}^i B_{a,k}^j &= \frac{(k+1)(\ell+1)}{(i+1)(j+1)} \delta_{i+j+2r,k+\ell+2} \delta_{i+r \geq \ell+2},
\end{aligned}$$

and we obtain that the left-hand side of the **BC** relation reads

$$\text{L.H.S. of } \mathbf{BC} = \frac{(k+1)(\ell+1)}{(i+1)(j+1)} \delta_{i+j+2r,k+\ell+2} (i-j)\delta_{i+j+r+1 \leq 0}. \tag{108}$$

Here, we see that under the condition $i+j+2r = k+\ell+2$, we have $i+j+r+1 \geq 1-r$. Therefore, if $r \leq 0$, (108) vanishes identically. But $r \geq 0$ also implies that $i+j+r+1 \leq 0$ cannot be satisfied for non-negative (i, j) . Hence, (108) vanishes identically (*i.e.* the **BC** relation is satisfied) for any $r \in \mathbb{Z}$.

Let us now assume $r \geq -1$ and analyze the **D** relation. As we have $A = 0$, it takes the form, for all $i > j \geq 0$.

$$\sum_{a \geq 0} \frac{(j-i)(a+1)}{(i+1)(j+1)} \delta_{i+j+r,a} D^a = 0$$

which is equivalent to the system of equations $D^{k+r} = 0$ for all $k \geq 1$.

Now we consider the case

$$\theta = \sum_{r \geq -1} t_r z^r (dz)^{-1}.$$

This results in a decomposition

$$X = \sum_{r \geq -1} t_r {}^r X, \quad X \in \{A, B, C\}.$$

As $(t_r)_{r \in \mathbb{Z}}$ are considered as formal parameters, while checking the three quadratic relations it is enough to check the coefficient of the monomial $t_r t_s$ with $r \neq s$. Indeed the coefficient of t_r^2 already displays the relations which $({}^r A, {}^r B, {}^r C)$ satisfy according to Step 1. We should therefore check that

$${}^r B_{j,a}^i \cdot {}^s B_{j,k}^a + {}^s B_{j,a}^i \cdot {}^s B_{j,k}^a + {}^r B_{k,a}^i \cdot {}^s B_{a,\ell}^j + {}^s B_{k,a}^i \cdot {}^r B_{a,\ell}^j + ({}^r C_{\ell,a}^i \cdot {}^s A_{a,k}^j + {}^s C_{\ell,a}^i \cdot {}^r C_{a,k}^j) - (i \leftrightarrow j) = 0 \quad (109)$$

and likewise for the two other relations. This is again checked by direct computations, which are very similar to the previous ones, thus omitted. Let us examine in this context the **D** relation. We get

$$\forall i > j \geq 0, \quad (i-j) \sum_{r \geq -1} t_r D^{i+j+r} = \sum_{r,s \geq -1} t_r t_s \frac{\delta_{i+j+r+s,0}}{2} (\sigma_{i+r-2} - \sigma_{j+r-2}).$$

One can check that the right-hand side always vanish. Therefore, the **D** relation is equivalent to

$$\forall i \geq 1, \quad \sum_{r \geq -1} t_r D^{r+i} = 0. \quad (110)$$

Therefore, if $t_k = 0$ for $k \in \{0, \dots, r_0\}$, $(D^i)_{i=0}^{r_0}$ can be chosen arbitrarily. □

Proof of Proposition 8.12 Due to the symmetry (23) it is sufficient to consider

$$\xi_k := \frac{(2k+1)}{z^{2k+2}} dz.$$

If we denote η_k the 1-forms used in Proposition 8.10, note that $\xi_k = \eta_{2k}$ is just a subset of the $(\eta_k)_{k \geq 0}$. So, the case $\theta \in \mathbb{C}[[z]](dz)^{-1}$ is covered by the previous Theorem with $r = 2s$ (excluding the statement about the **D** relation), by restricting to even indices. Indeed, although we have to sum over all indices a to check the relations, the terms with odd a are always zero since $X_{q,m}^p = 0$ whenever $p+q+m \not\equiv 0 \pmod{2}$ for any $X \in \{A, B, C\}$, and two of the indices are not summed over and always even by our restriction. This is not so for the **D** relation, because the indices a and b which are summed over appear in the same symbol.

The case $\theta \in (t_{-1} z^{-2} + \mathbb{C}[[z]])(dz)^{-1}$ needs special care, as the relations **BA** and **BB-AC** failed in the proof of Proposition 8.10 where indices of any parity were allowed.

We first focus on the warm-up case $\theta(z) = z^{-2}(dz)^{-1}$, for which

$$\begin{aligned} A_{j,k}^i &= \delta_{i,j,k,0}. \\ B_{j,k}^i &= \frac{2k+1}{2i+1} \delta_{i+j,k+1}. \\ C_{j,k}^i &= \frac{(2j+1)(2k+1)}{2i+1} \delta_{i,j+k+2}. \end{aligned}$$

We just have to check that the following specialization of (106) and (107)

$$2(i-j)\delta_{i+j<1}\delta_{i+j+k,\ell+1} = 0, \quad 2(i-j)\delta_{i+j+k+\ell+1,0}\delta_{i+j<1} = 0.$$

Checking the first one is the same as checking (106) for $r = -1$ in the previous proof. And it is obvious that the expression is always 0, as it forces $i = j = 0$ and thus $k + \ell + 1 = 0$. So, the three relations hold for $s = -1$ as well. The **D** relation reads

$$\delta_{i+j+s \geq 0}(i-j)(2(i+j+s)+1)D^{i+j+s} = \frac{\delta_{i+j+2s,0}}{4}(\tilde{\sigma}_{i+s-1} - \tilde{\sigma}_{j+s-1}),$$

with

$$\tilde{\sigma}_m = \sum_{a+b=m} (2a+1)(2b+1).$$

For $t_{-1} = 0$ we arrive to the same result, *i.e.* D^k is arbitrary for $k = 0, \dots, r$ and $D^k = 0$ for $r \geq k + 1$. For $t_{-1} \neq 0$, we obtain

$$D^k = \frac{\delta_{k,1}}{8},$$

where the non-zero value is fixed by $i = 2, j = 0$ and the fact that $\tilde{\sigma}_0 = 1$.

The proof for the more general case

$$\theta(z) = \sum_{s \geq -1} t_s z^{2s} (dz)^{-1}$$

is very close to the one in absence of \mathbb{Z}_2 -symmetry, thus omitted. Let us only examine the **D** relation in this case. We get

$$\forall i > j \geq 0, \quad (i-j) \sum_{s \geq -1} (2(i+j+s)+1)t_s D^{i+j+s} = \frac{t_{-1}}{4}(t_{-1}\delta_{i,2} + t_0\delta_{i,1}),$$

which is equivalent to

$$\begin{aligned} \sum_{s \geq -1} t_s (2s+3)D^{s+1} &= \frac{t_{-1}t_0}{4}, \\ \sum_{s \geq -1} t_s (2s+5)D^{s+2} &= u_{0,0} \frac{t_{-1}^2}{8}, \\ \sum_{s \geq -1} t_s (2s+2k+1)D^{s+k} &= 0, \quad \forall k \geq 3. \end{aligned}$$

□

8.4 Loop space of Frobenius algebras

Let \mathbb{A} be a (commutative) Frobenius algebra and recall the notations of Section 8.1. We choose an orthonormal basis $(e_\alpha)_\alpha$ of \mathbb{A} , *i.e.* $e_\alpha = e_\alpha^*$. Set $\mathcal{V} := \mathbb{A}[[z]]$. The proofs of Propositions 8.2 and 8.10-8.12 can easily be adapted to this setting. If $f \in \mathbb{A}[z^{-1}, z](dz)$, we define

$$\Phi(f) := \operatorname{Res}_{z \rightarrow 0} \varphi(f(z))$$

using the linear form $\varphi : \mathbb{A} \rightarrow \mathbb{C}$ provided by the Frobenius structure. Let $\xi_{i,\alpha}$ be linearly independent family of elements of \mathcal{V} , indexed by α and integers $i \geq 0$.

We declare

$$\begin{aligned}
A_{(j,\beta),(k,\gamma)}^{(i,\alpha)} &= \Phi(\xi_{i,\alpha}^* d\xi_{j,\beta}^* d\xi_{k,\gamma}^* \theta), \\
B_{(j,\beta),(k,\gamma)}^{(i,\alpha)} &= \Phi(\xi_{i,\alpha}^* d\xi_{j,\beta}^* \xi_{k,\gamma} \theta), \\
C_{(j,\beta),(k,\gamma)}^{(i,\alpha)} &= \Phi(\xi_{i,\alpha}^* \xi_{j,\beta} \xi_{k,\gamma} \theta),
\end{aligned} \tag{111}$$

where $\theta \in \mathbb{A}[z^{-1}, z][dz]^{-1}$.

Proposition 8.13 *Let $v_{(k,\alpha),(\ell,\beta)} = v_{(\ell,\beta),(k,\alpha)}$ be scalars indexed by basis elements α, β of \mathbb{A} and integers $k, \ell \geq 0$. Assume*

$$\theta = \sum_{\substack{r \geq -1 \\ \alpha}} t_{r,\alpha} z^r (dz)^{-1} e_\alpha$$

for some scalars $t_{r,\alpha}$, and choose

$$\xi_{k,\alpha} = \left(\frac{(k+1)e_\alpha dz}{z^{k+1}} + \sum_{\substack{\ell \geq 0 \\ \beta}} v_{(k,\alpha),(\ell,\beta)} z^\ell e_\beta \right) dz, \quad \xi_{k,\alpha}^* = \frac{z^{k+1}}{k+1} e_\alpha.$$

Then, the triple (A, B, C) given by (111) defines a classical Airy structure. Further, (A, B, C, D) is a quantum Airy structure iff

$$\forall i \geq 1 \quad \forall \alpha_1, \alpha_2, \quad \sum_{\substack{r \geq -1 \\ \alpha, \beta}} \varphi(e_{\alpha_1} e_{\alpha_2} e_\alpha e_\beta) t_{r,\beta} D^{(r+i,\alpha)} = 0.$$

Proposition 8.14 *Let $v_{(k,\alpha),(\ell,\beta)} = v_{(\ell,\beta),(k,\alpha)}$ be scalars indexed by basis elements α, β of \mathbb{A} and integers $k, \ell \geq 0$. Choose*

$$\xi_{k,\alpha} = \left(\frac{(2k+1)e_\alpha}{z^{2k+2}} + \sum_{\substack{\ell \geq 0 \\ \beta}} v_{(k,\alpha),(\ell,\beta)} z^{2\ell} e_\beta \right) dz, \quad \theta = \sum_{\substack{s \geq -1 \\ \alpha}} t_{s,\alpha} z^{2s} (dz)^{-1} e_\alpha.$$

Then, (A, B, C) given by (111) defines a classical Airy structure. Further, (A, B, C, D) then defines a quantum Airy structure iff, for any α_1, α_2

$$\begin{aligned}
\sum_{\substack{s \geq -1 \\ \alpha, \beta}} \varphi(e_{\alpha_1} e_{\alpha_2} e_\alpha e_\beta) (2s+3) t_{s,\alpha} D^{(s+1,\beta)} &= \sum_{\alpha, \beta} \varphi(e_{\alpha_1} e_{\alpha_2} H e_\alpha e_\beta) \frac{t_{-1,\alpha} t_{-1,\beta}}{8}, \\
\sum_{\substack{s \geq -1 \\ \alpha, \beta}} \varphi(e_{\alpha_1} e_{\alpha_2} e_\alpha e_\beta) (2s+5) t_{s,\alpha} D^{(s+2,\beta)} &= \sum_{\alpha, \beta} \left(\varphi(e_{\alpha_1} e_{\alpha_2} H e_\alpha e_\beta) \frac{t_{-1,\alpha} t_{0,\beta}}{4} \right. \\
&\quad \left. + \sum_{\gamma, \varepsilon} \varphi(e_{\alpha_1} e_{\alpha_2} e_\alpha e_\beta e_\gamma e_\varepsilon) u_{(0,\gamma),(0,\varepsilon)} \frac{t_{-1,\alpha} t_{-1,\beta}}{2} \right), \\
\sum_{\substack{s \geq -1 \\ \alpha, \beta}} \varphi(e_{\alpha_1} e_{\alpha_2} e_\alpha e_\beta) (2s+2i+1) t_{s,\alpha} D^{(s+i,\beta)} &= 0, \quad \forall i \geq 3,
\end{aligned} \tag{112}$$

where $H := \sum_\alpha e_\alpha^2$.

The proofs combine the two aspects of the proofs given in Sections 8.1 and 8.3, and hence are omitted. They rely on the fact that, each of the three terms in the three relations are already symmetric in i and j in the Frobenius algebra case. Note that, if we assume \mathbb{A} is semi-simple and

$(e_\alpha)_\alpha$ is an orthonormal basis such that $e_\alpha e_\beta = \delta_{\alpha\beta} e_\alpha$, the constraints on D can be rewritten, for all α as follows

$$\begin{aligned} \sum_{s \geq -1} (2s+3)t_{s,\alpha} D^{(s+1,\alpha)} &= \frac{t_{-1,\alpha^2}}{8}, \\ \sum_{s \geq -1} (2s+5)t_{s,\alpha} D^{(s+2,\alpha)} &= \frac{t_{-1,\alpha} t_{0,\alpha}}{4} + u_{(0,\alpha),(0,\alpha)} \frac{t_{-1,\alpha}^2}{2}, \\ \sum_{s \geq -1} (2s+2i+1)t_{s,\alpha} D^{(s+i,\alpha)} &= 0. \end{aligned}$$

Let us describe this classical Airy structure in the language of Section 3.2. On $W = \mathbb{A}((z^{-1}))$ we consider the following symplectic form

$$\omega(f(z), g(z)) = \operatorname{Res}_{z \rightarrow 0} \varphi(f(z) dg(z)).$$

Let $W = V^* \oplus V$ be a polarization of W , where V (resp. V^*) has basis $(\xi_{k,\alpha}(z))_{k,\alpha}$, respectively $(\xi_{k,\alpha}^*(z) := e_\alpha \frac{z^{k+1}}{k+1})_{k,\alpha}$, such that

$$\forall (k, l) \in \mathbb{N}^2, \omega(\xi_{k,\alpha}^*(z), \xi_{l,\beta}(z)) = \delta_{k,l} \delta_{\alpha,\beta}.$$

One defines a classical Airy structure given by the set of the infinitesimal symplectomorphisms $(\mathcal{L}_{k,\alpha})_{k \in \mathbb{N}}$, where

$$\forall k \in \mathbb{N}, \forall f \in W, \mathcal{L}_{k,\alpha} f(z) = \xi_{k,\alpha}^*(-z) \theta(-z) df(-z),$$

and $\theta(z) \in z^{-1} \mathbb{A}[[z]] \cdot (dz)^{-1}$.

Corollary 8.15 *The operators $(\mathcal{L}_k)_{k \geq 0}$ together with the above orthonormal basis of V and V^* define a classical Airy structure.*

9 Relation with the topological recursion of [24]

9.1 Comparison

The original setting of [24] for the topological recursion is the data of a spectral curve, *i.e.*

- a branched covering with simple ramification points $x : \Sigma \rightarrow \Sigma'$, where Σ' is an open subset of \mathbb{P}^1 .
- a meromorphic function y on Σ , such that the zeroes of dy are distinct from the zeroes of dx . We set $\omega_{0,1} := y dx$.
- a symmetric bidifferential $\omega_{0,2}$ on Σ^2 with a double pole at coinciding points and the following behavior in local coordinates

$$\omega_{0,2}(p_1, p_2) = \frac{dz(p_1) dz(p_2)}{(z(p_1) - z(p_2))^2} + O(1).$$

Such an object is sometimes called a *fundamental bidifferential of the second kind* on Σ .

We denote $\mathfrak{r} \subset \Sigma$ the set of the ramification points, *i.e.* the zeros of dx . As they are simple, we can find around each $r \in \mathfrak{r}$ a local coordinate z such that

$$x(p) = x(r) + \frac{z(p)^2}{2}.$$

Let $U \subseteq \Sigma$ be the disjoint union of small enough neighborhoods of the ramification points, in which $\iota : z \mapsto -z$ is a well-defined holomorphic involution. By construction $x \circ \iota = x$. We introduce the recursion kernel

$$K(p_0, p) = \frac{1}{2} \frac{\int_{\iota(p)}^p \omega_{0,2}(\cdot, p_0)}{\omega_{0,1}(p) - \omega_{0,1}(\iota(p))}.$$

It is defined for $(p_0, p) \in \Sigma \times U$ as a 1-form in p_0 and the inverse of a 1-form in p . For $2g - 2 + n > 0$, [24] defines by induction

$$\omega_{g,n}(p_1, I) := \sum_{r \in \mathfrak{r}} \operatorname{Res}_{p \rightarrow r} K(p_1, p) \left\{ \omega_{g-1, n+1}(p, \iota(p), I) + \sum_{\substack{g'+g''=g \\ J' \sqcup J''=I}}^* \omega_{g', 1+|J'|}(p, J') \omega_{g'', 1+|J''|}(\iota(p), J'') \right\}, \quad (113)$$

where $I = \{p_2, \dots, p_n\}$, and Σ^* means that we exclude the terms of the form $\omega_{0,1} \omega_{g,n}$ from the sum. Although (113) gives a special role to the variable p_1 , [24] proves inductively that $\omega_{g,n}(p_1, \dots, p_n)$ is symmetric under permutation of all the p_i 's, therefore for $L = \{p_1, \dots, p_\ell\}$ an unordered ℓ -tuple of variables in Σ , the notation $\omega_{h,\ell}(L)$ makes sense. Manifestly (113) produces differential forms whose only poles are located at the ramification points. In other words

$$\omega_{g,n} \in H^0(\Sigma^{\times n}, K^{\boxtimes n}(\star \mathfrak{r}))^{\mathfrak{S}_n}.$$

In [21], it is shown that the $\omega_{g,n}$ can be decomposed on a suitable family of meromorphic 1-forms. To be self-contained we review this proof, and make explicit the recursion following from (113) for the coefficients of the decomposition.

Definition 9.1 For $k \geq 0$ and $r \in \mathfrak{r}$, we define for $p_0 \in \Sigma$ the meromorphic 1-form

$$\xi_{k,r}(p_0) := \operatorname{Res}_{p \rightarrow r} \left(\int_r^p \omega_{0,2}(p_0, \cdot) \right) \frac{(2k+1)dz(p)}{z(p)^{2k+2}}. \quad (114)$$

We also define, for p_0 in a neighborhood of r in Σ

$$\xi_{k,r}^*(p_0) := \frac{z^{2k+1}(p_0)}{2k+1}, \quad \theta(p_0) := \frac{-2}{\omega_{0,1}(p_0) - \omega_{0,1}(\iota(p_0))},$$

and if p_0 is in a neighborhood of $r_0 \neq r$, we define $\xi_{k,r}^*(p_0) := 0$.

Lemma 9.1 For $2g - 2 + n > 0$, there exists a unique decomposition with finitely many non-zero terms

$$\omega_{g,n}(p_1, \dots, p_n) = \sum_{\substack{r_1, \dots, r_n \in \mathfrak{r} \\ k_1, \dots, k_n \geq 0}} W_{g,n} \begin{bmatrix} r_1 & \dots & r_n \\ k_1 & \dots & k_n \end{bmatrix} \prod_{i=1}^n \xi_{k_i, r_i}(p_i). \quad (115)$$

More precisely, one can show [21] that the coefficients in (115) with $\sum_i k_i > 3g - 3 + n$ vanish. For completeness, we also give a proof in Appendix B.

We can now compare with the quantum Airy structure of Section 8.4. We take $\mathbb{A} = \bigoplus_{r \in \mathfrak{r}} \mathbb{C}$ as the sum of trivial 1-dimensional Frobenius algebras, and we let $V = \mathbb{A}[[z]]$ be the vector space with a basis indexed by $k \geq 0$ and $r \in \mathfrak{r}$. As we assumed y is holomorphic and dy has no zero at \mathfrak{r} , we deduce that $\theta(p)$ has an expansion for $p \rightarrow r$ of the form

$$\theta(p) = \sum_{m \geq -1} t_{m,r} z^{2m}(p) (dz(p))^{-1}. \quad (116)$$

According to Proposition 8.14,

$$\begin{aligned}
A_{(k_2, r_2), (k_3, r_3)}^{(k_1, r_1)} &:= \operatorname{Res}_{p \rightarrow r_1} \xi_{k_1, r_1}^*(p) d\xi_{k_2, r_2}^*(p) d\xi_{k_3, r_3}^*(p) \theta(p), \\
B_{(k_2, r_2), (k_3, r_3)}^{(k_1, r_1)} &:= \operatorname{Res}_{p \rightarrow r_1} \xi_{k_1, r_1}^*(p) d\xi_{k_2, r_2}^*(p) \xi_{k_3, r_3}(p) \theta(p), \\
C_{(k_2, r_2), (k_3, r_3)}^{(k_1, r_1)} &:= \operatorname{Res}_{p \rightarrow r_1} \xi_{k_1, r_1}^*(p) \xi_{k_2, r_2}(p) \xi_{k_3, r_3}(p) \theta(p), \\
D^{(k, r)} &= \delta_{k,0} \left(\frac{t-1, r}{2} \varphi_{0,2} \begin{bmatrix} r & r \\ 0 & 0 \end{bmatrix} + \frac{t_0, r}{8} \right) + \delta_{k,1} \frac{t-1, r}{24}
\end{aligned} \tag{117}$$

is a quantum Airy structure. Here, $\varphi_{0,2} \begin{bmatrix} r & r \\ 0 & 0 \end{bmatrix}$ is a scalar, which corresponds to the constant term in the expansion of $\omega_{0,2}$ near $(p_1, p_2) = (r, r)$ in local coordinates $(z(p_1), z(p_2))$. One can then check that indeed D is a solution of the \mathbf{D} relation in the form (112). Substituting the expansion (115) in the residue formula (113) gives a recursion for the $W_{g,n}$, which is identical to KS recursion (6) for the Taylor coefficients $F_{g,n}$ of this quantum Airy structure. Since we can check (see the proof below) that the initial data are the same, this leads to

Proposition 9.2 *For $2g - 2 + n > 0$, $F_{g,n}((k_1, r_1), \dots, (k_n, r_n))$ computed by KS topological recursion for the quantum Airy structure (117) and $W_{g,n} \begin{bmatrix} r_1 & \dots & r_n \\ k_1 & \dots & k_n \end{bmatrix}$ computed by the topological recursion of [24], agree.*

Proof. We start by a preliminary study of the recursion kernel. If we expand $\omega_{0,2}(p_1, p_2)$ in local coordinates when p_i is in a neighborhood of $r_i \in \mathfrak{r}$ we get that

$$\omega_{0,2}(p_1, p_2) = \frac{\delta_{r_1, r_2} dz(p_1) dz(p_2)}{(z(p_1) - z(p_2))^2} + \sum_{\ell_1, \ell_2 \geq 0} \varphi_{0,2} \begin{bmatrix} r_1 & r_2 \\ \ell_1 & \ell_2 \end{bmatrix} z^{\ell_1}(p_1) z^{\ell_2}(p_2) dz(p_1) dz(p_2). \tag{118}$$

We find the following expansion for (114) when $p_0 \rightarrow r_0$ for some $r_0 \in \mathfrak{r}$

$$\xi_{k,r}(p_0) = \frac{(2k+1)\delta_{r,r_0}}{z^{2k+2}(p_0)} + (2k+1) \sum_{\ell \geq 0} \varphi_{0,2} \begin{bmatrix} r & r_0 \\ 2k & \ell \end{bmatrix} z^\ell(p_0) dz(p_0).$$

In particular, $\xi_{k,r}(p_0)$ has a pole of order $2k+2$ at $p_0 = r$, and is holomorphic elsewhere. We also find for p in a neighborhood of r

$$\frac{1}{2} \int_{\iota(p)}^p \omega_{0,2}(\cdot, p_0) = \sum_{\substack{k \geq 0 \\ r \in \mathfrak{r}}} \xi_{k,r}(p_0) \xi_{k,r}^*(p)$$

under the condition $|z(p_0)| > |z(p)|$ when p_0 is in the neighborhood of r . Here we have used that

$$\frac{1}{(z(p_0) - z(p))^2} = \sum_{\ell \geq 0} \frac{(\ell+1) z^\ell(p)}{z^{\ell+2}(p_0)}$$

for p_0, p in the neighborhood of the same r . To perform the residue computation in (113), we will need the expansion of the recursion kernel $K(p_0, p)$ around $p \rightarrow r$

$$K(p_0, p) = -\frac{1}{2} \sum_{k \geq 0} \xi_{k,r}(p_0) \xi_{k,r}^*(p) \theta(p), \quad \xi_{k,r}^*(p) \in O(z(p)^{2k-1}). \tag{119}$$

Let us start by computing $\omega_{0,3}$.

$$\omega_{0,3}(p_1, p_2, p_3) = \sum_{r \in \mathfrak{r}} \operatorname{Res}_{p \rightarrow r} K(p_1, p) \{ \omega_{0,2}(p, p_2) \omega_{0,2}(\iota(p), p_3) + \omega_{0,2}(\iota(p), p_2) \omega_{0,2}(p, p_3) \}.$$

Since $\theta(p) \in O(z^{-2}(p)(dz)^{-1})$, $K(p_1, p)$ has a simple pole at $p = r$. So, the residue selects the coefficient of $(dz(p))^2$ in $\{\dots\}$, and as $d\xi_{k,r}(p) = z^k(p)dz(p)$, we find that

$$\omega_{0,3}(p_1, p_2, p_3) = \delta_{k_1, k_2, k_3, 0} \sum_{r \in \mathfrak{t}} \left(\prod_{i=1}^3 d\xi_{0,r}(p_i) \right) \operatorname{Res}_{p \rightarrow r} \xi_{0,r}^*(p) d\xi_{0,r}^*(p) d\xi_{0,r}^*(p) \theta(p). \quad (120)$$

The factor $-\frac{1}{2}$ in front of (119) disappeared as (120) has two terms with equal contribution, and the $\iota(p)$ in one of the factor $\omega_{0,2}$ results into a minus sign in the local coordinate z . We therefore have proved (115) for $(g, n) = (0, 3)$, and can identify the coefficients $W_{0,3} \begin{bmatrix} r_1 & r_2 & r_3 \\ k_1 & k_2 & k_3 \end{bmatrix}$ with $A_{(k_2, r_2), (k_3, r_3)}^{(k_1, r_1)}$ introduced in (117) – these coefficients vanish unless $k_1 = k_2 = k_3 = 0$ and $r_1 = r_2 = r_3$.

Likewise we compute $\omega_{1,1}$. Examining the local behavior at ramification points, we find

$$\begin{aligned} \omega_{1,1}(p_1) &= \sum_{r \in \mathfrak{t}} \operatorname{Res}_{p \rightarrow r} K(p_1, p) \omega_{0,2}(p, \iota(p)) \\ &= \sum_{r \in \mathfrak{t}} \sum_{k=0}^1 \xi_{k,r}(p_1) \operatorname{Res}_{p \rightarrow r} \frac{z(p)^{2k+1}}{2(2k+1)} \left(\frac{t_{-1,r}}{z^2(p)} + t_0 + O(z^2(p)) \right) \left(\frac{1}{4z^2(p)} + \varphi_{0,2} \begin{bmatrix} r & r \\ 0 & 0 \end{bmatrix} + O(z^2(p)) \right) \\ &= \sum_{r \in \mathfrak{t}} \left(\frac{t_{-1,r}}{2} \varphi_{0,2} \begin{bmatrix} r & r \\ 0 & 0 \end{bmatrix} + \frac{t_{0,r}}{8} \right) \xi_{0,r}(p_1) + \frac{t_{-1,r}}{24} \xi_{1,r}(p_1), \end{aligned} \quad (121)$$

which proves (115) for $(g, n) = (1, 1)$ with $W_{1,1} \begin{bmatrix} r \\ k \end{bmatrix} = D^{(k,r)}$ given in (117).

Now let $2g - 2 + n > 2$ and assume the claim of Lemma 9.1 has been established for all (g', n') such that $2g' - 2 + n' < 2g - 2 + n$. Let $I = \{p_2, \dots, p_n\}$ an unordered $(n-1)$ -uple of variables in Σ . In equation (113) for $\omega_{g,n}(p_1, I)$, we denote $\omega_{g,n}^B$ the sum of terms in the right-hand side involving $\omega_{0,2}\omega_{g,n-1}$, and $\omega_{g,n}^C$ the sum of all the other terms. We have that

$$\omega_{g,n}^B(p_1, I) = \sum_{i=2}^n \operatorname{Res}_{p \rightarrow \mathfrak{t}} K(p_1, p) \left(\omega_{0,2}(p, p_i) \omega_{g,n-1}(\iota(p), I \setminus \{p_i\}) + \omega_{0,2}(\iota(p), p_i) \omega_{g,n-1}(p, I \setminus \{p_i\}) \right). \quad (122)$$

As $K(p_1, p)$ is invariant under $p \rightarrow \iota(p)$, the two terms give an equal contribution. The form (115) of $\omega_{g,n-1}$ by the induction hypothesis implies that

$$\omega_{g,n-1}(p, I \setminus \{p_i\}) = \frac{1}{2} \left(\omega_{g,n-1}(p, I \setminus \{p_i\}) - \omega_{g,n-1}(\iota(p), I \setminus \{p_i\}) \right) + O(z(p)dz(p)).$$

As $\omega_{0,2}(p, p_i)$ is holomorphic near $p \rightarrow r$, we deduce that replacing it with its odd part in (122) does not change the residue

$$\omega_{g,n}^B(p_1, I) = \sum_{i=2}^n \operatorname{Res}_{p \rightarrow \mathfrak{t}} K(p_1, p) \frac{1}{2} \left(\omega_{0,2}(\iota(p), p_i) - \omega_{0,2}(p, p_i) \right) \omega_{g,n-1}(p, I \setminus \{p_i\}).$$

We substitute in this formula, for p in the neighborhood of r

$$\frac{1}{2} \left(\omega_{0,2}(\iota(p), p_i) - \omega_{0,2}(p, p_i) \right) = - \sum_{k \geq 0} \xi_{k,r}(p_i) d\xi_{k,r}^*(p),$$

and the decomposition (115) for $\omega_{g,n-1}$. The result for $\omega_{g,n}^B$ decomposes like (115) with coefficients

$$W_{g,n}^B \begin{bmatrix} r_1 & \dots & r_n \\ k_1 & \dots & k_n \end{bmatrix} = \sum_{i=2}^n \sum_{k', r'} B_{(k_i, r_i), (k', r')}^{(k_1, r_1)} W_{g,n-1} \begin{bmatrix} r' & r_2 & \dots & r_n \\ k' & k_2 & \dots & k_n \end{bmatrix},$$

where

$$B_{(k_i, r_i), (k', r')}^{(k_1, r_1)} = \operatorname{Res}_{p \rightarrow r_1} \xi_{k_1, r_1}^*(p) d\xi_{k_i, r_i}^*(p) \xi_{k', r'}(p) \theta(p)$$

as given in (117). Due to the local behavior of the integrand, $B_{(k_2, r_2), (k_3, r_3)}^{(k_1, r_1)}$ vanishes when $r_1 \neq r_2$, or when $r_1 = r_2 \neq r_3$ and $k_1 + k_2 > 0$, or when $r_1 = r_2 = r_3$ and $k_1 + k_2 \geq k_3 + 1$. In particular these selection rules imply that there are finitely many non-zero $W_{g,n}^B$'s.

Let us turn to

$$\omega_{g,n}^C(p_1, I) = \operatorname{Res}_{p \rightarrow \mathfrak{t}} K(p_1, p) \left\{ \omega_{g-1, n+1}(p, \iota(p), I) + \sum_{\substack{g'+g''=g \\ J' \sqcup J'' = I}}^{**} \omega_{g', 1+|J'|}(p, J') \omega_{g'', 1+|J''|}(\iota(p), J'') \right\},$$

where \sum^{**} excludes the terms of the form $\omega_{0,1} \omega_{g,n}$ or $\omega_{0,2} \omega_{g,n-1}$. By induction hypothesis, we can directly substitute the decomposition (115) for all the ω 's involved in the left-hand side. We find that $\omega_{g,n}^C$ has a decomposition again of the form (115), with coefficients

$$\begin{aligned} W_{g,n}^C \begin{bmatrix} r_1 & \dots & r_n \\ k_1 & \dots & k_n \end{bmatrix} &= \frac{1}{2} \sum_{\substack{k', k'' \geq 0 \\ r', r'' \in \mathfrak{t}}} C_{(k', r'), (k'', r'')}^{(k_1, r_1)} \left(W_{g-1, n+1} \begin{bmatrix} r' & r'' & r_2 & \dots & r_n \\ k' & k'' & k_2 & \dots & k_n \end{bmatrix} \right. \\ &+ \left. \sum_{\substack{g'+g''=g \\ J' \sqcup J'' = \{2, \dots, n\}}}^{**} W_{g', 1+|J'|} \begin{bmatrix} r' & (r_j)_{j \in J'} \\ k' & (k_j)_{j \in J'} \end{bmatrix} W_{g'', 1+|J''|} \begin{bmatrix} r'' & (r_j)_{j \in J''} \\ k'' & (k_j)_{j \in J''} \end{bmatrix} \right), \end{aligned} \quad (123)$$

where

$$C_{(k_2, r_2), (k_3, r_3)}^{(k_1, r_1)} = \operatorname{Res}_{p \rightarrow r_1} \xi_{k_1, r_1}^*(p) \xi_{k_2, r_2}(p) \xi_{k_3, r_3}(p) \theta(p)$$

as given in (117). Due to the local behavior of the integrand, $C_{(k_2, r_2), (k_3, r_3)}^{(k_1, r_1)} = C_{(k_3, r_3), (k_2, r_2)}^{(k_1, r_1)}$ vanishes when $r_2, r_3 \neq r_1$ and $k_1 > 0$, or when $r_2 = r_1 \neq r_3$ and $k_1 \geq k_2 + 2$, or when $r_1 = r_2 = r_3$ and $k_1 \geq k_2 + k_3 + 3$. In particular, this implies that (123) contains only finitely many non-zero terms. We therefore have justified that $\omega_{g,n} = \omega_{g,n}^B + \omega_{g,n}^C$ has the form (115), and proved Lemma 9.1 by induction.

Since we have checked $F_{0,3} = A = W_{0,3}$ and $F_{1,1} = D = W_{1,1}$, and the recursive rules to build the $W_{g,n}$'s agree with the KS topological recursion (6) for the $F_{g,n}$'s, this entails Proposition 9.2. \square

More explicitly, in terms of coefficients of expansion of $\omega_{0,1}$ in (116) and $\omega_{0,2}$ in (118), the relevant quantum Airy structure is

$$\begin{aligned} A_{(k_2, r_2), (k_3, r_3)}^{(k_1, r_1)} &= \delta_{k_1, k_2, k_3, 0} \delta_{r_1, r_2, r_3} t_{-1, r_1}, \\ B_{(k_2, r_2), (k_3, r_3)}^{(k_1, r_1)} &= \frac{2k_3 + 1}{2k_1 + 1} \delta_{r_1, r_2} \left(\delta_{r_2, r_3} t_{k_3 - k_2 - k_1, r_1} + \delta_{k_1, k_2, 0} \varphi_{0,2} \begin{bmatrix} r_3 & r_1 \\ 2k_3 & 0 \end{bmatrix} \right), \\ C_{(k_2, r_2), (k_3, r_3)}^{(k_1, r_1)} &= \frac{(2k_3 + 1)(2k_2 + 1)}{2k_1 + 1} \left(\delta_{r_1, r_2, r_3} t_{1+k_2+k_3-k_1, r_1} + \sum_{m=0}^{1+k_3-k_1} \delta_{r_1, r_3} \varphi_{0,2} \begin{bmatrix} r_2 & r_1 \\ 2k_2 & 2m \end{bmatrix} t_{k_3-k_1-m, r_1} \right. \\ &+ \left. \sum_{m=0}^{1+k_2-k_1} \delta_{r_1, r_2} \varphi_{0,2} \begin{bmatrix} r_3 & r_1 \\ 2k_3 & 2m \end{bmatrix} t_{k_2-k_1-m, r_1} + \delta_{k_1, 0} \varphi_{0,2} \begin{bmatrix} r_2 & r_1 \\ 2k_2 & 0 \end{bmatrix} \varphi_{0,2} \begin{bmatrix} r_3 & r_1 \\ 2k_3 & 0 \end{bmatrix} t_{-1, r_1} \right). \end{aligned}$$

9.2 The point of view of Givental quantization of Lagrangian cones

One of the applications of the original topological recursion formalism is the study of Frobenius manifolds/cohomological field theories. In this setup, [18] established that the topological recursion applied to a specific local spectral curve is equivalent to Givental's quantization formalism [31] for computing the ancestor potential of a semi-simple cohomological field theory. This correspondence was obtained by a direct comparison of the result of the topological recursion and of Givental reconstruction procedure. In this section, we revisit this equivalence from the point of view of quantization of

Givental's Lagrangian cone [14, 32], giving it a stronger geometric explanation. We first review the Lagrangian cone formalism, following Coates and Givental.

Let V be a finite dimensional vector space equipped with a bilinear form (\cdot, \cdot) and a distinguished vector $\mathbf{1}$, and let $\mathcal{W} := V((z^{-1}))$ be the corresponding loop space equipped with the symplectic form⁹ $\tilde{\omega}$ defined by

$$\forall (f, g) \in \mathcal{W}^2, \quad \tilde{\omega}(f, g) := \frac{1}{2i\pi} \oint (f(-z), g(z)) dz.$$

Consider the polarization $\mathcal{W} = \mathcal{V}_+ \oplus \mathcal{V}_-$ where $\mathcal{V}_+ := V[[z]]$ and $\mathcal{V}_- := z^{-1}V[[z^{-1}]]$. Then the symplectic form gives an identification $(\mathcal{W}, \tilde{\omega}) \simeq (T^*\mathcal{V}_+, \tilde{\omega})$.

Parametrizing elements q of \mathcal{V}_+ by an infinite dimensional vector $\mathbf{t} := (t_k)_{k \in \mathbb{N}}$,

$$q(z) := \sum_{k=0}^{\infty} (t_k - \delta_{k,1} \mathbf{1}) z^k,$$

one defines the graph $\mathcal{L}_{\mathcal{F}}$ of the derivative of a function $\mathcal{F}(\mathbf{t})$ on \mathcal{V}_+ by

$$\mathcal{L}_{\mathcal{F}} := \{(p, q) \in T^*\mathcal{V}_+ : p = d_q \mathcal{F}(\mathbf{t})\}.$$

As a formal germ around $q = -z$, this defines a Lagrangian submanifold of $T^*\mathcal{V}_+$ and hence of $(\mathcal{W}, \tilde{\omega})$.

An interesting choice for such functions are genus 0 free energies coming from a CohFT, *i.e.* functions $F_0(\mathbf{t})$ satisfying the following three axioms. It is convenient to state them by choosing a basis $(e_\nu)_{\nu=1}^d$ of V and denoting $g_{\mu,\nu} := (e_\mu, e_\nu)$ and by $(g^{\mu,\nu})_{\mu,\nu}$ its inverse matrix. For $k \geq 0$, we denote $t_k := \sum_{\nu=1}^d t_k^\nu e_\nu$. With these notations, the three axioms defining a genus 0 free energy read as follows.

- The *dilaton equation*, which states that F_0 is homogenous of degree 2

$$2F_0(\mathbf{t}) = \sum_{k \in \mathbb{N}} \sum_{\nu=1}^d t_k^\nu \frac{\partial F_0(\mathbf{t})}{\partial t_k^\nu}. \quad (124)$$

- The *string equation*, which decomposes the action of $\frac{\partial}{\partial t_0^1}$ (the unit vector field)

$$\frac{\partial F_0}{\partial t_0^1} = \frac{1}{2}(t_0, t_0) + \sum_{k=0}^{\infty} \sum_{\nu=1}^d t_{k+1}^\nu \frac{\partial F_0}{\partial t_k^\nu}. \quad (125)$$

- The *topological recursion relations*¹⁰

$$\forall (\alpha, \beta, \gamma) \in \{1, \dots, d\}^3 \quad \forall (k, l, m) \in \mathbb{N}^3, \quad \frac{\partial^3 F_0}{\partial t_{k+1}^\alpha \partial t_l^\beta \partial t_m^\gamma} = \sum_{\mu, \nu=1}^d \frac{\partial^2 F_0}{\partial t_k^\alpha \partial t_0^\mu} g^{\mu,\nu} \frac{\partial^3 F_0}{\partial t_0^\nu \partial t_l^\beta \partial t_m^\gamma}. \quad (126)$$

The Lagrangians \mathcal{L}_{F_0} defined by such functions have a very nice characterization.

Theorem 9.3 [32] F_0 satisfies Eqn. (124), Eqn. (125) and Eqn. (126) if and only if \mathcal{L}_{F_0} is a Lagrangian cone with vertex at the origin and such that its tangent spaces L satisfy $zL = L$.

In addition, Givental described a large group of symmetries of the set of such cones.

⁹This is not the same symplectic form as in the example of Section 8.4. One can go from one to the other by a Laplace transform.

¹⁰This set of equations is different from the topological recursion of [24]. It is unfortunate that both names coincide.

Theorem 9.4 [32] *The twisted loop group G_{tw} , consisting of elements $M(z) \in \text{End}(V)[[z^{-1}]]$ such that $M^*(-z)M(z) = \text{Id}$ preserves the class of cones of Theorem 9.3.*

Note that the condition $M^*(-z)M(z) = \text{Id}$ implies that $M(z)$ defines a symplectomorphism of \mathcal{W} . The set of tangent spaces to such a cone carries the structure of a Frobenius manifold \mathcal{M} . For instance, this applies to (and was motivated by the application to) the genus 0 descendent or ancestor potentials of Gromov-Witten theory of a complex projective variety [14], to genus 0 correlation functions of cohomological field theories [35] and to quantum K -theory [14].

Conversely, if \mathcal{M} is a semi-simple Frobenius manifold, there is a notion of a descendent (resp. ancestor) potential $F_0^{\mathcal{M}}(\mathbf{t})$, which satisfy the axioms above. Here $(t^\nu)_\nu$ are local flat coordinates on \mathcal{M} , and we fix a point $m \in \mathcal{M}$ to identify $V := T_m\mathcal{M}$. We denote as before $(t'_k)_{k \geq 0}$ the linear coordinates on $V[[z]]$. Dubrovin – see e.g. [17] – associates to \mathcal{M} a Riemann-Hilbert problem on \mathbb{P}^1 . Its solution is an element $M_v(z)$ of the associated loop space depending on a point m of the Frobenius manifold¹¹, and it admits a Birkhoff factorization

$$M_v(z) = M_{v,\infty}(z)^{-1}M_{v,0}(z),$$

where $M_{v,0}(z)$ (resp. $M_{v,\infty}(z)$) is analytic and invertible for $|z| < 1$ (resp. $1 < |z| \leq \infty$). Combining Givental's analysis of the action of the twisted loop group [32] and Teleman's classification of semi-simple Frobenius manifolds [44], one can conclude that the cone defined by the graph of the genus zero descendent (resp. ancestor) potential of a $\dim V = N$ semi-simple Frobenius manifold is obtained by the action of the symplectomorphism $\gamma(v)M_v(z)$ (resp. $M_{v,0}(z)$) on the cone \mathcal{L}_N corresponding to the trivial theory of type $A_1^{\times N}$ where $\gamma(v)$ is a suitably chosen normalization factor.

Finally, Givental reconstruction procedure proved by Teleman through its classification can be expressed as the following quantization result.

Theorem 9.5 [32, 44] *If $\dim V = N$, then the descendant (resp. ancestor) potential of a semi-simple Frobenius manifold is obtained by quantizing the cone obtained by the action of the symplectomorphism $\gamma(v)M_v(z)$ (resp. $M_{v,0}(z)$) on the cone \mathcal{L}_N corresponding to the trivial theory of type $A_1^{\times N}$ where $\gamma(v)$ is a suitably chosen normalization factor.*

In order to be more explicit, let us describe \mathcal{L}_N . Let $F_0^{\text{KdV}}(\mathbf{t})$ be the genus 0 potential of the Gromov-Witten theory of a point, i.e. the genus 0 part of the logarithm of the partition function of the quantum Airy structure of Proposition 8.14 with all $v_{(k,i),(l,j)}$ vanishing and $\theta(z) = z^{-2} \cdot 1_V$. Then, after the identification by the dilaton shift $q_{k,i} = t_k^i - \delta_{k,1}$, one has indeed

$$\mathcal{L}_N := \left\{ (p, q) \in T^*\mathcal{V} : p = \sum_{i=1}^N d_q F_0^{\text{KdV}}(\mathbf{t}^i) \right\}.$$

The full partition function of this quantum Airy structure is the matrix Airy function of [34].

Because $M_{v,0}(z)$ is analytic for $|z| > 1$, the quantum structure corresponding to the ancestor's Lagrangian cone is obtained by the action of the operator $\exp\left(\frac{\hbar}{2} \sum_{k,\ell,i,j} u_{(k,i),(l,j)} \partial_{(k,i)} \partial_{(l,j)}\right)$ on the quantum Airy structure built from \mathcal{L}_N where

$$\frac{M_{v,0}^*(z_1)M_{v,0}(z_2) - \text{Id}}{z_1 + z_2} := \sum_{k,\ell \geq 0} (-1)^{k+\ell} U_{k,\ell} z_1^k z_2^\ell, \quad U_{k,\ell} e_i := \sum_{j=1}^N u_{(k,i),(l,j)} e_j.$$

¹¹This element and its factorization are unique if the Frobenius manifold admits an Euler vector field. Otherwise, one needs to fix the diagonal ambiguity by some other geometric condition.

This transformation preserves Airy structures and the topological recursion gives a way to compute the ancestor potential. Further, the local spectral curve is fixed by the symplectomorphism $M_{v,0}(z)$.

The action of the Givental's twisted loop group on Lagrangian cones is easily seen to coincide with the action of symplectomorphisms on Airy structures defined on \mathcal{W} . This leads to the equivalence of Givental quantization procedure and quantization of the corresponding Airy structure.

Corollary 9.6 *The ancestor potential of a semi-simple Frobenius manifold is obtained from the quantum Airy structure defined on the loop space \mathcal{W} from the action of the symplectomorphism $M_{v,0}(z)$ on the trivial quantum Airy structure whose partition function is the matrix Airy function.*

This quantization procedure is equivalent to Givental's quantization of the corresponding Lagrangian cone.

On the other hand, the action of the quantization of $M_{v,\infty}(z)^{-1}$ taking the ancestor potential to the descendant potential is just the multiplication of the partition function by $\exp\left(\frac{\hbar}{2} \sum_{k,\ell,i,j} s_{(k,i),(\ell,j)} x_{(k,i)} x_{(\ell,j)}\right)$ where

$$\frac{M_{v,\infty}^*(z_1)M_{v,\infty}(z_2) - \text{Id}}{1/z_1 + 1/z_2} := \sum_{k,\ell \geq 0} S_{k,\ell} z_1^{-k} z_2^{-\ell}, \quad S_{k,\ell} e_i := \sum_{j=1}^N s_{(k,i),(\ell,j)} e_j.$$

Even though it is a simple transformation, let us remark that this does not preserve the Airy structure property, since it adds a linear term in the x 's. This explains why the topological recursion does not directly provide the descendant potential, except in the cases where $M_{v,\infty}$ is trivial.

10 A topological recursion without branched covers

In this section, we explain the simple observation that the quantum Airy structures of Proposition 8.13 can be realized by a new variant of the topological recursion of [24]. We take as initial data

- a Riemann surface Σ .
- a meromorphic 1-form $\omega_{0,1}$ on Σ .
- a fundamental bidifferential of the second kind $\omega_{0,2}$ on Σ .
- a finite subset $\mathfrak{r} \subset \Sigma$, such that $\omega_{0,1}$ has at most simple zeroes at \mathfrak{r} – this allows poles of $\omega_{0,1}$ at \mathfrak{r} .
- a meromorphic 1-form $\omega_{1,1}$ on Σ , such that, for any $r \in \mathfrak{r}$, $(x(p) - x(r))^2 \frac{\omega_{1,1}(p)}{\omega_{0,1}(p)}$ is holomorphic around $p \rightarrow r$.

We define a recursion kernel

$$\tilde{K}(p_0, p) = \frac{\int_r^p \omega_{0,2}(\cdot, p_0)}{\omega_{0,1}(p)},$$

and for $2g - 2 + n > 0$ and $(g, n) \neq (1, 1)$, we make the inductive definition

$$\omega_{g,n}(p_1, I) = \sum_{r \in \mathfrak{r}} \text{Res}_{p \rightarrow r} \tilde{K}(p_0, p) \left\{ \omega_{g-1, n+1}(p, p, I) + \sum_{\substack{* \\ g'+g''=g \\ J' \sqcup J''=I}} \omega_{g', 1+|J'|}(p, J') \omega_{g'', 1+|J''|}(p, J'') \right\}. \quad (127)$$

We have to include $\omega_{1,1}$ in the initial data since $\omega_{0,2}(p,p)$, which would appear in (127) for $(g,n) = (1,1)$, does not make sense due to the double pole at coinciding point. As in Section 9.1, one can prove that the $\omega_{g,n}$ decompose on a basis of 1-forms.

Definition 10.1 For $k \geq 0$ and $r \in \mathfrak{r}$, we define for $p_0 \in \Sigma$ the meromorphic 1-form

$$\xi_{k,r}(p_0) := \left(\operatorname{Res}_{p \rightarrow r} \int_r^{p_0} \omega_{0,2}(\cdot, p_0) \right) \frac{(k+1)dx(p)}{(x(p) - x(r))^{k+1}}.$$

We also define, for p_0 in a neighborhood of r in Σ

$$\xi_{k,r}^*(p_0) := \frac{(x(p) - x(r))^{k+1}}{k+1}, \quad \theta(p) := \frac{1}{\omega_{0,1}(p)},$$

and if p_0 is in a neighborhood of $r_0 \neq r$, we define $\xi_{k,r}^*(p_0) := 0$.

Lemma 10.1 For $2g-2+n > 0$, there exists a unique decomposition with finitely many non-zero terms

$$\omega_{g,n}(p_1, \dots, p_n) = \sum_{\substack{r_1, \dots, r_n \in \mathfrak{r} \\ k_1, \dots, k_n \geq 0}} W_{g,n} \begin{bmatrix} r_1 & \dots & r_n \\ k_1 & \dots & k_n \end{bmatrix} \prod_{i=1}^n \xi_{k_i, r_i}(p_i). \quad (128)$$

□

The assumption on $\omega_{0,1}$ guarantees that $\theta(p)$ for $p \rightarrow r$ has an expansion of the form

$$\theta(p) = \sum_{m \geq -1} t_{m,r} z^m(p) (dx(p))^{-1}.$$

According to Proposition 8.13, we have a quantum Airy structure given by

$$\begin{aligned} A_{(k_2, r_2), (k_3, r_3)}^{(k_1, r_1)} &:= 0, \\ B_{(k_2, r_2), (k_3, r_3)}^{(k_1, r_1)} &:= \operatorname{Res}_{p \rightarrow r_1} \xi_{k_1}^*(p) d\xi_{k_2, r_2}^*(p) \xi_{k_3, r_3}(p) \theta(p), \\ C_{(k_2, r_2), (k_3, r_3)}^{(k_1, r_1)} &:= \operatorname{Res}_{p \rightarrow r_1} \xi_{k_1}^*(p) \xi_{k_2, r_2}(p) \xi_{k_3, r_3}(p) \theta(p), \\ D^{(k_1, r_1)} &:= W_{1,1} \begin{bmatrix} r_1 \\ k_1 \end{bmatrix}, \end{aligned} \quad (129)$$

and by comparison of KS-TR and the recursive relation for $W_{g,n}$'s ensuing from (127) we obtain that

Proposition 10.2 For $2g-2+n > 0$, $F_{g,n}((k_1, r_1), \dots, (k_n, r_n))$ computed by KS topological recursion for the quantum Airy structure (129), and $W_{g,n} \begin{bmatrix} r_1 & \dots & r_n \\ k_1 & \dots & k_n \end{bmatrix}$, agree. □

We omit the proof of Lemma 10.1 and Proposition 10.2, as it is similar to Lemma 9.1 and Proposition 9.2, in fact simpler due to the absence of the involution. Note that the assumption made on $\omega_{1,1}$ is equivalent to the **D** relation.

11 Dynamics on (colored) Young diagrams

11.1 Setting

In this section we show that a quantum Airy structure on $V = \mathbb{C}[[z]]$ (or $V = \mathbb{C}^d[[z]]$, or $z\mathbb{C}^d[[z^2]]$, etc.) gives a recursion on (colored) Young diagrams, which are in correspondence with the monomials

that can appear in the Taylor expansion of the partition function. We first formulate abstractly the recursion on Young diagrams, and relate it to quantum Airy structure in Proposition 11.1 below. Please see also [5], where this dynamics on Young diagrams in some special cases was given.

Let d be a positive integer. For a Young diagram $\lambda = (\lambda_1 \geq \dots \geq \lambda_{\ell(\lambda)})$, we consider λ_i as column heights. We denote $\text{Col}(\lambda)$ the set of columns, and $|\lambda|$ be the number of boxes.

Definition 11.1 *A d -coloring of a Young diagram λ is a map $\text{Col}(\lambda) \rightarrow \{1, \dots, d\}$. A column type is an ordered pair $(k, \alpha) \in \mathbb{N}_{>0} \times \{1, \dots, d\}$, where k is the height and α is the color. We denote $\text{Aut } \lambda$ the group of permutations of columns respecting column types.*

We use the notation $N_{k,\alpha}(\lambda)$ for the number of columns of type (k, α) , hence

$$|\lambda| = \sum_{k \geq 1} \sum_{\alpha=1}^d N_{k,\alpha}(\lambda)k, \quad |\text{Aut } \lambda| = \prod_{\substack{k \geq 1 \\ 1 \leq \alpha \leq d}} N_{k,\alpha}(\lambda)!$$

Definition 11.2 *Let $\mathfrak{y}_{g,n}^{(d)}$ be the set of d -colored Young diagrams λ such that*

$$\ell(\lambda) = n \quad \text{and} \quad |\lambda| \leq (2g - 2 + n)r, \quad (130)$$

and $\tilde{\mathfrak{y}}_{g,n}^{(d)}$ the set of such d -colored Young diagrams together with the choice of a column type – remembered by a label “1”.

We denote $s : \tilde{\mathfrak{y}}_{g,n}^{(d)} \rightarrow \mathfrak{y}_{g,n}^{(d)}$ the map which forgets the label “1”. We have an injective linear map

$$S : \begin{array}{ccc} \mathbb{C}[\mathfrak{y}_{g,n}^{(d)}] & \longrightarrow & \mathbb{C}[\tilde{\mathfrak{y}}_{g,n}^{(d)}] \\ \lambda & \longmapsto & \sum_{\tilde{\lambda} \in s^{-1}(\lambda)} \tilde{\lambda}, \end{array}$$

that is, each Young diagram is mapped to the sum (linear combination with unit coefficients) of the same Young diagrams differing only by placing the label “1” on all types of columns present in this diagram. We now define two unary operations on diagrams from $\mathfrak{y}_{g,n}^{(d)}$ which results in two linear maps

$$\Delta_B : \mathbb{C}[\mathfrak{y}_{g,n}^{(d)}] \longrightarrow \mathbb{C}[\tilde{\mathfrak{y}}_{g,n+1}^{(d)}], \quad \Delta_C : \mathbb{C}[\mathfrak{y}_{g-1,n+1}^{(d)}] \longrightarrow \mathbb{C}[\tilde{\mathfrak{y}}_{g,n}^{(d)}],$$

and a binary operation on ordered pairs of colored Young diagrams, which results in the bilinear map

$$\Delta_C^{(2)} : \mathbb{C}[\mathfrak{y}_{g_1,n_1}^{(d)}] \otimes \mathbb{C}[\mathfrak{y}_{g_2,n_2}^{(d)}] \longrightarrow \mathbb{C}[\tilde{\mathfrak{y}}_{g_1+g_2,n_1+n_2-1}^{(d)}].$$

The data for our recursion will be either finite or semi-infinite complex tensors $B = (B_{(k_2,\alpha_2),(k_3,\alpha_3)}^{(k_1,\alpha_1)})$ and $C = (C_{(k_2,\alpha_2),(k_3,\alpha_3)}^{(k_1,\alpha_1)})$, where $(k_i, \alpha_i) \in \mathbb{N}_{>0} \times \{1, \dots, d\}$ characterize the possible column types. We assume in the semi-infinite case that the entries of B vanish whenever $k_1 + k_2 > k_3 + r$, and the entries of C vanish whenever $k_1 > k_2 + k_3 + r$. This guarantees that all sums appearing below are finite. The bound (130) on the number of boxes of our Young diagrams are tailored to this property of B and C .

11.2 The operations

The first unary operation $\Delta_B : \mathbb{C}[\mathfrak{y}_{g,n}^{(d)}] \rightarrow \mathbb{C}[\tilde{\mathfrak{y}}_{g,n+1}^{(d)}]$ is defined by the following rule. It is a sum over $\alpha_3 \in \{1, \dots, d\}$ followed by a sum over all possible column types in λ of color α_3 . The terms of this

sum are obtained by replacing a column of the selected type with two new columns of colors α_1 and α_2

$$\Delta_B \left(\begin{array}{c} \alpha_3 \\ \updownarrow \\ k_3 \end{array} \right) = \sum_{\substack{k_1, k_2 \geq 1 \\ k_1 + k_2 \leq k_3 + r}} \sum_{\alpha_1, \alpha_2 = 1}^d B_{(k_2, \alpha_2), (k_3, \alpha_3)}^{(k_1, \alpha_1)} N_{k_2, \alpha_2}(\lambda) \quad k_1 \downarrow \begin{array}{c} \alpha_1 \\ \updownarrow \\ 1 \end{array} \begin{array}{c} \alpha_2 \\ \updownarrow \\ k_2 \end{array} . \quad (131)$$

In this operation, we place a label “1” on the column type (k_1, α_1) in the resulting Young diagram.

For the second unary operation $\Delta_C : \mathbb{C}[\mathcal{Y}_{g-1, n+1}] \rightarrow \mathbb{C}[\tilde{\mathcal{Y}}_{g, n}]$, we proceed analogously, except now we sum over ordered pairs $((k_2, \alpha_2), (k_3, \alpha_3))$ of column types in λ . The terms are obtained by replacing the ordered pair of column of each type

$$\Delta_C \left(\begin{array}{c} \alpha_2 \quad \alpha_3 \\ \updownarrow \quad \updownarrow \\ k_2 \quad k_3 \end{array} \right) = \sum_{1 \leq k_1 \leq k_2 + k_3 + r} \sum_{\alpha_1 = 1}^d \frac{1}{2} C_{(k_2, \alpha_2), (k_3, \alpha_3)}^{(k_1, \alpha_1)} \begin{array}{c} \alpha_1 \\ \updownarrow \\ 1 \end{array} \begin{array}{c} k_1 \end{array} , \quad (132)$$

with a new column of color α_1 .

The binary operation $\Delta_C^{(2)} : \mathbb{C}[\mathcal{Y}_{g_1, n_1}] \times \mathbb{C}[\mathcal{Y}_{g_2, n_2}] \rightarrow \mathbb{C}[\tilde{\mathcal{Y}}_{g_1 + g_2, n_1 + n_2 - 1}]$ is obtained in a similar way to Δ_C , but fusing the two Young diagrams. More precisely, $\Delta_C^{(2)}(\lambda, \lambda')$ is the sum over all column types (k_2, α_2) in λ and column types (k_3, α_3) in λ' , of the following contribution. We erase one column of the selected type in λ and λ' , fuse the two Young diagrams, and insert a column of type (k_1, α_1) with a label “1” and a weight

$$\frac{1}{2} C_{(k_2, \alpha_2), (k_3, \alpha_3)}^{(k_1, \alpha_1)}.$$

These terms are then summed over $(k_1, \alpha_1) \in \mathbb{N}_{>0} \times \{1, \dots, d\}$ such that $k_1 \leq k_2 + k_3 + r$ to give $\Delta_C^{(2)}(\lambda, \lambda')$.

11.3 Evaluation and relation to quantum Airy structures

Let $\mathcal{S}^{(d)}$ be the \mathbb{C} -algebra of symmetric functions in infinite number of variables $x_{\alpha, i}$, $i \in \mathbb{N}_{\geq 0}$ and $\alpha \in \{1, \dots, d\}$. The power sums $p_{k, \alpha} := \sum_{i \geq 0} x_{\alpha, i}^k$ give a linear basis for $\mathcal{S}^{(d)}$. If λ is a Young diagram with a d -coloring we denote

$$P_\lambda := \prod_{\substack{k \geq 1 \\ 1 \leq \alpha \leq d}} p_{k, \alpha}^{N_{k, \alpha}(\lambda)}.$$

We define the linear evaluation map

$$\text{ev} : \begin{array}{ccc} \mathbb{C}[\mathcal{Y}_{g, n}^{(d)}] & \longrightarrow & \mathcal{S}^{(d)} \\ \lambda & \longmapsto & |\text{Aut } \lambda|^{-1} P_\lambda \end{array} .$$

The map ev is obviously injective. Let $k_1, \alpha_1 | \dots | k_\ell, \alpha_\ell$ be the Young diagram with columns of height k_i and color α_i . We remark that $\mathcal{S}^{(d)}$ is isomorphic – via Taylor expansions – to the algebra of polynomial functions on

$$V := z\mathbb{C}^d[[z]] \cong \bigoplus_{\substack{k \geq 1 \\ 1 \leq \alpha \leq d}} \mathbb{C} \cdot \underline{k, \alpha}. \quad (133)$$

Proposition 11.1 *Assume that (A, B, C, D) defines a quantum Airy structure on V given by (133), and assume that $A_{(k_2, \alpha_2), (k_3, \alpha_3)}^{(k_1, \alpha_1)} = 0$ whenever $k_1 + k_2 + k_3 > r$, and $D^{(k, \alpha)}$ vanishes whenever $k > r$.*

Then there exists unique $\Omega_{g,n} \in \mathbb{C}[Y_{g,n}]$ indexed by $g \geq 0$ and $n \geq 1$ satisfying $2g - 2 + n > 0$, such that

$$\Omega_{0,3} = \sum_{(k_i, \alpha_i)_{i=1}^3} A_{(k_2, \alpha_2), (k_3, \alpha_3)}^{(k_1, \alpha_1)} \underline{k_1, \alpha_1 | k_2, \alpha_2 | k_3, \alpha_3}, \quad (134)$$

$$\Omega_{1,1} = \sum_{k, \alpha} D^{(k, \alpha)} \underline{k, \alpha}, \quad (135)$$

and for $2g + n \geq 0$

$$S(\Omega_{g,n}) = \Delta_B(\Omega_{g,n-1}) + \Delta_C(\Omega_{g-1,n+1}) + \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n}} \Delta_C^{(2)}(\Omega_{g_1,n_1}, \Omega_{g_2,n_2}). \quad (136)$$

Moreover the coefficients of the partition function of the quantum Airy structure are $F_{g,n} = \text{ev}(\Omega_{g,n})$.

Vice versa, if the dynamics of Young diagrams is governed by relations (131), (132) endowed with the initial conditions (134) and (135) and we require the result of this action to have the form $S(\Omega_{g,n})$ for some $\Omega_{g,n} \in \mathbb{C}[Y_{g,n}]$ for all g and n , i.e., this result must belong to the image of the mapping S for all g and n , then the partition function of these correlation functions exists and is annihilated by the L_i given by (1).

Let us comment on this formalism. The operations Δ_B, Δ_C and $\Delta_C^{(2)}$ introduce some asymmetry in the treatment of the column types, tracked by the label “1”. The linear map S discards this label by summing over all underlying colored Young diagrams. For given (A, B, C, D) , we would like to define a dynamic on (colored) Young diagrams by the formulae (134)-(136) – note that B and C enter in the definition of Δ_B, Δ_C and $\Delta_C^{(2)}$. However, at each step the right-hand side of (136) is an expression in terms of Young diagrams with a label “1”. As S is an injection, there is at most one expression in colored Young diagrams $\Omega_{g,n}$ satisfying (136). Such a $\Omega_{g,n}$ does exist if and only if the right-hand side of (136) produces a function on labeled colored Young diagrams which lies in the image of the linear map S – i.e. it gives a symmetric function on V when evaluated. This is true only if the quadruple (A, B, C, D) satisfies some conditions. The first part of Proposition 2.4 shows that a sufficient condition for the right-hand side to be in the image of S is that (A, B, C, D) defines a quantum Airy structure. In this case, (134)-(136) just mimicks, at the level of functions on Young diagrams, the recursive computation of the partition function of the quantum Airy structure. The proof is straightforward and thus omitted. Let us prove the inverse statement of Proposition 11.1.

Proof. For simplicity, we replace the multi-index (k, α) merely by k . The consideration below is general and does not depend on details of the model. It is also insensitive to whether we are in a finite or infinite-dimensional situation.

We first identify the coefficients of $F_{g,n}$ with the Taylor coefficients at $\xi = 0$ of a function $S_g(\xi)$

$$F_{g,n}(k_1, \dots, k_n) = \left. \frac{\partial^n S_g}{\partial \xi_{k_1} \partial \xi_{k_2} \dots \partial \xi_{k_n}} \right|_{\xi_i=0}.$$

It is convenient to interpret $\Omega_{g,n}$ as symmetric differentials

$$\Omega_{g,n} := \sum_{\{k\}} F_{g,n}(k_1, \dots, k_n) d\xi_{k_1} \dots d\xi_{k_n}.$$

We introduce an auxiliary object

$$\bar{\Omega} := \sum_{2g-2+n>0} \hbar^{2g-2+n} \sum_{\{k_i\}_{i=1}^n} \frac{\partial^n S_g}{\partial \xi_{k_1} \dots \partial \xi_{k_n}} d\xi_{k_1} \dots d\xi_{k_n},$$

where *we do not impose the constraint* $\xi_i = 0$. We segregate the term proportional to $d\xi_s d\xi_{k_1} \cdots d\xi_{k_n}$ without a priori symmetrization with respect to the index s . The condition that the right-hand side is actually fully symmetric with respect to permutations of all indices including s implies that it must be of the form

$$d\bar{\Omega} = \sum_s \sum_{2g-2+n>0} \hbar^{2g-2+n} \sum_{\{k_i\}_{i=1}^n} \frac{\partial^{n+1} S_g}{\partial \xi_s \partial \xi_{k_1} \cdots \partial \xi_{k_n}} d\xi_s d\xi_{k_1} \cdots d\xi_{k_n},$$

which is fully symmetric by construction.

In the right-hand side we have several terms. Let us segregate the coefficients of $d\xi_s d\xi_{k_1} \cdots d\xi_{k_n}$. Our strategy is to push the whole collection of partial derivatives $\frac{\partial^n}{\partial \xi_{k_1} \cdots \partial \xi_{k_n}}$ outside the action of the other operators. At the end of the calculations we set all $\xi_i = 0$, thus obtaining the original TR relations.

- (1) In the term corresponding to $C_{q,p}^s$ we remove two differentials $d\xi_q$ and $d\xi_p$ and replace them by $d\xi_s$. The corresponding term proportional to $d\xi_s d\xi_{k_1} \cdots d\xi_{k_n}$ reads

$$\begin{aligned} C_{q,p}^s & \sum_{\substack{g_1+g_2=g \\ I \cup J = \{1, \dots, n\}}} \frac{\partial}{\partial \xi_q} \frac{\partial^{|I|} S_{g_1}}{\partial \xi_{k_{\alpha_1}} \cdots \partial \xi_{k_{\alpha_{|I|}}}} \frac{\partial}{\partial \xi_p} \frac{\partial^{|J|} S_{g_2}}{\partial \xi_{k_{\beta_1}} \cdots \partial \xi_{k_{\beta_{|J|}}}} + \frac{\partial^2}{\partial \xi_q \partial \xi_p} \frac{\partial^n S_g}{\partial \xi_{k_1} \cdots \partial \xi_{k_n}} \\ & = \frac{\partial^n}{\partial \xi_{k_1} \cdots \partial \xi_{k_n}} \left[C_{q,p}^s \sum_{g_1+g_2=g} \frac{\partial S_{g_1}}{\partial \xi_q} \frac{\partial S_{g_2}}{\partial \xi_p} + C_{q,p}^s \frac{\partial^2 S_{g-1}}{\partial \xi_q \partial \xi_p} \right]. \end{aligned}$$

- (2) In the term corresponding to $B_{q,p}^s$ we erase $d\xi_p$ and add $d\xi_s$ and $d\xi_q$ and multiply the result by N_q —the number of times the index q appears in the set $\{k_1, \dots, k_n\}$. For the action of this operation not to vanish, the index q has to be found at least once in the set k_1, \dots, k_n , say, $q = k_a$ and the corresponding coefficient is $\frac{\partial^{n-1} S_g}{\partial \xi_{k_1} \cdots \widehat{\partial \xi_{k_a}} \cdots \partial \xi_{k_n}}$. Here, the hat denotes the omission of the corresponding term. In order to collect the set of partial derivatives with respect to all ξ_{k_i} , $i = 1, \dots, n$ we use the following trick. We write

$$N_{k_a} \frac{\partial^n S_g}{\partial \xi_p \partial \xi_{k_1} \cdots \widehat{\partial \xi_{k_a}} \cdots \partial \xi_{k_n}} = \frac{\partial^n}{\partial \xi_{k_1} \cdots \partial \xi_{k_n}} \left[\xi_{k_a} \frac{\partial S_g}{\partial \xi_p} \right] + N_{k_a} \xi_{k_a} \frac{\partial^{n+1} S_g}{\partial \xi_p \partial \xi_{k_1} \cdots \partial \xi_{k_n}}, \quad (137)$$

where $N_{k_a} = N_q$ is exactly the proper coefficient appearing in the TR relations (131). The second term in the right-hand side of (137) vanishes, when we impose the condition $\xi_i = 0$ at the end of calculations. The remaining term reads

$$\frac{\partial^n}{\partial \xi_{k_1} \cdots \partial \xi_{k_n}} \left[\sum_{q,p} B_{q,p}^s \xi_q \frac{\partial S_g}{\partial \xi_p} \right]. \quad (138)$$

- (3) The last two terms describe the lowest order terms of TR, not determined by recursion formulae, namely $F_{0,3}$ and $F_{1,1}$. For the first term, we just use that

$$\frac{\partial^3 S_0}{\partial \xi_s \partial \xi_q \partial \xi_p} = A_{q,p}^s \frac{\partial^2}{\partial \xi_q \partial \xi_p} [\xi_q \xi_p] + O(\xi_i), \quad \text{where } A_{q,p}^s = \left. \frac{\partial^3 S_0}{\partial \xi_s \partial \xi_q \partial \xi_p} \right|_{\xi_i=0}$$

and for the second we get that

$$\frac{\partial S_1}{\partial \xi_s} = D_s + O(\xi_s) \quad \text{where } D_s = \left. \frac{\partial S_1}{\partial \xi_s} \right|_{\xi_s=0}.$$

Combining all terms together, we obtain the following statement.

The coefficient of $\hbar^{2g+n-2} d\xi_s d\xi_{k_1} \cdots d\xi_{k_n}$ in the TR relations is given by the following expression

$$\frac{\partial^n}{\partial \xi_{k_1} \cdots \partial \xi_{k_n}} \left[\hbar^{-1} \frac{\partial S_g}{\partial \xi_s} - \sum_{q,p} B_{q,p}^s \xi_q \frac{\partial S_g}{\partial \xi_p} - \sum_{q,p} C_{q,p}^s \left(\sum_{g_1+g_2=g} \frac{\partial S_{g_1}}{\partial \xi_q} \frac{\partial S_{g_2}}{\partial \xi_p} + \frac{\partial^2 S_{g-1}}{\partial \xi_q \partial \xi_p} \right) - \delta_{2g+n,3} \left(\sum_{q,p} A_{q,p}^s \xi_q \xi_p + D_s \right) \right] \Big|_{\xi_i=0} = 0. \quad (139)$$

Because relation (139) holds for all sets of external partial derivatives $\frac{\partial^n}{\partial \xi_{k_1} \cdots \partial \xi_{k_n}}$ and the expression in square brackets depends neither on n nor on the set $\{k_i\}_{i=1}^n$, whereas the quantities S_g are defined to be formal power series in ξ_i , we conclude that this expression is identically zero for all ξ_i in an open neighborhood of the set of initial values $\xi_i = 0$. Thus the set of TR relations is equivalent to the set of (A, B, C, D) -differential constraints L_s imposed on the partition function \mathcal{Z} . \square

This proposition applies in particular to the quantum Airy structure of Proposition 8.13 and 8.14, and *a fortiori* to the one underlying the topological recursion of [24] according to Proposition 9.2 and its new, branched cover-free version Proposition 10.2.

12 A list of problems

By way of conclusion, we collect a few problems opened throughout the article – a disjoint list of problems was put forward in [36].

Problem 12.1 *Complete the classification of finite-dimensional quantum Airy structures based on semi-simple Lie algebras.*

This is likely to be a case study of the candidate modules listed in Proposition 6.9. One may wonder in particular whether $\mathfrak{sl}_2(\mathbb{C})$ is the only simple Lie algebra supporting a quantum Airy structure, and if not, if the resulting classification has a geometric meaning.

Problem 12.2 *What is the field theoretic meaning of the quantum Airy structure/partition function attached to a non-commutative Frobenius manifold?*

In the commutative case, we have found that the partition function computes the 2d TQFT amplitudes (Lemma 8.6). It is desirable to have a similar interpretation, maybe involving open-closed 2d TQFTs [37, 39] – these are indeed in correspondence with pairs of commutative and non-commutative Frobenius algebras together with some morphisms between them [3]. Independently, one may wonder if the amplitudes of open-closed 2d TQFTs can be computed from quantum Airy structures.

We gave in Section 4 the basic definitions of moduli spaces of classical and quantum Airy structures. The geometry of these spaces is worth studying. In particular, the translations define commuting flows (although maybe not independent) on them. In this direction, one may wonder if those spaces carry integrable systems.

Problem 12.3 *Study the algebraic geometry of the moduli spaces of classical and quantum Airy structures.*

The Lagrangian cones studied by Givental and Coates are quadratic Lagrangians in a symplectic space $T^*V[[z]]$, therefore seem to be close to the setting for classical Airy structures. For those cones

related to semi-simple Frobenius manifolds, one could use the action of the twisted loop group to bring such Lagrangians in the form of a standard Lagrangian cone which is known to describe an Airy structure (whose partition function is a product of the matrix Airy function of [34]), and therefore the original cone does correspond to a classical Airy structure. This is however rather indirect.

Problem 12.4 *Can one associate directly to Givental's Lagrangian cones a classical Airy structure (or some generalization, e.g. dropping the assumption that the differential operators are at most quadratic), without the semi-simplicity assumption?*

Problem 12.5 *Given a quantum Airy structure on V , when does S_0 defines the structure of a (germ of a) Frobenius manifolds at 0 on V_0 ? Or, more generally, if V is infinite-dimensional and contains a distinguished finite-dimensional subspace V_0 , when does the restriction of S_0 to V_0 defines a (germ of a) Frobenius manifolds at 0 on V_0 ?*

We have checked that the S_0 of the quantum Airy structure of Proposition 8.2 on $V =$ a Frobenius algebra, does define the prepotential of a germ of a Frobenius manifold at 0 in V . We also know that this is true for the quantum Airy structure corresponding (see Section 9.1) to TR for compact spectral curves. In this case S_0 is the prepotential of the Hurwitz space equipped with its usual Frobenius structure [16, ?]. It would be interesting to know whether this is still true for the loop space examples of Section 8.4 especially when \mathbb{A} is non semi-simple.

The setting of quantum Airy structure is very much restricted to the case of a symplectic space isomorphic to T^*V . Yet, some works indicate that TR should be related to quantization of moduli spaces, which are curved Kähler manifolds.

Problem 12.6 *Can one construct interesting families of quantum Airy structure from (curved) symplectic manifolds, or from Kähler manifolds?*

A Asymptotics of special functions

A.1 Cardinality of \mathfrak{T}_g

In Proposition 5.3, the partition function for quantum Airy structures with $A = B = 0$ was computed as a sum over the set \mathfrak{T}_g of rooted trivalent trees with g leaves. Let

$$N_g := \sum_{T \in \mathfrak{T}_g} \frac{1}{|\text{Aut } T|}.$$

Applying Proposition 5.3 results in the one-dimensional quantum Airy structure

$$L = \hbar \partial_x - \frac{\hbar}{2} \partial_x^2 - \hbar$$

yields $Z = \exp\left(\sum_{g \geq 1} \hbar^{g-1} N_g\right)$. The differential equation $L \cdot Z = 0$ is easy to solve

$$Z = \exp\left(\frac{x}{\hbar} (1 - \sqrt{1 - 2\hbar})\right).$$

We deduce

$$N_g = \frac{(2g)!}{(2g-1) \cdot 2^g \cdot g!^2}.$$

A.2 The Bairy function

The full asymptotic expansion for the Bairy function is known in closed form

$$\text{Bi}(z) = \frac{e^{\frac{2}{3}z^{\frac{3}{2}}}}{\pi^{\frac{1}{2}} z^{\frac{1}{4}}} \left(\sum_{m \geq 0} \alpha_m z^{-\frac{3m}{2}} \right), \quad \alpha_m = \frac{6^m \Gamma(m + \frac{1}{6}) \Gamma(m + \frac{5}{6})}{2\pi m!}.$$

The first few values are $\alpha_0 = 1$, $\alpha_1 = \frac{5}{6}$, $\alpha_2 = \frac{1}{2!} \frac{385}{36}$, $\alpha_3 = \frac{1}{3!} \frac{85085}{216}$, etc. However, to our knowledge there is no closed formula for the asymptotic expansion of $\ln \text{Bi}(z)$, which are more directly related to the value of $F_{g,m}$. We know it is of the form

$$\ln \text{Bi}(z) = \frac{2}{3} z^{\frac{3}{2}} - \frac{\ln z}{4} - \frac{\ln \pi}{2} + \sum_{g \geq 1} \frac{\beta_g}{6g \cdot 8^g} z^{-\frac{3g}{2}}$$

where $(\beta_g)_{g \geq 1}$ is the integer sequence defined by the initial data $\beta_1 = 0$, $\beta_2 = 5$ and the recurrence relation

$$\beta_g = 6(g-1)\beta_{g-1} + \sum_{h=2}^{g-2} \beta_h \beta_{g-h}$$

which can easily be obtained from the Airy differential equation. The first values are

g	β_g
2	5
3	60
4	1105
5	27120
6	828250
7	30220800
8	1282031525
9	61999046400
10	3366961243750
11	202903221120000
12	13437880555850250
13	970217083619328000
14	75849500508999712500
15	6383483988812390400000

A.3 The Whittaker function

The partition function for the general Airy structure in dimension 1 (Equation 92) has as Taylor coefficients

$$\begin{aligned}
F_{0,3} &= \theta_A \\
F_{0,4} &= 3\theta_A\theta_B \\
F_{0,5} &= 3\theta_A(\theta_A\theta_C + 4\theta_B^2) \\
F_{0,6} &= 15\theta_A\theta_B(3\theta_A\theta_C + 4\theta_B^2) \\
F_{0,7} &= 45\theta_A(\theta_A^2\theta_C^2 + 12\theta_A\theta_B^2\theta_C + 8\theta_B^4) \\
F_{1,1} &= D \\
F_{1,2} &= \frac{1}{2}\theta_A\theta_C + \theta_B D \\
F_{1,3} &= \frac{5}{2}\theta_A\theta_B\theta_C + \theta_A\theta_C D + 2\theta_B^2 D \\
F_{1,4} &= 3\theta_A^2\theta_C^2 + \frac{27}{2}\theta_A\theta_B^2\theta_C + 12\theta_A\theta_B\theta_C D + 6\theta_B^3 D \\
F_{1,5} &= \frac{111}{2}\theta_A^2\theta_B\theta_C^2 + 9\theta_A^2\theta_C^2 D + 84\theta_A\theta_B^3\theta_C + 24\theta_B^2 D \\
F_{2,1} &= \theta_C(\frac{1}{4}\theta_A\theta_C + \frac{1}{2}\theta_B D + \frac{1}{2}D^2) \\
F_{2,2} &= \frac{3}{2}\theta_A\theta_B\theta_C + \theta_A\theta_C D + \frac{3}{2}\theta_B^2 D + \frac{3}{2}\theta_B D^2 \\
F_{2,3} &= 2\theta_A^2\theta_C^2 + \frac{39}{4}\theta_A\theta_B^2\theta_C + \frac{21}{2}\theta_A\theta_B\theta_C D + \frac{3}{2}\theta_A\theta_C D^2 + 6\theta_B^3 D + 6\theta_B^2 D^2 \\
F_{3,1} &= (\theta_B + D)\theta_C^2(\frac{3}{4}\theta_A\theta_C + \frac{3}{4}\theta_B D + \frac{1}{4}D^2) \\
F_{3,2} &= \theta_C^2(\frac{9}{8}\theta_A^2\theta_C^2 + \frac{45}{8}\theta_A\theta_B^2\theta_C + 8\theta_A\theta_B\theta_C D + 2\theta_A\theta_C D^2 + \frac{15}{4}\theta_B^3 D + \frac{25}{4}\theta_B^2 D^2 + 10\theta_B D^3)
\end{aligned}$$

They are enumerating the trivalent graphs appearing when unfolding the topological recursion formula (6). For low values of (g, n) this can be directly checked in Figure 2.

B Vanishing results for quantum Airy structures on loop spaces

Let \mathbb{A} be a Frobenius manifold. We consider the quantum Airy structure on $\mathbb{A}[[z]] = \mathbb{A} \otimes \mathbb{C}[[z]]$ given by Proposition 8.10. The label α will index a basis of \mathbb{A} , and $i \geq 0$ the natural basis of $\mathbb{C}[[z]]$.

Lemma B.1 *Assume $\theta(z) = t_{r_0} z^{r_0} + O(z^{r_0+1})$ with $t_{r_0} \neq 0$ for some $r_0 \geq -1$.*

- If $r_0 = -1$, the partition function is $Z = 1$, i.e. all $F_{g,n}$ vanish.
- If $r_0 \geq 0$, for any g, n such that $2g - 2 + n > 0$, we have $F_{g,n}((i_1, \alpha_1), \dots, (i_n, \alpha_n)) = 0$ whenever $\sum_{m=1}^n i_m > 2g - 2 + r_0(2 - n)$.

Now we consider the \mathbb{Z}_2 -symmetric version of a quantum Airy structure on $\mathbb{A}[[z^2]]$ given by Proposition 8.12. Here the label $i \geq 0$ indexes the natural basis of $\mathbb{C}[[z^2]]$.

Lemma B.2 Assume $\theta(z) = t_{s_0} z^{2s_0} + O(z^{2(s_0+1)})$ with $t_{s_0} \neq 0$ for some $s_0 \geq -1$.

- If $s_0 = -1$, for any g, n such that $2g - 2 + n > 0$, we have $F_{g,n}((i_1, \alpha_1), \dots, (i_n, \alpha_n)) = 0$ whenever $\sum_{m=1}^n i_m > 3g - 3 + n$.
- If $s_0 \geq 0$, for any g, n such that $2g - 2 + n > 0$, we have $F_{g,n}((i_1, \alpha_1), \dots, (i_n, \alpha_n)) = 0$ whenever $\sum_{m=1}^n i_m > g - 1 + s_0(2 - n)$.

Proof of Proposition B.1. We give the proof in detail for $\mathbb{A} = \mathbb{C}$, as the argument and the result are similar for general Frobenius algebras. We also remark that the conjugation by $\exp\left(\frac{\hbar}{2} u_{a,b} \partial_a \partial_b\right)$ preserves the vanishing property, so it is sufficient to prove it when

$$\xi_k = \frac{k+1}{z^{k+2}} dz.$$

The corresponding quantum Airy structure has $A = 0$ and

$$B_{j,k}^i = \sum_{r \geq r_0} \frac{2k+1}{2i+1} t_r \delta_{i+j+r,k}, \quad (140)$$

$$C_{j,k}^i = \sum_{r \geq r_0} \frac{(2k+1)(2j+1)}{2i+1} t_r \delta_{i+r,j+k+2}. \quad (141)$$

So, $B_{j,k}^i = 0$ unless $i + j + r_0 \leq k$ and $C_{j,k}^i = 0$ unless $i + r_0 \leq j + k + 2$. We also know from Section 8.3 that $D^k = 0$ unless $k \leq r_0$.

The fact that $A = 0$ already guarantees that $F_{0,n} = 0$ for all n . If $r_0 = -1$, we furthermore have $D = 0$. Since A and D are initial data for the recursion formula (6), it therefore can only produce $F_{g,n} = 0$ for all (g, n) .

We now assume $r_0 \geq 0$, and examine the conditions under which $F_{g,n}$ for $2g - 2 + n > 0$ are possibly non-zero. We already know that $F_{0,3}(i, j, k) = A_{j,k}^i = 0$, and that $F_{1,1}(i) = D^i$ vanishes unless $i \leq r_0$. We have

$$F_{1,2}(i, j) = B_{j,a}^i D^a,$$

and it will vanish unless the two conditions $i + j + r_0 \leq a$ and $a \leq r_0$ are satisfied. This imposes $i + j \leq 0$, hence $i = j = 0$. We also have

$$F_{2,1}(i) = \frac{1}{2} C_{a,b}^i (F_{1,2}(a, b) + D^a D^b).$$

The first term vanishes unless the two conditions $i + r_0 \leq a + b + 2$ and $a = b = 0$ are satisfied, hence unless $i \leq 2 - r_0$. The second term vanishes unless the three conditions $i + r_0 \leq a + b + 2$, $a \leq r_0$ and $b \leq r_0$ are satisfied, hence unless $i \leq r_0 + 2$. As $r_0 \geq 0$, we deduce that $F_{2,1}(i)$ vanishes unless $i \leq r_0 + 2$. These three cases $(g, n) \in \{(1, 1), (1, 2), (2, 1)\}$ are sharply compatible with a linear bound

$$\sum_{m=1}^n i_m \leq d_{g,n} = \alpha n + \beta g + \gamma, \quad \text{fixing } (\alpha, \beta, \gamma) = (-r_0, 2, 2r_0 - 2).$$

Note that this bound also holds for $g = 0$ and all $n \geq 3$, as then $d_{g,n} < 0$. Now that we have guessed a candidate for $d_{g,n}$, let us prove by induction on $2g - 2 + n > 0$ that indeed

$$\sum_{m=1}^n i_m > d_{g,n} \implies F_{g,n}(i_1, \dots, i_n) = 0.$$

We already know it is true for $\chi = 2g - 2 + n \leq 2$. Assume it is true for all (g', n') such that $0 < 2g' - 2 + n' < \chi_0$, and pick (g, n) such that $\chi_0 = 2g - 2 + n$. The recursion gives

$$\begin{aligned} F_{g,n}(i_1, \dots, i_n) &= \sum_{m=2}^n B_{i_m, a}^{i_1} F_{g, n-1}(a, i_2, \dots, \widehat{i_m}, \dots, i_n) \\ &+ \frac{1}{2} C_{a,b}^i \left(F_{g-1, n+1}(a, b, i_2, \dots, i_m) + \sum_{\substack{g'+g''=g \\ n'+n''=n+1}} F_{g', n'}(a, \dots) F_{g'', n''}(b, \dots) \right). \end{aligned} \quad (142)$$

We have not written the details of the sum, but the only thing to know is that in each term, all indices $(i_m)_{m=2}^n$ appear once. Denote $[I] = \sum_{m=1}^n i_m$. Using the induction hypothesis – which is valid for all terms in the right-hand side of (142) – and the vanishing rules for B and C , we find that $F_{g,n}(i_1, \dots, i_n)$ vanish unless one of the following three conditions is satisfied

- $[I] \leq d_{g, n-1} - r_0 = d_{g, n}$,
- $[I] \leq d_{g-1, n+1} + 2 - r_0 = d_{g, n}$,
- $[I] \leq d_{g', n'} + d_{g'', n''} + 2 - r_0 = d_{g, n}$.

This proves the claim by induction. We also see that, as the bound was sharp for the three basic cases, it remains sharp throughout the induction.

Proof of Proposition B.2. As before we can restrict to the case $\mathbb{A} = \mathbb{C}$ and

$$\xi_k = \frac{(2k+1)}{z^{2k+2}} dz,$$

which correspond to

$$\begin{aligned} A_{j,k}^i &= \sum_{s \geq s_0} \frac{1}{2i+1} \delta_{i+j+k+s+1, 0} t_s, \\ B_{j,k}^i &= \sum_{s \geq s_0} \frac{2k+1}{2i+1} \delta_{i+j+s, k} t_s, \\ C_{j,k}^i &= \sum_{s \geq s_0} \frac{(2k+1)(2j+1)}{2i+1} \delta_{i+s, j+k+1} t_s. \end{aligned}$$

So, $B_{j,k}^i = 0$ unless $i+j+s_0 \leq k$, and $C_{j,k}^i = 0$ unless $i+s_0 \leq j+k+1$.

For $s_0 = -1$, we have $F_{0,3}(i, j, k) = A_{j,k}^i = 0$ unless $i = j = k = 0$, and we have $D^i = 0$ unless $i \geq 2$. We compute

$$F_{1,2}(i, j) = B_{j,a}^i D^a + \frac{1}{2} C_{a,b}^i A_{a,b}^j. \quad (143)$$

The first term vanishes unless $i+j-1 \leq a$ and $a \leq 1$, that is unless $i+j \leq 2$. The second term vanishes unless $i-1 \leq a+b+1$ and $j = a = b = 0$, that is unless $i+j \leq 2$. Hence, $F_{1,2}(i, j)$ vanishes unless $i+j \leq 2$. The three cases $(g, n) \in \{(0, 3), (1, 1), (1, 2)\}$ are sharply compatible with a linear bound

$$\sum_{m=1}^n i_m \leq d_{g,n} = \alpha n + \beta g + \gamma, \quad \text{fixing } (\alpha, \beta, \gamma) = (1, 3, -3). \quad (144)$$

An induction similar to the proof of Lemma B.1 proves (144) for any (g, n) such that $2g - 2 + n > 0$.

Now consider the case $s_0 \geq 0$. We then have $A = 0$, hence $F_{g,n} = 0$ for all n . From Section 8.3 we already know that $F_{1,1}(i) = D^i$ vanishes unless $i \leq s_0$. We compute

$$F_{1,2}(i, j) = B_{j,a}^i D^a,$$

and it vanishes unless $i + j + s_0 \leq a$ and $a \leq s_0$, hence unless $i + j \leq 0$. We also compute

$$F_{2,1}(i) = \frac{1}{2} C_{a,b}^i (F_{1,2}(a, b) + D^a D^b).$$

The first term vanishes unless the two conditions $i + s_0 \leq a + b + 1$ and $a + b \leq 0$ are satisfied, hence unless $i \leq 1 - s_0$. The second term vanishes unless the three conditions $i + s_0 \leq a + b + 1$ and $a \leq s_0$ and $b \leq s_0$ are satisfied, hence unless $i \leq s_0 + 1$. As $s_0 \geq 0$, we deduce that $F_{2,1}(i)$ vanishes unless $i \leq s_0 + 1$. The three cases $(g, n) \in \{(1, 1), (1, 2), (2, 1)\}$ are compatible with the linear bound

$$\sum_{m=1}^n i_m \leq d_{g,n} = \alpha n + \beta g + \gamma, \quad \text{fixing } (\alpha, \beta, \gamma) = (-s_0, 1, 2s_0 - 1). \quad (145)$$

Again, an induction similar to the proof of Lemma B.1 proves (145) for any (g, n) such that $2g - 2 + n > 0$. \square

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