Schwarz Reflection Principle and Boundary Uniqueness for J-Complex Curves

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Abstract

We establish the Schwarz Reflection Principle for J-complex discs attached to a real analytic J-totally real submanifold of an almost complex manifold with real analytic J. As a second result a boundary uniqueness theorem for J-complex discs with Lipschitz-continuous J is obtained. We also prove the precise regularity of J-complex discs attached to a J-totally real submanifold [1].

1. Introduction

1.1. Reflection Principle

Denote by Δ the unit disc in $\mathbb{C}$, by $\mathbf{S}$ - the unit circle. Let $\gamma \subset \mathbf{S}$ be a non-empty open subarc of $\mathbf{S}$.

Theorem 1 (Reflection Principle). *Let (X, J) be a real analytic almost complex manifold and W a real analytic J-totally real submanifold of X. Let $u : \Delta \to X$ be a J-holomorphic map continuous up to γ and such that $u(\gamma) \subset W$. Then u extends to a neighborhood of γ as a (real analytic) J-holomorphic map.*

The case of integrable J is due to H. A. Schwarz [Sw]. Indeed, one can find local holomorphic coordinates in a neighborhood of $u(p)$ for a taken $p \in \gamma$ such that $W = \mathbb{R}^n$ in these coordinates and now the Schwarz Reflection Principle applies. In our case there is no such reflection, since a general almost complex structure doesn't admits any local (anti)-holomorphic maps. But the extension result still holds.

One can put Theorem 1 into a more general form of Carathéodory, [Ca]. For this recall that the cluster set $\mathsf{cl}(u, \gamma)$ of u at γ consists of all limits $\lim_{k\to\infty} u(\zeta_k)$ for all sequences $\{\zeta_k\} \subset \Delta$ converging to γ. In [CGS] it was proved that if the cluster set $\mathsf{cl}(u, \gamma)$ of a J-holomorphic map $u : \Delta \to X$ is compactly contained in a totally real submanifold W then u smoothly extends to γ. Therefore we derive the following

Corollary 1. *In the conditions of the Theorem 1 the assumption of continuity of u up to γ and $u(\gamma) \subset W$ one can replace by the assumption that u is bounded and the cluster set $\mathsf{cl}(u, \gamma)$ is compactly contained in W.*

[1]Key-words: almost complex structure, totally real manifold, holomorphic disc, reflection principle.

1.2. Boundary Uniqueness

As in the case of holomorphic functions in $\mathbb{C}$ the Reflection Principle of Theorem 1 has as its immediate consequence the boundary uniqueness for J-holomorphic maps. Namely, if two such maps coincide on an non-empty arc on the boundary they coincide everywhere in the disc. Our second observation in this paper is that this property of J-holomorphic maps doesn't requires the real analyticity of J.

Theorem 2 (Boundary Uniqueness). *Let $u,v:\Delta\to X$ be J-holomorphic maps into an almost complex manifold (X,J). Suppose that J is Lipschitz-continuous and that u and v are of class $L^{1,p}$ for some $p>2$ up to an non-empty open arc $\gamma\subset\mathbf{S}$. If $u(z)=v(z)$ for $z\in\gamma$ then $u\equiv v$.*

The statement of Theorem 2 is optimal in the sense that for $J\in\mathcal{C}^{\alpha}$ one can construct two distinct J-holomorphic maps which coincide on a non-empty open subset of the disc. The boundary $L^{1,p}$-regularity assumption could be, eventually, weakened, see problem section §8.

1.3. Boundary Regularity

For the proof of our Reflection Principle we need to study not only real analytic boundary values but also the smooth ones (with finite smoothness). For our method to work we need an exact regularity and a certain kind of uniqueness of smooth J-complex discs attached to a J-totally real submanifold.

Theorem 3. *Let $u:(\Delta,\gamma)\to(X,W)$ be a J-holomorphic map of class $L^{1,2}\cap\mathcal{C}^0(\Delta\cup\gamma)$. Then:*

(i) for any integer $k\geqslant 0$ and real $0<\alpha<1$ if $J\in\mathcal{C}^{k,\alpha}$ and $W\in\mathcal{C}^{k+1,\alpha}$ then u is of class $\mathcal{C}^{k+1,\alpha}$ on $\Delta\cup\gamma$;

(ii) for $k\geqslant 1$ the condition $u\in L^{1,2}\cap\mathcal{C}^0(\Delta\cup\gamma)$ and $u(\gamma)\subset W$ can be replaced by the assumption that u is bounded and the cluster set $\mathsf{cl}(u,\gamma)$ is compactly contained in W.

Remark 1 If J is of class $\mathcal{C}^0$ and W of $\mathcal{C}^1$ then $u\in\mathcal{C}^{\alpha}$ up to γ for all $0<\alpha<1$. This was proved in [IS1], Lemma 3.1.

For integrable J the result of Theorem 3 is due to E. Chirka [Ch]. For non-integrable J weaker versions of this Theorem were obtained in [CGS, GS, MS]. Namely, the $\mathcal{C}^{k,\alpha}$-regularity of u up to γ was achieved there under the same assumptions.

1.4. Proofs

Though the interior analyticity of J-holomorphic discs in analytic almost complex manifolds follows from classical results on elliptic regularity in the real analytic category (see, for instance, [BJS]), the real analyticity up to the boundary does not follows directly from the known results since we do not deal with a boundary problem of the Dirichlet type. The direct application of the reflection principle (in the form of Vekua, for example) also leads to technical complications because of the non-linearity of the Cauchy-Riemann operator on an almost complex manifold. So our approach is different and is based on the reduction of the boundary regularity to a non-linear Riemann-Hilbert type problem.

This paper is organized in the following way.

1. First, in §3, using [IS1] we prove the case $k = 0$ of Theorem 3, *i.e.,* we prove the $\mathcal{C}^{1,\alpha}$-regulatity of u if $J \in \mathcal{C}^{\alpha}$ and $W \in \mathcal{C}^{1,\alpha}$. In §4 we use this techniques to prove the Theorem 2.
2. In §5 we get from [CGS] the Hölder regularity of J-holomorphic maps with cluster sets on J-totally real submanifolds provided $J \in \mathcal{C}^{1,\alpha}$ and $W \in \mathcal{C}^{2,\alpha}$.
3. In §6 we prove the solvability and uniqueness of our Riemann-Hilbert type problem in smooth, i.e., $\mathcal{C}^{k,\alpha}$ category thus obtaining Theorem 3 for the case $k \geqslant 1$.
4. In §7 we adapt our method to the real analytic case. Then the uniqueness, both in smooth and in real analytic categories gives the proof of the Reflection Principle of Theorem 1.
5. We end up with the formulation of open questions in §8.

We would like to express our gratitude to J.-F. Barraud who turned our attention to the question of extendability of J-holomorphic maps through totally real submanifolds in real analytic category.

2. Preliminaries

Denote by $J_{st} = i\mathsf{Id}$ the standard complex structure of $\mathbb{C}^n$ (as well as of $\mathbb{C}$ and of Δ). Let u be a $\mathcal{C}^1$-map (or $\mathcal{C}^0 \cap L^{1,1}_{\mathsf{loc}}$-map) from Δ into an almost complex manifold (X, J). Recall that u is called *J-holomorphic* if for every $\zeta \in \Delta$

$$du(\zeta) \circ J_{st} = J(u(\zeta)) \circ du(\zeta) \tag{2.1}$$

as mappings of tangent spaces $T_\zeta \Delta \to T_{u(\zeta)} X$ (in the $L^{1,1}$-case the condition (2.1) should be satisfied a.e.). The image $u(\Delta)$ is called a *J-complex* disc. Every almost complex manifold (X, J) of complex dimension n can be locally viewed as the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ equipped with an almost complex structure which is a small deformation of J_{st}. To see this fix a point $p \in X$, choose a coordinate system such that $p = 0$, make an $\mathbb{R}$-linear change of coordinates in order to have $J(0) = J_{st}$ and rescale, i. e., consider $J(tz)$ for $t > 0$ small enough. Then the equation (2.1) of J-holomorphicity of a map $u : \Delta \longrightarrow B$ can be written in local coordinates ζ on Δ and z on $\mathbb{C}^n$ as the following first order quasilinear system of partial differential equations

$$u_{\bar\zeta} - A_J(u)\overline{u_\zeta} = 0, \tag{2.2}$$

where $A_J(z)$ is the complex $n \times n$ matrix of the operator whose composite with complex conjugation is equal to the endomorphism $(J_{st} + J(z))^{-1}(J_{st} - J(z))$ (which is an anti-linear operator with respect to the standard structure J_{st}). Since $J(0) = J_{st}$, we have $A_J(0) = 0$. So in a sufficiently small neighborhood of the origin the norm $\| A_J \|_{L^\infty}$ is also small which implies the ellipticity of the system (2.2).

Let W be a real submanifold in an almost complex manifold (X,J). Similarly to the integrable case, W is called *J-totally real* if $T_pW \cap J(T_pW) = \{0\}$ at every point p of W. If n is the complex dimension of X, any totally real submanifold of X is locally contained in a totally real submanifold of real dimension n. So in what follows we assume that W is n-dimensional.

We recall some classical integral transformations. Let Ω be a bounded domain with $\mathcal{C}^\infty$ boundary in $\mathbb{C}$. Denote by T_Ω^{CG} the Cauchy-Green transform in Ω:

$$\left(T_\Omega^{CG}h\right)(\zeta) = \frac{1}{2\pi i}\int\int_\Omega \frac{h(\tau)d\tau \wedge d\overline{\tau}}{\tau - \zeta}. \tag{2.3}$$

Denote also by

$$K_\Omega h(\zeta) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{h(\tau)d\tau}{\tau - \zeta}. \tag{2.4}$$

the Cauchy integral.

Proposition 2.1 *For every integer $k \geq 0$ and real $0 < \alpha < 1$:*

(i) $T_\Omega^{CG} : \mathcal{C}^{k,\alpha}(\bar{\Omega}) \longrightarrow \mathcal{C}^{k+1,\alpha}(\bar{\Omega})$ is a bounded linear operator and $(T_\Omega^{CG}h)_{\overline{\zeta}} = h$ for any $h \in \mathcal{C}^{k,\alpha}(\Omega)$;

(ii) if h is a real analytic function on Ω, then $T_\Omega^{CG}h$ is real analytic on Ω. Furthermore, if we reperesent a real analytic function h in the form $h = h(\zeta,\overline{\zeta})$ viewing ζ and $\overline{\zeta}$ as independent variables, then for any $\zeta \in \Omega$ we have

$$T_\Omega^{CG}h(\zeta) = H(\zeta,\overline{\zeta}) - K_\Omega H(\zeta), \tag{2.5}$$

where H is a primitive of h with respect to $\overline{\zeta}$. If $\zeta \in \mathbb{C}\backslash\overline{\Omega}$, then

$$T_\Omega^{CG}h(\zeta) = -K_\Omega H(\zeta). \tag{2.6}$$

The proof of (i) is contained, for instance, in [Ve], Theorem 1.32; for the statement (ii) see [Ve], p.26, in fact this is nothing but the Cauchy-Green formula. (2.6) follows from Plemelj - Sokhotskiy formula. We shall also need the Schwarz integral transform on Δ:

$$\left(T^{SW}h\right)(\zeta) = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{\tau + \zeta}{\tau - \zeta}\cdot\frac{h(\tau)}{\tau}d\tau. \tag{2.7}$$

Given a smoothly bounded domain $\Omega \subset \mathbb{C}$ denote by $\mathcal{O}^{1,\alpha}(\Omega)$ the Banach space of holomorphic (with respect to the standard structure) maps $g : \Omega \longrightarrow \mathbb{C}^n$ of class $\mathcal{C}^{1,\alpha}(\bar{\Omega})$. This space is equipped with the norm $\parallel g \parallel_{\mathcal{C}^{1,\alpha}(\bar{\Omega})}$.

Proposition 2.2 *$T^{SW} : \mathcal{C}^{k,\alpha}(\mathbf{S}) \longrightarrow \mathcal{O}^{k,\alpha}(\Delta)$ and $K_\Omega : \mathcal{C}^{k,\alpha}(\partial\Omega) \longrightarrow \mathcal{O}^{k,\alpha}(\Omega)$ are bounded linear operators. For any real-valued function $\psi \in \mathcal{C}^{k,\alpha}(\mathbf{S})$ one has*

$$\Re(T^{SW}\psi)|_{\mathbf{S}} = \psi \tag{2.8}$$

and

$$\Im(T^{SW}\psi)(0) = 0.$$

For the proof see, for instance, [Ve], Theorem 1.10.

3. Reflection Principle-I: Boundary $\mathcal{C}^{1,\alpha}$-Regularity

In this subsection we shall use a version of a Reflection Principle proposed in [IS1] to prove the case $k=0$ of the Theorem 3. It will also serve us as a preparatory material for the proof of the boundary uniqueness statement of Theorem 2 in the next section. Therefore the structure J is supposed to be of class $\mathcal{C}^{\alpha}$ only. A J-totally real submanifold W of X will be supposed to have $\mathcal{C}^{1,\alpha}$-regularity.

First we make a suitable change of coordinates.

Lemma 3.1 *One can find coordinates in a neighborhood V of $p\in W$ such that in these coordinates $V=\mathbb{R}^{2n}$, $W=\mathbb{R}^n$, $J|_{\mathbb{R}^n}=J_{\mathsf{st}}$ and $J(x,y)-J_{\mathsf{st}}=O(||y||^{\alpha})$.*

Proof. After a change of coordinates of class $\mathcal{C}^{1,\alpha}$ we can suppose that in some neighborhood of $p=0$ our manifold W coincides with $\mathbb{R}^n$. Next we are looking for a $\mathcal{C}^{1,\alpha}$-diffeomorphism $\varphi=(\varphi_1,...\varphi_{2n})$ in a neighborhood of the origin such that

1) $\varphi_j(x,0)=x_j$ for $j=1,...,n$;
2) $\varphi_j(x,0)=0$ for $j=n+1,...,2n$;
3) $\frac{\partial\varphi}{\partial y_j}(x,0)=J(x,0)\left(\frac{\partial}{\partial x_j}\right)$ for $j=1,...,n$.

Such $\mathcal{C}^{1,\alpha}$-diffeomorphism exists due to the Trace theorem, see [Tr]. In new coordinates given by φ we shall clearly have $W=\mathbb{R}^n$, $J|_{\mathbb{R}^n}=J_{\mathsf{st}}$ and $J(x,y)-J_{\mathsf{st}}=O(||y||^{\alpha})$ due to $\mathcal{C}^{\alpha}$-regularity of J.

□

In what follows the disc Δ with an arc γ on its boundary $\mathbf{S}$ will be suitable for us to change by the upper half-disc $\Delta^+=\{\zeta:\Re\zeta>0\}$ and the segment $(-1,1)$. Let $u:(\Delta^+,\gamma)\to(X,W)$ be a J-holomorphic map of class $W^{1,p}$ up to $\gamma=(-1,1)$ for some $p>2$.

The following lemma will prove the case $k=0$ of the Theorem 3.

Lemma 3.2 *Let J be of class $\mathcal{C}^{\alpha}$ and W of class $\mathcal{C}^{1,\alpha}$. Let $u:(\Delta^+,\gamma)\to(X,W)$ be J-holomorphic of class $\mathcal{C}^0\cap L^{1,2}$ up to γ. Then u is of class $\mathcal{C}^{1,\alpha}$ up to γ.*

Proof. We can assume that $W=\mathbb{R}^n$ and $J(x,y)-J_{\mathsf{st}}=O(||y||^{\alpha})$. On the trivial bundle $\Delta^+\times\mathbb{R}^{2n}\to\Delta^+$ we consider the following linear complex structure: $J_u(z)[\xi]=J(u(z))[\xi]$ for $\xi\in\mathbb{R}^{2n}$ and $z\in\Delta^+$. At this point we stress that J_u is defined only on $\Delta^+\times\mathbb{R}^{2n}$. Denote by τ the standard conjugation in $\Delta\subset\mathbb{C}$ as well as the standard conjugation in $\mathbb{R}^{2n}=\mathbb{C}^n$. Now we extend J_u to $\Delta\times\mathbb{R}^{2n}$ by setting

$$\tilde{J}_u(z)[\xi]=-\tau J_{u(\tau z)}[\tau\xi] \text{ for} z\in\Delta \text{ and } \xi\in\mathbb{R}^{2n}. \tag{3.1}$$

We consider now u as a section (over Δ^+) of the trivial bundle $E=\mathbb{R}^{2n}\times\Delta\to\Delta$ and endow E with the complex structure $\tilde{J}_u$. Complex structure $\tilde{J}_u$ defines a $\bar\partial$-operator $\bar\partial_{\tilde{J}_u}w=\partial_xw+\tilde{J}_u\partial_yw$ on $L^{1,p}$-sections of E for all $1\leqslant p<\infty$ (for this only continuity of $\tilde{J}_u$ is needed). Remark that u is $\tilde{J}_u$-holomorphic on Δ^+. By F we denote the totally real subbundle $\Delta\times\mathbb{R}^n\to\Delta$ of E.

Definition 1 *Define the "extension by reflection" operator* $\mathsf{ext} : L^1(\Delta^+, E) \to L^1(\Delta, E)$ *by setting*

$$\mathsf{ext}(w)(z) = \tau w(\tau z) \tag{3.2}$$

for $z \in \Delta^-$ *and* $w \in L^1(\Delta^+, E)$. *We shall also write* $\tilde{w}$ *for* $\mathsf{ext}(w)$.

Note that if w is continuous up to γ and takes on γ values in the subbundle F then $\mathsf{ext}(w)$ stays continuous. By the reflection principle of Theorem 1.1 from [IS1] we know that $\mathsf{ext} : L^{1,p}(\Delta^+, E, F) \to L^{1,p}(\Delta, E)$ is a continuous operator for all $1 \leqslant p < \infty$ and that $\bar{\partial}_{\tilde{J}_u}\tilde{w} = 0$ if $\bar{\partial}_{J_u} w = 0$. Let $\tilde{u}$ be the extension of u, in particular, $\tilde{u}$ is $\tilde{J}_u$-holomorphic on Δ of class $L^{1,2}(\Delta)$. Lemma 3.1 insures that $\tilde{u} \in L^{1,p}_{loc}$ for all $p < \infty$. In tarticular $\tilde{u} \in \mathcal{C}^\beta$ for every $\beta < 1$. Therefore $\tilde{J}_u$ is of class $\mathcal{C}^\delta$, where $\delta = \alpha\beta$.

Set $v = \rho\tilde{u}$, where ρ is a cut-off function equal to 1 in $\Delta_{\frac{1}{2}}$. Then from (2.2) we get

$$v_{\bar{\zeta}} - A_J(\tilde{u})\overline{v_{\zeta}} = g, \tag{3.3}$$

where the function $g = \left[\rho_{\bar{\zeta}} - \rho_{\zeta} A_J(\tilde{u})\right]\tilde{u}$ is of class $\mathcal{C}^\delta(\Delta)$. Observe that $v_{\bar{\zeta}} - A_J(\tilde{u})\overline{v_{\zeta}} = \left(v - T^{CG}_{\Delta} A_J(\tilde{u})\overline{v_{\zeta}}\right)_{\bar{\zeta}}$. Here T^{CG}_{Δ} denotes the Cauchy-Green integral on Δ. Elliptic regularity implies that $v - T^{CG}_{\Delta} A_J(\tilde{u})\overline{v_{\zeta}}$ is of class $\mathcal{C}^{1,\delta}$ and invertibility of the operator $\mathsf{Id} - T^{CG}_{\Delta} A_J(\tilde{u})\overline{\frac{\partial}{\partial z}}$ in $\mathcal{C}^{1,\delta}$ gives that v is in $\mathcal{C}^{1,\delta}$.

One can repeat this step once more to get $\mathcal{C}^{1,\alpha}$-regularity of v on Δ and therefore of u up to γ. □

4. Boundary Uniqueness

We interrupt the proof of Theorems 1 and 3 in order to show how the method of Reflection Principle-I can be applied for proving the boundary uniqueness of Theorem 2.

Let's start with some reductions. The problem is local, therefore we shall suppose that u and v are both defined and of class $L^{1,p}$ in the upper half-disc $\Delta^+ = \{z \in D : \Im z > 0\}$ up to $\gamma = (-1,1)$ for some $p > 2$. Following §3 on the trivial bundle $\Delta^+ \times \mathbb{R}^{2n} \to \Delta^+$ we consider two complex structures $J_u(z)[\xi] = J(u(z))[\xi]$ for $\xi \in \mathbb{R}^{2n}, z \in \Delta^+$ and $J_v(z)[\xi] = J(v(z))[\xi]$. Again we stress that J_u (and J_v) are defined only on $\Delta^+ \times \mathbb{R}^{2n}$. We extend J_u and J_v to $(\Delta \setminus (-1,1)) \times \mathbb{R}^{2n}$ and get $\tilde{J}_u$ and $\tilde{J}_v$. Using the extension operator (3.2) we get $\tilde{u}$ and $\tilde{v}$ - extensions onto $\Delta \setminus (-1.1)$ of u and v respectively. In particular $\tilde{u}$ is $\tilde{J}_u$-holomorphic and $\tilde{v}$ is $\tilde{J}_v$-holomorphic.

Remark 2 Let's stress at this point the difference between our setting and that of §3. Since we have no totally real boundary conditions here our objects extend only to $\Delta \setminus (-1,1)$ and not to the whole of Δ.

Remark also that we can suppose that $u(0) = v(0) = 0$, $J(0) = J_{\mathsf{st}}$ and, rescaling as in §2, we can suppose that $\|J - J_{\mathsf{st}}\|_{L^\infty}$ is small. Therefore $\left\|\tilde{J}_u - J_{\mathsf{st}}\right\|_{L^\infty}$ and $\left\|\tilde{J}_v - J_{\mathsf{st}}\right\|_{L^\infty}$ are as small as we wish.

4.1. $\overline{\partial}$-Lemmas

Lemma 4.1 *For $\tilde{J}_u, \tilde{v}, \tilde{u}$ defined above there exists an $h \in L^p_{loc}(\Delta)$ such that for almost all $z \in \Delta \setminus (-1,1)$ one has*

$$|\overline{\partial}_{\tilde{J}_{u(z)}}(\tilde{v}-\tilde{u})(z)| \leqslant h(z)|(\tilde{v}-\tilde{u})(z)|. \tag{4.1}$$

In particular, the weak derivative $\overline{\partial}_{\tilde{J}_u}(\tilde{v}-\tilde{u})$, taken on $\Delta \setminus (-1,1)$, belongs to $L^p_{loc}(\Delta)$.

Proof. First we check this inequality for $\Im z > 0$. In that case we have

$$\begin{aligned}\overline{\partial}_{J_u}(v-u) = \overline{\partial}_{J_u} v = \partial_x v + J(u(z))\partial_y v = \partial_x v + [J(u(z)) - J(v(z))]\partial_y v + \\ +J(v(z))\partial_y v = \overline{\partial}_{J_v} v + [J(u(z)) - J(v(z))]\partial_y v = [J(u(z)) - J(v(z))]\partial_y v.\end{aligned} \tag{4.2}$$

Since J is Lipschitz and v is of class $L^{1,p}$, we get from (4.2) for $\Im z > 0$

$$|\overline{\partial}_{J_u}(v-u)(z)| \leqslant h(z)|(v-u)(z)| \text{ a.e.}$$

with $h(z) = ||J||_{Lip} \cdot |\nabla v(z)| \in L^p$. Let now $\Im z < 0$. Then

$$\begin{aligned}\overline{\partial}_{\tilde{J}_{u(z)}}(\tilde{v}-\tilde{u})(z) = \overline{\partial}_{\tilde{J}_{u(z)}}\tilde{v} = \partial_x \tilde{v} + \tilde{J}_u(z)\partial_y \tilde{v} = \left[\tilde{J}(u(z)) - \tilde{J}(v(z))\right]\partial_y \tilde{v} = \\ = -\tau\left[J(u(\bar{z})) - J(v(\bar{z}))\right]\overline{\partial_y \tilde{v}}.\end{aligned}$$

Therefore

$$|\overline{\partial}_{\tilde{J}_{u(z)}}(\tilde{v}-\tilde{u})(z)| \leqslant h(\bar{z})|(v-u)(\bar{z})| = h(z)|(\tilde{v}-\tilde{u})(z)|,$$

where we put $h(z) = h(\bar{z})$ for $z \in \Delta^-$. The right nahd side of (4.1) is a function from $L^p_{loc}(\Delta)$ and we can conclude that $\overline{\partial}_{\tilde{J}_u}(\tilde{v}-\tilde{u}) \in L^p_{loc}(\Delta)$.

□

Set $J = \tilde{J}_u$. It is a bounded complex linear structure on our trivial bundle $E = \Delta \times \mathbb{R}^{2n}$. That means that $J \in L^\infty(\Delta, \mathsf{End}(\mathbb{R}^{2n}))$ and, in particular, it is defined almost everywhere (for example, it is not defined on $(-1,1)$!). Structure J defines on E the following "$\overline{\partial}$-type" operator $\overline{\partial}_J : L^{1,p}(E) \to L^p(E)$:

$$\overline{\partial}_J w = \frac{1}{2}\left(\frac{\partial w}{\partial x} + J(z)\frac{\partial w}{\partial y}\right). \tag{4.3}$$

This operator is well defined, continuos and satisfies the identity

$$\overline{\partial}_J(fw) = \overline{\partial} f \cdot w + f\overline{\partial}_J w \tag{4.4}$$

for a function f and a section w. Here and later on $\overline{\partial} = \overline{\partial}_{J_{\mathsf{st}}}$ stands for the standard Cauchy-Riemann operator in the space of functions (sections) which are clear from the context.Note that multiplication by "i" of sections of E should be understood as $iw := Jw$ in order for (4.4) to be true.

Lemma 4.2 *Let $w : \Delta \to \mathbb{C}^n$ be a continuous map such that $w|_{(-1.1)} \equiv 0$. Assume that for some $g \in L^1_{\mathsf{loc}}(\Delta, \mathbb{C}^n)$ the equation $\overline{\partial}_J w = g$ holds (in the weak sense) in $\Delta \setminus (-1,1)$. Then $\overline{\partial}_J w = g$ holds (in the weak sense) in the whole disc Δ.*

Proof. Recall that the equality $\overline{\partial} w = g$ in $\Delta \setminus (-1,1)$ in the weak sense means that for every smooth test (vector) function $\psi(z)$ with compact support in $\Delta \setminus (-1,1)$ one has

$$\int_\Delta (w \overline{\partial} \psi + g\psi) dx \wedge dy = 0,$$

and similarly for the weak equality in Δ. Multiplying w by a cut-off function we can suppose that the equation $\overline{\partial}_J w = g$ holds in the weak sense in $\Delta \setminus [-\frac{1}{2}, \frac{1}{2}]$ (with some other $g \in L^1_{loc}(\Delta, \mathbb{C}^n)$).

Now let ψ has compact support in the whole disc Δ. Fix a sequence of smooth functions ϕ_n in Δ with the following properties:

- $0 \le \phi_n \le 1$, $\|d\phi_n\|_{L^\infty(\Delta)} \le 2n$;
- $\phi_n(z) \equiv 0$ for $|y| \le \frac{1}{n}, |x| \le \frac{5}{8}$, $\phi_n(z) \equiv 1$ for $|y| \ge \frac{2}{n}$ or $|x| \ge \frac{7}{8}$.

Then

$$\int_\Delta (w \overline{\partial}_J (\varphi_n \psi) + g\varphi_n \psi) dx \wedge dy = 0$$

and $\lim_{n\to\infty} \varphi_n \psi = \psi$ in the weak L^∞-sense. This means, in particular, that one has

$$\lim_{n\to\infty} \int_\Delta g\varphi_n \psi dx \wedge dy = \int_\Delta g\psi dx \wedge dy$$

for any $g \in L^1_{loc}(\Delta, \mathbb{C}^n)$. Further,

$$\int_\Delta w \overline{\partial} (\varphi_n \psi) dx \wedge dy = \int_\Delta (\psi w \overline{\partial} \varphi_n + w\phi_n \overline{\partial}_J \psi) dx \wedge dy$$

and

$$\lim_{n\to\infty} \int_\Delta w\phi_n \overline{\partial}_J \psi dx \wedge dy = \int_\Delta w \overline{\partial}_J \psi dx \wedge dy.$$

The crucial point of the proof is the estimation of

$$\lim_{n\to\infty} \int_\Delta \psi w \overline{\partial} \varphi_n dx \wedge dy.$$

Here we obtain

$$\int_\Delta |\psi w \overline{\partial} \varphi_n| dx \wedge dy \le \|\psi\|_{L^\infty(\Delta)} \cdot \|w\|_{L^2(A_n)} \cdot \left\|\overline{\partial} \varphi_n\right\|_{L^2(A_n)}, \tag{4.5}$$

where $A_n = \{|x| \leqslant \frac{7}{8}, |y| \leqslant \frac{2}{n}\}$. Now observe that $\left\|\overline{\partial}\varphi_n\right\|_{L^2(A_n)}$ is bounded by $C\sqrt{n}$ and that $\|w\|_{L^2(A_n)} = o(\frac{1}{\sqrt{n}})$. Therefore the right hand side of (4.1) tends to zero as $n \to \infty$.

We obtain the equality $\int_\Delta (w\overline{\partial}_J \psi + g\psi) dx \wedge dy = 0$ for the given ψ. Since ψ was arbitrary, we conclude the assertion of the Lemma.

□

Remark 3 From Lemmas 4.1 and 4.2 we conclude that $w := \tilde{u} - \tilde{v}$ possesses the following properties:

- $w \in L^p_{loc}(\Delta)$ (in fact $w \in \mathcal{C}^\alpha(\Delta)$ with $\alpha = 1 - \frac{2}{p}$);
- $\overline{\partial}_J w \in L^1_{loc}(\Delta)$;
- Since $(-1,1)$ is of measure zero we see that (4.1) writes as

$$|\overline{\partial}_J w| \leqslant h|w| \text{ a.e. in } \Delta \text{ with some } h \in L^p_{loc}; \tag{4.6}$$

  and, since w is bounded, this implies that
- $\overline{\partial}_J w \in L^p_{loc}(\Delta)$.

4.2. Generalized Giraud-Calderon-Zygmund Inequality

We would like to conclude from these items that $w = \tilde{v} - \tilde{u} \in L^{1,p}_{loc}(\Delta)$. Would $J = \tilde{J}_{\tilde{u}}$ be the standard complex structure this statement is called the Giraud-Calderon-Zygmund Inequality (or estimate): *for all $1 < p < \infty$ there is a constant G_p such that for all $w \in L^p(\Delta, \mathbb{C}^n)$*

$$\left\|(\partial \circ T^{CG}_{\mathbb{C}})(w)\right\|_{L^p(\Delta)} \leq G_p \cdot \|w\|_{L^p(\Delta)}. \tag{4.7}$$

Therefore our next step will be to prove the generalization of (4.7) to $\overline{\partial}$-type operators with bounded coefficients.

Extend our complex linear structure J from $\Delta \times \mathbb{R}^{2n}$ to linear complex structure on $\mathbb{C} \times \mathbb{R}^{2n}$ by setting $J|_{\mathbb{C}\setminus\Delta} = J_{\mathsf{st}}$. Remember that by Remark 3 $\|J - J_{\mathsf{st}}\|_{L^\infty}$ is as small as we wish. Then for $w \in L^p(\mathbb{C}, \mathbb{R}^{2n})$ it holds that

$$\begin{aligned}\left\|(\overline{\partial}_J \circ T^{CG}_{\mathbb{C}} - \overline{\partial}_{J_{\mathsf{st}}} \circ T^{CG}_{\mathbb{C}})w\right\|_{L^p(\Delta)} &\leq \|J - J_{\mathsf{st}}\|_{L^\infty(\Delta)} \cdot \left\|d(T^{CG}_{\mathbb{C}} w)\right\|_{L^p(\Delta)} \leq \\ &\leq \|J - J_{\mathsf{st}}\|_{L^\infty(\Delta)} (1 + G_p) \|w\|_{L^p(\Delta)},\end{aligned} \tag{4.8}$$

where G_p is the constant from (4.7). For the standard structure in $\mathbb{C}^n$ the operator $\overline{\partial}_{J_{\mathsf{st}}} \circ T^{CG}_{\mathbb{C}} : L^p(\mathbb{C}, \mathbb{C}^n) \to L^p(\mathbb{C}, \mathbb{C}^n)$ is identity. So from (4.8) we see that there exists $\varepsilon_p = \frac{1}{1+G_p}$ such that if $\|J - J_{\mathsf{st}}\|_{L^\infty(\Delta)} < \varepsilon_p$, then $\overline{\partial}_J \circ T^{CG}_{\mathbb{C}} : L^p(\mathbb{C}, \mathbb{C}^n) \to L^p(\mathbb{C}, \mathbb{C}^n)$ is an isomorphism. Moreover, since $\overline{\partial}_J \circ T^{CG}_{\mathbb{C}} = \overline{\partial}_{J_{\mathsf{st}}} \circ T^{CG}_{\mathbb{C}} + (\overline{\partial}_J - \overline{\partial}_{J_{\mathsf{st}}}) \circ T^{CG}_{\mathbb{C}}$, we have

$$(\overline{\partial}_J \circ T^{CG}_{\mathbb{C}})^{-1} = \left[\mathsf{Id} + (\overline{\partial}_J - \overline{\partial}_{J_{\mathsf{st}}}) \circ T^{CG}_{\mathbb{C}}\right]^{-1} = \sum_{n=0}^{\infty} (-1)^n [(\overline{\partial}_J - \overline{\partial}_{J_{\mathsf{st}}}) \circ T^{CG}_{\mathbb{C}}]^n. \tag{4.9}$$

This shows, in particular, that $(\overline{\partial}_J \circ T^{CG}_{\mathbb{C}})^{-1}$ does not depend on $p > 1$ provided that $\|J - J_{\mathsf{st}}\|_{L^\infty(\Delta)} < \varepsilon_p$. Now we shall prove the following statement, which can be viewed as a generalization of the Giraud-Calderon-Zygmund estimate.

Lemma 4.3 *For any $p > 2$, any $w \in L^p(\Delta, \mathbb{R}^{2n})$ with compact support in Δ, any bounded J with $\|J - J_{\mathsf{st}}\|_{L^\infty(\Delta)} < \varepsilon_p$ conditions*

i) $\overline{\partial}_J w \in L^p(\Delta, \mathbb{R}^{2n})$;

ii) $|\overline{\partial}_J w| \leqslant h|w|$ a.e. in Δ for some $h \in L^p_{loc}$; imply $dw \in L^p(\Delta, \mathbb{R}^{2n})$ and therefore $w \in L^{1,p}(\Delta, \mathbb{R}^{2n})$.

Proof. First take smoothing w_ε of u by convolution. Put $h_\varepsilon = w_\varepsilon - T_{CG}\circ\bar\partial_{J_{\mathsf{st}}} w_\varepsilon$. Then $\bar\partial_{J_{\mathsf{st}}} h_\varepsilon = 0$. So h_ε is holomorphic and descends at infinity. Thus $h_\varepsilon \equiv 0$, which implies $w_\varepsilon = (T_{CG}\circ\bar\partial_{J_{\mathsf{st}}})w_\varepsilon$. Write

$$\|dw_\varepsilon\|_{L^p(\Delta)} \leqslant C\left\|\bar\partial_{J_{\mathsf{st}}} w_\varepsilon\right\|_{L^p(\Delta)} = \left\|(\bar\partial_J\circ T^{CG}_{\mathbb{C}})^{-1}(\bar\partial_J\circ T^{CG}_{\mathbb{C}})\bar\partial_{J_{\mathsf{st}}} w_\varepsilon\right\|_{L^p(\Delta)} =$$

$$= \left\|(\bar\partial_J\circ T^{CG}_{\mathbb{C}})^{-1}(\bar\partial_J w_\varepsilon\right\|_{L^p(\Delta))} \leq C\sum_{n=0}^{\infty}\left\|(\bar\partial_J - \bar\partial_{J_{\mathsf{st}}})\circ T^{CG}_{\mathbb{C}}\right\|_p^n\cdot\left\|\bar\partial_J w_\varepsilon\right\|_{L^p(\Delta)} \leq$$

$$\leq C\cdot\left\|\bar\partial_J w_\varepsilon\right\|_{L^p(\Delta)}, \tag{4.10}$$

provided that $\|J - J_{\mathsf{st}}\|_{L^\infty} < \varepsilon_p$. But (4.6) implies $|\bar\partial_J w_\varepsilon| \leqslant |(hw)_\varepsilon|$ pointwise and therefore the right hand side of (4.10) is bounded in $L^p(\Delta)$.

Now w_ε converges in L^p to w, therefore dw_ε converges to dw in the sense of distributions. Take a subsequence of dw_ε weakly converging in L^p to conclude that $dw \in L^p$ and therefore $w \in L^{1,p}$.

□

4.3. Uniqueness

We see that our w is a $L^{1,p}$-section of a $L^{1,p}$-bundle E which is equipped with a $\bar\partial$-type operator $\bar\partial_J$. By Lemma 1.2.3 from [IS2] there is an $L^{1,p}$-frame of E making from E a holomorphic bundle and from $\bar\partial_J$ the usual $\bar\partial$-operator, *i.e.*, there exists a local $L^{1,p}$-frame $e_1,...,e_n$ which is holomorphic with respect to $\bar\partial_J$: $\bar\partial_J e_k = 0$.

But in this new frame (4.6) again writes as

$$|\bar\partial w(z)| \leqslant h(z)|w(z)| \text{ a.e. }, \tag{4.11}$$

but now with the standard $\bar\partial$. We are precisely in the assumptions of Lemma 1.4.1 from [IS2] which says that nonzero solutions of differential inequalities of the type (4.11) have only isolated zeroes. In our case $w = 0$ on the real axis. Therefore $w = \tilde v - \tilde u \equiv 0$ and Theorem 2 is proved.

□

5. Cluster Sets on Totally Real Submanifolds

The case $k = 0$ of the Theorem 3 is proved in §3 and we restrict ourselves in the future with $k \geqslant 1$. Fix an almost complex manifold (X,J) with J of class $\mathcal{C}^{1,\alpha}$ and a J-totally real submanifold W of class $\mathcal{C}^{2,\alpha}$. Let $u:\Delta\to X$ be a bounded J-holomorphic map of the unit disc into X. Suppose that $\mathsf{cl}(u,\gamma) \Subset W$, where γ is some non-empty open subarc of the boundary.

We use the Proposition 4.1 from [CGS] and observe that u is in Sobolev class $L^{1,p}$ up to γ for all $p < 4$. In particular u is $\mathcal{C}^\beta$-regular up to γ with $\beta = 1 - \frac{2}{p}$ (this means for all $\beta < \frac{1}{2}$). Lemma 3.2 implies now the following

Corollary 5.1 *Let $J \in \mathcal{C}^{1,\alpha}$ and $W \in \mathcal{C}^{2,\alpha}$. If $u:(\Delta^+,\gamma)\to(X,W)$ is a bounded J-holomorphic map with $\mathsf{cl}(u,\gamma)\Subset W$ then $u\in\mathcal{C}^{1,\alpha}(\Delta^+\cup\gamma)$.*

Let's stress here that $\mathcal{C}^{1,\alpha}$ is not the optimal regularity of u, it should be $\mathcal{C}^{2,\alpha}$. This will be achieved in the next section §6.

6. Riemann-Hilbert Problem

As it was already told we need to prove the regularity (and uniqueness) of smooth discs first. This will also imply the exact, *i.e.*, $\mathcal{C}^{k+1,\alpha}$-regularity up to the boundary of J-complex discs (if $J \in \mathcal{C}^{k,\alpha}$ and $W \in \mathcal{C}^{k+1,\alpha}$). Therefore we proceed in this section with the proof of Theorem 3 from the Introduction for the case $k \geqslant 1$. The proof will be given in two steps. Denote by $\mathbf{S}^+ = \{e^{i\theta} : \theta \in]0,\pi[\}$ the upper semi-circle, which will serve us for time being as γ.

6.1. Small deformations of the standard structure

First we consider the following special case. Let $W = i\mathbb{R}^n = \{z = x + iy : x = 0\}$ and let $J \in \mathcal{C}^{k,\alpha}$, $k \geqslant 1$, be a small deformation of J_{st}. Fix also a J_{st}-holomorphic map $u^0 : \Delta \longrightarrow \mathbb{C}^n$ of class $\mathcal{C}^{k+1,\alpha}(\overline{\Delta})$ such that $u^0(\mathbf{S}^+) \subset i\mathbb{R}^n$ (so that u^0 extends holomorphically to a neighborhood of $\mathbf{S}^+$ by the classical Schwarz Reflection Principle). For J close enough to J_{st} we will establish the boundary $\mathcal{C}^{k+1,\alpha}$-regularity of a J-holomorphic disc u close enough to u^0 satisfying the boundary condition $u(\mathbf{S}^+) \subset i\mathbb{R}^n$.

Therefore for J close enough to J_{st} we study the solutions of (2.2) satisfying the boundary condition

$$\Re u|_{\mathbf{S}^+} = 0. \tag{6.1}$$

For every positive integer k denote by $\mathcal{C}^{k,\alpha}_0(\mathbf{S})$ the Banach space of ($\mathbb{R}^n$ -valued) functions $\varphi \in \mathcal{C}^{k,\alpha}(\mathbf{S})$ vanishing on $\mathbf{S}^+$. This space is equipped with the standard norm $\| \varphi \|_{\mathcal{C}^{k,\alpha}(\mathbf{S})}$. Set now $\varphi^0 := \Re u^0|\mathbf{S}$. $\varphi^0 \in \mathcal{C}^{k+1,\alpha}_0(\mathbf{S})$ because $\Re u^0|_{\mathbf{S}^+} = 0$. We replace the condition (6.1) for the solutions of the partial differential equation (2.2) by the condition

$$\Re u|_{\mathbf{S}} = \varphi, \tag{6.2}$$

where $\varphi \in \mathcal{C}^{k+1,\alpha}_0(\mathbf{S})$. Therefore we consider the following boundary value problem

$$\begin{cases} u_{\overline{\zeta}} - A_J(u)\overline{u}_{\overline{\zeta}} = 0, \\ \Re u|_{\mathbf{S}} = \varphi, \\ \Im u(0) = a, \end{cases} \tag{6.3}$$

for the given initial data $\varphi \in \mathcal{C}^{k+1,\alpha}_0(\mathbf{S})$, $a \in \mathbb{R}^n$.

Lemma 6.1 *Suppose $k \geqslant 1$. If J is close enough to J_{st} in $\mathcal{C}^{k,\alpha}$-norm then for every $1 \leqslant l \leqslant k$:*

(i) there exists a neighborhood U of φ^0 in $\mathcal{C}^{l+1,\alpha}_0(\mathbf{S})$, a neighborhood U' of $a^0 := \Im\varphi^0(0)$ in $\mathbb{R}^n$ and a neighborhood V of u^0 in $\mathcal{C}^{l+1,\alpha}(\bar{\Delta})$ such that for each $\varphi \in U$ and $a \in U'$ the boundary problem (6.3) admits a unique solution $u \in V$;

(ii) *the unit disc Δ can be replaced in part (i) of the present Lemma by any bounded simply connected domain Ω with $\mathcal{C}^\infty$ boundary and $\mathbf{S}^+$ can be replaced by any open arc.*

Proof. The part (ii) follows from (i) by the Riemann mapping theorem and the classical theorems on the boundary regularity of conformal maps. So it suffices just to prove the part (i). Consider the operator

$$L_J : \mathcal{C}^{l+1,\alpha}(\bar\Delta) \longrightarrow \mathcal{C}^{l,\alpha}(\bar\Delta) \times \mathcal{C}^{l+1,\alpha}(\mathbf{S}) \times \mathbb{R}^n$$

defined by

$$L_J : u \mapsto \begin{pmatrix} u_{\bar\zeta} - A_J(u)\overline{u}_{\bar\zeta} \\ \Re u|_{\mathbf{S}} \\ \Im u(0) \end{pmatrix}.$$

L_J smoothly depends on the parameter J. Denote by $\dot L_J(u)$ the Fréchet derivative of L_J at u. $\dot L_J$ is continuous on the couple (J,u) and at J_{st}-holomorphic u^0 the derivative $\dot L_{J_{\mathsf{st}}}(u^0)$ is particulary simple:

$$\dot L_{J_{\mathsf{st}}}(u^0) : \mathcal{C}^{l+1,\alpha}(\overline{\Delta}) \longrightarrow \mathcal{C}^{l,\alpha}(\overline{\Delta}) \times \mathcal{C}^{l+1,\alpha}(\mathbf{S}) \times \mathbb{R}^n$$
$$\dot L_{J_{\mathsf{st}}}(u^0) : \dot u \mapsto \begin{pmatrix} \dot u_{\bar\zeta} \\ \Re \dot u|_{\mathbf{S}} \\ \Im \dot u(0) \end{pmatrix}.$$

Let's see that $\dot L_{J_{\mathsf{st}}}(u^0)$ is an isomorphism. Indeed, given $h \in \mathcal{C}^{l,\alpha}(\overline{\Delta})$, $\psi \in \mathcal{C}^{l+1,\alpha}(\mathbf{S})$ and $a \in \mathbb{R}^n$ then the function

$$\dot u = T^{CG}_\Delta h - i\Im\left(T^{CG}_\Delta h(0)\right) + ia + T^{SW}(\psi - \Re(T^{CG}_\Delta h)|\mathbf{S})$$

is of class $\mathcal{C}^{l+1,\alpha}(\overline{\Delta})$ and satisfies the equation

$$\dot L_{J_{\mathsf{st}}}(u^0)(\dot u) = \begin{pmatrix} h \\ \psi \\ a \end{pmatrix}.$$

Uniqueness of $\dot u$ is obvious. Therefore by the Implicit Function Theorem every L_J is a $\mathcal{C}^1$-diffeomorphism of neighborhoods of u^0 in $\mathcal{C}^{l+1,\alpha}(\bar\Delta)$ and of $(0,\varphi_0,a_0)$ in $\mathcal{C}^{l,\alpha}(\overline{\Delta}) \times \mathcal{C}^{l+1,\alpha}(\mathbf{S}) \times \mathbb{R}^n$. Since $\mathcal{C}^{l+1,\alpha}_0(\mathbf{S})$ is a closed subspace of $\mathcal{C}^{l+1,\alpha}(\mathbf{S})$ the Lemma 5.1. follows. □

6.2. General case

We consider a J-holomorphic map $u : \Delta \to X$ such that $u(\mathbf{S}^+) \subset W$, where W is of class $\mathcal{C}^{k+1,\alpha}$ and $J \in \mathcal{C}^{k,\alpha}$, $k \geqslant 1$. First of all we point out that in view of the classical results on the interior regularity of J-holomorphic maps we can assume that the map u is of class

$\mathcal{C}^{k+1,\alpha}$ in Δ, see [BJS, MS, Sk]. By Corollary 5.1 (or Lemma 3.2) we can suppose that u is $\mathcal{C}^{1,\alpha}$ also on $\mathbf{S}^+$.

The statement of Lemma is local, so fixing $e^{i\theta_0} \in \mathbf{S}^+$ and shrinking a neighborhood of the point $p = u(e^{i\theta_0})$, we reduce the general case to the case of a small deformation of J_{st}. More precisely, we proceed as follows.

A neighborhood of $e^{i\theta_0}$ in $\bar{\Delta}$ we see now as semi-disc $\Delta^+ = \{\zeta \in \Delta : \Im\zeta > 0\} \cup \gamma$, where $\gamma = (-1,1)$. Shrinking a bit we build a domain Ω in the upper half-plane with $\mathcal{C}^\infty$ boundary $\partial\Omega$ such that (after delating Ω) the real interval $(-1,1)$ is contained in $\partial\Omega$. Now we consider the real interval $\gamma = (-1,1)$ instead of the upper semi-circle ($e^{i\theta_0}$ becomes zero in these new coordinates). Furthermore we assume that X is the unit ball of $\mathbb{C}^n$ equipped with an almost complex structure $J \in \mathcal{C}^{k,\alpha}$ with $J(0) = J_{st}$ and $T_0(W) = i\mathbb{R}^n$ and $p = u(0) = 0$. Since W is locally $\mathcal{C}^{k+1,\alpha}$ diffeomorphic to its tangent space at the origin, pushing J forward by this diffeomorphism we can suppose that $W = i\mathbb{R}^n$ and we preserve the previous assumptions. In particular, J remains in class $\in \mathcal{C}^{k,\alpha}$. We stress that $\Re u$ is of class $\mathcal{C}^{k+1,\alpha}$ on $\partial\Omega$ because u is $\mathcal{C}^{k+1,\alpha}$ on $\partial\Omega \setminus \gamma$ and $\Re u|_\gamma \equiv 0$.

The map u being of class $\mathcal{C}^{1,\alpha}(\Delta^+ \cup \gamma)$ admits the expansion $u(\zeta) = b\zeta + o(|\zeta|)$ near the origin. Note that the linear term of the expansion is $\mathbb{C}$-linear because $J(0) = J_{\mathsf{st}}$. For $t > 0$ consider the structures $J_t(z) = J(tz)$. They tend to J_{st} in the $\mathcal{C}^{k,\alpha}$ norm on any compact subset of $\mathbb{C}^n$ as t tends to 0. The maps $u^t(\zeta) = (t^{-1} \circ u)(t\zeta)$ are J_t-holomorphic and tend to the map $u^0 : \zeta \longrightarrow b\zeta$ on any compact subset of the closed upper semi-plane $\{\Im\zeta \geq 0\}$ as $t \longrightarrow 0$. We view the map u^0 as a J_{st}-holomorphic map.

Since all maps $\varphi^t := \Re u^t|\partial\Omega$ vanish on $\gamma = (-1,1)$ they are of class $\mathcal{C}_0^{k+1,\alpha}(\partial\Omega) := \{\varphi \in \mathcal{C}^{k+1,\alpha}(\partial\Omega, \mathbb{R}^n) : \varphi|_\gamma \equiv 0\}$. Applying Lemma 6.1 to u^0 and φ^t for t small enough, we obtain by the uniqueness statement of this Lemma that maps u^t are of class $\mathcal{C}^{k+1,\alpha}(\overline{\Omega})$ for t small enough. This proves the case $k \geqslant 1$ of Theorem 3.

□

Remark 4 Note that we used the uniqueness statement of Lemma 5.1 both for $l = 1$ and $l = k$.

7. Reflection Principle-II: Real Analytic Case

We turn now to the proof of the Reflection Principle of Theorem 1. As in the smooth category we proceed in two steps.

7.1. Small deformations of the standard structure

Here we consider the case when J is a small real analytic deformation of J_{st} and $W = i\mathbb{R}^n$. First we introduce suitable Banach spaces of real analytic functions using the complexification.

Denote by $\Delta^2 = \Delta \times \Delta$ the standard bidisc in $\mathbb{C}^2$. We define the space $\mathcal{C}^{1,\alpha}_\omega(\Delta)$ consisting of functions u (or $\mathbb{C}^n$-valued maps) of class $\mathcal{C}^{1,\alpha}(\bar{\Delta})$ with the following properties:

(i) u is a sum of a power series $u(\zeta) = \sum_{kl} u_{kl}\zeta^k\overline{\zeta}^l$ for $\zeta \in \overline{\Delta}$.

(ii) The "polarization" $\hat{u}$ of u defined by $\hat{u}(\zeta,\xi) = \sum_{kl} u_{kl}\zeta^k\xi^l$ is a function holomorphic on Δ^2 and of class $\mathcal{C}^{1,\alpha}(\overline{\Delta}^2)$.

(iii) The mixed derivative $\frac{\partial^2 \hat{u}}{\partial\zeta\partial\xi}$ is of class $\mathcal{C}^{\alpha}(\overline{\Delta}^2)$.

We define the norm of u as following:

$$\|u\|_{\mathcal{C}^{1,\alpha}_\omega(\Delta)} = \|\hat{u}\|_{\mathcal{C}^{1,\alpha}(\overline{\Delta}^2)} + \left\| \frac{\partial^2 \hat{u}}{\partial\zeta\partial\xi} \right\|_{\mathcal{C}^{\alpha}(\overline{\Delta}^2)}.$$

Since u is the restriction of $\hat{u}$ onto the totally real diagonal $\{\xi = \bar{\zeta}\}$ the polarization $\hat{u}$ is uniquely determined by u and therefore $\mathcal{C}^{1,\alpha}_\omega(\Delta)$ equipped with this norm is a Banach space.

Remark 5 One has the following continuous inclusion $\mathcal{O}^{1,\alpha}(\Delta) \subset \mathcal{C}^{1,\alpha}_\omega(\Delta)$: for $u \in \mathcal{O}^{1,\alpha}(\Delta)$ the corresponding $\hat{u}$ is simply $\hat{u}(\zeta,\xi) = u(\zeta)$. Really, for such $\hat{u}$ one has $\frac{\partial^2 \hat{u}}{\partial\zeta\partial\xi} = 0$.

We denote by $\mathcal{C}^{1,\alpha}_\omega(\partial\Delta^+)$ the space of real functions φ on $\partial\Delta^+$ such that there exists a function $v \in \mathcal{O}^{1,\alpha}(\Delta)$ satisfying the condition $\Re v|\partial\Delta^+ = \varphi$. In particular such function φ is real analytic on the interval $(-1,1)$. The holomorphic function v is unique up to an imaginary constant so we always assume that $\Im v(0) = 0$. We define the norm of φ as a $\mathcal{C}^{1,\alpha}$ norm of the corresponding function v on $\bar{\Delta}$. Then $\mathcal{C}^{1,\alpha}_\omega(\partial\Delta^+)$ equipped with this norm, is a Banach space.

Furthermore, denote by $\mathcal{C}^{1,\alpha}(\partial\Delta^+)$ the space of real continuous functions on $\partial\Delta^+$ which are of class $\mathcal{C}^{1,\alpha}$ on the closed upper semi-circle and on the interval $[-1,1]$. Finally we denote by we denote by $\mathcal{C}^{1,\alpha}_0(\partial\Delta^+)$ the space of real functions of class $\mathcal{C}^{1,\alpha}(\partial\Delta^+)$ vanishing on the interval $[-1,1]$. The following statement is a consequence of the reflection principle.

Lemma 7.1 *The space* $\mathcal{C}^{1,\alpha}_0(\partial\Delta^+)$ *is a subspace of* $\mathcal{C}^{1,\alpha}_\omega(\partial\Delta^+)$.

Proof. Let φ be a function of class $\mathcal{C}^{1,\alpha}_0(\partial\Delta^+)$. Solving the Dirichlet problem for φ in the upper semi-disc, we obtain a harmonic function h in Δ^+ continuous on $\overline{\Delta}^+$ such that $h|_{\partial\Delta^+} = \varphi$. Since h vanishes on $[-1,1]$ it extends harmonically on Δ by the classical reflection principle for harmonic functions. Namely, its extension h^* is defined by $h^*(\zeta) = -h(\overline{\zeta})$ for ζ in the lower semi-disc Δ^-. Thus we obtain a function $\tilde{h}$ harmonic on Δ and continuous on $\overline{\Delta}$. Since the restriction φ of h on the closed upper semi-circle is a function of class $\mathcal{C}^{1,\alpha}$, it follows easily by the definition of the reflection h^* that the restriction $\tilde{\varphi} := \tilde{h}|_{\partial\Delta}$ of $\tilde{h}$ on $\partial\Delta$ is a function of class $\mathcal{C}^{1,\alpha}(\partial\Delta)$. Then the Schwarz integral $T^{SW}\tilde{\varphi}$ gives by Proposition (2.2) a function of class $\mathcal{O}^{1,\alpha}(\Delta)$ whose real part coincides with $\tilde{h}$. □

Lemma 7.2 *If* $u \in \mathcal{C}^{1,\alpha}_\omega(\Delta)$ *then* $\Re u|_{\partial\Delta^+} \in \mathcal{C}^{1,\alpha}_\omega(\partial\Delta^+)$.

Proof. Let $\hat{u}(\zeta,\xi) = \sum u_{kl}\zeta^k\xi^l$ be the polarization of u holomorphic in the bidisc Δ^2 (that is $u(\zeta) = \hat{u}(\zeta,\overline{\zeta})$). Then the function $h(\zeta) = \hat{u}(\zeta,\zeta)$ is of class $\mathcal{O}^{1,\alpha}(\Delta)$ and $h|_{[-1,1]} = u|_{[-1,1]}$. Denote by φ the restriction of $\Re(u-h)$ to $\partial\Delta^+$. Then $\varphi \in \mathcal{C}^{1,\alpha}_0(\partial\Delta^+)$ and by Lemma 7.1 there exists a function $v \in \mathcal{O}^{1,\alpha}(\Delta)$ such that $\Re v|\partial\Delta^+ = \varphi$. Since the function $h+v$ is of class $\mathcal{O}^{1,\alpha}(\Delta)$, its real part gives the desired extension of the function $\Re u|\partial\Delta^+$.

□

We suppose everywhere below that our almost complex structure J (and therefore A_J in the equation for J holomorphic curves) is a real analytic function given by a convergent power series $\sum a_{kl}z^k\overline{z}^l$ with the radius of convergence big enough. The equation (2.2) on Δ can be rewritten in the form

$$(u - T_\Delta^{CG}A_J(u)\overline{u}_{\overline{\zeta}})_{\overline{\zeta}} = 0, \tag{7.1}$$

where T_Δ^{CG} denotes the Cauchy - Green transform on Δ. Define the map

$$\Phi_J : \mathcal{C}^{1,\alpha}(\bar{\Delta}) \to \mathcal{C}^{1,\alpha}(\bar{\Delta}),$$

as

$$\Phi_J : u \mapsto u - T_\Delta^{CG}A_J(u)\overline{u}_{\overline{\zeta}}. \tag{7.2}$$

Equation (7.1) means that u is J-holomorphic if and only if $\Phi_J u$ is holomorphic with respect to J_{st}.

Lemma 7.3 *For J close to J_{st} the operator Φ_J establishes a diffeomorphism of neighborhoods of zero in the space $\mathcal{C}_\omega^{1,\alpha}(\Delta)$.*

Proof. First we prove that Φ_J maps the space $\mathcal{C}_\omega^{1,\alpha}(\Delta)$ to itself. Given function $u \in \mathcal{C}_\omega^{1,\alpha}(\Delta)$ denote the function $A_J(u)\overline{u}_{\overline{\zeta}}$ by h. We need to prove that $T_\Delta^{CG}h$ belongs to $\mathcal{C}_\omega^{1,\alpha}(\Delta)$. Consider the polarization $\hat{h}(\zeta,\xi) = \hat{h}(\zeta,\xi)$ of h. By Proposition 2.1 we have the representation

$$T_\Delta^{CG}h(\zeta) = \hat{H}(\zeta,\overline{\zeta}) - \frac{1}{2\pi i}\int_{\partial\Delta}\frac{\hat{H}(\tau,\overline{\tau})d\tau}{\tau-\zeta}. \tag{7.3}$$

where

$$\hat{H}(\zeta,\xi) = \int_{[0,\xi]}\hat{h}(\zeta,\omega)d\omega \tag{7.4}$$

is a primitive of $\hat{h}$ with respect to ξ. Let's study the primitive (7.4) of $\hat{h}$ first. We point out that the function $\hat{h}$ is of class $\mathcal{C}^{0,\alpha}(\overline{\Delta}^2)$. Furthermore, the condition (iii) of the definition of the space $\mathcal{C}_\omega^{1,\alpha}(\Delta)$ implies that $\partial\hat{h}/\partial\zeta$ is of class $\mathcal{C}^{0,\alpha}(\overline{\Delta}^2)$. Now the derivation of the integral (7.4) with respect to ζ and ξ gives that $\hat{H}$ satisfies conditions (i), (ii), (iii) of the definition of the space $\mathcal{C}_\omega^{1,\alpha}(\Delta)$.

By Proposition 2.2 the Cauchy integral in the right hand side of (6.3) represents a function of class $\mathcal{O}^{1,\alpha}(\Delta)$ and so also belongs to the space $\mathcal{C}_\omega^{1,\alpha}(\Delta)$.

Thus we obtain that $\Phi_J(u)$ belongs to $\mathcal{C}_\omega^{1,\alpha}(\Delta)$. Since the Fréchet derivative of Φ_J with respect to u at $u=0$ and $J=J_{st}$ is the identity map, the lemma follows from the inverse mapping theorem.

□

Hence Φ_J is a diffeomorphism between neighborhoods of zero in the manifolds of J-holomorphic and J_{st}-holomorphic maps of class $\mathcal{C}_\omega^{1,\alpha}(\bar{\Delta})$. In particular, J-holomorphic discs form a Banach submanifold in $\mathcal{C}_\omega^{1,\alpha}(\Delta)$ in a neighborhood of zero. We denote this manifold as $\mathcal{O}_{\omega,J}^{1,\alpha}(\Delta)$.

Remark 6 Note that $\mathcal{O}^{1,\alpha}_{\omega,J_{\mathsf{st}}}(\Delta) = \mathcal{O}^{1,\alpha}(\Delta)$.

We use the notation $\mathcal{O}^{1,\alpha}_{\omega,J,0}(\Delta)$ for the submanifold of such $u \in \mathcal{O}^{1,\alpha}_{\omega,J}(\Delta)$ that $\Re u|_{[-1.1]} \equiv 0$. By $R_{\partial\Delta^+}$ denote "taking real part and restriction to $\partial\Delta^+$" operator. $\Phi_J\left(\mathcal{O}^{1,\alpha}_{\omega,J,0}(\Delta)\right)$ is the diffeomorphic image of $\mathcal{O}^{1,\alpha}_{\omega,J,0}(\Delta)$ under Φ_J.

One has the following commutative diagram:

$$\begin{array}{ccccccc} \mathcal{C}^{1,\alpha}_{\omega}(\Delta) & \supset & \mathcal{O}^{1,\alpha}(\Delta) & \stackrel{\Phi_J^{-1}}{\longrightarrow} & \mathcal{O}^{1,\alpha}_{\omega,J}(\Delta) & & \\ & & \uparrow i & & \uparrow i & & \\ \mathcal{C}^{1,\alpha}_{\omega}(\Delta) & \supset & \Phi_J\left(\mathcal{O}^{1,\alpha}_{\omega,J,0}(\Delta)\right) & \stackrel{\Phi_J^{-1}}{\longrightarrow} & \mathcal{O}^{1,\alpha}_{\omega,J,0}(\Delta) & \stackrel{R_{\partial\Delta^+}}{\longrightarrow} & \mathcal{C}^{1,\alpha}_{0}(\partial\Delta^+) \end{array} , \tag{7.5}$$

where both i-s are natural imbeddings.

For an unknown map u from $\Phi_J(\mathcal{O}^{1,\alpha}_{\omega,J,0}(\Delta))$ and given $\varphi \in \mathcal{C}^{1,\alpha}_0(\partial\Delta^+)$ consider the system

$$\begin{cases} R_{\partial\Delta^+}\Phi_J^{-1}u = \varphi, \\ \Im\Phi_J^{-1}u(0) = a. \end{cases} \tag{7.6}$$

Fix a J_{st}-holomorphic map $u^0 \in \mathcal{O}^{1,\alpha}(\Delta)$ such that $\Re u^0|_{[-1,1]} \equiv 0$ and set $\varphi^0 = \Re u^0|_{\partial\Delta^+}$.

Lemma 7.4 *For real analytic J close enough to J_{st} in $\mathcal{C}^{1,\alpha}$-norm there exists a neighborhood U of φ^0 in $\mathcal{C}^{1,\alpha}_0(\partial\Delta^+)$, a neighborhood U' of $a^0 := \Im u^0(0)$ in $\mathbb{R}^n$ and a neighborhood V of u^0 in $\mathcal{O}^{1,\alpha}(\Delta)$ such that for $\varphi \in U$ and $a \in U'$ the system (7.6) admits a unique solution $u \in V \cap \Phi_J(\mathcal{O}^{1,\alpha}_{\omega,J,0}(\Delta))$.*

Proof. The surjectivity condition for the operator obtained by the linearization of (7.6) at $u = u^0$ and $J = J_{\mathsf{st}}$ is reduced to the resolution of the system

$$R_{\partial\Delta^+}\dot{u} = \psi, \tag{7.7}$$

$$\Im\dot{u}(0) = a,$$

for an arbitrary given function $\psi \in \mathcal{C}^{1,\alpha}_0(\partial\Delta^+)$, arbitrary $a \in \mathbb{R}^n$ and an unknown map $\dot{u} \in \mathcal{O}^{1,\alpha}(\Delta)$. By Lemma 7.1 we obtain a solution for any given right hand side of (7.7). The uniqueness of $\dot{u}$ is obvious. Now the Implicit Function Theorem implies the desired statement.

□

7.2. General case

Theorem 1 now follows similarly to the smooth case (but using this smooth case!). We replace the unit disc by the upper semi-disc Δ^+ and $\mathbf{S}^+$ by the interval $(-1,1)$. By the classical results on the interior regularity of pseudo-holomorphic maps we can assume that the map u is real analytic in a neighborhood of $\bar{\Delta}^+ \backslash (-1,1)$. We can assume that X is the unit ball of $\mathbb{C}^n$ equipped with a real analytic almost complex structure J with $J(0) = J_{st}$ and that $W = i\mathbb{R}^n$ and $u(0) = 0$.

Our map u is of class $\mathcal{C}^{1,\alpha}(\overline{\Delta}^+)$ and admits the expansion $u(\zeta) = b\zeta + o(|\zeta|)$ near the origin. For $t > 0$ consider the real analytic structures $J_t(z) = J(tz)$. They tend to J_{st} as t tends to 0. Maps $u^t(\zeta) = t^{-1}u(t\zeta)$ are J_t-holomorphic and tend to the map $u^0 : \zeta \longrightarrow b\zeta$ as $t \longrightarrow 0$ which is viewed as a real analytic J_{st}-holomorphic map.

Applying Lemma 7.4 to u^0 and $\varphi^t := \Re u^t|\partial\Omega$ for t small enough, we obtain by the uniqueness statements of Lemma 7.4 and Lemma 6.1 that the maps u^t are real analytic for t small enough. This implies finally that u extends as a real analytic map past $(-1, 1)$. Since it satisfies the real analytic condition (2.1) on an open set, the extension is a J-holomorphic map. This proves the Theorem 1.

□

Remark 7 In order to apply Lemma 6.1 correctly one should take a domain Ω in Δ^+ with $\mathcal{C}^\infty$ boundary $\partial\Omega$ such that the real interval $(-1, 1)$ is contained in $\partial\Omega$.

8 Open Questions

At the end we would like to turn the attention of a reader to some open questions.

8.1. Real Analyticity up to the Boundary

Real analytic extension of solutions of elliptic PDE-s with real analytic data across a part of the boundary were studied in numerous papers and books, let's cite the following ones: [K, M, Mu, SS, Yu]. However, the linearization of (2.2) is the standard Cauchy-Riemann operator and it is not properly elliptic. Therefore the results in §6.7 and 6.8 in [M] are not applicable in our case, see Definition 6.1.2 in [M] and discussion there. .

1. Our Theorem 1 suggests that these results could be generalized to more general elliptic systems, which includes (2.2) as a partial case.

One can try to save the situation in the following way. Applying (2.1) to to tangent vectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ one gets

$$\frac{\partial u}{\partial x} + J(u)\frac{\partial u}{\partial y} = 0$$

and

$$\frac{\partial u}{\partial y} - J(u)\frac{\partial u}{\partial x} = 0.$$

Differentiating the first equation with respect to x, the second with respect to y and adding the results one gets

$$\Delta u + \left\langle \nabla J, \frac{\partial u}{\partial x} \right\rangle \frac{\partial u}{\partial y} - \left\langle \nabla J, \frac{\partial u}{\partial y} \right\rangle \frac{\partial u}{\partial x} = 0. \tag{8.8}$$

That operator satisfies conditions of the Theorem 6.7.6$'$ but fails to satisfy the boundary conditions, see definitions and discussion on pp. 209-212, especially Definition 6.1.3.

2. This may mean that also for second order real analytic elliptc systems there should be more general analyticity theorems, which include the system (8.8) with boundary conditions like "to belong to some distinguished submanifold of $\mathbb{R}^{2n}$" along some part of the boundary.

8.2. Reflection Principle

The following couple of questions are more complex analytic by their nature.

3. Let (X,J) be a real analytic almost complex manifold and W a real analytic J-totally real submanifold of X. Let C^+ be J-complex curve in $X\setminus W$. Does there exists a neighborhood V of W and a J-complex curve C^- in $V\setminus W$ (reflection of C^+) such that $\overline{(C^+\cup C^-)}\cap V$ is a J-complex curve in V?

For integrable J the answer is yes and is due to H. Alexander, see [A].

4. The following question is a particular case of the previous one. Let C be a J-complex curve in the complement of a point. Will its closure $\bar{C}$ be a J-complex curve?

5. This question was communicated to us by J.-C. Sikorav. Define a J-holomorphic map as a differentiable map $u:\Delta\to X$ such that (2.1) is satisfied at every point. Prove that $u\in W^{1,2}_{loc}$ (and therefore u is a J-holomorphic map in the usual sense).

8.3. Boundary Uniqueness

There are several natural questions concerning the boundary uniqueness problem for pseudoholomorphic maps.

1. Does Theorem 2 hold under weaker assumptions on the boundary regularity of u and v? Namely, if they are only continuous up to the boundary and coinside there on the set of positive linear measure?

2. We wish that Lemma 4.3 could be enhanced to the estimate of the form

$$\|dw\|_{L^p(\Delta)} \leq C\cdot \left\|\bar{\partial}_J w\right\|_{L^p(\Delta)} \tag{8.9}$$

where $C = C(p, \|J-J_{\mathsf{st}}\|_{L^\infty}) = \sum_{n=1}^{\infty}\|J-J_{\mathsf{st}}|_{L^\infty}(1+G_p)\|^n$, provided that $\|J-J_{\mathsf{st}}\|_{L^\infty} < \varepsilon_p$.

References

[A] ALEXANDER H.: *Continuing* 1*-dimensional Analytic Sets.* Math. Ann. **191** (1971), 143-144.

[BJS] BERS L., JOHN F., SCHECHTER M.: *Partial differential equations.* J. Wiley and Sons (1964).

[Ca] CARATHÉODORY C.: *Zum Schwarzschen Spiegelungsprinzip.* Comm. Math. Helv. **19**, n 1 (1946), 263-278.

[Ch] CHIRKA E.: *Regularity of boundaries of analytic sets.* Math. USSR Sbornik **43** (1983), 291-335.

[CGS] COUPET B., GAUSSIER H., SUKHOV A.: *Fefferman's mapping theorem on almost complex manifold in complex dimension two.* Math. Z. **250** (2005), 59-90.

[GS] GAUSSIER H., SUKHOV A.: *On the geometry of model almost complex manifolds with boundary.* Math. Z. **254** (2006), 567-589.

[IR] IVASHKOVICH S., ROSAY J.-P.: *Schwarz-type lemmas for solutions of $\bar{\partial}$-inequalities and complete hyperbolicity of almost complex manifolds.* Annales Inst. Fourier **54**, (2004) 2387-2435.

[IS1] IVASHKOVICH S., SHEVCHISHIN V: *Reflection Principle and J-Complex Curves with Boundary on Totally Real Immersions.* Communications in Contemporary Mathematics **4**, 65-106 (2002).

[IS2] IVASHKOVICH S., SHEVCHISHIN V: *Structure of the moduli space in a neighorhood of a cusp-curve and meromorphic hulls.* Invent. math. **136**, 571-602 (1999).

[K] KRAFT R.: *Analiticity and reflectivity for first order systems of elliptic type in two independant variables.* J. Math. Anal. Appl. **29**, 1-17 (1970).

[MS] MCDUFF D., SALAMON D.: *J-holomorphic curves and symplectic topology.* AMS Colloquium Publ., 52, AMS, Providence, RI, (2004).

[M] MORREY, C.: *Multiple integrals in the calculus of variations.* Springer Verlag, (1966).

[Mu] MÜLLER F.: *On the continuation of solutions for elliptic equations in two variables.* Ann. Inst. H. Poincaré, **19**, 745-776 (2002).

[Sk] SIKORAV J.-C.: *Some properties of holomorphic curves in almost complex manifolds.* In "Holomorphic curves in symplectic geometry". Ed. M. Audin, J. Lafontaine. Birkhäuser, Progress in Mathematics v.117, Ch. V, 165-189 (1994).

[SS] STERNIN B., SHATALOV V.: *Continuation of solutions to elliptic equations and localization of singularities.* Lect. Notes Math. **150**, 237-59 (1992).

[Sw] SCHWARZ H.A.: *Über einige Abbildungsaufgaben.* Journal für reine und angewandte Mathematik **70**, 105-120 (see pages 106-107). See also Gesammelte mathematische Abhandlungen, Springer (1892), 66-67. Or the Second Edition, Bronx, N.Y., Chelsea Pub. Co. (1972).

[Tr] TRIEBEL, H.: *Theory of Function Spaces.* Birkhäuser (1983).

[Ve] VEKUA I.N.: *Generalized analytic functions. Fizmatgiz, Moscow* (1959); English translation - Pergamon Press, London, and Addison-Welsey, Reading, Massachuset (1962).

[Yu] YU C.L.: *Reflection principle for systems of first order elliptic equations with analytic coefficients* , Trans. Amer. Math. Soc. **164** (1972), 489 - 501.

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