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by

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SURJECTIVITY OF CERTAIN WORD MAPS ON $PSL(2,\mathbb{C})$ AND $SL(2,\mathbb{C})$

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ABSTRACT. Let F_2 be a free group on two generators, $F^{(1)}, F^{(2)}$ its first and second derived subgroups. We show that if $w \in F^{(1)} \setminus F^{(2)}$, then the corresponding word map $PSL(2,\mathbb{C})^2 \to PSL(2,\mathbb{C})$ is surjective. We also describe certain words maps that are surjective on $SL(2,\mathbb{C})$.

1. Introduction

The surjectivity of word maps on groups became recently a vivid topic: the review on the latest activities may be found in [Se], [Ku], [BGaK], [KBKP].

Let $w \in F_d$ be an element of the free group F_d on d generators $g_1, \dots, g_d$:

$$w = \prod_{i=1}^{n} g_{n_i}^{m_i}, \ 1 \leq n_i \leq d.$$

For a group G by the same letter w we shall denote the corresponding word map $w : G^d \to G$ defined as a non-commutative product by the formula

$$w(x_1, \dots, x_d) = \prod_{i=1}^{k} x_{n_i}^{m_i}.$$

We call $w(x_1, \dots, x_d)$ both *a word in d letters* if considered as an element of a free group and *a word map in d letters* if considered as the corresponding map $G^d \to G$.

We assume that it is reduced, i.e. $n_i \neq n_{i+1}$ for every $1 \leq i \leq k-1$ and $m_i \neq 0$ for $1 \leq i \leq k$.

Let k be a field and $G = H(k)$ a connected semisimple algebraic linear group. Then the image

$$w_G := w(G^d) = \{z \in G \ : z = w(x_1, \dots, x_d) \text{ for some } (x_1, \dots, x_d) \in G^d\}$$

is a Zariski dense subset of $H(k)$ if the word w is not identity (([Bo],[La]).

In [Ku] formulated is the following Question.

Question 2.1 (i). Assume that w is not a power of another reduced word and $G = H(k)$ a connected semisimple algebraic linear group.

Is w surjective when $k = \mathbb{C}$ is a field of complex numbers and H is of adjoint type?

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According to [Ku], Question 2.1(i) is still open, even in the simplest case $G = PSL(2, C)$, even for words in two letters.

We consider word maps in two letters on groups $G = SL(2, \mathbb{C})$ and $\tilde{G} = PSL(2, \mathbb{C})$. Put $F := F_2$. We describe certain words $w \in F$ such that the corresponding word maps are surjective on G and/or $\tilde{G}$.

If $w(x, y) = x^n$ then w is surjective on G if and only if n is odd (see ([Ch1],[Ch2]). Indeed, the element

$$x = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

is not a square in $SL(2, \mathbb{C})$. Since only the elements with $tr(x) = -2$ may be outside w_G ([Ch1],[Ch2]), the induced by w map $\tilde{w}$ is surjective on $\tilde{G}$.

Assume that a word map $w(x, y) : G^2 \to G$ is defined by the formula

$$w(x, y) = \prod_{i=1}^{k} x^{a_i} y^{b_i}.$$

We call $w_i = x^{a_i} y^{b_i}$ *a syllable* of w and k the complexity of w.

We will use the following notation:

- $\mathbb{C}^n_{x_1,\ldots,x_n}$ n-dimensional complex affine space with coordinates $x_1, \ldots, x_n$;
- $s = tr(x),\ t = tr(y),\ u = tr(xy)$, for two matrices $x, y \in G = SL(2, \mathbb{C})$;
- $\pi : G \times G \to \mathbb{A}^3_{s,t,u,}$ is a map $\pi(x, y) = (tr(x), tr(y), tr(xy))$.
- $$id = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For every word $w(x, y) : G^2 \to G$ defined are the trace polynomials $P_w(s, t, u) = tr(w(x, y))$ and $Q_w = tr(w(x, y)y)$ in three variables $s = tr(x),\ t = tr(y),\ u = tr(xy)$. ([FK], [Go],[Vo]).

In other words, the maps

$$\varphi_w : G^2 \to G^2,\ \varphi_w(x, y) = (w(x, y), y)$$

and

$$\psi_w : \mathbb{C}^3_{s,t,u} \to \mathbb{C}^3_{s,t,u},\ \psi_w(s, t, u) = (P_w(s, t, u), t, Q_w(s, t, u))$$

may be included into the following commutative diagram:

$$\begin{array}{ccc} G \times G & \xrightarrow{\varphi} & G \times G \\ \pi\downarrow & & \pi\downarrow \\ \mathbb{C}^3_{s,t,u} & \xrightarrow{\psi} & \mathbb{C}^3_{s,t,u} \end{array} \tag{1}$$

For details, one can be referred to ([BGK],[BGaK]).

This diagram immediately implies

Lemma 1.1. *For every word $w(x,y) \neq id$ the image $w(G)$ contains every semi-simple element $z \in G$ with $a = tr(z) \neq \pm 2$.*

Proof. Indeed, let
$$\Sigma = \{(s,t,u) \mid P_w(s,t,u) = tr(z) = a\}.$$
Since $\Sigma \neq \emptyset$, and π is a surjective map ([Go]), there is a pair $(x_0, y_0) \in G^2$ such that $tr(w(x_0,y_0)) = a$. Since $a \neq \pm 2$, z is conjugate to $z_0 = w(x_0,y_0)$: there is $v \in G$ such that $vz_0v^{-1} = z$. Hence $w(vx_0v^{-1}, vy_0v^{-1}) = z$. □

Thus, in order to check whether the word map w is surjective on G (or on $\tilde{G}$) it is sufficient to check whether the elements z with $tr(z) = \pm 2$ (or the elements z with $tr(z) = 2$, respectively) are in the image.

2. Surjectivity on $PSL(2,\mathbb{C})$

Consider a word map $w(x,y) = x^{a_1}y^{b_1}\dots x^{a_k}y^{b_k}$, where $a_i \neq 0$ and $b_i \neq 0$, for all $i = 1,...,k$. Denote $A(w) = \sum_{i=1}^k a_i$, $B(w) = \sum_{i=1}^k b_i$. Let $\tilde{w} : \tilde{G}^2 \to \tilde{G}$ be the induced word map on $\tilde{G}$.

Assume that $A := A(w) \neq 0$. Then the word map $w_A(x,y) = x^A$ is surjective on $\tilde{G}$. Thus, considering pairs $\{(x, id)\}$ we get that $\tilde{w}(\tilde{G}^2) = \tilde{G}$. Similarly, if $B := B(w) \neq 0$, we have $\tilde{w}(\tilde{G}^2) = \tilde{G}$.

If $A(w) = B(w) = 0$, then $w \in F^{(1)} = [F,F]$. Since $F^{(1)}$ is a free group generated by elements $w_{n,m} = [x^n, y^m]$, $n \neq 0$, $m \neq 0$ ([Ser], Chapter 1, §1.3), the word w with $A(w) = B(w) = 0$ may be written as a (noncommutative) product (with $s_i \neq 0$)

$$w = \prod_1^r w_{n_i,m_i}^{s_i}. \tag{2}$$

Moreover, the shortest (reduced) representation of this kind is unique. We denote by $S_w(n,m)$ the number of appearances of $w_{n,m}$ in representation (2) of w and by $R_w(n,m)$ the sum of exponents at all the appearances. We denote by $Supp(w)$ the set of all pairs (n,m) such that $w_{n,m}$ appears in the product. For example, if $w = w_{1,1}w_{2,2}^5w_{1,1}^{-1}$, then

$$Supp(w) = \{(1,1),(2,2)\}; S_w(1,1) = 2, S_w(2,2) = 1, R_w(1,1) = 0, R_w(2,2) = 5.$$

The subgroup
$$F^{(2)} = [F^{(1)}, F^{(1)}] = \{w \in F^{(1)} | R_w(n,m) = 0 \text{ for all } (n,m) \in Supp(w)\}.$$

Example 2.1. The Engel word $e_n = \underbrace{[...[x,y],y],...y]}_{n \quad times}$ belongs to $F^{(1)} \setminus F^{(2)}$ (see also [ET]).

Indeed, the direct computation shows that

$$yw_{n,m} = yx^ny^mx^{-n}y^{-m} = yx^ny^{-1}x^{-n} \cdot x^nyy^mx^{-n}y^{-m}y^{-1} \cdot y = w_{n,1}^{-1}w_{n,m+1}y, \tag{3}$$

(4) $yw_{n,m}^{-1} = y \cdot y^m x^n y^{-m} x^{-n} = y^{(m+1)} x^n y^{-(m+1)} x^{-n} \cdot x^n y x^{-n} y^{-1} \cdot y = w_{n,m+1}^{-1} w_{n,1} y.$

Let us prove by induction that $|R_{e_n}(1,n)| = 1$, $S_{e_n}(1,n) = 1$ and $S_{e_n}(r,m) = 0$ if $r \neq 1$ or $m > n$.

Indeed $e_1 = w_{1,1}$. Assume that the claim is valid for all $k \leq n$. We have $e_{n+1} = e_n y e_n^{-1} y^{-1}$. Using (3), (4) me can move y toward y^{-1}, changing places of y with its right neighbour $w_{1,m}$, one change at each step. By induction assumption, only $w_{1,m}$ appear in e_n, and for all of them but one $m < n$. Thus at each step we will get factors $w_{1,m+1}$ and $w_{1,1}$ with appropriate powers, and at each step but one $m < n$. There will be precisely one change with $w_{1,n}$ which will provide precisely one appearance of $w_{1,n+1}$. At the last step we will get product of words of type $w_{1,m}$ with proper powers and $y \cdot y^{-1}$ at the end. Thus the claim will remain to be valid for $n+1$.

Theorem 2.2. *The word map defined by a word $w \in F^{(1)} \setminus F^{(2)}$ is surjective on $PSL(2,\mathbb{C})$.*

Remark 2.3. In [ET] the words from $F^{(1)} \setminus F^{(2)}$ are proved to be surjective on $SU(n) \times SU(n)$.

Proof. We have only to prove that a matrix

$$\begin{pmatrix} 1 & K \\ 0 & 1 \end{pmatrix} \tag{5}$$

for a non-zero $K \neq 0$ is in the image.

Let us take

$$x = \begin{pmatrix} \lambda & c \\ 0 & \frac{1}{\lambda} \end{pmatrix}, \tag{6}$$

$$y = \begin{pmatrix} \mu & d \\ 0 & \frac{1}{\mu} \end{pmatrix}, \tag{7}$$

Then

$$x^n = \begin{pmatrix} \lambda^n & c \cdot h_{|n|}(\lambda) sgn(n) \\ 0 & \frac{1}{\lambda^n} \end{pmatrix}, \tag{8}$$

$$y^m = \begin{pmatrix} \mu^m & d \cdot h_{|m|}(\mu) sgn(m) \\ 0 & \frac{1}{\mu^m} \end{pmatrix}, \tag{9}$$

Here *sgn* is the *signum* function, and (see [BG], Lemma 5.2)

$$h_n(\zeta) = \frac{\zeta^{2n} - 1}{\zeta^{n-1}(\zeta^2 - 1)}. \tag{10}$$

.

Note that $h_n(1) = n$.

Direct computations show that

$$x^n y^m = \begin{pmatrix} \lambda^n \mu^m & d \cdot \lambda^n sgn(m) h_{|m|}(\mu) + c \cdot sgn(n) h_{|n|}(\lambda)\mu^{-m} \\ 0 & \lambda^{-n}\mu^{-m} \end{pmatrix}. \tag{11}$$

$$x^{-n} y^{-m} = \begin{pmatrix} \lambda^{-n} \mu^{-m} & -d \cdot \lambda^{-n} sgn(m) h_{|m|}(\mu) - c \cdot sgn(n) h_{|n|}(\lambda)\mu^{m} \\ 0 & \lambda^{n}\mu^{m} \end{pmatrix}. \tag{12}$$

$$w_{n,m}(x,y) = \begin{pmatrix} 1 & f(c,d,n,m,) \\ 0 & 1 \end{pmatrix}, \tag{13}$$

where

$$f(c,d,n,m) = ch_{|n|}(\lambda) sgn(n) \lambda^n (1-\mu^{2m}) + dh_{|m|}(\mu) sgn(m) \mu^m (\lambda^{2n} - 1). \tag{14}$$

Hence,

$$w(x,y) = \prod_1^r w_{n_i,m_i}^{s_i}(x,y) = \begin{pmatrix} 1 & F_w(c,d,\lambda,\mu) \\ 0 & 1 \end{pmatrix}, \tag{15}$$

where

$$F_w(c,d,\lambda,\mu) = \sum_1^r s_i f(c,d,n_i,m_i) = c\Phi_w(\lambda,\mu) + d\Psi_w(\lambda,\mu)$$

and

$$\Phi_w(\lambda,\mu) = \sum_{(\alpha,\beta)\in Supp(w)} R_w(\alpha,\beta) sgn(\alpha)(1-\mu^{2\beta}) \frac{(\lambda^{2|\alpha|}-1)\lambda^\alpha}{\lambda^{|\alpha|-1}(\lambda^2-1)}, \tag{16}$$

$$\Psi_w(\lambda,\mu) = \sum_{(\alpha,\beta)\in Supp(w)} R_w(\alpha,\beta) sgn(\beta)(\lambda^{2\alpha}-1) \frac{(\mu^{2|\beta|}-1)\mu^\beta}{\mu^{|\beta|-1}(\mu^2-1)}. \tag{17}$$

(Since the order of factors in w is not relevant for (16) and (17) , we use here α, β instead of n_i, m_i to simplify the formulas).

The function $F_w(c,d,\lambda,\mu) = c\Phi(\lambda,\mu) + d\Psi(\lambda,\mu)$, where c, d may be chosen arbitrary, therefore it is sufficient to prove that at least one of $\Phi(\lambda,\mu)$ or $\Psi(\lambda,\mu)$ is not identically zero.

Lemma 2.4. *If $\Phi_w(\lambda,\mu) \equiv 0$ then $R_w(\alpha,\beta) = 0$ for all $(\alpha,\beta) \in Supp(w)$.*

Proof. We use induction by the number of elements $|Supp(w)|$ in $Supp(w)$ for the word w. If $Supp(w)$ contains only one pair (α,β), then there is nothing to prove:

$$\Phi(\lambda,\mu) = R_w(\alpha,\beta) h_{|\alpha|}(\lambda) sgn(\alpha) \lambda^\alpha (1-\mu^{2\beta}).$$

Assume that for words v with $|Supp(v)| = l$ it is proved. Let w be such a word that $|Supp(w)| = l + 1$.

Let $n := max\{\alpha \ |(\alpha, \beta) \in Supp(w)\}$.

Case 1. $n > 0$.

We have

$$\Phi_w(\lambda, \mu) = \sum_{(\alpha,\beta)\in Supp(w)} R_w(\alpha, \beta) sgn(\alpha)(1 - \mu^{2\beta}) \frac{(\lambda^{2|\alpha|} - 1)\lambda^{\alpha}}{\lambda^{|\alpha|-1}(\lambda^2 - 1)} =$$

$$\sum_{(\alpha,\beta)\in Supp(w)} R_w(\alpha, \beta) sgn(\alpha)(1 - \mu^{2\beta})\lambda^{a-|a|+1}(1 + \lambda^2 + \cdots + \lambda^{2(|\alpha|-1)}).$$

It means that the coefficient of $\lambda^{2|n|-1}$ in rational function $\Phi_w(\lambda, \mu)$ is

$$p(\mu) = \sum_{(n,\beta)\in Supp(w)} R_w(n, \beta)(1 - \mu^{2\beta}).$$

Hence, if $\Phi_w(\lambda, \mu) \equiv 0$, then $p(\mu) \equiv 0$, and all $R_w(n, \beta) = 0$ for all β.

That means that $\Phi_w(\lambda, \mu) = \Phi_v(\lambda, \mu)$, where v is such a word that may be obtained from $w(x, y) = \prod_1^r w_{n_i,m_i}^{s_i}(x, y)$ by taking away every appearance of $w_{n,\beta}$:

$$v = \prod_{\substack{1 \\ n_i \neq n}}^{r} w_{n_i,m_i}^{s_i}(x, y).$$

But $|Supp(v)| \leq l$ and by induction assumption Lemma is valid in this case.

Case 2. $n < 0$. Let $-n' := min\{\alpha \ |(\alpha, \beta) \in Supp(w)\}$ We proceed as in Case 1 with $-n'$ instead of n : the coefficient of $\lambda^{-2n'+1}$ is $q(\mu) = \sum_{(-n',\beta)\in Supp(w)} R_w(-n', \beta)(1-\mu^{2\beta})$. If $\Phi_w(\lambda, \mu) \equiv 0$, then $q(\mu) \equiv 0$, and all $R_w(-n', \beta) = 0$ for all β. Once more, we may replace w by a word v with $|Supp(v)| \leq l$. □

We have proven, that if $w \notin F^{(2)}$ and x, y are defined by (6),(7), then

$$w(x, y) = \begin{pmatrix} 1 & F_w(c, d, \lambda, \mu) \\ 0 & 1 \end{pmatrix},$$

where $F_w(c, d, \lambda, \mu)$ is not an identically zero function. Thus, there are elements of the form

$$\begin{pmatrix} 1 & K \\ 0 & 1 \end{pmatrix}$$

for a $K \neq 0$ in the image $w(G^2)$. □

3. Surjectivity on $SL(2,\mathbb{C})$

We maintain notation of Section 2.

Lemma 3.1. *Assume that* $w = x^{a_1}y^{b_1}\dots x^{a_k}y^{b_k},\quad a_i\neq 0,\ b_i\neq 0,\ i+1,...,k\ A=\sum a_i \neq 0$ *or* $B=\sum b_i\neq 0$ *and* x, y *are defined by* (6), (7) *respectively. Then*

$$w(x,y)=\begin{pmatrix}\lambda^A\mu^B & \tilde{F}_w(c,d,\lambda,\mu)\\ 0 & \lambda^{-A}\mu^{-B}\end{pmatrix}, \tag{18}$$

where

$$\tilde{F}_w(c,d,\lambda,\mu)=c\tilde{\Phi}_w(\lambda,\mu)+d\tilde{\Psi}_w(\lambda,\mu)$$

and

$$\tilde{\Phi}_w(\lambda,\mu)=\sum_1^k sgn(a_i)h_{|a_i|}(\lambda)\frac{\lambda^{\sum_{j<i}a_i}\mu^{\sum_{j<i}b_i}}{\lambda^{\sum_{j>i}a_i}\mu^{\sum_{j\geq i}b_i}} \tag{19}$$

$$\tilde{\Psi}_w(\lambda,\mu)=\sum_1^k sgn(b_i)h_{|b_i|}(\mu)\frac{\lambda^{\sum_{j\leq i}a_i}\mu^{\sum_{j<i}b_i}}{\lambda^{\sum_{j>i}a_i}\mu^{\sum_{j>i}b_i}} \tag{20}$$

Proof. We use induction on the complexity k of the word w. Using (11), we get

$$x^{a_1}y^{b_1}=\begin{pmatrix}\lambda^{a_1}\mu^{b_1} & d\cdot\lambda^{a_1}sgn(b_1)h_{|b_1|}(\mu)+c\cdot sgn(a_1)h_{|a_1|}(\lambda)\mu^{-b_1}\\ 0 & \lambda^{-a_1}\mu^{-b_1}\end{pmatrix}. \tag{21}$$

Thus for $k=1$ the Lemma is valid. Assume that it is valid for $k'<k$. Let $u=x^{a_1}y^{b_1}\dots x^{a_{k-1}}y^{b_{k-1}}$ and $w=ux^{a_k}y^{b_k}$.

By induction assumption,

$$u(x,y)=\begin{pmatrix}\lambda^{A-a_k}\mu^{B-b_k} & \tilde{F}_u(c,d,\lambda,\mu)\\ 0 & \lambda^{-A+a_k}\mu^{-B+b_k}\end{pmatrix}.$$

From (11) we get

$$x^{a_k}y^{b_k}=\begin{pmatrix}\lambda^{a_k}\mu^{b_k} & d\cdot\lambda^{a_k}sgn(b_k)h_{|b_k|}(\mu)+c\cdot sgn(a_k)h_{|a_k|}(\lambda)\mu^{-b_k}\\ 0 & \lambda^{-a_k}\mu^{-b_k}\end{pmatrix}.$$

Multiplying matrices u and $x^{a_k}y^{b_k}$ we get

$\tilde{F}_w(c,d,\lambda,\mu)=\lambda^{A-a_k}\mu^{B-b_k}(d\cdot\lambda^{a_k}sgn(b_k)h_{|b_k|}(\mu)+c\cdot sgn(a_k)h_{|a_k|}(\lambda)\mu^{-b_k})+\tilde{F}_u(c,d,\lambda,\mu)\lambda^{-a_k}\mu^{-b_k}.$

Thus, the induction assumption implies that

$$\tilde{\Phi}_w(\lambda,\mu)=sgn(a_k)h_{|a_k|}(\lambda)\mu^{-b_k}\lambda^{A-a_k}\mu^{B-b_k}+\sum_1^{k-1}sgn(a_i)h_{|a_i|}(\lambda)\frac{\lambda^{\sum_{j<i}a_i}\mu^{\sum_{j<i}b_i}}{\lambda^{\sum_{j=i+1}^k a_i}\mu^{\sum_{j=i}^k b_i}}$$

$$=\sum_1^k sgn(a_i)h_{|a_i|}(\lambda)\frac{\lambda^{\sum_{j<i}a_i}\mu^{\sum_{j<i}b_i}}{\lambda^{\sum_{j>i}a_i}\mu^{\sum_{j\geq i}b_i}}.$$

$$\tilde{\Psi}_w(\lambda,\mu) = sgn(b_k)h_{|b_k|}(\mu)\lambda^{a_k}\lambda^{A-a_k}\mu^{B-b_k} + \sum_1^{k-1} sgn(b_i)h_{|b_i|}(\mu)\frac{\lambda^{\sum_{j\le i} a_i}\mu^{\sum_{j<i} b_i}}{\lambda^{\sum_{j=i+1}^k a_i}\mu^{\sum_{j=i+1}^k b_i}}$$

$$= \sum_1^k sgn(a_i)h_{|a_i|}(\lambda)\frac{\lambda^{\sum_{j\le i} a_i}\mu^{\sum_{j<i} b_i}}{\lambda^{\sum_{j>i} a_i}\mu^{\sum_{j>i} b_i}}.$$

□

Assume now that for $K \neq 0$ the matrices

(22) $$\begin{pmatrix} -1 & K \\ 0 & -1 \end{pmatrix}$$

are not in the image. That means that $\tilde{\Phi}_w(\lambda,\mu) \equiv 0$ and $\tilde{\Psi}_w(\lambda.\mu) \equiv 0$ on the curve

$$C = \{\lambda^A\mu^B = -1\} \subset \mathbb{C}^2_{\lambda,\mu}.$$

Denote:

$$A_i = \sum_{j\le i} a_i;\ \ B_i = \sum_{j<i} b_i.$$

Multiplying (19) and (20) by $\lambda^A\mu^B$ we see that on C the following relations are valid:

(23) $$\tilde{\Phi}_w(\lambda,\mu) = -\sum_1^k sgn(a_i)h_{|a_i|}(\lambda)\lambda^{2A_i-a_i}\mu^{2B_i}$$

(24) $$\tilde{\Psi}_w(\lambda,\mu) = -\sum_1^k sgn(b_i)h_{|b_i|}(\mu)\lambda^{2A_i}\mu^{\sum 2B_i+b_i}$$

In particular, on C

(25) $$\tilde{\Phi}_w(1,\mu) = -\sum_1^k a_i\mu^{2B_i},$$

(26) $$\tilde{\Psi}_w(\lambda,1) = -\sum_1^k b_i\lambda^{2A_i}.$$

Lemma 3.2. *Assume that $A \neq 0$ and the word map w is not surjective. Then*

$$\sum_1^k b_i\gamma^{2A_i} = 0$$

for every root γ of equation

$$q(z) := z^A + 1 = 0.$$

Assume that
If $B \neq 0$ and the word map w is not surjective, then

$$\sum_1^k a_i \delta^{2B_i} = 0$$

for every root δ of equation

$$p(z) := z^B + 1 = 0.$$

Proof. Indeed, in these cases, respectively, the pairs $(\gamma, 1)$ and $(1, \delta)$ belong to the curve C. We have to use only (27), (26), respectively . □

Corollary 3.3. *Let $2B_i = k_i B + T_i$, where k_i are integers and $0 \leq T_i < B \neq 0$. If w is not surjective, then for every $0 \leq T < B$*

$$\sum_{i:T_i=T} a_i (-1)^{k_i} = 0 \tag{27}$$

Proof. Indeed in this case

$$0 = \sum_1^k a_i \delta^{2B_i} = \sum_{T=0}^{B-1} \delta^T \Big(\sum_{i:T_i=T} a_i (-1)^{k_i} \Big)$$

for any root δ of equation

$$p(z) = z^B + 1 = 0.$$

Since $p(z)$ has no multiple roots, it implies that $p(z)$ divides the polynomial

$$p_1(z) := \sum_{T=0}^{B-1} x^T \Big(\sum_{i:T_i=T} a_i (-1)^{k_i} \Big) = 0.$$

But since degree of $p(z)$ is bigger than degree of $p_1(z)$ that can be only if $p_1(z) \equiv 0$. □

Corollary 3.4. *If all b_i are positive, then the word map w is either surjective or the square of another word $v \neq id$.*

Proof. In this case every $0 \leq 2B_i < 2B$. If w is not surjective, $p_1(z) \equiv 0$ by Corollary 3.3. Thus for every $0 \leq T < B$ there are at most two indexes i such that $2B_i = k_i B + T$, and the corresponding $k_i = 0$ or $k_i = 1$. Since $a_i \neq 0$, $p_1(z) \equiv 0$ implies that for every i there is j such that $a_i - a_j = 0$ and $T_i = T_j, 2B_i = 2B_j + B$. Since the sequence of B_i is increasing, it means that $k = 2l$,

$$0 = 2B_1, \ B = 2B_{l+1};$$

$$B + 2B_2 = 2B_{l+2};$$

$$\dots$$

$$B + 2B_l = 2B_{2l} = 2B - 2b_k.$$

Thus, $a_i = a_{i+l}$. On the other hand, $b_s = B_{s+1} - B_s = B_{s+l+1} - B_{s+l} = b_{s+l}$. Therefore the word is the square of $v = x^{a_1}y^{b_1}\dots x^{a_l}y^{b_l}$. □

Corollary 3.5. *If all b_i are negative, then the word map of the word w is either surjective or the square of another word $v \neq id$.*

Proof. We may change y to $z = y^{-1}$ and apply Corollary 3.5 to the word $w(x, z)$. □

Corollary 3.6. *If all a_i are positive, then the word map of the word w is either surjective or the square of another word $v \neq id$.*

Proof. Consider $v = x^{-1}, \ z = y^{-1}$, a word

$$w'(z, v) = w(x, y)^{-1} = y^{-b_k}x^{-a_k}\dots y^{-b_1}x^{-a_1} = z^{b_k}v^{a_k}\dots z^{b_1}v^{a_1},$$

and apply Corollary 3.5 to the word $w'(z, v)$. □

4. The word $v(x, y) = [[x, [x, y]], [y[x, y]]]$

In this section we provide an example of a word v that is surjective though it belongs to $F^{(2)}$. The interesting feature of this word is the following: if we consider it as a polynomial in Lie algebra $\mathfrak{sl}_2$, ($[x, y]$ being the Lie bracket) then it is not surjective ([BGKP], Example 4.9).

Theorem 4.1. *The word $v(x, y) = [[x, [x, y]], [y[x, y]]]$ is surjective on $SL(2, \mathbb{C})$ (and, consequently, on $PSL(2, \mathbb{C})$).*

Proof. As it was shown in Lemma 1.1, for every $z \in SL(2, \mathbb{C})$ with $tr(z) \neq \pm 2$ there are $x, y \in SL(2, \mathbb{C})^2$ such that $v(x, y) = z$.

Assume now that $a = \pm 2$. We have to show that there are matrices x, y in $SL(2, \mathbb{C})$, such that

$$v(x, y) := \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$$

has the following properties :

- $q_{12} + q_{22} = \pm 2$;
- $q_{12} \neq 0$.

We may look for these pairs among the matrices $x = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$ and $y = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

In the following MAGMA calculations $C = [x, y]$, $D = [[x, y], x]$, $B = [[x, y], y]$, $A = [D, B]$.

Ideal I in the polynomial ring $Q[b, c, d, t]$ is defined by conditions $det(x) = 1, tr(A) = 2$. Ideal J in the polynomial ring $Q[b, c, d, t]$ is defined by conditions $det(x) = 1, tr(A) = -2$. These are ideals of affine subsets $T_+ \subset SL(2)^2$ and $T_- \subset SL(2)^2$ respectively in affine variety $SL(2)^2$.

The computations show that q_{12} does not vanish identically on T_+ or T_-.

```
 >  Q:=Rationals();
> R<t,b,c,d>:=PolynomialRing(Q,4);
> X:=Matrix(R,2,2,[0,b,c,d]);
> Y:=Matrix(R,2,2,[ 1,t,0,1]);
>  X1:= Matrix(R,2,2,[d,-b,-c,0]);
> Y1:=Matrix(R,2,2,[1,-t,0,1]);
> C:=X*Y*X1*Y1;
>  p11:=C[1,1];
>  p12:=C[1,2];
>  p21:=C[2,1];
>  p22:=C[2,2];
> C1:=Matrix(R,2,2,[p22,-p12,-p21,p11]);
> D:=C*X*C1*X1;
>
>
> d11:=D[1,1];
> d12:=D[1,2];
> d21:=D[2,1];
> d22:=D[2,2];
> D1:=Matrix(R,2,2,[d22,-d12,-d21,d11]);
>
> B:=C*Y*C1*Y1;
>
>
> b11:=B[1,1];
> b12:=B[1,2];
> b21:=B[2,1];
> b22:=B[2,2];
> B1:=Matrix(R,2,2,[b22,-b12,-b21,b11]);
>
> A:=D*B*D1*B1;
>
> TA:=Trace(A);
>
> q12:=A[1,2];
> I:=ideal<R|b*c+1,TA-2>;
>
> IsInRadical(q12,I);
false
> J:=ideal<R|b*c+1,TA+2>;
>
> IsInRadical(q12,J);
```

```
false
>
```

It follows that the function q_{12} does not vanish identically on the sets T_+ and T_-, hence, there are pairs with $tr(v(x,y)) = 2, v(x,y) \neq id$, and $tr(v(x,y)) = -2, v(x,y) \neq -id$. □

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