BRANCHED CRYSTALS AND CATEGORY $\mathcal{O}$

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Introduction

The theory of crystal bases introduced by Kashiwara in [6] to study the category of integrable representations of quantized Kac–Moody Lie algebras has been a major development in the combinatorial approach to representation theory. In particular Kashiwara defined the tensor product of crystal bases and showed that it corresponded to the tensor product of representations. Later, in [7] he defined the abstract notion of a crystal, the tensor product of crystals and showed that the tensor product was commutative and associative.

In this paper, using some ideas from [4] and [9], we give a definition of branched crystals adapted to the study of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ of the quantized enveloping algebra of $\mathfrak{sl}_2$ which coincides with the usual definition of crystals for the integrable modules. In [6], Kashiwara essentially defined a crystal basis for the Verma modules. However, it is not hard to see that his tensor product rule, even for the case of $\mathfrak{sl}_2$, does not give the decomposition (as a direct sum of indecomposable modules) of the corresponding tensor product of modules. Also, the restriction of the basis to a particular color does not reflect the decomposition of the module as a direct sum of indecomposable modules for the corresponding $\mathfrak{sl}_2$. We define a notion of the tensor product of branched crystals which extends Kashiwara's definition and the definition in [9] and prove in a combinatorial manner that the tensor product decomposes in the same way as the corresponding representations. Using this, we are then able to prove that the tensor product is both associative and commutative. It is not hard to generalize the definition of branched crystals to the higher rank case although the connection with $\mathcal{O}$ is probably much harder to establish. However, the results of Section 1 make it plausible that this is essentially the only possible theory of crystals for $\mathcal{O}$ which would satisfy the requirement that the crystal corresponding to a representation V, when restricted to a particular color, is the crystal of V regarded as a module for $\mathbf{U}_q(\mathfrak{sl}_2)$.

The paper is organized as follows. In Section 1 we define the notion of a (one–colored) branched crystal, its indecomposable components and classify the indecomposable branched crystals. In Section 2 we define the tensor product of two branched crystals and show that the result is also a branched crystal. Finally, we identify the indecomposable components (with multiplicities) in the tensor product of indecomposable branched crystals. In Section 3 we make a connection with the representation theory of $\mathbf{U}_q(sl_2)$. Using the results of Section 1 we see that given a module V in $\mathcal{O}$, we can associate to it in a purely formal but natural and unique way a branched crystal $B(V)$ so that direct sums are preserved. Further, the results of Section 2 make it clear that this association preserves tensor products, i.e., $B(V \otimes W) \cong B(V) \otimes B(W)$ for all $V, W \in \mathcal{O}$.

Acknowledgements: We thank J. Greenstein for bringing [9] to our attention and for useful discussions. D.J. thanks the Max-Planck-Institut für Mathematik in Bonn for hospitality.

1. Branched Crystals

In this section we introduce the notion of a (one colored) branched crystal. These combinatorial objects are analogous to the notion of normal crystals defined by Kashiwara in [7].

1.1.

Definition. A branched crystal is a nonempty set B together with maps $\tilde{e}, \tilde{f} : B \sqcup \{0\} \to B \sqcup \{0\}$, $\mathrm{wt} : B \to \mathbf{Z}$, $\varepsilon, \varphi : B \to \mathbf{Z}$ satisfying the following axioms:

(i) $\tilde{e}0 = \tilde{f}0 = 0$ and if $\tilde{e}b \in B$ (resp.$\tilde{f}b \in B$), then $\mathrm{wt}(\tilde{e}b) = \mathrm{wt}(b) + 2$ (resp. $\mathrm{wt}(\tilde{f}b) = \mathrm{wt}(b) - 2$)),
(ii) if $\tilde{f}b \in B$, then $\tilde{e}\tilde{f}b = b$,
(iii) $\tilde{e}b \neq \tilde{e}b'$ if $b, b' \in B$ are such that $\tilde{e}b, \tilde{e}b' \in B$ but $\tilde{f}\tilde{e}b \neq b$ and $\tilde{f}\tilde{e}b' \neq b'$,
(iv) $\varepsilon(b) = \min\{s \in \mathbf{Z}^+ : \tilde{f}\tilde{e}^{s+1}b \neq \tilde{e}^s b\}$,
(v) $\varphi(b) = \mathrm{wt}(b) + \varepsilon(b)$. If there exists $l > 0$ such that $\tilde{f}^{l-1}b \neq 0$ and $\tilde{f}^l b = 0$, then $l = \varphi(b) + 1$. Further, if $b \in B$ is such that $\tilde{e}b \in B$ but $\tilde{f}\tilde{e}b \neq b$, then $\varphi(\tilde{e}b) = 0$.

An element $b \in B$ is called a branch point if $\tilde{e}b \in B$ but $\tilde{f}\tilde{e}b \neq b$ and B^{br} denotes the set of branch points in B. Note that $\tilde{f}b \notin B^{br}$ for $b \in B$. A subcrystal of B is a subset B' of B such that B' together with the restrictions of the maps $\mathrm{wt}, \tilde{e}, \tilde{f}, \varepsilon, \varphi$ is a branched crystal. Given $b \in B$, let B_b be the subcrystal

$$B_b = \{\tilde{f}^r \tilde{e}^s b : r, s \in \mathbf{Z}^+\}.$$

We say that B_b is cyclic on b. If $\tilde{e}b = 0$, then B_b has no branch points.

Given branched crystals $(B, \mathrm{wt}, \tilde{e}, \tilde{f}, \varepsilon, \varphi)$ and $(B', \mathrm{wt}, \tilde{e}, \tilde{f}, \varepsilon, \varphi)$ a map $\Phi : B \sqcup \{0\} \to B' \sqcup \{0\}$ is called a morphism of crystals if

(i) $\Phi(0) = 0$,
(ii) if $b \in B$ and $\Phi(b) \in B'$, then $\mathrm{wt}(\Phi(b)) = \mathrm{wt}(b)$ and $\varepsilon(\Phi(b)) = \varepsilon(b)$,
(iii) if $b \in B$ and $\Phi(b), \Phi(\tilde{f}b) \in B'$ (resp. $\Phi(\tilde{e}b) \in B'$), then $\Phi(\tilde{f}b) = \tilde{f}\Phi(b)$ (resp. $\Phi(\tilde{e}b) = \tilde{e}\Phi(b)$).

A morphism is strict if it satisfies

(iii') $\Phi(\tilde{f}b) = \tilde{f}\Phi(b)$ and $\Phi(\tilde{e}b) = \tilde{e}\Phi(b)$ for all $b \in B$.

An injective morphism is called an embedding and an isomorphism is a bijective morphism.

1.2. Given $b, b' \in B$, say that $b \sim b'$ iff $B_b \cap B_{b'} \neq \emptyset$. It follows easily from Definition 1.1 (ii) that if $\tilde{e}b = \tilde{e}b' = 0$ then $b \sim b'$ iff $b = b'$.

Proposition. *Let $b, b' \in B$. Then $b \sim b'$ iff there exist $l, l' \in \mathbf{Z}^+$ such that $\tilde{e}^l b = \tilde{e}^{l'} b' \in B$. Further, $\sim$ is an equivalence relation on B.*

Proof. The first statement is obvious from Defintion 1.1(ii). To prove the second it is only neccesary to check that $\sim$ is transitive. Choose nonnegative integers l_i, $1 \leq i \leq 4$, such that

$$\tilde{e}^{l_1} b_1 = \tilde{e}^{l_2} b_2, \quad \tilde{e}^{l_3} b_2 = \tilde{e}^{l_4} b_3.$$

If $l_2 \geq l_3$, then

$$\tilde{e}^{l_4 + l_2 - l_3} b_3 = \tilde{e}^{l_2} b_2 = \tilde{e}^{l_1} b_1,$$

and hence $B_{b_1} \cap B_{b_3} \neq \emptyset$. The case $l_2 < l_3$ is similar and the lemma is proved. ∎

An equivalence class of B with respect to $\sim$ is a subcrystal and is called an indecomposable component of B. A crystal B is indecomposable if $b \sim b'$ for all $b, b' \in B$. Thus, an indecomposable component of B is a maximal indecomposable subcrystal of B.

Corollary.

(i) *An indecomposable component of a branched crystal contains at most one element b such that $\tilde{e}b = 0$.*
(ii) *Suppose that B_i, $i \in I$, are a family of indecomposable subcrystals of a branched crystal B such that $B = \cup_{i\in I} B_i$ and $B_i \cap B_j = \emptyset$ if $i, j \in I$, $i \neq j$. Then B_i, $i \in I$, are the indecomposable components of B.*

1.3.

Proposition. *Let B be a branched crystal and assume that $B^{br} \neq \emptyset$. Let $b \in B^{br}$, then*

(i) $\mathrm{wt}(b) \leq -2$,
(ii) $\tilde{e}^l b$ *is not a branch point for all* $0 < l \leq -\mathrm{wt}(b) - 1$ *and* $\tilde{e}^{-\mathrm{wt}(b)-1} b \neq 0$,
(iii) $\tilde{e}^{-\mathrm{wt}(b)} b = 0$,
(iv) $\tilde{f}^l b \neq 0 \quad \forall\, l \geq 0$.

Proof. If b is a branch point then $\tilde{e}b \in B$ and by Definition 1.1(i),(iv),(v) we have

$$0 \leq \varepsilon(\tilde{e}b) = -\mathrm{wt}(\tilde{e}b) = -\mathrm{wt}(b) - 2$$

proving (i). If $\mathrm{wt}(b) = -2$, then $\mathrm{wt}(\tilde{e}b) = 0 = \varepsilon(\tilde{e}b)$. The first equality together with (i) implies that $\tilde{e}b \notin B^{br}$ and the second equality implies that $\tilde{e}^2 b = 0$ by Definition 1.1(iv). Suppose now that $\mathrm{wt}(b) < -2$. It follows by induction on l that $\varepsilon(\tilde{e}^l b) = \varepsilon(\tilde{e}b) - (l-1) = -\mathrm{wt}(b) - 1 - l > 0$ for $0 < l < -\mathrm{wt}(b) - 1$. Hence $\tilde{e}^l b \notin B^{br}$ and $\tilde{e}^{l+1} b \neq 0$. Further, $\varepsilon(\tilde{e}^{-\mathrm{wt}(b)-1} b) = 0$ and $\mathrm{wt}(\tilde{e}^{-\mathrm{wt}(b)-1} b) = -\mathrm{wt}(b) - 2 > 0$. This proves (ii). It also follows that $\tilde{e}^{-\mathrm{wt}(b)} b = 0$. If $l \in \mathbf{Z}^+$ is minimal such that $\tilde{f}^l b = 0$ then by Definition 1.1 (v) we have $l = \varphi(b) + 1 = \mathrm{wt}(b) + 1$. But this is impossible since by (i) the right hand side is negative. $\square$

Corollary.

(i) *An indecomposable branched crystal has at most one branch point.*
(ii) *Every indecomposable crystal has one element b such that $\tilde{e}b = 0$.*
(iii) *If $\varphi(b) \geq 0$, then $\tilde{e}^l b \notin B^{br}$ for all $l \geq 0$.*

Proof. Let B be an indecomposable branched crystal and assume that $b_1, b_2 \in B^{br}$, $b_1 \neq b_2$. It follows from Proposition 1.3(iii) and Corollary 1.2 that

$$\tilde{e}^{-\mathrm{wt}(b_1)-1} b_1 = \tilde{e}^{-\mathrm{wt}(b_2)-1} b_2$$

and hence $\mathrm{wt}(b_1) = \mathrm{wt}(b_2)$. Applying $\tilde{f}$ and using Proposition 1.3(ii) we get $\tilde{e}b_1 = \tilde{e}b_2$. But this is impossible by Definition 1.1(iii). This proves (i). If $B^{br} \neq \emptyset$, (ii) follows from Proposition1.3(iii). If $B^{br} = \emptyset$, then $\varepsilon(b) = \varepsilon(\tilde{e}b) + 1 > 0$ for all $b \in B$ such that $\tilde{e}b \in B$. Hence $\tilde{e}^{\varepsilon(b)+1} b = 0$ for all $b \in B$. To prove (iii), note that $\varphi(b) < 0$ for all $b \in B^{br}$. Therefore, it suffices to show that if $\varphi(b) \geq 0$ then $\varphi(\tilde{e}^l b) \geq 0$ for all $l \geq 0$ such that $\tilde{e}^l b \in B$. This is easily done inductively. $\square$

1.4. The main result of this section is:

Theorem. Let B be an indecomposable branched crystal. Then B is isomorphic to a branched crystal in one of the four infinite families defined below. Let $r, s \in \mathbf{Z}, r \geq 0$.

$B(V(r)) = \{b_j : 0 \leq j \leq r\}$,

$$\tilde{f}b_j = b_{j+1}, \qquad \tilde{e}b_j = b_{j-1}, \qquad \mathrm{wt}(b_j) = r - 2j, \qquad \varepsilon(b_j) = j, \qquad \varphi(b_j) = r - j.$$

$B(M(s)) = \{b_j : j \geq 0\}$,

$$\tilde{f}b_j = b_{j+1}, \qquad \tilde{e}b_j = b_{j-1}, \qquad \mathrm{wt}(b_j) = s - 2j, \qquad \varepsilon(b_j) = j, \qquad \varphi(b_j) = s - j$$

$B(T(r)) = \{b_{(j)} : j \geq 0\} \sqcup \{b_j : j \geq 0\}$, where $\tilde{e}b_0 = b_{(r)}$, and

$$\tilde{f}b_{(j)} = b_{(j+1)}, \qquad \tilde{e}b_{(j)} = b_{(j-1)}, \qquad \mathrm{wt}(b_{(j)}) = r - 2j, \qquad \varepsilon(b_{(j)}) = j, \qquad \varphi(b_{(j)}) = r - j,$$

$$\tilde{f}b_j = b_{j+1}, \qquad \tilde{e}b_{j+1} = b_j, \qquad \mathrm{wt}(b_j) = -r - 2 - 2j, \qquad \varepsilon(b_j) = j, \qquad \varphi(b_j) = -r - 2 - j.$$

$B(M(r)^\sigma) = \{b_{(j)} : 0 \leq j \leq r\} \sqcup \{b_j : j \geq 0\}$, where $\tilde{e}b_0 = b_{(r)}$, and

$$\tilde{f}b_{(j)} = b_{(j+1)}, \qquad \tilde{e}b_{(j)} = b_{(j-1)}, \qquad \mathrm{wt}(b_{(j)}) = r - 2j, \qquad \varepsilon(b_{(j)}) = j, \qquad \varphi(b_{(j)}) = r - j.$$

$$\tilde{f}b_j = b_{j+1}, \qquad \tilde{e}b_{j+1} = b_j, \qquad \mathrm{wt}(b_j) = -r - 2 - 2j, \qquad \varepsilon(b_j) = j, \qquad \varphi(b_j) = -r - 2 - j.$$

Proof. It is not hard to verify that the elements of the families given in the theorem satisfy the conditions for branched crystals. Now let B be an indecomposable branched crystal. Suppose first that $B^{br} = \emptyset$ and choose $b \in B$ such that $\tilde{e}b = 0$. Then, $B_b = \{\tilde{f}^r b : r \in \mathbf{Z}^+\}$ is a subcrystal of B. Since $\varepsilon(\tilde{f}^r b) = r$ for all $r \in \mathbf{Z}^+$ such that $\tilde{f}^r b \in B$, it follows that B_b is isomorphic to $B(M(\mathrm{wt}(b)))$ if $\tilde{f}^r b \neq 0$ for all $r \in \mathbf{Z}^+$, and to $B(V(\mathrm{wt}(b)))$ if $\tilde{f}^{\mathrm{wt}(b)+1} b = 0$. It remains to show that $B = B_b$. But this is clear, for given $b' \in B$ there exists $s \in \mathbf{Z}^+$ with $\tilde{e}^s b' = 0$ or equivalently $\tilde{e}^{s-1} b' = b$ and hence $\tilde{f}^{s-1} b = b'$. If $B^{br} = \{b\}$, set $\tilde{e}^{-\mathrm{wt}(b)-1} b = \bar{b}$ and let $b' \in B \backslash \{b\}$. If $\tilde{e}^s b' = b$ for some $s \in \mathbf{Z}^+$, then $b' = \tilde{f}^s b \in B_b$. Otherwise $\tilde{e}^s b' \neq b$ for all $s \in \mathbf{Z}^+$ and $\tilde{e}^k b' = \bar{b}$ for some $k \in \mathbf{Z}^+$. Hence $b' = \tilde{f}^k \bar{b} \in B_b$. It is not hard to see now that B_b is isomorphic to $B(T(-\mathrm{wt}(b) - 2))$ or to $B(M(-\mathrm{wt}(b) - 2)^\sigma)$. $\square$

We conclude this section with some additional results which are needed later.

1.5. Assume that $\{B_j\}_{j \in J}$ is a family of branched crystals. There exists an obvious structure of a branched crystal on the disjoint union $\sqcup_{j \in J} B_j$ which we denote by $\oplus_{j \in J} B_j$. Namely the maps wt, $\tilde{e}$, $\tilde{f}, \varepsilon, \varphi$ on $\oplus_{j \in J} B_j$ are defined by requiring their restriction to B_j to be the corresponding map on B_j. The canonical inclusion $\iota_j : B_j \to \oplus_{j \in J} B_j$ is a strict embedding of branched crystals. This proves,

Proposition. *Let B be a branched crystal and let B_j, $j \in J$ be the indecomposable components of B. Then, $B = \oplus_{j \in J} B_j$. Furthermore, if $\Phi : B \to \oplus_{l \in L} B'_l$ is an isomorphism of branched crystals with B'_l indecomposable, then there exists a bijective map $\phi : J \to L$ such that $\Phi|_{B_j} : B_j \to B'_{\phi(j)}$, $j \in J$ is an isomorphism of branched crystals.*

1.6. Given a branched crystal B, set

$$B^{hw} = \{b \in B\backslash(\cup_{b'\in B^{br}} B_{b'}) : \tilde{e}b = 0\}$$

and

$$B^{br,\sigma} = \{b \in B^{br} : \tilde{f}\tilde{e}b = 0\}.$$

The following is easily seen as a consequence of Theorem 1.4.

Proposition. *Let B be a branched crystal. The indecomposable components of B are $B_{\mathbf{b}}$, where*

$$\mathbf{b} \in B^{hw} \cup B^{br}.$$

Moreover we have isomorphisms of branched crystals,

$$\begin{aligned}
&B_{\mathbf{b}} \cong B(M(\mathrm{wt}(\mathbf{b}))), \quad \textit{if} \;\; \mathbf{b} \in B^{hw}, \quad \tilde{f}B_{\mathbf{b}} \subset B,\\
&B_{\mathbf{b}} \cong B(V(\mathrm{wt}(\mathbf{b}))), \quad \textit{if} \;\; \mathbf{b} \in B^{hw}, \quad \tilde{f}^{\mathrm{wt}(b)+1}\mathbf{b} = 0,\\
&B_{\mathbf{b}} \cong B(T(-\mathrm{wt}(\mathbf{b}) - 2)), \quad \textit{if} \;\; \mathbf{b} \in B^{br}\backslash B^{br,\sigma},\\
&B_{\mathbf{b}} \cong B(M(-\mathrm{wt}(\mathbf{b}) - 2)^{\sigma}), \quad \textit{if} \;\; \mathbf{b} \in B^{br,\sigma}.
\end{aligned}$$

1.7.

Lemma. *Let B be a branched crystal and let $b, b' \in B$. Then $B_b \cap B_{b'} \neq \emptyset$ iff $B_{b'} \subset B_b$ or vice versa.*

Proof. Suppose $B_b \cap B_{b'} \neq \emptyset$. Let $a = \tilde{e}^{\varepsilon(b)}b$ and $a' = \tilde{e}^{\varepsilon(b')}b'$. Then $B_b = B_a$ and $B_{b'} = B_{a'}$. We have three cases:

(i) $a, a' \in B^{br}$,
(ii) $\tilde{e}a = \tilde{e}a' = 0$,
(iii) $a \in B^{br}$ and $\tilde{e}a' = 0$ or vice versa.

It follows from Corollary 1.3(i) (resp. Corollary 1.2) that $a = a'$ in case (i) (resp. case (ii)). Similarly in case (iii) we have $\tilde{e}^{-\mathrm{wt}(a)-1}a = a'$, hence $B_{a'} \subset B_a$. $\square$

2. A tensor product rule for branched crystals

2.1. We now define an analogue for branched crystals of Kashiwara's tensor product rule for crystals. Extending [9], we also define $\psi : B \to \mathbf{Z}$ as follows:

$$\psi(b) = \max\{\varepsilon(b), \varepsilon(b) - \varphi(b) - 1\}.$$

Definition. Given two branched crystals B, B', let $B \otimes B'$ be the set $B \times B'$ equipped with the maps $\tilde{e}, \tilde{f} : (B \sqcup \{0\}) \times (B' \sqcup \{0\}) \to (B \times B') \sqcup \{0\}$ and $\mathrm{wt} : B \times B' \to \mathbf{Z}$ defined below.

$$\mathrm{wt}(b \otimes b') = \mathrm{wt}(b) + \mathrm{wt}(b')$$

$$\begin{aligned}
\tilde f(b\otimes b') &= \tilde f b\otimes b',\ \text{ if } b,b' \text{ satisfy } \mathbf{F}1,\\
&= b\otimes \tilde f b',\ \text{ if } b,b' \text{ satisfy } \mathbf{F}1',\\
&= \tilde f^{\varphi(b)+1} b\otimes \tilde e^{\varphi(b)} b',\ \text{ if } b,b' \text{ satisfy } \mathbf{F}2,\\
\tilde e(b\otimes b') &= \tilde e b\otimes b',\ \text{ if } b,b' \text{ satisfy } \mathbf{E}1,\\
&= b\otimes \tilde e b',\ \text{ if } b,b' \text{ satisfy } \mathbf{E}1',\\
&= \tilde e^{\varphi(b')+1} b\otimes \tilde f^{\varphi(b')} b',\ \text{ if } b,b' \text{ satisfy } \mathbf{E}2,\\
&= \tilde f^{-\mathrm{wt}(b)-\mathrm{wt}(b')-2} b\otimes \tilde e^{-\mathrm{wt}(b)-\mathrm{wt}(b')-1} b',\ \text{ if } b,b' \text{ satisfy } \mathbf{E}2',\\
&= 0 \text{ otherwise,}
\end{aligned}$$

where we say that b, b' satisfy

(i) the condition $\mathbf{F}1$ if $\varphi(b) < 0$ or $\psi(b') < \varphi(b)$,
(ii) the condition $\mathbf{F}1'$ if $\tilde f b' \neq 0$ and $\psi(b') \geq \varphi(b) \geq 0$,
(iii) the condition $\mathbf{F}2$ if $\tilde f b' = 0$ and $\psi(b') \geq \varphi(b) \geq 0$,
(iv) the condition $\mathbf{E}1$ if either $\mathrm{wt}(b) < \varphi(b) < -1$ or $\psi(b') \leq \varphi(b)$,
(v) the condition $\mathbf{E}1'$ if $\psi(b') > \varphi(b) \geq 0$,
(vi) the condition $\mathbf{E}2$ if either $\varphi(b) = -1, \tilde f^{\varphi(b')+1} b' = 0$, and $0 \leq \varphi(b') \leq \varepsilon(b) - 1$ or $b \in B^{br}, \nexists a' \in B'^{br}$ such that $b' = \tilde e^l a'$ for some $l \geq 0$, and $0 \leq \varphi(b') \leq -\mathrm{wt}(b) - 2$,
(vii) the condition $\mathbf{E}2'$ if either $\varphi(b) = -1, \tilde e b \in B$, and $-\mathrm{wt}(b) - 1 \leq \varphi(b') \leq -\mathrm{wt}(b) - 2 + \varepsilon(b')$, or $\varphi(b) < 0, \varepsilon(b) = 0$, and $-\mathrm{wt}(b) - 1 \leq \varphi(b') \leq -\mathrm{wt}(b) - 2 + \varepsilon(b')$.

Here we understand $0 \otimes b' = b \otimes 0 = 0$ and define $\tilde e 0 = \tilde f 0 = 0$. Finally, set

$$\begin{aligned}
\varepsilon(b\otimes b') &= \infty,\ \text{ if } \tilde f \tilde e^{s+1}(b\otimes b') = \tilde e^s(b\otimes b') \text{ for all } s\geq 0,\\
&= \min\{s\in \mathbf{Z}^+ : \tilde f \tilde e^{s+1}(b\otimes b') \neq \tilde e^s(b\otimes b')\},\ \text{ otherwise}
\end{aligned}$$

and

$$\varphi(b\otimes b') = \mathrm{wt}(b\otimes b') + \varepsilon(b\otimes b').$$

2.2. The next few subsections are devoted to the proof the following theorem.

Theorem. Let B, B' be branched crystals and assume that B_i, $i \in I$ and B'_j, $j \in J$ are the indecomposable components of B and B', respectively. Then $B \otimes B'$ is a branched crystal and $B \otimes B' = \oplus_{i,j} B_i \otimes B'_j$.

Assuming that $B \otimes B'$ is a branched crystal the second statement is proved as follows. Let $\tilde e_{B,B'} : (B \times B') \sqcup \{0\} \to (B \times B') \sqcup \{0\}$ and $\tilde e_{B_i,B'_j} : (B_i \times B'_j) \sqcup \{0\} \to (B_i \times B'_j) \sqcup \{0\}$ be defined as above. It is clear that for all $(b, b') \in B_i \times B'_j$ we have $\tilde e_{B,B'}(b \otimes b') = \tilde e_{B_i,B'_j}(b \otimes b') \in (B_i \times B'_j) \sqcup \{0\}$ (and similarly for $\tilde f$). It is now immediate that $B \otimes B' = \oplus_{i,j} B_i \otimes B'_j$.

2.3. We proceed now with the proof of Theorem 2.2. It is clear that the condition (i) of Definition 1.1 is satisfied.

For (ii), let $b \in B$, $b' \in B'$ be such that $\tilde f(b \otimes b') \in B \otimes B'$. If b, b' satisfy $\mathbf{F}1$ (resp. $\mathbf{F}1'$) then it is obvious that $\tilde f b, b'$ (resp. $b, \tilde f b'$) satisfy $\mathbf{E}1$ (resp. $\mathbf{E}1'$). If b, b' satisfy $\mathbf{F}2$, we have

$$\varphi(\tilde f^{\varphi(b)+1} b) = -1, \quad \varphi(\tilde e^{\varphi(b)} b') = \varphi(b), \quad \varepsilon(\tilde f^{\varphi(b)+1} b) = \varepsilon(b) + \varphi(b) + 1.$$

Here we used Corollary 1.3(iii). Hence $\tilde f^{\varphi(b)+1} b, \tilde e^{\varphi(b)} b'$ satisfy $\mathbf{E}2$ and $\tilde e \tilde f(b \otimes b') = b \otimes b'$. This proves (ii).

Let us now check condition (iv). Let $b \in B, b' \in B'$ and $\bar{B}, \bar{B}'$ be the indecomposable components they belong to respectively. It is evident that $\tilde{e}^l(b\otimes b') \in (\bar{B}\times\bar{B}')\sqcup\{0\}$. We claim that $\varepsilon(b\otimes b') \in \mathbf{Z}^+$ and, therefore, condition (iv) of Definition 1.1 is satisfied. To prove the claim it suffices to check that there exists $l \in \mathbf{Z}^+$ such that $\tilde{e}^l(b\otimes b') = 0$. If that was not the case we would have $\mathrm{wt}(\tilde{e}^l(b\otimes b')) = \mathrm{wt}(b\otimes b')+2l$ for all $l \in \mathbf{Z}^+$. But this is impossible since the weight function is clearly bounded from above on $\bar{B}\times\bar{B}'$.

2.4. To prove (iii) and (v), we begin with the following proposition which characterizes the branch points in $B \otimes B'$.

Proposition. *Suppose that B, B' are branched crystals. Let $b \in B$, $b \in B'$. Then*

(a) $\tilde{f}(b \otimes b') = 0$ iff b, b' satisfy **F**2 *and $\tilde{f}^{\varphi(b)+1}b = 0$.*
(b) Let b, b' be such that $\tilde{e}(b \otimes b') \neq 0$. Then $\tilde{f}\tilde{e}(b \otimes b') \neq b \otimes b'$ iff one of the following happens:
  (i) *$b' \in B'^{br}$ and b, b' satisfy* **E**$1'$,
  (ii) *$b \in B^{br}$ and b, b' satisfy* **E**2,
  (iii) *b, b' satisfy* **E**$2'$,
  (iv) *$\varphi(b) = -\mathrm{wt}(b') - 2 \geq 0$ and $\varphi(b') < -1$ (or equivalently, $\psi(b') = \varphi(b) + 1$ and $\varphi(b') < -1$).*

Proof. Part (a) is obvious. For part (b), we first prove that $\tilde{f}\tilde{e}(b\otimes b') \neq b\otimes b'$ if (i)–(iv) happens. This is obvious in cases (i) and (ii). For (iii) it is also obvious unless $-\mathrm{wt}(b) - \mathrm{wt}(b') - 2 = 0$. In this case we need to show that **F**$1'$ does not apply to $b\otimes \tilde{e}b'$. This is true since $\varphi(b) < 0$. In case (iv) we see that **E**$1'$ applies to b, b' and, hence, we need to show that **F**$1'$ does not apply to $b \otimes \tilde{e}b'$. But this is clear since $\varphi(b') < -1$ implies $\psi(\tilde{e}b') = \psi(b') - 2$. The converse is proved similarly. □

Corollary. *Let $b \in B$, $b' \in B'$ satisfy $\tilde{e}(b \otimes b') \neq 0$ and $\tilde{f}\tilde{e}(b \otimes b') \neq b \otimes b'$. Then $\tilde{f}^l(b \otimes b') \in B \otimes B'$ for all $l \geq 0$.*

Proof. Suppose that b, b' satisfy condition (b)(i) of the proposition. Then, by Proposition 1.3 we see that $\tilde{f}^l b' \in B'$ for all $l \geq 0$. Since $\psi(f^l b') > \psi(b)$ it follows that b and $f^l b'$ satisfy **F**$1'$ for all $l \geq 0$ and hence the corollary follows in this case. Suppose now that b, b' satisfy **E**2. Then, $\tilde{f}^l b \in B$ for all $l > 0$ and $\tilde{f}^l b$, b' satisfy **F**1 and hence $\tilde{f}^l(b \otimes b') = \tilde{f}^l b \otimes b' \in B$. A similar computation proves the result if b, b' satisfy condition (b)(iii). Finally, if we have $\varphi(b) = -\mathrm{wt}(b') - 2 \geq 0$ and $\varphi(b') < -1$, then by Definition 1.1(v) we have that $\tilde{f}^l b' \in B'$ for all $l \geq 0$ and since

$$\psi(b') = -\mathrm{wt}(b') - 1 \geq -\mathrm{wt}(b') - 2 = \varphi(b) \geq 0,$$

it follows that b and $\tilde{f}^l b'$ always satisfy **F**$1'$ and the corollary is proved. □

2.5. We can now prove that condition (v) of Definition 1.1 is satisfied. Let $b \otimes b'$ be such that $\tilde{e}(b \otimes b') \neq 0$ and $\tilde{f}\tilde{e}(b \otimes b') \neq b \otimes b'$. Suppose $b' \in B'^{br}$ and b, b' satisfy **E**$1'$. Then $b, \tilde{e}^l b'$ satisfy **E**$1'$ for all $0 \leq l \leq -\varphi(b) - \mathrm{wt}(b') - 1$ and **F**$1'$ for all $2 \leq l \leq -\varphi(b) - \mathrm{wt}(b') - 1$. Therefore,

$$\tilde{e}^l(b \otimes b') = b \otimes \tilde{e}^l b' \neq 0, \quad \text{for all} \quad 1 \leq l \leq -\varphi(b) - \mathrm{wt}(b') - 1$$

$$\tilde{f}\tilde{e}^{l+1}(b \otimes b') = \tilde{e}^l(b \otimes b') \quad \text{for all} \quad 1 \leq l \leq -\varphi(b) - \mathrm{wt}(b') - 2.$$

Next, using **E**1 and **F**1 we have

$$\tilde{e}^{-\varphi(b)-\mathrm{wt}(b')-1+l}(b \otimes b') = \tilde{e}^l b \otimes \tilde{e}^{-\varphi(b)-\mathrm{wt}(b')-1} b' \neq 0 \quad \text{for all} \quad 0 \leq l \leq \varepsilon(b),$$

$$\tilde{f}\tilde{e}^{-\varphi(b)-\mathrm{wt}(b')-1+l}(b \otimes b') = \tilde{e}^{-\varphi(b)-\mathrm{wt}(b')-2+l}(b \otimes b') \quad \text{for all} \quad 1 \leq l \leq \varepsilon(b),$$

and since $\tilde{e}^{\varepsilon(b)+1}b = 0$,

$$\tilde{e}^{-\varphi(b)-\mathrm{wt}(b')+\varepsilon(b)}(b \otimes b') = 0.$$

This proves that
$$\varepsilon(\tilde{e}(b\otimes b')) = -\mathrm{wt}(b) - \mathrm{wt}(b') - 2,$$
i.e that $\varphi(\tilde{e}(b\otimes b')) = 0$. A similar computation takes care of the case $b \in B^{br}$ and b, b' satisfy $\mathbf{E}2$. Now suppose b, b' satisfy $\mathbf{E}2'$, then
$$\tilde{e}(b\otimes b') = \tilde{f}^{-\mathrm{wt}(b')-\mathrm{wt}(b)-2} b \otimes \tilde{e}^{-\mathrm{wt}(b')-\mathrm{wt}(b)-1} b',$$
and using $\mathbf{E}1$ and $\mathbf{F}1$, it follows that
$$\tilde{e}^{l+1}(b\otimes b') = \tilde{e}^l \tilde{f}^{-\mathrm{wt}(b')-\mathrm{wt}(b)-2} b \otimes \tilde{e}^{-\mathrm{wt}(b')-\mathrm{wt}(b)-1} b' \quad \text{for all} \quad 0 \le l \le -\mathrm{wt}(b') - \mathrm{wt}(b) - 2,$$
$$\tilde{f}\tilde{e}^{l+1}(b\otimes b') = \tilde{e}^l(b\otimes b') \quad \text{for all} \quad 1 \le l \le -\mathrm{wt}(b') - \mathrm{wt}(b) - 2.$$
Since none of the rules $\mathbf{E}1, \mathbf{E}1', \mathbf{E}2, \mathbf{E}2'$ apply to $b \otimes \tilde{e}^{-\mathrm{wt}(b')-\mathrm{wt}(b)-1} b'$ we conclude that $\varepsilon(\tilde{e}(b\otimes b')) = -\mathrm{wt}(b') - \mathrm{wt}(b) - 2$, i.e. $\varphi(\tilde{e}(b\otimes b')) = 0$. Finally, consider the case $\varphi(b) = -\mathrm{wt}(b') - 2 \ge 0$ and $\varphi(b') < -1$. This time, we find that $\tilde{e}(b\otimes b') = b \otimes \tilde{e}b'$ and using $\mathbf{E}1$ and $\mathbf{F}1$ we have
$$\tilde{e}^{l+1}(b\otimes b') = \tilde{e}^l b \otimes \tilde{e}b' \ne 0 \quad \text{for all} \quad 0 \le l \le \varepsilon(b),$$
$$\tilde{f}\tilde{e}^{l+1}(b\otimes b') = \tilde{e}^l(b\otimes b') \quad \text{for all} \quad 1 \le l \le \varepsilon(b),$$
and $\tilde{e}^{\varepsilon(b)+2}(b\otimes b') = 0$.

Hence $\varepsilon(\tilde{e}(b\otimes b')) = \varepsilon(b)$. We must check that $\varepsilon(b) = -\mathrm{wt}(b) - \mathrm{wt}(b') - 2$, but this follows from $\varphi(b') < -1$ and $\psi(b') = \varphi(b) + 1$.

To complete the proof of (v), we must show that if l is minimal such that $\tilde{f}^l(b\otimes b') = 0$ then $l = \varphi(b\otimes b') + 1$. It clearly suffices to consider the case $l = 1$. By Proposition 2.4 we know that b, b' satisfy $\mathbf{F}2$. Then using $\mathbf{E}1'$ and $\mathbf{F}1'$ we see that
$$\tilde{e}^l(b\otimes b') = b \otimes \tilde{e}^l b' \ne 0, \quad \text{for all} \quad 0 \le l \le \varepsilon(b') - \varphi(b),$$
and
$$\tilde{f}\tilde{e}^l(b\otimes b') = \tilde{e}^{l-1}(b\otimes b'), \quad \text{for all} \quad 1 \le l \le \varepsilon(b') - \varphi(b).$$
Then, using $\mathbf{E}1$ and $\mathbf{F}1$ we get
$$\tilde{e}^{\varepsilon(b')-\varphi(b)+l}(b\otimes b') = \tilde{e}^l b \otimes \tilde{e}^{\varepsilon(b')-\varphi(b)} b' \ne 0, \quad \text{for all} \quad 0 \le l \le \varepsilon(b),$$
$$\tilde{f}\tilde{e}^{\varepsilon(b')-\varphi(b)+l}(b\otimes b') = \tilde{e}^{\varepsilon(b')-\varphi(b)+l-1}(b\otimes b'), \quad \text{for all} \quad 1 \le l \le \varepsilon(b),$$
and $\tilde{e}^{\varepsilon(b')-\varphi(b)+\varepsilon(b)+1}(b\otimes b') = 0$. Hence, $\varepsilon(b\otimes b') = \varepsilon(b') - \varphi(b) + \varepsilon(b) = -\mathrm{wt}(b') - \mathrm{wt}(b)$, i.e., $\varphi(b\otimes b') = 0$.

2.6. Finally, we must prove that condition (iii) of Definition 1.1 holds:

Proposition. *Suppose that $b_i \in B$, $b_i' \in B'$ are distinct elements such that $\tilde{e}(b_i \otimes b_i') \in B \otimes B'$ and $\tilde{f}\tilde{e}(b_i \otimes b_i') \ne (b_i \otimes b_i')$ for $i = 1, 2$. Then*
$$\tilde{e}(b_1 \otimes b_1') \ne \tilde{e}(b_2 \otimes b_2').$$

Proof. Assume $\tilde{e}(b_1 \otimes b_1') = \tilde{e}(b_2 \otimes b_2')$. We consider four cases depending on the various possibilities for the pairs b_1, b_1' given Proposition 2.4.

Case 1. b_1, b_1' satisfy $\mathbf{E}1'$ and $b_1' \in B'^{br}$. If b_2, b_2' also satisfy $\mathbf{E}1'$ and $b_2' \in B'^{br}$ then we get $b_1 = b_2$ and $\tilde{e}b_1' = \tilde{e}b_2'$ which implies that $b_1' = b_2'$. If b_2, b_2' satisfy $\mathbf{E}2$ with $b_2 \in B^{br}$, then we get
$$b_1 = \tilde{e}^{\varphi(b_2')+1} b_2, \quad \tilde{e}b_1' = \tilde{f}^{\varphi(b_2')} b_2',$$

i.e., $b_2' = \tilde{e}^{\varphi(b_2')+1}b_1'$. But $\varphi(b_2) = \mathrm{wt}(b_2) \leq -2$ by Proposition 1.3, so we get a contradiction to the fact that b_2, b_2' satisfy $\mathbf{E}2$. Next, suppose that b_2, b_2' satisfy $\mathbf{E}2'$. This gives,

$$b_1 = \tilde{f}^{-(\mathrm{wt}(b_2)+\mathrm{wt}(b_2')+2)}b_2,$$

which gives $\varphi(b_1) = \varphi(b_2) + \mathrm{wt}(b_2) + \mathrm{wt}(b_2') + 2 < 0$, contradicting the fact that $\varphi(b_1) \geq 0$. Finally, suppose that b_2, b_2' satisfy $\varphi(b_2) = -\mathrm{wt}(b_2) - 2 \geq 0$ and $\varphi(b_2') < -1$. In particular, this means that b_2, b_2' satisfy $\mathbf{E}1'$ and hence we get $b_1 = b_2$ and $\tilde{e}b_1' = \tilde{e}b_2'$. Since $b_1' \in B'^{br}$ this implies that $b_2' \notin B'^{br}$ and so $\tilde{f}\tilde{e}b_1' = b_2'$, which gives

$$\varphi(b_2') = \mathrm{wt}(b_1') + \varepsilon(\tilde{f}\tilde{e}b_1') = \mathrm{wt}(b_1') + \varepsilon(\tilde{e}b_1') + 1 = -1,$$

since $\varepsilon(\tilde{e}b_1') = -\mathrm{wt}(b_1') - 2$ which contradicts $\varphi(b_2') < -1$.

Case 2. b_1, b_1' satisfy $\mathbf{E}2$ and $b_1 \in B^{br}$. If b_2 and b_2' also satisfy $\mathbf{E}2$ and $b_2 \in B^{br}$, assume without loss of generality that $\varphi(b_1') \geq \varphi(b_2')$. Since, $\tilde{e}^{\varphi(b_1')+1}b_1 = \tilde{e}^{\varphi(b_2')+1}b_2$, we get by using Proposition 1.3 that $\tilde{e}b_1 = \tilde{f}^{\varphi(b_1')-\varphi(b_2')}\tilde{e}b_2$. But this means that,

$$0 = \varphi(\tilde{e}b_1) = -\varphi(b_1') + \varphi(b_2'),$$

which implies $b_1 = b_2$. Since we also have $\tilde{f}^{\varphi(b_1')-\varphi(b_2')}b_1' = b_2'$, it follows that $b_1' = b_2'$.

Assume next that b_2 and b_2' satisfy $\mathbf{E}2'$. We get $\tilde{e}^{\varphi(b_1')+1}b_1 = \tilde{f}^{-\mathrm{wt}(b_2')-\mathrm{wt}(b_2)-2}b_2$ and hence $b_2 = \tilde{e}^{\varphi(b_1')-\mathrm{wt}(b_2')-\mathrm{wt}(b_2)-1}b_1$. This implies that

$$\varphi(b_2) = \varphi(\tilde{e}b_1) + \varphi(b_1') - \mathrm{wt}(b_2') - \mathrm{wt}(b_2) - 2 \geq 0,$$

which is again a contradiction. Finally, let us assume that $\varphi(b_2) = -\mathrm{wt}(b_2') - 2 \geq 0$ and $\varphi(b_2') < -1$, then $\tilde{e}(b_2 \otimes b_2') = b_2 \otimes \tilde{e}b_2'$ and so we get $\tilde{e}^{\varphi(b_1')+1}b_1 = b_2$, which gives $\varphi(b_2) = \varphi(b_1') < 0$ which is again a contradiction.

Case 3. b_1 and b_1' satisfy $\mathbf{E}2'$. If b_2 and b_2' also satisfy $\mathbf{E}2'$, then we get $\tilde{f}^{-\mathrm{wt}(b_1)-\mathrm{wt}(b_1')-2}b_1 = \tilde{f}^{-\mathrm{wt}(b_2)-\mathrm{wt}(b_2')-2}b_2$. Since $\mathrm{wt}(b_1) + \mathrm{wt}(b_1') = \mathrm{wt}(b_2) + \mathrm{wt}(b_2')(= l)$, it follows immediately that $b_1 = b_2$. Since $\tilde{e}^{l-1}b_1' = \tilde{e}^{l-1}b_2'$ and $\varphi(b_i') \geq 0$, it follows from Corollary 1.3(iii) that $b_1' = b_2'$.

If b_2 and b_2' satisfy $\varphi(b_2) = -\mathrm{wt}(b_2') - 2 \geq 0$ and $\varphi(b_2') < -1$, we get $b_2 = \tilde{f}^{-\mathrm{wt}(b_1)-\mathrm{wt}(b_1')-2}b_1$. This means that $\tilde{e}^{-\mathrm{wt}(b_1)-\mathrm{wt}(b_1')-2}b_2 = b_1$ and that $\tilde{e}^{l}b_2 \notin B^{br}$ for all $0 \leq l \leq -\mathrm{wt}(b_1) - \mathrm{wt}(b_1') - 2$ and hence $\varphi(b_1) = \varphi(b_2) - \mathrm{wt}(b_1) - \mathrm{wt}(b_1') - 2 \geq 0$ which is a contradiction.

Case 4 The remaining case is when b_i and b_i' satisfy $\varphi(b_i) = -\mathrm{wt}(b_i') - 2 \geq 0$ and $\varphi(b_i') < -1$, $i = 1, 2$. Again we get $b_1 = b_2$ and $\tilde{e}b_1' = \tilde{e}b_2'$. This means that the only time that it is not obvious that $b_1' = b_2'$ is when $b_1' \in B'^{br}$ and $b_2' \notin B'^{br}$(or vice versa). But then we get $b_2' = \tilde{f}\tilde{e}b_1'$ and so $\varphi(b_2') = -1$ since $\varphi(\tilde{e}b_1') = 0$. But this is again a contradiction. □

The proof of Theorem 2.2 is now complete.

2.7.

Proposition. *Let B, B' be branched crystals and let $b \in B$, $b' \in B'$. Then $\tilde{e}(b \otimes b') = 0$ iff one of the following holds:*

(i) $\varepsilon(b) = 0, b \notin B^{br}$ *(resp.* $\varepsilon(b') = 0, b' \notin B'^{br}$*), and* b, b' *satisfy* $\mathbf{E}1$ *(resp.* $\mathbf{E}1'$*),or*
(ii) b, b' *do not satisfy* $\mathbf{E}1$, $\mathbf{E}2$, $\mathbf{E}1'$, $\mathbf{E}2'$.

Moreover, if (ii) holds, then either $\varepsilon(b) = 0$ *or* $\varphi(b) = -1$

Proof. Observe that if b, b' satisfy **E**1 (resp. **E**1′) then $\tilde{e}(b \otimes b') = 0$ iff $\tilde{e}b = 0$ (resp. $\tilde{e}b' = 0$). It is also easy to see that if b, b' satisfy none of the conditions $\mathbf{E}1, \mathbf{E}1', \mathbf{E}2, \mathbf{E}2'$, then either $\varepsilon(b) = 0$ or $\varphi(b) = -1$. It remains to show that if b, b' satisfy **E**2 or **E**2′, then $\tilde{e}(b \otimes b') \neq 0$. If b, b' satisfy **E**2, observe that the upper bound for $\varphi(b')$ implies that $\tilde{e}^{\varphi(b')+1}b \neq 0$ and we are done. If b, b' satisfy **E**2′ then, since $\varphi(b) < 0$, we have $\tilde{f}^l b \neq 0$ for all $l \geq 0$. We need to show that $\tilde{e}^{-\mathrm{wt}(b)-\mathrm{wt}(b')-1}b' \neq 0$. It suffices to check that $-\mathrm{wt}(b) - \mathrm{wt}(b') - 1 \leq \varepsilon(b')$. But this is equivalent to $\varphi(b') \geq -\mathrm{wt}(b) - 1$. □

2.8. Let B, B' be indecomposable branched crystals and $\mathbf{B} = B \otimes B'$. We conclude this section by writing down the indecomposable components of $\mathbf{B}$ in a case by case fashion. By Proposition 1.6 it suffices to compute the sets $\mathbf{B}^{br}, \mathbf{B}^{br,\sigma}, \mathbf{B}^{hw}$ which is done by using Propositions 2.4 and 2.7 and Lemma 1.7. In the following we use the notation of Theorem 1.4. Also we understand $p \in \mathbf{Z}^+$ and use the convention $B(M(-1)) = B(T(-1))$ when convenient.

Case 1. Suppose that $B = B(M(s)), B' = B(V(r))$.

$$\begin{aligned}
&\mathbf{B}_{b_{s+1}\otimes b'_{r-p}} \cong B(T(r+s-2p)), 2s+2 \leq 2p \leq r+s+1, \text{ if } s \geq 0,\\
&\mathbf{B}_{b_0\otimes b'_{r+s+1-p}} \cong B(T(r+s-2p)), 0 \leq 2p \leq r+s+1, \text{ if } s < 0,\\
&\mathbf{B}_{b_0\otimes b'_p} \cong B(M(r+s-2p)), 0 \leq p \leq \min\{r,s\},\\
&\mathbf{B}_{b_0\otimes b'_p} \cong B(M(r+s-2p)), r+s+2 \leq p \leq r.
\end{aligned}$$

Case 2. $B = B(T(s)), B' = B(V(r))$.

$$\begin{aligned}
&\mathbf{B}_{b_{(s+1)}\otimes b'_{r-p}} \cong B(T(r+s-2p)), 2s+2 \leq 2p \leq r+s+1,\\
&\mathbf{B}_{b_0\otimes b'_{r-p}} \cong B(T(r+s-2p)), 0 \leq 2p \leq r+s+1, \text{ if } s < r,\\
&\mathbf{B}_{b_0\otimes b'_{r-p}} \cong B(T(r+s-2p)), 0 \leq p \leq r, \text{ if } s \geq r.
\end{aligned}$$

Case 3. $B = B(M(s)^\sigma), B' = B(V(r))$.

$$\begin{aligned}
&\mathbf{B}_{b_0\otimes b'_{r-p}} \cong B(T(r+s-2p)), 2s+2 \leq 2p \leq r+s+1,\\
&\mathbf{B}_{b_0\otimes b'_{r-p}} \cong B(M(r+s-2p)^\sigma), 0 \leq p \leq \min\{r,s\}.
\end{aligned}$$

Case 4. $B = B(V(r)), B' = B(M(s)^\sigma)$.

$$\begin{aligned}
&\mathbf{B}_{b_{r+s-2p}\otimes b'_{p-s}} \cong B(T(r+s-2p)), 2s+2 \leq 2p \leq r+s,\\
&\mathbf{B}_{b_{r-p}\otimes b'_0} \cong B(M(r+s-2p)^\sigma), 0 \leq p \leq \min\{r,s\},\\
&\mathbf{B}_{b_0\otimes b'_{(r-s-1)/2}} \cong B(M(-1)).
\end{aligned}$$

Case 5. $B' = B(M(r)), B = B(M(s))$.

$$\begin{aligned}
&\mathbf{B}_{b_{r+s-2p}\otimes b'_{p+1}} \cong B(T(r+s-2p)), 2s+2 \le 2p \le r+s,\\
&\mathbf{B}_{b_{r+1}\otimes b'_{s-p}} \cong B(T(r+s-2p)), 2r+2 \le 2p \le r+s+1, \text{ if } r \ge 0,\\
&\mathbf{B}_{b_0\otimes b'_{r+s+1-p}} \cong B(T(r+s-2p)), 0 \le 2p \le r+s+1, \text{ if } r < 0,\\
&\mathbf{B}_{b_0\otimes b'_p} \cong B(M(r+s-2p)), 0 \le p \le \min\{r,s\},\\
&\mathbf{B}_{b_0\otimes b'_p} \cong B(M(r+s-2p)), p \ge r+s+2, \text{ if } r < 0,\\
&\mathbf{B}_{b_{r+1}\otimes b'_{p-r-1}} \cong B(M(r+s-2p)), p \ge \max\{r,s\}+1, \text{ if } r \ge 0,\\
&\mathbf{B}_{b_p\otimes b'_0} \cong B(M(r+s-2p)), r+s+2 \le p \le r, \text{ if } s < 0,\\
&\mathbf{B}_{b_0\otimes b'_{(r+s+1)/2}} \cong B(M(-1)), \text{ if } r \ge s.
\end{aligned}$$

Case 6. $B = B(M(r)), B' = B(M(s)^\sigma)$.

$$\begin{aligned}
&\mathbf{B}_{b_{r-p}\otimes b'_0} \cong B(T(r+s-2p)), 0 \le p \le s,\\
&\mathbf{B}_{b_{r+s-2p}\otimes b'_{p-s}} \cong B(T(r+s-2p)), 2s+2 \le 2p \le r+s,\\
&\mathbf{B}_{b_{r+1}\otimes b'_{(s-p)}} \cong B(T(r+s-2p)), 2r+2 \le 2p \le r+s+1, \text{ if } r \ge 0,\\
&\mathbf{B}_{b_0\otimes b'_{(r+s+1-p)}} \cong B(T(r+s-2p)), 0 \le 2p \le r+s+1, \text{ if } r < 0,\\
&\mathbf{B}_{b_{r+1}\otimes b'_{p-r-s-2}} \cong B(M(r+s-2p)), p \ge r+s+2, \text{ if } r \ge 0,\\
&\mathbf{B}_{b_0\otimes b'_{(p)}} \cong B(M(r+s-2p)), r+s+2 \le p \le s, \text{ if } r < 0,\\
&\mathbf{B}_{b_0\otimes b'_{p-s-1}} \cong B(M(r+s-2p)), p \ge s+1, \text{ if } r < 0,\\
&\mathbf{B}_{b_0\otimes b'_{(r-s-1)/2}} \cong M(-1), \text{ if } r \ge 0.
\end{aligned}$$

Case 7. $B = B(M(s)^\sigma), B' = B(M(r))$.

$$\begin{aligned}
&\mathbf{B}_{b_0\otimes b'_{r-p}} \cong B(T(r+s-2p)), 0 \le 2p \le r+s+1, \text{ if } s < r,\\
&\mathbf{B}_{b_0\otimes b'_{r-p}} \cong B(T(r+s-2p)), 0 \le p \le r, \text{ if } s \ge r,\\
&\mathbf{B}_{b_{(r+s-2p)}\otimes b'_{p+1}} \cong B(T(r+s-2p)), 2r+2 \le 2p \le r+s,\\
&\mathbf{B}_{b_0\otimes b'_{p-s-1}} \cong B(M(r+s-2p)), p \ge r+s+2, \text{ if } r \ge 0,\\
&\mathbf{B}_{b_0\otimes b'_{p-s-1}} \cong B(M(r+s-2p)), p \ge s+1, \text{ if } r < 0,\\
&\mathbf{B}_{b_{(p)}\otimes b'_0} \cong B(M(r+s-2p)), s+r+2 \le p \le s,\\
&\mathbf{B}_{b_{(0)}\otimes b'_{(r+s+1)/2}} \cong B(M(-1)), \text{ if } s \ge r.
\end{aligned}$$

Case 8. $B = B(M(r)^\sigma), B' = B(M(s)^\sigma)$.

$$
\begin{aligned}
&\mathbf{B}_{b_{(r+s-2p)}\otimes b'_{p-s}} \cong B(T(r+s-2p)), 2s+2 \le 2p \le r+s,\\
&\mathbf{B}_{b_0\otimes b'_{(s-p)}} \cong B(T(r+s-2p)), 2r+2 \le 2p \le r+s+1,\\
&\mathbf{B}_{b_{(r-p)}\otimes b'_0} \cong B(M(r+s-2p)^\sigma), 0 \le p \le \min\{r,s\},\\
&\mathbf{B}_{b_0\otimes b'_{(p-r-1)}} \cong B(M(r+s-2p)), \max\{r,s\}+1 \le p \le r+s+1,\\
&\mathbf{B}_{b_0\otimes b'_{p-r-s-2}} \cong B(M(r+s-2p)), p \ge r+s+2,\\
&\mathbf{B}_{b_{(0)}\otimes b'_{(r-s-1)/2}} \cong B(M(-1)).
\end{aligned}
$$

Case 9. $B = B(M(s)), B' = B(T(r))$.

$$
\begin{aligned}
&\mathbf{B}_{b_{s-p}\otimes b'_0} \cong B(T(r+s-2p)), 0 \le p \le \min\{r,s\},\\
&\mathbf{B}_{b_{r+s-2p}\otimes b'_{p-r}} \cong \mathbf{B}_{b_{r+s-2p}\otimes b'_{(p+1)}} \cong B(T(r+s-2p)), 2r+2 \le 2p \le r+s,\\
&\mathbf{B}_{b_{s+1}\otimes b'_{(r-p)}} \cong B(T(r+s-2p)), 2s+2 \le 2p \le r+s+1,\\
&\mathbf{B}_{b_0\otimes b'_{(r+s+1-p)}} \cong B(T(r+s-2p)), 0 \le 2p \le r+s+1, \text{ if } s<0,\\
&\mathbf{B}_{b_{s+1}\otimes b'_{p-r-s-2}} \cong B(M(r+s-2p)), p \ge r+s+2, \text{ if } s \ge 0,\\
&\mathbf{B}_{b_{s+1}\otimes b'_{(p-s-1)}} \cong B(M(r+s-2p)), p \ge \max\{r,s\}+1, \text{ if } s \ge 0,\\
&\mathbf{B}_{b_0\otimes b'_{(p)}} \cong B(M(r+s-2p)), p \ge r+s+2, \text{ if } s<0,\\
&\mathbf{B}_{b_0\otimes b'_{(p-s-1)}} \cong B(M(r+s-2p)), p \ge s+1, \text{ if } s<0,\\
&\mathbf{B}_{b_0\otimes b'_{(s-r-1)/2}} \cong \mathbf{B}_{b_0\otimes b'_{((r+s+1)/2)}} \cong B(M(-1)), \text{ if } s \ge r.
\end{aligned}
$$

Case 10. $B = B(T(r)), B' = B(M(s))$.

$$
\begin{aligned}
&\mathbf{B}_{b_0\otimes b'_{s-p}} \cong B(T(r+s-2p)), 0 \le 2p \le r+s+1, \text{ if } s>r,\\
&\mathbf{B}_{b_0\otimes b'_{s-p}} \cong B(T(r+s-2p)), 0 \le p \le s, \text{ if } s \le r,\\
&\mathbf{B}_{b_{(r+1)}\otimes b'_{s-p}} \cong B(T(r+s-2p)), 2r+2 \le 2p \le r+s+1,\\
&\mathbf{B}_{b_{(r+s-2p)}\otimes b'_{p+1}} \cong B(T(r+s-2p)), 2s+2 \le 2p \le r+s,\\
&\mathbf{B}_{b_0\otimes b'_{p-r-1}} \cong B(M(r+s-2p)), p \ge r+s+2, \text{ if } s \ge 0,\\
&\mathbf{B}_{b_0\otimes b'_{p-r-1}} \cong B(M(r+s-2p)), p \ge r+1, \text{ if } s<0,\\
&\mathbf{B}_{b_{(r+1)}\otimes b'_{p-r-1}} \cong B(M(r+s-2p)), p \ge \max\{r,s\}+1,\\
&\mathbf{B}_{b_{(p)}\otimes b'_0} \cong B(M(r+s-2p)), s+r+2 \le p \le r,\\
&\mathbf{B}_{b_{(0)}\otimes b'_{(r+s+1)/2}} \cong B(M(-1)), \text{ if } s \le r.
\end{aligned}
$$

Case 11. $B = B(T(r)), B' = B(T(s))$.

$$\begin{aligned}
&\mathbf{B}_{b_{(r+s-2p)}\otimes b'_{(p+1)}} \cong \mathbf{B}_{b_{(r+s-2p)}\otimes b'_{p-s}} \cong B(T(r+s-2p)), 2s+2 \leq 2p \leq r+s,\\
&\mathbf{B}_{b_{(r-p)}\otimes b'_0} \cong B(T(r+s-2p)), 0 \leq p \leq s,\\
&\mathbf{B}_{b_{(r+1)}\otimes b'_{(s-p)}} \cong \mathbf{B}_{b_0\otimes b'_{(s-p)}} \cong B(T(r+s-2p)), 2r+2 \leq 2p \leq r+s+1,\\
&\mathbf{B}_{b_{(r+1)}\otimes b'_{(p-r-1)}} \cong \mathbf{B}_{b_0\otimes b'_{(p-r-1)}} \cong B(M(r+s-2p)), p \geq s+1,\\
&\mathbf{B}_{b_{(r+1)}\otimes b'_{p-r-1}} \cong \mathbf{B}_{b_0\otimes b'_{p-r-1}} \cong (M(r+s-2p)), p \geq r+1,\\
&\mathbf{B}_{b_{(0)}\otimes b'_{((r+s+1)/2)}} \cong \mathbf{B}_{b_{(0)}\otimes b'_{(r-s-1)/2}} \cong B(M(-1)), \text{ if } r \geq s.
\end{aligned}$$

Case 12. $B = B(T(r)), B' = B(M(s)^\sigma)$.

$$\begin{aligned}
&\mathbf{B}_{b_{(r+s-2p)}\otimes b'_{p-s}} \cong B(T(r+s-2p)), 2s+2 \leq 2p \leq r+s,\\
&\mathbf{B}_{b_{(r-p)}\otimes b'_0} \cong B(T(r+s-2p)), 0 \leq p \leq \max\{r,s\},\\
&\mathbf{B}_{b_0\otimes b'_{(s-p)}} \cong \mathbf{B}_{b_{(r+1)}\otimes b'_{(s-p)}} \cong B(T(r+s-2p)), 2r+2 \leq 2p \leq r+s+1,\\
&\mathbf{B}_{b_0\otimes b'_{(p-r-1)}} \cong B(M(r+s-2p)), p \geq r+1, \text{ if } r \geq s,\\
&\mathbf{B}_{b_0\otimes b'_{(p-r-1)}} \cong B(M(r+s-2p)), s+1 \leq p \leq r+s+1, \text{ if } s > r,\\
&\mathbf{B}_{b_0\otimes b'_{p-r-s-2}} \cong \mathbf{B}_{b_{(r+1)}\otimes b'_{p-r-s-2}} \cong B(M(r+s-2p)), p \geq r+s+2,\\
&\mathbf{B}_{b_{(0)}\otimes b'_{(r-s-1)/2}} \cong B(M(-1)), \text{ if } r \geq s.
\end{aligned}$$

Case 13. $B = B(M(s)^\sigma), B' = B(T(r))$.

$$\begin{aligned}
&\mathbf{B}_{b_{(r+s-2p)}\otimes b'_{(p+1)}} \cong \mathbf{B}_{b_{(r+s-2p)}\otimes b'_{p-r}} \cong B(T(r+s-2p)), 2r+2 \leq 2p \leq r+s,\\
&\mathbf{B}_{b_{(s-p)}\otimes b'_0} \cong B(T(r+s-2p)), 0 \leq p \leq r,\\
&\mathbf{B}_{b_0\otimes b'_{(r-p)}} \cong B(T(r+s-2p)), 2s+2 \leq 2p \leq r+s+1,\\
&\mathbf{B}_{b_0\otimes b'_{(p-s-1)}} \cong B(M(r+s-2p)), p \geq r+1,\\
&\mathbf{B}_{b_0\otimes b'_{p-s-1}} \cong (M(r+s-2p)), p \geq s+1,\\
&\mathbf{B}_{b_{(0)}\otimes b'_{((r+s+1)/2)}} \cong B(M(-1)), \text{ if } r \leq s.
\end{aligned}$$

The cases $B = B(V(r))$ and $B' \in \{B(V(s)), B(M(s)), B(T(s))\}$ were done in [6], [9]. We give it here for completeness.

Case 14. $B = B(V(r))$, $B' = B(V(s))$.

$$\mathbf{B}_{b_0\otimes b'_p} \cong B(V(r+s-2p)), 0 \leq p \leq \min\{r,s\}.$$

Case 15. $B = B(V(r))$, $B' = B(M(s))$.

$$\begin{aligned}
&\mathbf{B}_{b_{r+s-2p}\otimes b'_{p+1}} \cong B(T(r+s-2p)), 2s+2 \le 2p \le r+s,\\
&\mathbf{B}_{b_0\otimes b'_p} \cong B(M(r+s-2p)), 0 \le p \le \min\{r,s\},\\
&\mathbf{B}_{b_p\otimes b'_0} \cong B(M(r+s-2p)), r+s+2 \le p \le r,\\
&\mathbf{B}_{b_0\otimes b'_{(r+s+1)/2}} \cong B(M(-1)),\ \text{if } r \ge s.
\end{aligned}$$

Case 16. $B = B(V(r))$, $B' = B(T(s))$.

$$\begin{aligned}
&\mathbf{B}_{b_{r+s-2p}\otimes b'_{(p+1)}} \cong \mathbf{B}_{b_{r+s-2p}\otimes b'_{p-s}} \cong B(T(r+s-2p)), 2s+2 \le 2p \le r+s,\\
&\mathbf{B}_{b_{r-p}\otimes b'_0} \cong B(T(r+s-2p)), 0 \le p \le \min\{r,s\},\\
&\mathbf{B}_{b_0\otimes b'_{((r+s+1)/2)}} \cong \mathbf{B}_{b_0\otimes b'_{(r-s-1)/2}} \cong B(M(-1)),\ \text{if } r \ge s.
\end{aligned}$$

3. The category $\mathcal{O}$ and branched crystals.

We recall the definition and properties of the category $\mathcal{O}$ of modules for $\mathbf{U}_q(sl_2)$ in 3.1–3.4. The relevant results can be found in [1], [2], [5], and [8].

3.1. Let $\mathbf{C}(q)$ be the field of rational functions in an indeterminate q and let $\mathbf{U}_q(\mathfrak{sl}_2)$ be the quantized enveloping algebra of $\mathfrak{sl}_2$ over $\mathbf{C}(q)$, i.e., the algebra generated by elements $e, f, k^{\pm 1}$ and relations

$$kk^{-1} = 1 = k^{-1}k, \quad kek^{-1} = q^2 e, \quad kfk^{-1} = q^{-2} f,$$
$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

It is well–known that $\mathbf{U}_q(\mathfrak{sl}_2)$ is a Hopf algebra. Let $\sigma : \mathbf{U}_q(\mathfrak{sl}_2) \to \mathbf{U}_q(\mathfrak{sl}_2)$ be the involutive antiautomorphism obtained by extending the assignment

$$\sigma(e) = f, \quad \sigma(f) = e, \quad \sigma(k) = k^{-1}, \quad \sigma(q) = q^{-1}.$$

Let $\Omega \in \mathbf{U}_q(\mathfrak{sl}_2)$ be the quantum Casimir element.

3.2. Recall that a module M of $\mathbf{U}_q(\mathfrak{sl}_2)$ is said to be a weight module of type 1 if we can write

$$M = \oplus_{r\in\mathbf{Z}} M_r, \qquad M_r = \{m \in M : km = q^r m\}.$$

Let $\mathcal{O}$ be the category of type 1 $\mathbf{U}_q(\mathfrak{sl}_2)$–modules M such that $\dim M_r < \infty$ for all $r \in \mathbf{Z}$, and such that there exists an integer n depending on M such that $M_r = 0$ for all $r \ge n$. It is obvious that $\mathcal{O}$ is an abelian category and that it is closed under taking tensor products. Given any $M \in \mathcal{O}$ let M^σ be the subspace of $M^* = \mathrm{Hom}_{\mathbf{C}(q)}(M, \mathbf{C}(q))$ consisting of elements m^* such that $m^*(M_r) = 0$ for all but finitely many $r \in \mathbf{Z}$. The formula

$$(um^*)(m) = m^*(\sigma(u)m), \quad u \in U_q(\mathfrak{sl}_2),\ m^* \in M^*,\ m \in M,$$

defines an action of $\mathbf{U}_q(\mathfrak{sl}_2)$ on M^σ and it is easy to check that $M^\sigma \in \mathcal{O}$. The module M^σ is called the restricted dual of M and we say M is self–dual if $M \cong M^\sigma$. The assignment $M \mapsto M^\sigma$ is an exact contravariant functor on $\mathcal{O}$ and , if $M, N \in \mathcal{O}$, then $(M \otimes N)^\sigma \cong N^\sigma \otimes M^\sigma$. Given $M \in \mathcal{O}$, and $c \in \mathbf{C}(q)$, let

$$M[c] = \{m \in M : (\Omega - c)^n m = 0, \ \ \forall\, n >> 0\}.$$

Clearly $M[c]$ is a $\mathbf{U}_q(\mathfrak{sl}_2)$–submodule of M, $M = \oplus_{c\in\mathbf{C}(q)} M[c]$, and $M[c] \neq 0$ implies $c = (q^{r+1} + q^{-r-1})(q-q^{-1})^{-2}$ for some $r \in \mathbf{Z}$.

3.3. Given $r \in \mathbf{Z}$, $r \geq -1$, let $\mathcal{O}_r$ be the abelian subcategory consisting of modules $M \in \mathcal{O}$ such that

$$M = M[(q^{r+1} + q^{-r-1})(q-q^{-1})^{-2}].$$

Since $\sigma(\Omega) = \Omega$ it follows that $M \in \mathcal{O}_r$ iff $M^\sigma \in \mathcal{O}_r$. Further,

$$\mathcal{O} = \oplus_{r\geq -1} \mathcal{O}_r$$

and any object in $\mathcal{O}_r$ is isomorphic to a direct sum of indecomposable objects in $\mathcal{O}_r$. The last statement is easily seen by an induction on $\dim(M_r) + \dim(M_{-r-2})$, $r \geq -1$.

3.4. We now describe the indecomposable objects in $\mathcal{O}_r$. Given $s \in \mathbf{Z}$, let $M(s)$ be the Verma module with highest weight s and highest weight vector m_s. Namely, $M(s)$ is the $\mathbf{U}_q(\mathfrak{sl}_2)$–module generated by an element m_s with relations:

$$em_s = 0, \quad km_s = q^s m_s.$$

Let $V(s)$ be the unique irreducible quotient of $M(s)$. If $r \geq 0$, then

$$M(-r-1) \cong V(-r-1) \quad \text{and} \quad \dim V(r) = r+1.$$

For $r \in \mathbf{Z}^+$, let $T(r)$ be the $\mathbf{U}_q(\mathfrak{sl}_2)$–module generated by an element t_r satisfying the relations:

$$e^{r+2} t_r = 0, \quad kt_r = q^{-r-2} t_r, \; (\Omega - (q^{r+1} + q^{-r-1})(q-q^{-1})^{-2})^2 t_r = 0.$$

Note that $\dim(T(r))_{-r-2p} = 2$ if $p > 0$ and $\dim T(r)_{r-2p} = 1$ if $0 \leq p \leq r$.

For $r \in \mathbf{Z}^+$, the modules $M(-r-1)$, $T(r)$, and $V(r)$ are self dual. Let $m_r^* \in M(r)^\sigma$ be such that $m_r^*(m_r) = 1$ and $m_r^*(M(r)_s) = 0$ if $s \neq r$. Then

$$U_q(\mathfrak{sl}_2) m_r^* \cong V(r).$$

The following proposition is proved in [8].

Proposition. *Let $r \in \mathbf{Z}^+$. The modules $M(r)$, $M(r)^\sigma$, $M(-r-2)$, $T(r)$, and $V(r)$ are precisely the indecomposable modules in $\mathcal{O}_r$ while the only indecomposable module in $\mathcal{O}_{-1}$ is $M(-1)$.*

Corollary. *Let $r \in \mathbf{Z}^+$ and $V \in \mathcal{O}_r$. There exist unique integers $n_V(T(r))$, $n_V(M(r))$, etc, such that,* $V \cong T(r)^{\oplus n_V(T(r))} \oplus M(r)^{\oplus n_V(M(r))} \oplus M(-r-2)^{\oplus n_V(M(-r-2))} \oplus V(r)^{\oplus n_V(V(r))} \oplus (M(r)^\sigma)^{\oplus n_V((M(r))^\sigma)}$.

3.5. It follows that the tensor product of any two objects of $\mathcal{O}$ can be written uniquely as a direct sum of indecomposable modules. We give these decompositions explicitly. It is enough to write down the formulas for modules V in the set

$\{V(r) \otimes V(s),\ V(r) \otimes M(s),\ V(r) \otimes T(s),\ T(r) \otimes M(s),\ T(r) \otimes T(s),\ M(r) \otimes M^\sigma(s),\ M(r) \otimes M(s)\}$

since the other decompositions can be obtained by using σ. Our proof actually establishes that the tensor product in $\mathcal{O}$ is commutative, although this can be established as usual by using the R–matrix.

Case 1. If $V = V(r) \otimes V(s)$, $r, s \geq 0$ then it is well–known that $V(r) \otimes V(s) \cong \bigoplus_{p=0}^{\min\{r,s\}} V(r+s-2p)$.
Case 2. In all other cases, the element f acts freely on V and hence we can write,

$$V \cong \mathbf{M}_{<-1} \oplus \mathbf{T} \oplus \mathbf{M}_{\geq 0}, \tag{3.1}$$

where

$$\mathbf{M}_{<-1} = \bigoplus_{s\geq 0} M(-s-2)^{\oplus n_V(M(-s-2))}, \quad \mathbf{T} = \bigoplus_{r\geq -1} T(r)^{\oplus n_V(T(r))}, \quad \mathbf{M}_{\geq 0} = \bigoplus_{r\geq 0} M(r)^{\oplus n_V(M(r))}, \tag{3.2}$$

where we set $T(-1) = M(-1)$.

Case 2(a) Suppose that $V = M(r) \otimes M(s)$ and $r, s \geq 0$. Then,

$$\mathbf{M}_{\geq 0} = \bigoplus_{p=0}^{\min\{r,s\}} M(r+s-2p).$$

To see this, observe that (3.1) implies that

$$V^\sigma \cong \bigoplus_{\ell \geq 0} (M(\ell)^\sigma)^{\oplus n_V(M(\ell))} \oplus \mathbf{T} \oplus \mathbf{M}_{<-1}.$$

It is clear that the submodule

$$V^\sigma_{tor} = \{v \in V^\sigma : f^\ell v = 0, \quad \ell > 0\}$$

of V is isomorphic to the submodule $\bigoplus_{\ell \geq 0} V(\ell)^{\oplus n_V(M(\ell))}$ of $\bigoplus_{\ell \geq 0} (M(\ell)^\sigma)^{\oplus n_V(M(\ell))}$. On the other hand, it is not hard to see that

$$V^\sigma_{tor} \cong V(r) \otimes V(s).$$

It follows immediately that

$$n_V(M(r+s-2p)) = 1, \quad 0 \leq p \leq \min\{r,s\},$$

and $n_V(M(\ell)) = 0$ otherwise. Since $\dim(M(r) \otimes M(s))_{r+s-2p} = p+1$ for all $p \geq 0$, a simple dimension counting now implies that

$$\mathbf{T} = \bigoplus_{p=\min\{r,s\}+1}^{[\frac{r+s+1}{2}]} T(r+s-2p), \quad \text{and} \quad \mathbf{M}_{<-1} = \bigoplus_{p=\max\{r,s\}+1}^{\infty} M(r+s-2p).$$

Case 2(b) If $V = V(r) \otimes M(s)$ with $r, s \geq 0$, then

$$\begin{aligned}
\mathbf{M}_{<-1} &= 0, \\
\mathbf{M}_{\geq 0} &= \bigoplus_{p=0}^{\min\{r,s\}} M(r+s-2p), \\
\mathbf{T} &= \bigoplus_{p=s+1}^{[\frac{r+s+1}{2}]} T(r+s-2p), \quad \text{if} \quad r > s, \\
&= 0, \quad \text{if } r \leq s.
\end{aligned}$$

This can be proved similarly, see also [3].

Case 2(c). If $V \notin \{M(r) \otimes M(s), V(r) \otimes M(s) : r, s \in \mathbf{Z}^+\}$, then f acts freely on both V and V^σ and hence $\mathbf{M}_{\geq 0} = 0$. This means that $V_p \cong \mathbf{T}_p$ for all $p \geq -1$, and so by comparing the dimension of the weight spaces on both sides of (3.1) we get

$$n_V(T(p)) = \dim V_p - \dim V_{p+2}, \quad \forall\ p \geq -1,$$

and, for $\ell \geq 0$,

$$n_V(M(-\ell-2)) = \dim V_{-\ell-2} - \sum_{p=0}^{[\frac{\ell-1}{2}]} n_V(M(-\ell+2p)) - 2\sum_{p=0}^{[\frac{\ell}{2}]} n_V(T(\ell-2p)) - \sum_{p>0} n_V(T(\ell+2p)).$$

Using these formulas, we write down the non–zero multiplicities in all cases. Below we assume $p \in \mathbf{Z}^+$.

(i) $V = V(r) \otimes M(s)$ with $s < 0$.

$$n_V(T(r+s-2p)) = 1, \text{ if } 2p \leq r+s+1, \quad n_V(M(r+s-2p)) = 1, \text{ if } r+s+2 \leq p \leq r.$$

(ii) $V = M(r) \otimes M(s)$ with $r < 0$ or $s < 0$.

$$n_V(T(r+s-2p)) = 1, \text{ if } 2p \leq r+s+1, \quad n_V(M(r+s-2p)) = 1, \quad \text{if } p \geq r+s+2.$$

(iii) $V = V(r) \otimes T(s)$.

$$n_V(T(r+s-2p)) = 1, \text{ if } p \leq \min\{r,s\}, \quad n_V(T(r+s-2p)) = 2, \text{ if } 2s+2 \leq 2p \leq r+s+1.$$

(iv) $V = M(r) \otimes M(s)^\sigma$.

$$n_V(T(r+s-2p)) = 1, \text{ if } 2p \leq r+s+1, \quad n_V(M(r+s-2p)) = 1, \text{ if } p \geq r+s+2.$$

(v) $V = T(r) \otimes M(s)$.

$$n_V(T(r+s-2p)) = 1, \text{ if } 2p \leq \min\{2r, r+s+1\}, \quad n_V(T(r+s-2p)) = 2, \text{ if } 2r+2 \leq 2p \leq r+s+1,$$

and

(a) $s \geq 0$:

$$n_V(M(r+s-2p)) = 1, \text{ if } \max\{r,s\}+1 \leq p \leq r+s+1, \quad n_V(M(r+s-2p)) = 2, \text{ if } p \geq r+s+2,$$

(b) $s < 0$:

$$n_V(M(r+s-2p)) = 1, \text{ if } r+s+2 \leq p \leq r, \quad n_V(M(r+s-2p)) = 2, \text{ if } p \geq r+1.$$

(vi) $V = T(r) \otimes T(s) \cong T(r) \otimes M(s) \oplus T(r) \otimes M(-s-2)$.

3.6. Given $V \in \mathcal{O}$, let $B(V)$ be the branched crystal

$$B(V) = \bigoplus_W B(W)^{\oplus n_V(W)},$$

where the sum is over the indecomposable objects $W \in \mathcal{O}$. A straightforward comparison between sections 3.5 and 2.8 shows the following.

Theorem. For all $U, V, W \in \mathcal{O}$ we have:

(i) $B(U) \otimes B(V) \cong B(U \otimes V) = B(V \otimes U) \cong B(V) \otimes B(U)$,
(ii) $B(U) \otimes (B(V) \otimes B(W)) \cong B(U \otimes (V \otimes W)) = B((U \otimes V) \otimes W) \cong (B(U) \otimes B(V)) \otimes B(W))$.

In particular, the tensor product of branched crystals is commutative and associative.

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