Operator Algebras on Cuspidal Wedges

Bert-Wolfgang Schulze, Boris Sternin*
and Victor Shatalov*

Max-Planck-Arbeitsgruppe
"Partielle Differentialgleichungen und
komplexe Analysis"
Universität Potsdam
Am Neuen Palais 10
14469 Potsdam
Germany

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

Germany

*
Moscow State University
Department of Mathematics
Vorob'evy Gory
119899 Moscow

Russia

MPI 96-137

Operator Algebras on Cuspidal Wedges

Bert-Wolfgang Schulze,
MPAG "Analysis"
Universität Potsdam
e-mail: schulze@mpg-ana.uni-potsdam.de

&

Boris Sternin*, and Victor Shatalov*
Moscow State University
e-mail: boris@sternin.msk.su

October 14, 1996

Abstract

Equations on manifolds with wedges consisting of cusp-type points are investigated. The corresponding calculus of peudodifferential operators is constructed and finiteness theorems (Fredholm property) are established.

Keywords: manifolds with singularities, cusp, wedge, noncommutative analysis, left ordered representation, ellipticity, regularizer, finiteness theorem, Fredholm property.

*Supported by Max-Planck-Gesellshaft zur Förderung der Wissenschaften e.V., Arbeitsgruppe "Partielle Differentialgleichungen und komplexe Analysis" and Deutsche Forschungsgemeinschaft.

Contents

Introduction

The present paper is a continuation of a sery of author's works on the investigation of differential operators on manifolds with cusp-type singularities.

In the previous papers [1], [2] we have investigated the case of isolated singularities. Here we extend our results to the case when the singularity set is a closed smooth manifold without boundary.

The problem we deal with is the construction of the corresponding algebra which is called by us a local cusp-wedge algebra (LCW-algebra). The latter is an algebra of operators concentrated in a neighborhood of the edge. The construction of the corresponding algebra on the whole manifold is now quite standard task which can be carried out, for example, by means of the technique used in the book [3].

While constructing LCW-algebra one can use two different approaches. One of them is connected with consideration of operators with coefficients in the algebra of ψDO's on a smooth manifold (on the base of the corresponding model cusp), and the other uses the point local cusp algebra (LC-algebra) as the algebra of coefficients.

Such an approach was used in the situation of conical-wedge singularities in [3], in the Sobolev problems for submanifolds with multidimensional singularities [4] and is, in present, a common one in the considered field.

It is worth noting that the first approach is undoubtedly more simple. At the same time, the merit of the second approach together with its matrix version is that it is applicable to more wide range of problems. We note also that, similar to our previous paper [2], we use Maslov's noncommutative analysis method [5], [6] for the construction of the mentioned algebras.

After the appearence of papers [1], [2], Professor V. Maz'ya have attracted our attension to papers [7], [8]. In these papers the Dirichlet problem for elliptic operator with constant coefficients in a domain with boundary having singularities of the caspidal type was investigated, and the *principal (dominant)* term of asymptotic expansion of a solution to such a problem in a neighborhood of the singular point was obtained[1]. The authors are grateful to Professor V. Maz'ya for his remark.

1 Preliminaries

This paper is aimed at the consideration of differential equations on manifolds with edges consisting of cusp-type points. More precisely, we consider a pair (M, X) of topological spaces such that[2]:

- $M \setminus X$ is an open C^∞-manifold.
- X is a compact C^∞-manifold without boundary.
- There exists a neighborhood U of the manifold X such that

$$U \simeq C \times X,$$

where C is an open cusp

$$C = \{[0,1) \times \Omega\} / \{\{0\} \times \Omega\}$$

with a smooth C^∞-manifold Ω as a base.

[1] We recall that in papers [1], [2] the authors obtain the *full* asymptotic expansion of solutions including *recessive* terms as well.

[2] For simplicity, we suppose that the singularity manifod X has only one component; this will not be essential in the sequel.

- C^∞-structure on $C_0 \times X$ coincides with that on $M \setminus X$. Here

$$C_0 = (0,1) \times \Omega$$

is the cusp C without its vertex.

On the manifold M, we consider differential operators with *cusp-type wedge degeneracy* of order k, that is the operators having the form[3]:

$$\hat{H} = H\left(r, \omega, x, -ir^{k+1}\frac{\partial}{\partial r}, -i\frac{\partial}{\partial \omega}, -ir^{k+1}\frac{\partial}{\partial x}\right) \tag{1}$$

near X, where the function

$$H\left(r, \omega, x, \xi_r, \xi_\omega, \xi_x\right) \tag{2}$$

is a polynomial of order m with coefficients smooth up to $r = 0$. Function (2) will be referred as the *symbol* of the operator $\hat{H}$ given by (1). We remark that the case when the manifold X consists of a single point of cusp-type degeneracy was considered in [2], and for $k = 0$ we obtain operators with *(conical) edge degeneracy* (see. e. g. [3]).

One must have in mind that the definition of differential operator (1) via its symbol (2) is not clear until we fix the *order of action* of arguments in (1) which are operators with nontrivial commutation relations. For example, if

$$H\left(r, \omega, x, \xi_r, \xi_\omega, \xi_x\right) = \sum_{j+|\alpha|+|\beta|\le m} a_{j\alpha\beta}\left(r, \omega, x\right) \xi_r^j \xi_\omega^\alpha \xi_x^\beta,$$

then

$$\begin{aligned} &H\left(\overset{4}{r}, \overset{4}{\omega}, \overset{4}{x}, \overset{1}{\overline{-ir^{k+1}\frac{\partial}{\partial r}}}, \overset{3}{\overline{-i\frac{\partial}{\partial \omega}}}, \overset{2}{\overline{-ir^{k+1}\frac{\partial}{\partial x}}}\right) \\ &= \sum_{j+|\alpha|+|\beta|\le m} a_{j\alpha\beta}\left(r, \omega, x\right) \left(-i\frac{\partial}{\partial \omega}\right)^\alpha \left(-ir^{k+1}\frac{\partial}{\partial x}\right)^\beta \left(-ir^{k+1}\frac{\partial}{\partial r}\right)^j, \end{aligned}$$

[3]More precisely, we should consider operators of the form

$$r^{-(k+1)m} H\left(r, \omega, x, -ir^{k+1}\partial/\partial r, -i\partial/\partial \omega, -ir^{k+1}\partial/\partial x\right).$$

However, in the local considerations the factor $r^{-(k+1)m}$ is inessential, and we omit it.

but

$$H\left(\overset{4}{r},\overset{4}{\omega},\overset{4}{x},\overset{2}{\overline{-ir^{k+1}\frac{\partial}{\partial r}}},\overset{3}{\overline{-i\frac{\partial}{\partial \omega}}},\overset{1}{\overline{-ir^{k+1}\frac{\partial}{\partial x}}}\right)$$

$$= \sum_{j+|\alpha|+|\beta|\le m} a_{j\alpha\beta}(r,\omega,x)\left(-i\frac{\partial}{\partial\omega}\right)^{\alpha}\left(-ir^{k+1}\frac{\partial}{\partial r}\right)^{j}\left(-ir^{k+1}\frac{\partial}{\partial x}\right)^{\beta}.$$

Here the indices over operators define the action of these operators (*Feynmann indices*, see [5], [6]): the operator with smaller index acts on the function earlier than that with larger index.

Our aim is to introduce an operator algebra including operators of the type (1) as well as regularizers for such operators. Similar to the paper [2], it is natural to construct such an algebra in the framework of the *noncommutative analysis*, that is, to construct elements of the algebra as functions of operators

$$\begin{aligned} A_1 &= -ir^{k+1}\frac{\partial}{\partial r}, \\ A_2 &= -ir^{k+1}\frac{\partial}{\partial x}, \\ B_1 &= r, \\ B_2 &= x, \end{aligned} \tag{3}$$

with some concrete ordering.

There are two ways of constructing the operator algebra corresponding to operators of the form (1). Both of them are based on the consideration of functions of operators with operator-valued symbols.

A. The first way is to consider functions on the local model

$$\{[0,1)\times X\times\Omega\}/\{\{0\}\times X\times\Omega\} \tag{4}$$

of the manifold M near the submanifold X as functions on the direct product $[0,1)\times X$ with values in a function space on the manifold Ω. In this case operator (1) can be written down in the form[4]

$$\hat{H}\left(\overset{3}{B_1},\overset{3}{B_2},\overset{1}{A_1},\overset{2}{A_2}\right), \tag{5}$$

[4]The concrete ordering of operators in (5) is chosen for convenience of computation of the left ordered representation of these operators.

where the function $\hat{H}(y,x,\eta,\xi)$ takes its values in the algebra of (pseudo)differential operators on the manifold Ω.

B. The second way is to consider functions on (4) as functions on the manifold X with values in a function space on the model cusp

$$C = \{[0,1) \times \Omega\} \,/\, \{\{0\} \times \Omega\}$$

(this space will be a *weighted Sobolev space* $E^{s}_{\sigma,\gamma}$ introduced in the paper [2]). Then the operators from our future algebra will have the form

$$\overset{3}{\hat{H}}\left(\overset{2}{B_2}, \overset{1}{A_2}\right), \tag{6}$$

where $\hat{H}(x,\xi)$ is a function with values in the LC-algebra constructed in [2]:

$$\hat{H}(x,\xi) = \hat{H}_1\left(\overset{2}{B_1}, \overset{1}{A_1}; x, \xi\right) \tag{7}$$

with some function $\hat{H}_1(y,\eta;x,\xi)$ with values in the algebra of (pseudo)differential operators on the manifold Ω. In this approach, it is necessary to define explicitly not only the order of action of operators $\overset{2}{B_2}, \overset{1}{A_2}$ in (6) but also of coefficients of the operator-valued symbol $\hat{H}(x,\xi)$, since the operator

$$A_2 = -ir^{k+1}\frac{\partial}{\partial x} \tag{8}$$

does not commute with operators from the LC-algebra. In such operator-valued treatment, the operator of multiplication by r^{k+1} in (8) must be viewed as an operator in the corresponding space of functions on the local cusp C. We remark also that operators of the form (6) with symbols (7) can be written down as functions of operators (3):

$$\overset{3}{\hat{H}}\left(\overset{2}{B_2}, \overset{1}{A_2}\right) = \hat{H}_1\left(\overset{4}{B_1}, \overset{3}{A_1}; \overset{2}{B_2}, \overset{1}{A_2}\right)$$

but with ordering different from that in (5).

In the two subsequent sections we shall introduce these algebras and derive the corresponding commutation formulas.

2 Operator algebra (first version)

2.1 Function spaces

Let $\mathcal{H}$ be a Hilbert space scale. This means that

$$\mathcal{H} = \{\mathcal{H}^s,\ s \in \mathbf{Z}\},$$

the spaces $\mathcal{H}^s$ are compactly embedded to one another:

$$\mathcal{H}^s \subset \mathcal{H}^{s-1},$$

and each $\mathcal{H}^s$ forms a dense set in $\mathcal{H}^{s-1}$. We suppose also that the space

$$\mathcal{H}^\infty = \bigcap_s \mathcal{H}^s$$

is dense in $\mathcal{H}^s$ for any s. We denote by $\|\cdot\|_s$ the norm in the space $\mathcal{H}^s$.

Let

$$D : \mathcal{H} \to \mathcal{H}$$

be a generating operator of the scale $\mathcal{H}$, that is, an operator

$$D : \mathcal{H}^\infty \to \mathcal{H}^\infty$$

such that it can be continued up to the isomorphism

$$D : \mathcal{H}^s \to \mathcal{H}^{s-1}.$$

Then the norms in spaces $\mathcal{H}^s$ can be given by

$$\|u\|_s = \|D^s u\|_0 .$$

In what follows we shall use the Sobolev space scale $H^s(\Omega)$ as $\mathcal{H}^s$. Then the operator D is given by

$$D = (1 + \Delta)^{1/2},$$

where Δ is a positive Beltrami-Laplace operator on Ω corresponding to some Riemannian structure.

Consider the space scale

$$E^s = E^s([0,\infty) \times X, \mathcal{H})$$

consisting of functions on $[0,\infty)\times X$ with values in $\mathcal{H}$ with the finite norm

$$\begin{aligned}|||u|||_s^2 &= \sum_{i,j,k\geq 0; i+j+k\leq s} \int_0^\infty \int_X \left\| A_1^i A_2^j D^k u(r,x)\right\|_0^2 \frac{dr dx}{r^{k+1}} \\ &= \sum_{i,j,k\geq 0; i+j+k\leq s} \int_0^\infty \int_X \left\| A_1^i A_2^j u(r,x)\right\|_k^2 \frac{dr dx}{r^{k+1}}. \end{aligned} \qquad (9)$$

As it was mentioned above, we shall use some function space scale on the manifold Ω as $\mathcal{H}$, so that the elements of the spaces E^s can be treated as functions of the local model

$$\{[0,\infty)\times X\times\Omega\} \,/\, \{\{0\}\times X\times\Omega\}$$

of the manifold M near the singularity manifold X.

The space $E^0([0,\infty)\times X,\mathcal{H})$ defined by norm (9) is an L_2-space with special weight $r^{-(k+1)}$ chosen in such a way that all the operators (3) be symmetric operators with respect to the corresponding scalar product. However, not all these operators are self-adjoint in the space E_0. Namely, the operator A_1 is just symmetric, not self-adjoint operator in this space.

Similar to the paper [2], we modify operators (3) in such a way that they become self-adjoint without changing these operators in a neighborhood of $r=0$. Let $\varphi(r)$ be a C^∞-function (defined on the whole axis $\mathbf{R}$) coinciding with the function r^{k+1} near the origin and equal to a constant for large values of r. We shall consider the modification of operators (3) of the form

$$\begin{aligned} A_1 &= -i\varphi(r)\frac{\partial}{\partial r}, \\ A_2 &= -i\varphi(r)\frac{\partial}{\partial x}, \\ B_1 &= r, \\ B_2 &= x. \end{aligned} \qquad (10)$$

The reader can verify that all the operators (10) are self-adjoint in the space E^0 (see similar considerations in the above cited paper [2]) if this space is also modified in the appropriate way. Namely, we shall use the norms

$$\begin{aligned}|||u|||_s^2 &= \sum_{i,j,k\geq 0; i+j+k\leq s} \int_0^\infty \int_X \left\| A_1^i A_2^j D^k u(r,x)\right\|_0^2 \frac{dr dx}{\varphi(r)} \\ &= \sum_{i,j,k\geq 0; i+j+k\leq s} \int_0^\infty \int_X \left\| A_1^i A_2^j u(r,x)\right\|_k^2 \frac{dr dx}{\varphi(r)} \end{aligned}$$

instead of norms (9). The proof of the self-adjointness of operators (10) in the space E^0 can be performed now by the direct computation of the corresponding unitary groups

$$\exp(itA_j),\ \exp(itB_j),\ j = 1,2.$$

Below, we shall use also the space scale $E^s_{\sigma,\gamma}$ defined by the norm

$$|||u|||_{s,\sigma,\gamma} = |||e^{\sigma\psi(r)}r^{-\gamma}u|||_s,$$

where $\psi(r)$ is a function on $\mathbf{R}_+$ such that $\psi'(r) = \varphi^{-1}(r)$.

2.2 Functions of noncommuting operators

As it was explained above, we shall construct the local algebra near X as an algebra of functions of the four non-commuting operators (10)

$$\hat{H} = \hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right) \tag{11}$$

with some operator-valued symbol $\hat{H}(y, x, \eta, \xi)$ with values in the algebra

$$\mathcal{L}(\mathcal{H}, \mathcal{H})$$

of continuous operators in $\mathcal{H}$. We recall that an operator

$$\hat{G} : \mathcal{H}^\infty \to \mathcal{H}^\infty$$

is said to be a continuous operator in the scale $\mathcal{H}$ if it extends up to a continuous operator

$$\hat{G} : \mathcal{H}^s \to \mathcal{H}^{s-m}$$

for every $s \in \mathbf{R}_+$. The minimal value of m for which the latter operator is continuous is called an *order* of this operator.

The definition of operators of the type (11) closely follows constructions of the paper [2]. Namely, we define the operator $\hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right)$ via its symbol $\hat{H}(y, x, \eta, \xi)$ as

$$\hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right) = \int e^{it_1B_1}e^{it_2B_2}e^{it_3A_2}e^{it_4A_1}\tilde{H}(t_1, t_2, t_3, t_4)\, dt_1dt_2dt_3dt_4,$$

where $\tilde{H}(t_1, t_2, t_3, t_4)$ is the Fourier transform of the symbol $\hat{H}(y, x, \eta, \xi)$ in all its variables.

Let us describe the symbol classes which will be used in sequel.

Definition 1 By $S^m(\mathcal{H})$, we denote the space of functions $\hat{H}(y,x,\eta,\xi)$ such that the following estimate

$$\left\| D_y^j D_\eta^l D_x^\alpha D_\xi^\beta \hat{H}(y,x,\eta,\xi)\, u \right\|_s \leq C_{jl\alpha\beta} \left\| \left(1+\eta^2+\xi^2+D^2\right)^{\frac{m-l-|\beta|}{2}} u \right\|_s$$

takes place for any j, l, α, β, and $u \in \mathcal{H}^{s+m-l-|\beta|}$.

Then, similar to the results of the paper [2], one can prove the following statement:

Theorem 1 *Let $\hat{H}(y,x,\eta,\xi)$ be a symbol from $S^m(\mathcal{H})$. Then the corresponding operator*

$$\hat{H}\left(\overset{3}{B}_1, \overset{3}{B}_2, \overset{1}{A}_1, \overset{2}{A}_2\right) \;:\; E^s([0,\infty)\times X, \mathcal{H}) \;\rightarrow\; E^{s-m}([0,\infty)\times X, \mathcal{H})$$

is continuous.

2.3 Commutation relations and the composition law

In this subsection, we prove that the set $Op([0,\infty)\times X, \mathcal{H})$ of operators of the form

$$\hat{H}\left(\overset{3}{B}_1, \overset{3}{B}_2, \overset{1}{A}_1, \overset{2}{A}_2\right) \tag{12}$$

with symbols $\hat{H}(y,x,\eta,\xi) \in S^m(\mathcal{H})$, where m can depend on the symbol, form an algebra. To do this, we must prove that for any two operators $\hat{H}_1$ and $\hat{H}_2$ of the form (12) the composition

$$\hat{H}_1 \circ \hat{H}_2 = \left[\!\left[\hat{H}_1\left(\overset{3}{B}_1, \overset{3}{B}_2, \overset{1}{A}_1, \overset{2}{A}_2\right) \right]\!\right] \circ \left[\!\left[\hat{H}_2\left(\overset{3}{B}_1, \overset{3}{B}_2, \overset{1}{A}_1, \overset{2}{A}_2\right) \right]\!\right] \tag{13}$$

can be represented as a function of the ordered tuple $\left(\overset{3}{B}_1, \overset{3}{B}_2, \overset{1}{A}_1, \overset{2}{A}_2\right)$ of operators. Besides, we shall show that if

$$\hat{H}_1(y,x,\eta,\xi) \in S^{m_1}(\mathcal{H}),\; \hat{H}_2(y,x,\eta,\xi) \in S^{m_2}(\mathcal{H}),$$

then the symbol $\hat{H}(y,x,\eta,\xi)$ of composition (13) belongs to the symbol space $S^{m_1+m_2}(\mathcal{H})$, and present the closed formula for this symbol via $\hat{H}_1(y,x,\eta,\xi)$ and $\hat{H}_2(y,x,\eta,\xi)$.

To do this, we shall use the *ordered representation method* [6]. Namely, in accordance to this method to derive the composition formula, one have to compute only formulas for symbols of the compositions

$$\begin{array}{c} B_1 \left[\!\left[\hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right) \right]\!\right], \\ B_2 \left[\!\left[\hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right) \right]\!\right], \\ A_1 \left[\!\left[\hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right) \right]\!\right], \\ A_2 \left[\!\left[\hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right) \right]\!\right] \end{array} \qquad (14)$$

of generators B_1, B_2, A_1, A_2 with an arbitrary operator $\hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right)$. We denote by l_{B_1}, l_{B_2}, l_{A_1}, l_{A_2} the operators on the symbol space such that

$$\begin{aligned} l_{B_j}\left[\hat{H}(y,x,\xi,\eta)\right] &= smbl\left\{B_j \left[\!\left[\hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right) \right]\!\right]\right\}, \ j = 1,2, \\ l_{A_j}\left[\hat{H}(y,x,\xi,\eta)\right] &= smbl\left\{A_j \left[\!\left[\hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right) \right]\!\right]\right\}, \ j = 1,2. \end{aligned}$$

These operators are called the *operators of the left ordered representation.*

Let us proceed with the process of computations.

First of all, we write down the commutation relations

$$\begin{aligned} [A_1, B_1] &= -i\varphi(B_1), \\ [A_1, A_2] &= -i\varphi'(B_1)A_2, \\ [A_2, B_2] &= -i\varphi(B_1) \end{aligned} \qquad (15)$$

It is evident that

$$B_1 \left[\!\left[\hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right) \right]\!\right] = \overset{3}{B_1}\ \hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right)$$

and

$$B_2 \left[\!\left[\hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right) \right]\!\right] = \overset{3}{B_2}\ \hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right),$$

so that the symbols of the first two operators in (14) are

$$\begin{aligned} l_{B_1}\left(\hat{H}(y,x,\eta,\xi)\right) &= y\hat{H}(y,x,\eta,\xi), \\ l_{B_2}\left(\hat{H}(y,x,\eta,\xi)\right) &= x\hat{H}(y,x,\eta,\xi). \end{aligned}$$

Let us compute the symbol of the third operator in (14). We have

$$A_1 \left[\!\!\left[\hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right)\right]\!\!\right] = \overset{5}{A_1}\, \hat{H}\left(\overset{4}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right).$$

Due to the commutation formula (see [6, p. 62]) we obtain

$$\begin{aligned} A_1 \left[\!\!\left[\hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right)\right]\!\!\right] &= \overset{4}{A_1}\, \hat{H}\left(\overset{5}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right) \\ &+ \overset{5}{\overline{[A_1, B_1]}}\, \frac{\delta \hat{H}}{\delta y}\left(\overset{6}{B_1}, \overset{4}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right). \end{aligned}$$

Since the commutator

$$[A_1, B_1] = -i\varphi(B_1)$$

(see (15)) commutes with the operator B_1, we obtain

$$\begin{aligned} A_1 \left[\!\!\left[\hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right)\right]\!\!\right] &= \overset{4}{A_1}\, \hat{H}\left(\overset{5}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right) \\ &-i\varphi\left(\overset{3}{B_1}\right) \frac{\partial \hat{H}}{\partial y}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right). \end{aligned} \tag{16}$$

Later on, since the operators A_1 and B_2 commute, we can interchange the order of action of these operators on the right in the latter formula. Then, again by the commutation formula, we get

$$\begin{aligned} \overset{4}{A_1}\, \hat{H}\left(\overset{5}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right) &= \overset{3}{A_1}\, \hat{H}\left(\overset{5}{B_1}, \overset{4}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right) = \overset{1}{A_1}\, \hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right) \\ &+ \overset{3}{\overline{[A_1, A_2]}}\, \frac{\delta \hat{H}}{\delta \xi}\left(\overset{5}{B_1}, \overset{5}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}, \overset{4}{A_2}\right). \end{aligned}$$

The commutator

$$[A_1, A_2] = -i\varphi'(B_1)\, A_2$$

(see (15)) commutes with A_2, and we arrive at the relation

$$\begin{aligned} \overset{4}{A_1}\, \hat{H}\left(\overset{5}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right) &= \overset{1}{A_1}\, \hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right) \\ &-i\varphi'\left(\overset{3}{B_1}\right) \overset{2}{A_2}\, \frac{\partial \hat{H}}{\partial \xi}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right). \end{aligned} \tag{17}$$

Collecting formulas (16) and (17), we finally obtain

$$A_1 \left[\!\left[\hat{H}\left(\overset{3}{B}_1, \overset{3}{B}_2, \overset{1}{A}_1, \overset{2}{A}_2\right) \right]\!\right] = \overset{1}{A}_1 \; \hat{H}\left(\overset{3}{B}_1, \overset{3}{B}_2, \overset{1}{A}_1, \overset{2}{A}_2\right)$$
$$-i\varphi\left(\overset{3}{B}_1\right) \frac{\partial \hat{H}}{\partial y}\left(\overset{3}{B}_1, \overset{3}{B}_2, \overset{1}{A}_1, \overset{2}{A}_2\right) - i\varphi'\left(\overset{3}{B}_1\right) \overset{2}{A}_2 \frac{\partial \hat{H}}{\partial \xi}\left(\overset{3}{B}_1, \overset{3}{B}_2, \overset{1}{A}_1, \overset{2}{A}_2\right),$$

so that the symbol of the third composition in (14) equals

$$\begin{aligned} l_{A_1}\left(\hat{H}(y,x,\eta,\xi)\right) &= \eta \hat{H}(y,x,\eta,\xi) - i\varphi(y) \frac{\partial \hat{H}}{\partial y}(y,x,\eta,\xi) \\ &\quad - i\varphi'(y)\, \xi \frac{\partial \hat{H}}{\partial \xi}(y,x,\eta,\xi). \end{aligned}$$

Similar (but more simple) computations show that the symbol of the last operator in (14) is

$$l_{A_2}\left(\hat{H}(y,x,\eta,\xi)\right) = \xi \hat{H}(y,x,\eta,\xi) - i\varphi(y) \frac{\partial \hat{H}}{\partial x}(y,x,\eta,\xi).$$

So, the operators of the left ordered representation are

$$\begin{aligned} l_{B_1} &= y, \\ l_{B_2} &= x, \\ l_{A_1} &= \eta - i\varphi'(y)\, \xi \frac{\partial}{\partial \xi} - i\varphi(y) \frac{\partial}{\partial y}, \\ l_{A_2} &= \xi - i\varphi(y) \frac{\partial}{\partial x}. \end{aligned} \tag{18}$$

Now, using the composition theorem from the book [6, p. 98], we arrive to the following statement:

Theorem 2 *Let $\hat{H}_j(y,x,\eta,\xi)$ be symbols from $S^{m_j}(\mathcal{H})$, $j = 1,2$. Then the symbol $\hat{H}(y,x,\eta,\xi)$ of the composition $\hat{H}_1 \circ \hat{H}_2$ of the operators*

$$\begin{aligned} \hat{H}_1 &= \hat{H}_1\left(\overset{3}{B}_1, \overset{3}{B}_2, \overset{1}{A}_1, \overset{2}{A}_2\right), \\ \hat{H}_2 &= \hat{H}_2\left(\overset{3}{B}_1, \overset{3}{B}_2, \overset{1}{A}_1, \overset{2}{A}_2\right) \end{aligned}$$

belongs to the space $S^{m_1+m_2}(\mathcal{H})$ *and is given by the formula*

$$\hat{H}(y,x,\eta,\xi) = \hat{H}_1\left(\overset{3}{\overline{l_{B_1}}},\overset{3}{\overline{l_{B_2}}},\overset{1}{\overline{l_{A_1}}},\overset{2}{\overline{l_{A_2}}}\right)\hat{H}_2(y,x,\eta,\xi),$$

where l_{B_1}, l_{B_2}, l_{A_1}, *and* l_{A_2} *are operators of the left ordered representation of the tuple* $\left(\overset{3}{B_1},\overset{3}{B_2},\overset{1}{A_1},\overset{2}{A_2}\right)$ *given by formulas* (18).

This theorem shows that the set $Op([0,\infty)\times X,\mathcal{H})$ *forms an algebra.* In fact, this algebra is an algebra with involution. The following statement describes the symbol of the adjoint operator.

Theorem 3 *If* $\hat{H}(y,x,\eta,\xi)$ *is a symbol from* $S^m(\mathcal{H})$, *then the adjoint operator to*

$$\hat{H} = \hat{H}\left(\overset{3}{B_1},\overset{3}{B_2},\overset{1}{A_1},\overset{2}{A_2}\right)$$

is given by

$$\hat{H}^* = \hat{H}^*\left(\overset{1}{B_1},\overset{1}{B_2},\overset{3}{A_1},\overset{2}{A_2}\right),$$

where $\hat{H}^*(y,x,\eta,\xi)$ *is the symbol adjoint to* $\hat{H}(y,x,\eta,\xi)$ *in the sense of the space* $\mathcal{H}_0$.

The proof of this statement is clear.

2.4 Elliptic elements and their regularizers

Let us consider some element

$$\hat{H} = \hat{H}\left(\overset{3}{B_1},\overset{3}{B_2},\overset{1}{A_1},\overset{2}{A_2}\right) \in Op([0,\infty)\times X,\mathcal{H}),$$

and let us try to construct an inverse to this element within the algebra

$$Op([0,\infty)\times X,\mathcal{H}).$$

To do this, we shall try to find an element $\hat{R}\left(\overset{3}{B_1},\overset{3}{B_2},\overset{1}{A_1},\overset{2}{A_2}\right)$ such that[5]

$$\left[\!\left[\hat{H}\left(\overset{3}{B_1},\overset{3}{B_2},\overset{1}{A_1},\overset{2}{A_2}\right)\right]\!\right] \circ \left[\!\left[\hat{R}\left(\overset{3}{B_1},\overset{3}{B_2},\overset{1}{A_1},\overset{2}{A_2}\right)\right]\!\right] = \mathbf{1}$$

[5]By $[\![\cdot]\!]$ we denote the so-called *autonomous brackets*, see [6].

(that is, to find the right inverse for this element.) Applying Theorem 2 to the latter relation, we arrive at the following equation for the symbol $\hat{R}(y,x,\eta,\xi)$ of this inverse element:

$$\hat{H}\left(\overset{3}{\overline{l_{B_1}}},\overset{3}{\overline{l_{B_2}}},\overset{1}{\overline{l_{A_1}}},\overset{2}{\overline{l_{A_2}}}\right)\hat{R}(y,x,\eta,\xi)=1.$$

Clearly, the latter equation cannot be, in general, solved in the explicit way. However, taking in mind that we are intended to construct a regularizer of the element $\hat{H}$ rather than the exact inverse element, we can search for the operator $\hat{R}$ in the form of the asymptotic series (in what follows we shall clearly use the corresponding truncated series) in the filtration $S^m(\mathcal{H})$

$$\hat{R}(y,x,\eta,\xi)\simeq\sum_{j=0}^{\infty}\hat{R}_j(y,x,\eta,\xi),$$

where $\hat{R}_j(y,x,\eta,\xi)\in S^{-m-j}(\mathcal{H})$. Here m is an order of the operator $\hat{H}$, so that $\hat{H}(y,x,\eta,\xi)\in S^m(\mathcal{H})$. Taking into account relations (18), we obtain

$$\hat{H}\left(\overset{3}{y},\overset{3}{x},\overset{1}{\overline{\eta-i\varphi'(y)\xi\frac{\partial}{\partial\xi}-i\varphi(y)\frac{\partial}{\partial y}}},\overset{2}{\overline{\xi-i\varphi(y)\frac{\partial}{\partial x}}}\right)\sum_{j=0}^{\infty}\hat{R}_j(y,x,\eta,\xi)=\mathbf{1}. \quad (19)$$

Let us consider the operators of the left ordered representation used in the latter relations as arguments of the function $\hat{H}(y,x,\eta,\xi)$. One can see that the first summand η of the operator

$$l_{A_1}=\eta-i\varphi'(y)\xi\frac{\partial}{\partial\xi}-i\varphi(y)\frac{\partial}{\partial y} \quad (20)$$

takes symbols from $S^m(\mathcal{H})$ into $S^{m+1}(\mathcal{H})$ whereas all the rest terms preserve the index m of the symbol space. Similar, the first summand ξ in

$$l_{A_2}=\xi-i\varphi(y)\frac{\partial}{\partial x} \quad (21)$$

enlarges the index m by one, and the second preserves it. So, the first summands in (20) and (21) are operators of the first order with respect to the filtration $\{S^m(\mathcal{H})\}$ in m, whereas all the rest terms of these operators are operators of zeroth order in the same sense. Therefore, one can try to expand the operator on the left in (19) in the Taylor series at the point (η,ξ).

To do this, we use the *Taylor formula* for functions of noncommutative operators. We recall (see [6, p. 58]) that this formula has the form

$$\begin{aligned} f(C) &= \sum_{k=0}^{N} \frac{1}{k!} f^{(k)}\left(\overset{2}{A}\right) \overset{1}{\overline{\left[\left[(C-A)^{k}\right]\right]}} \\ &\quad + \frac{1}{N!} \overset{2}{\overline{\left[\left[(C-A)^{N+1}\right]\right]}} \left(\frac{\partial}{\partial x}\right)^{N} \frac{\delta f}{\delta x}\left(\overset{1}{C}, \overset{3}{A}\right) \end{aligned}$$

for any two noncommuting operators C and A and any smooth function $f(x)$.

Let us apply this formula for the third argument of the operator on the right in (19). Here

$$C = \eta - i\varphi'(y)\,\xi\partial/\partial\xi - i\varphi(y)\,\partial/\partial y, \; A = \eta$$

Since $[C, A] = 0$, we have

$$\begin{aligned} &\hat{H}\left(\overset{3}{y}, \overset{3}{x}, \overset{1}{\overline{\eta - i\varphi'(y)\,\xi\frac{\partial}{\partial\xi} - i\varphi(y)\,\frac{\partial}{\partial y}}}, \overset{2}{\overline{\xi - i\varphi(y)\,\frac{\partial}{\partial x}}}\right) \\ = &\sum_{k=0}^{N} \frac{1}{k!} \frac{\partial^{k} H}{\partial\eta^{k}} \left(\overset{3}{y}, \overset{3}{x}, \overset{3}{\eta}, \overset{2}{\overline{\xi - i\varphi(y)\,\frac{\partial}{\partial x}}}\right) \overset{1}{\overline{\left[\left[-i\varphi'(y)\,\xi\frac{\partial}{\partial\xi} - i\varphi(y)\,\frac{\partial}{\partial y}\right]\right]^{k}}} \\ &+ \frac{1}{N!} \overset{2}{\overline{\left[\left[-i\varphi'(y)\,\xi\frac{\partial}{\partial\xi} - i\varphi(y)\,\frac{\partial}{\partial y}\right]\right]^{N+1}}} \left(\frac{\partial}{\partial\eta}\right)^{N} \frac{\delta H}{\delta\eta}\left(\overset{6}{y}, \overset{5}{x}, \overset{3}{\eta},\right. \\ &\left. \overset{1}{\overline{\eta - i\varphi'(y)\,\xi\frac{\partial}{\partial\xi} - i\varphi(y)\,\frac{\partial}{\partial y}}}, \overset{4}{\overline{\xi - i\varphi(y)\,\frac{\partial}{\partial x}}}\right). \end{aligned}$$

Now, applying the same formula for operators in the right-hand side of the latter relation with respect to the fourth argument, we obtain finally

$$\begin{aligned} &\hat{H}\left(\overset{3}{y}, \overset{3}{x}, \overset{1}{\overline{\eta - i\varphi'(y)\,\xi\frac{\partial}{\partial\xi} - i\varphi(y)\,\frac{\partial}{\partial y}}}, \overset{2}{\overline{\xi - i\varphi(y)\,\frac{\partial}{\partial x}}}\right) \\ = &\sum_{j\geq 0, k\geq 0, j+k\leq N} \frac{1}{k!j!} \frac{\partial^{k+j} H}{\partial\eta^{k}\partial\xi^{j}} \left(\overset{3}{y}, \overset{3}{x}, \overset{3}{\eta}, \overset{3}{\xi}\right) \overset{2}{\overline{\left(-i\varphi(y)\,\frac{\partial}{\partial x}\right)^{j}}} \end{aligned}$$

$$\times \overset{1}{\overline{\left[\left[-i\varphi'(y)\,\xi\frac{\partial}{\partial \xi}-i\varphi(y)\frac{\partial}{\partial y}\right]\right]^{k}}}+\mathcal{R}_N,$$

where the remainder $\mathcal{R}_N$ equals

$$\begin{aligned}\mathcal{R}_N \;=\; & \sum_{k=0}^{N}\frac{1}{k!\,(N-k)!}\left(\frac{\partial}{\partial \xi}\right)^{N-k}\frac{\delta}{\delta \xi}\frac{\partial^k H}{\partial \eta^k}\left(\overset{5}{y},\overset{5}{x},\overset{5}{\eta},\overset{4}{\overline{\xi-i\varphi(y)\frac{\partial}{\partial x}}},\overset{2}{\xi}\right)\\
\times \quad & \overset{3}{\overline{\left(-i\varphi(y)\frac{\partial}{\partial x}\right)^{N-k+1}}}\;\overset{1}{\overline{\left[\left[-i\varphi'(y)\,\xi\frac{\partial}{\partial \xi}-i\varphi(y)\frac{\partial}{\partial y}\right]\right]^{k}}}\\
& +\frac{1}{N!}\overset{2}{\overline{\left[\left[-i\varphi'(y)\,\xi\frac{\partial}{\partial \xi}-i\varphi(y)\frac{\partial}{\partial y}\right]\right]^{N+1}}}\left(\frac{\partial}{\partial \eta}\right)^{N}\frac{\delta H}{\delta \eta}\left(\overset{6}{y},\overset{5}{x},\overset{3}{\eta},\right.\\
& \left.\overset{1}{\overline{\eta-i\varphi'(y)\,\xi\frac{\partial}{\partial \xi}-i\varphi(y)\frac{\partial}{\partial y}}},\overset{4}{\overline{\xi-i\varphi(y)\frac{\partial}{\partial x}}}\right).\end{aligned}$$

Let us apply the obtained expansion to equation (19). We have

$$\left[\sum_{j\geq 0,k\geq 0,j+k\leq N}\frac{1}{k!j!}\frac{\partial^{k+j}\hat{H}}{\partial \eta^k\partial \xi^j}\left(\overset{3}{y},\overset{3}{x},\overset{3}{\eta},\overset{3}{\xi}\right)\overset{2}{\overline{\left(-i\varphi(y)\frac{\partial}{\partial x}\right)^{j}}}\right.$$
$$\left.\overset{1}{\overline{\left[\left[-i\varphi'(y)\,\xi\frac{\partial}{\partial \xi}-i\varphi(y)\frac{\partial}{\partial y}\right]\right]^{k}}}+\mathcal{R}_N\right]\sum_{l=0}^{\infty}\hat{R}_l(y,x,\eta,\xi)=\mathbf{1}.$$

The latter equation can be rewritten as a recurrent system for the components $\hat{R}_l(y,x,\eta,\xi)$ of the symbol $\hat{R}(y,x,\eta,\xi)$:

$$\begin{aligned}\hat{H}(y,x,\eta,\xi)\,\hat{R}_0(y,x,\eta,\xi) &= \mathbf{1},\\
\hat{H}(y,x,\eta,\xi)\,\hat{R}_j(y,x,\eta,\xi) &= \sum_{l=1}^{j}\hat{P}_l\hat{R}_{j-l}(y,x,\eta,\xi),\; j\geq 1,\end{aligned}\tag{22}$$

where the operators $\hat{P}_l$ have the form

$$\hat{P}_l=\sum_{j+k=l}\frac{1}{k!j!}\frac{\partial^{k+j}\hat{H}}{\partial \eta^k\partial \xi^j}\left(\overset{3}{y},\overset{3}{x},\overset{3}{\eta},\overset{3}{\xi}\right)\overset{2}{\overline{\left(-i\varphi(y)\frac{\partial}{\partial x}\right)^{j}}}\;\overset{1}{\overline{\left[\left[-i\varphi'(y)\,\xi\frac{\partial}{\partial \xi}-i\varphi(y)\frac{\partial}{\partial y}\right]\right]^{k}}}.$$

It is easy to see that the operator $\hat{P}_l$ has $m-l$-th order in the scale $S^m(\mathcal{H})$.

From system (22), one can see that if the symbol $\hat{H}(y,x,\eta,\xi)$ is invertible as an operator in the scale $\mathcal{H}$, then there exist a unique solution to (22). So, we arrive at the affirmation:

Proposition 1 *Let N be a positive integer. Then, under the above assumption,*

$$\hat{H}\left(\overset{3}{B_1},\overset{3}{B_2},\overset{1}{A_1},\overset{2}{A_2}\right)\hat{R}^N\left(\overset{3}{B_1},\overset{3}{B_2},\overset{1}{A_1},\overset{2}{A_2}\right)=\mathbf{1}+\hat{Q}^N\left(\overset{3}{B_1},\overset{3}{B_2},\overset{1}{A_1},\overset{2}{A_2}\right),$$

where

$$\hat{R}^N(y,x,\eta,\xi)=\sum_{j=0}^{N}\hat{R}_j(y,x,\eta,\xi),$$

and the symbol of the operator $\hat{Q}^N$ has the form

$$\hat{Q}^N(y,x,\eta,\xi)=\sum_{j+k=N+1}\sum_{l}\hat{Q}^N_{1l}\left(\overset{3}{y},\overset{3}{x},\overset{3}{\eta},\overset{3}{\xi}\right)\overset{2}{\overline{\left(-i\varphi(y)\frac{\partial}{\partial x}\right)^j}}$$
$$\times\overset{1}{\overline{\left[\left[-i\varphi'(y)\,\xi\frac{\partial}{\partial\xi}-i\varphi(y)\frac{\partial}{\partial y}\right]\right]^k}}\hat{Q}^N_{2l}(y,x,\eta,\xi),$$

the inner sum being finite and the sum of orders of operators $\hat{Q}^N_{1l}$ and $\hat{Q}^N_{2l}$ equals $m-N-1$.

Remark 1 From the duality reasons, one can also construct the left regularizer for the operator in question.

Now we remark that the conjugation of the operator $\hat{H}=\hat{H}\left(\overset{3}{B_1},\overset{3}{B_2},\overset{1}{A_1},\overset{2}{A_2}\right)$ with the help of the power r^γ leads us to the operator $\hat{H}_\gamma$ given by[6]

$$\hat{H}_\gamma=r^\gamma\hat{H}r^{-\gamma}=\hat{H}\left(\overset{3}{B_1},\overset{3}{B_2},\overset{1}{\overline{A_1+i\gamma\frac{\varphi(r)}{r}}},\overset{2}{A_2}\right).$$

Again using the Taylor formula for noncommuting operators with respect to the third argument of the function $\hat{H}$, one can see that the of the operators $\hat{H}$ and $\hat{H}_\gamma$

[6]From now on we suppose the symbol $\hat{H}(y,x,\eta,\xi)$ to be a holomorphic function in (η,ξ).

coincide at $r = 0$ modulo terms from $S^{m-1}(\mathcal{H})$ (we consider here the case $k > 0$; for $k = 0$ the above conjugation leads to the shift of the variable η by $i\gamma$ in the complex plane). Since the multiplication by r^γ is an isomorphism of functional spaces

$$r^\gamma \,:\, E^s = E^s_{0,0} \to E^s_{0,\gamma},$$

all our considerations are valid in the spaces $E^s_{0,\gamma}$ as well.

To formulate the corresponding finiteness theorem, we have to globalize some above introduced notions. First of all, we denote by $E^s_{\sigma,\gamma}(M)$ the space of functions on M such that:

- $u \in H^s_{\mathrm{loc}}(M \setminus X)$;
- $\psi u \in E^s_{\sigma,\gamma}$ for some cut-off function ψ equal to 1 near X with support in a sufficiently small neighborhood of X.

Consider a differential operator $\hat{H}$ of order m on the manifold M having the form

$$\hat{H} = r^{-(k+1)m} H\left(r, \omega, x, -ir^{k+1}\frac{\partial}{\partial r}, -i\frac{\partial}{\partial \omega}, -ir^{k+1}\frac{\partial}{\partial x}\right)$$

near the edge X. Then it is easy to see that this operator is continuous as an operator in spaces

$$\hat{H} \,:\, E^s_{\sigma,\gamma}(M) \to E^{s-m}_{\sigma,\gamma-(k+1)m}(M). \tag{23}$$

Definition 2 The operator (23) is called to be *elliptic* if

- $\hat{H}$ is elliptic on $M \setminus X$;
- the corresponding symbol $\hat{H}(y, x, \eta, \xi)$ is invertible for small y in a neighborhood of the edge X for all values of η with $\operatorname{Im}\eta = \sigma$.

Now, similar to the paper [2], Proposition 1 together with Remark 1 yields the following result:

Theorem 4 *Let the operator $\hat{H}$ be elliptic in the sense of Definition 2. Then the operator (23) possesses the Fredholm property for $\sigma = 0$.*

The proof of the stated theorem goes quite similar to the corresponding theorem in [2], and we omit it. We remark only, that during the proof of this theorem, one has to examine the improvement of the operator $\hat{Q}^N$ in the variable r. To do this, one have take in mind that the operator A_2 which is substituted in a symbol instead of the variable ξ has the form

$$A_2 = -i\varphi(r)\frac{\partial}{\partial x},$$

and, hence, the application of this operator to a function improves the decrease of this function as $r \to 0$ by r^{k+1}. By this reason, the operator

$$\left[\left(-i\varphi'(y)\,\xi\frac{\partial}{\partial\xi} - i\varphi(y)\frac{\partial}{\partial y}\right)\hat{F}\right]\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{A_1}, \overset{2}{A_2}\right)$$

improves the power decrease of a function by r^{k+1} for any symbol $\hat{F}(y,x,\eta,\xi)$.

The last question which we have to consider in this section is the investigation of the operator $\hat{H}$ in spaces $E^s_{\sigma,\gamma}(M)$ for $\sigma \neq 0$. To do this, we calculate the conjugate $\hat{H}_\sigma$ of the operator $\hat{H}$ with the help of the exponent $\exp\{-\sigma\psi(r)\}$, where, as above, $\psi'(r) = \varphi^{-1}(r)$:

$$\hat{H}_\sigma = e^{\sigma\psi(r)}\hat{H}e^{-\sigma\psi(r)} = \hat{H}\left(\overset{3}{B_1}, \overset{3}{B_2}, \overset{1}{\overline{A_1 - i\sigma}}, \overset{2}{A_2}\right).$$

Since the multiplication by $\exp\{-\sigma\psi(r)\}$ is an isomorphism of function spaces

$$\exp\{-\sigma\psi(r)\} \;:\; E^s_{0,\gamma} \to E^s_{\sigma,\gamma},$$

we arrive at the following statement:

Theorem 5 *Let $\hat{H}$ be an elliptic differential operator with cusp-edge degeneracy. Then the operator* (23) *possesses the Fredholm property.*

Remark 2 One can use also the weights $\sigma(x)$ and $\gamma(x)$ depending on x. Then the conjugation will be made with the help of the function $r^{\gamma(x)}\exp\{-\sigma(x)\psi(r)\}$. The details are left to the reader.

2.5 Examples

In conclusion of this section we shall present two examples. In the first example we consider an operator which satisfies all conditions of Theorem 5 so included in the situation of applicability of the developed theory. The second illustrates the principal difference between situation with isolated singularities from the wedge-situation and the nesesity of developing another approach to wedge problems so that such operators could be included into the theory.

Example 1. Clearly, it is sufficient to consider the local form of the operator near the edge and to show that for such kind of operator the ellipticity condition is valid.

Consider the (local) wedge of the form

$$W = S^1 \times C,$$

where

$$C = \left(S^1 \times [0,1)\right) / \left(S^1 \times \{0\}\right)$$

is a direct circular cone. Consider the operator

$$\hat{H} = \left(r^{k+1}\frac{\partial}{\partial r}\right)^2 + a(x)\frac{\partial^2}{\partial \varphi^2} + \left(r^{k+1}\right)^2 - 1$$

on W. Then it is evident that the (operator-valued) symbol

$$\hat{H}\left(y,x,\eta,\xi\right) = a(x)\frac{\partial^2}{\partial \varphi^2} - \left(\xi^2 + \eta^2 + 1\right)$$

is an invertible operator on S^1, so that all conditions of Theorem 5 are valid.

Example 2. Consider the Beltrami-Laplace equation on the model cusp-type wedge, that is, on the direct product of circular simple cusp and the real line $\mathbf{R}_x$:

$$\left\{r^{-4}\left[\left(r^2\frac{\partial}{\partial r}\right)^2 + a\left(x\right)\frac{\partial^2}{\partial \varphi^2}\right] + \frac{\partial^2}{\partial x^2}\right\}u$$

$$= r^{-4}\left\{\left(r^2\frac{\partial}{\partial r}\right)^2 + a\left(x\right)\frac{\partial^2}{\partial \varphi^2} + \left(r^2\frac{\partial}{\partial x}\right)^2\right\}u = f \tag{24}$$

with smooth positive function $a\left(x\right)$, then we shall see that the corresponding symbol

$$\hat{H}\left(y,x,\eta,\xi\right) = a\left(x\right)\frac{\partial^2}{\partial \varphi^2} - \left(\eta^2 + \xi^2\right) \tag{25}$$

is not invertible for $\xi = \eta = 0$. Actually, the equation

$$a\left(x\right)\frac{\partial^2}{\partial \varphi^2}u\left(\varphi\right) = f\left(\varphi\right). \tag{26}$$

has the function

$$u_0\left(x\right) \equiv 1$$

as the generator of its kernel. So, the theory developed in the previous section is inapplicable to this equation. What is more, the introduction of exponential weights cannot improve the situation. Actually, the condition under which we can construct a regularizer in the weighted spaces $E^s_{\sigma,\gamma}$ is the invertibility of the symbol (25) on the line $\operatorname{Re}\eta = \sigma$. Putting $\eta = \zeta + i\sigma$, $\zeta \in \mathbf{R}$, we obtain the symbol

$$\hat{H}(y, x, \zeta + i\sigma, \xi) = a(x)\frac{\partial^2}{\partial\varphi^2} - (\zeta^2 + \xi^2) + \sigma^2 - 2i\zeta\sigma.$$

This operator is not invertible at points (ζ, ξ) such that

$$\begin{aligned} \zeta &= 0, \\ \xi &= \pm\sqrt{\sigma^2 - k^2 a(x)} \text{ for some } k \in \mathbf{Z}. \end{aligned}$$

3 Operator algebra (second version)

3.1 Commutaion relations and composition formula

Let A_1, A_2, B_1, B_2 be as above:

$$\begin{aligned} A_1 &= -i\varphi(r)\frac{\partial}{\partial r}, \\ A_2 &= -i\varphi(r)\frac{\partial}{\partial x}, \\ B_1 &= r, \\ B_2 &= x. \end{aligned} \tag{27}$$

The commutation relations and the properties of unitary groups corresponding to operators (27) were investigated in Section 2. Denote by $Op_c(C)$ the local algebra on the cusp

$$C = [0, \infty) \times \Omega / \{0\} \times \Omega,$$

consisting of operators of the form

$$\hat{H}\left(\overset{2}{B_1}, \overset{1}{A_1}\right),$$

where $\hat{H}(y, \eta)$ is a C^∞-function with values in the space of pseudodifferential operators on the manifold Ω satisfying the estimate

$$\left\| D_y^\alpha D_\eta^\beta \hat{H}(y, \eta) u \right\|_s \le C_{\alpha\beta} \left\| \left(1 + \eta^2 + \Delta\right)^{\frac{m-|\beta|}{2}} u \right\|_s .$$

with some m for any function $u \in H^{s+m-|\beta|}(\Omega)$. Here Δ is a positive Beltrami-Laplace operator on Ω.

Now denote by

$$M_{\mathrm{loc}} = X \times C$$

a local wedge. Consider a function

$$\hat{F}(x,\xi) \in C^{\infty}(T^{*}X, Op_{c}(C)) \tag{28}$$

subject to the estimates

$$\left\| D_x^{\alpha} D_{\xi}^{\beta} \hat{F}(x,\xi)\, u \right\|_{s,\sigma,\gamma} \le C_{\alpha\beta} \left\| \left(1 + |\xi|^2 + A_1^2 + \Delta\right)^{\frac{m-|\beta|}{2}} u \right\|_{s,\sigma,\gamma}$$

for any function $u \in E^{s}_{\sigma,\gamma}(C)$. The space of such functions (with different values of m) we denote by $S^m(M_{\mathrm{loc}})$. We want to assign the operator

$$\hat{F}\left(\overset{2}{B_2}, \overset{1}{A_2}\right) \tag{29}$$

to this function by substituting the ordered tuple $\left(\overset{2}{B_2}, \overset{1}{A_2}\right)$ instead of variables (x,ξ). This procedure is, however, ambiguous, since the operator $A_2 = -i\varphi(r)\,\partial/\partial x$ *does not commute with the elements of algebra* $Op_c(C)$ *of coefficients.* So, to avoid this ambiguity, one must specify in the explicit way the order of action not only for operators B_2 and A_2, but also for values of the function (29) itself. We shall use the following ordering:

$$\overset{3}{\hat{F}}\left(\overset{2}{B_2}, \overset{1}{A_2}\right). \tag{30}$$

The exact definition of operator (30) is

$$\overset{3}{\hat{F}}\left(\overset{2}{B_2}, \overset{1}{A_2}\right) = \int \tilde{F}(\tau_1,\tau_2)\, e^{i\tau_1 B_2} e^{i\tau_2 A_2}\, d\tau_1 d\tau_2, \tag{31}$$

where $\tilde{F}(\tau_1,\tau_2)$ is the Fourier transform of function (28). It is easy to see that if

$$\hat{F}(x,\xi) = \hat{G}\left(\overset{2}{B_1}, \overset{1}{A_1}, x, \xi\right),$$

then

$$\overset{3}{\hat{F}}\left(\overset{2}{B_2}, \overset{1}{A_2}\right) = \hat{G}\left(\overset{4}{B_1}, \overset{3}{A_1}, \overset{2}{B_2}, \overset{1}{A_2}\right) = \hat{G}\left(\overset{3}{B_1}, \overset{2}{A_1}, \overset{3}{B_2}, \overset{1}{A_2}\right) \tag{32}$$

since A_1 and B_2 commute with each other. We remark that this ordering *differs* from that used in the previous section.

We do not present here the corresponding continuability theorems for operators of the form (30) since formula (32) reduces the definition of such operators to the case of functions of four operators (27).

Let us begin establishing the composition formula for operators of the form (31). First, we now compute the operators of left ordered representation for the tuple $\left(\overset{2}{B_2}, \overset{1}{A_2}\right)$ on the set of operator-valued symbols from $Op_c\,(C)$. Since the operator B_2 commutes with elements of $Op_c\,(C)$, we have

$$B_2 \left[\!\left[\overset{3}{\hat{F}} \left(\overset{2}{B_2}, \overset{1}{A_2}\right) \right]\!\right] = \overset{2}{B_2}\, \overset{3}{\hat{F}} \left(\overset{2}{B_2}, \overset{1}{A_2}\right),$$

and, hence, the operator l_{B_2} equals

$$l_{B_2} = x.$$

Consider now the composition

$$A_2 \left[\!\left[\overset{3}{\hat{F}} \left(\overset{2}{B_2}, \overset{1}{A_2}\right) \right]\!\right].$$

The operator A_2 does not commute with $Op_c\,(C)$ since it contains the factor $\varphi\,(r)$ which has to be considered as an operator on a model cusp. However, we have

$$\begin{aligned} A_2 \left[\!\left[\overset{3}{\hat{F}} \left(\overset{2}{B_2}, \overset{1}{A_2}\right) \right]\!\right] &= -i\varphi\,(r) \frac{\partial}{\partial x} \left[\!\left[\overset{3}{\hat{F}} \left(\overset{2}{B_2}, \overset{1}{A_2}\right) \right]\!\right] \\ = \varphi\,(r) \left[\!\left[\overset{3}{\overline{-i\frac{\partial}{\partial x}\overset{4}{\hat{F}}}} \left(\overset{2}{B_2}, \overset{1}{A_2}\right) \right]\!\right] &= \overset{4}{\hat{G}} \left(\overset{2}{B_2}, \overset{3}{A_2}, \overset{1}{A_2}\right), \end{aligned} \tag{33}$$

where

$$\hat{G}\,(x, \xi', \xi'') = \xi' \left(\mathrm{ad}_{\varphi(r)}\, \hat{F}\right)(x, \xi''),$$

and

$$\left(\mathrm{ad}_{\varphi(r)}\, \hat{F}\right)(x, \xi'') = \varphi\,(r)\, \hat{F}\,(x, \xi'')\, \varphi^{-1}\,(r).$$

Commuting the operators $\overset{3}{A_2}$ and $\overset{2}{B_2}$ on the right in (33) in the usual way, we obtain

$$\begin{aligned} A_2 \left[\!\!\left[\overset{3}{\hat F}\left(\overset{2}{B_2},\overset{1}{A_2}\right)\right]\!\!\right] &= \overset{1}{A_2}\,\overset{3}{\overline{\left(\mathrm{ad}_{\varphi(r)}\hat F\right)}}\left(\overset{2}{B_2},\overset{1}{A_2}\right) \\ &\quad - i\,\overset{2}{\overline{\varphi(r)}}\,\overset{3}{\overline{\frac{\partial}{\partial x}\left(\mathrm{ad}_{\varphi(r)}\hat F\right)}}\left(\overset{2}{B_2},\overset{1}{A_2}\right). \end{aligned}$$

So, the operator l_{A_2} of the left ordered representation is

$$l_{A_2} = \mathrm{ad}_{\varphi(r)}\left(\xi - iR_{\varphi(r)}\frac{\partial}{\partial x}\right) = \mathrm{ad}_{\varphi(r)}\xi - iL_{\varphi(r)}\frac{\partial}{\partial x},$$

where $R_{\varphi(r)}$ $(L_{\varphi(r)})$ is the operator of right (left) multiplication by $\varphi(r)$.

Remark 3 Let us write down the action of operator $\mathrm{ad}_{\varphi(r)}$ to a function $\hat F\left(\overset{2}{B_1},\overset{1}{A_1}\right)$ with $\hat F(y,\eta) \in S^m$ (which may depend on additional parameters (x,ξ).) We have

$$\mathrm{ad}_{\varphi(r)}\hat F\left(\overset{2}{B_1},\overset{1}{A_1}\right) = \hat F\left(\overset{2}{\overline{\mathrm{ad}_{\varphi(r)}B_1}},\overset{1}{\overline{\mathrm{ad}_{\varphi(r)}A_1}}\right) = \hat F\left(\overset{2}{B_1},\overset{1}{\overline{A_1 + i\varphi'(r)}}\right).$$

Using the Taylor formula for noncommutative operators with respect to the second argument, we obtain

$$\begin{aligned} \mathrm{ad}_{\varphi(r)}\hat F\left(\overset{2}{B_1},\overset{1}{A_1}\right) &= \hat F\left(\overset{2}{B_1},\overset{1}{\overline{A_1 + i\varphi'(r)}}\right) \\ &= \hat F\left(\overset{2}{B_1},\overset{1}{A_1}\right) + i\,\overset{1}{\overline{\varphi'(B_1)}}\,\frac{\partial \hat F}{\partial \eta}\left(\overset{2}{B_1},\overset{1}{A_1}\right) + \dots \end{aligned}$$

so *the symbol of the operator* $\hat F\left(\overset{2}{B_1},\overset{1}{A_1}\right)$ *is not changed modulo* S^{m-1} *by application of* $\mathrm{ad}_{\varphi(r)}$ at least if $k > 0$.

So, the operator $\mathrm{ad}_{\varphi(r)}\,\xi$ can be represented in the form

$$\mathrm{ad}_{\varphi(r)}\,\xi = \xi\left(1 + \sum_{j=1}^{N}\hat P_j + \hat P'_N\right), \tag{34}$$

where the operators $\hat P_j$ diminish the index m of the symbol by j in the scale $S^m\,(M_{\mathrm{loc}})$ and contain at least j factors $\varphi'(r)$ (the same is valid for $\hat P'_N$).

Now we can formulate the main theorem of this subsection.

Theorem 6 *The following composition formula is valid:*

$$smbl\left\{\left[\!\left[\overset{3}{\hat{F}}\left(\overset{2}{B_2},\overset{1}{A_2}\right)\right]\!\right]\circ\left[\!\left[\overset{3}{\hat{G}}\left(\overset{2}{B_2},\overset{1}{A_2}\right)\right]\!\right]\right\}=\overset{3}{\hat{F}}\left(\overset{2}{\overline{l_{B_2}}},\overset{1}{\overline{l_{A_2}}}\right)\hat{G}(x,\xi)$$

This theorem states that the set of operators $\overset{3}{\hat{F}}\left(\overset{2}{B_2},\overset{1}{A_2}\right)$ for all $\hat{F}(x,\xi)\in S^m(M_{\mathrm{loc}})$ forms an algebra. This algebra will be referred below as a *local wedge cusp-type algebra (LWC-algebra)* and denoted by $Op(M_{\mathrm{loc}})$. The following affirmation shows that this is an algebra with involution.

Theorem 7 *If $\hat{H}(x,\xi)$ is a symbol from $S^m(M_{\mathrm{loc}})$, then the adjoint operator to*

$$\overset{3}{\hat{H}}\left(\overset{2}{B_2},\overset{1}{A_2}\right)$$

is given by

$$\overset{1}{\hat{H}^*}\left(\overset{2}{B_2},\overset{3}{A_2}\right),$$

where the star denotes the conjugation in the algebra of coefficients.

3.2 Regularizers and finiteness theorem

In this subsection, we shall construct a regularizer for the equation

$$\overset{3}{\hat{H}}\left(\overset{2}{B_2},\overset{1}{A_2}\right)u=f.$$

As usual, we search for a regularizer in the form

$$\hat{R}=\overset{3}{\hat{R}}\left(\overset{2}{B_2},\overset{1}{A_2}\right),$$

that is,

$$\left[\!\left[\overset{3}{\hat{H}}\left(\overset{2}{B_2},\overset{1}{A_2}\right)\right]\!\right]\circ\left[\!\left[\overset{3}{\hat{R}}\left(\overset{2}{B_2},\overset{1}{A_2}\right)\right]\!\right]=\mathbf{1}. \tag{35}$$

Using the result of Theorem 6, for the symbol of the regularizer we obtain the equation

$$\overset{3}{\hat{H}}\left(\overset{2}{\overline{l_{B_2}}},\overset{1}{\overline{l_{A_2}}}\right)\hat{R}(x,\xi)=\mathbf{1},$$

or

$$\overset{3}{\hat{H}}\left(\overset{2}{x}, \overline{\mathrm{ad}_{\varphi(r)}\xi - iL_{\varphi(r)}\frac{\partial}{\partial x}}^{1}\right)\hat{R}(x,\xi) = \mathbf{1}.$$

Using again the Taylor formula for noncommuting operators together with (34) and expanding $\hat{R}(x,\xi)$ in the asymptotic series with respect to the scale $S^m(M_{\mathrm{loc}})$

$$\hat{R}(x,\xi) \simeq \sum_{j=0}^{\infty} \hat{R}_j(x,\xi),$$

where $\hat{R}_j(x,\xi) \in S^{m-j}(M_{\mathrm{loc}})$, we obtain the following recurrent system of equation for the functions $\hat{R}_j(x,\xi)$:

$$\begin{aligned} \hat{H}(x,\,\xi)\,\hat{R}_0(x,\xi) &= \mathbf{1}, \\ \hat{H}(x,\,\xi)\,\hat{R}_k(x,\xi) &= \sum_{j=1}^{k} \mathcal{F}_j \hat{R}_{k-j}(x,\xi), \end{aligned}$$

where $\mathcal{F}_j$ are operators of order j in the scale $S^m(M_{\mathrm{loc}})$ containing j factors $\varphi(r)$ or $\varphi'(r)$.

In the case when the operator $\hat{H}(x,\,\xi)$ is invertible, all the latter equations are solvable and we obtain a local regularizer for the operator $\overset{3}{\hat{H}}\left(\overset{2}{B_2}, \overset{1}{A_2}\right)$.

Similar to the previous section, the left regularizer can be constructed from the duality reasons.

To globalize the above considerations, we again consider a differential operator $\hat{H}$ on the manifold M with the degeneration of the cusp type of order k on the edge X. This operator acts in spaces (23) as a continuous operator.

Definition 3 The operator (23) is called to be *elliptic* if:

- $\hat{H}$ is elliptic on $M \setminus X$;
- the corresponding symbol $\hat{H}(x,\xi)$ is invertible in a neighborhood of the edge X for all real values of ξ as an operator in the corresponding weighted cusp-type Sobolev spaces $E^{s}_{\sigma,\gamma}$.

Then the following statement is valid.

Theorem 8 *Suppose that operator* (23) *is elliptic. Then this operator possesses the Fredholm property.*

3.3 Example

Let us show that the operator considered in the end of the previous section is elliptic in this version as well, that is, satisfies the conditions of Theorem 8. We recall that this operator has the form

$$\hat{H} = \left(r^{k+1}\frac{\partial}{\partial r}\right)^2 + a(x)\frac{\partial^2}{\partial\varphi^2} + \left(r^{k+1}\right)^2 - 1$$

on the wedge

$$W = S^1 \times \left(S^1 \times [0,1]\right) / \left(S^1 \times \{0\}\right).$$

The operator-valued symbol of the operator under consideration is

$$\hat{H}(x,\xi) = \left(r^{k+1}\frac{\partial}{\partial r}\right)^2 + a(x)\frac{\partial^2}{\partial\varphi^2} - \left(\xi^2 + 1\right).$$

This operator is clearly invertible since it is strictly negative in the corresponding Hilbert space. So, the conditions of Theorem 8 are fulfilled.

4 Extension of the operator algebra

4.1 Main definitions

Let us now consider the case when the symbol $\hat{H}(x,\xi)$ of operator (23) is not invertible everywhere on $T^*\mathbf{R}^n$. In fact, we suppose that $\hat{H}(x,\xi)$ is a Fredholm operator everywhere on $T^*(\mathbf{R}^n)$, but this operator can even have a nonvanishing index. In this case one needs to equip the corresponding operator with the appropriate *boundary* and *coboundary* operators, that is, to include this operator into the matrix operator of the form

$$\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}.$$

Here:

- D_{11} is an operator on the manifold M_{loc} of the form

  $$D_{11} = \overset{3}{\hat{H}}\left(\overset{2}{B_2}, \overset{1}{A_2}\right)$$

  with symbol $\hat{H}(x,\xi)$ being a function on $T^*\mathbf{R}^n$ with values in the local cusp algebra. These operators will be called *exterior*.

- D_{21} is a "*boundary*" operator. Such operators must map the space of functions on M_{loc} in the space of sections of some finite-dimensional bundle. If we shall treat a function space on M_{loc} as an infinite-dimensional bundle with function spaces on the model cusp as fibers, then the simplest operator of this kind is a bundle homomorphism, taking a function on the model cusp $\mathcal{F}(C)$ whose fibers are function spaces on the model cusp C to a finite-dimensional vector space. It is clear that each component of the resulting vector viewed as a function on $\mathcal{F}(C)$, determines a functional over this space. Hence, each operator of this type can be represented in the form

$$\begin{aligned} D_{21}[v(r,\omega,x)] &= \left(\int\limits_C v(r,\omega,x)\,\psi_j(r,\omega,x)\,r^{n-1}\,dr d\omega,\ j=1.\dots N_1\right) \\ &= (\langle v(\cdot,x),\psi_j(\cdot,x)\rangle,\ j=1.\dots N_1) \end{aligned}$$

and is determined by a tuple of functions

$$\{\psi_j(r,\omega,x),\ j=1.\dots N_1\}$$

belonging to the dual space to $\mathcal{F}(C)$. Such operators are denoted by P_ψ and called *projectors*. They map sections of the infinite-dimensional bundle $\mathcal{F}(C)$ to sections of finite-dimensional bundle $\mathcal{B}$. The general *boundary operators* are, by definition, operators of the form

$$P_\psi \circ \overset{3}{\hat{G}}\left(\overset{2}{B_2}, \overset{1}{A_2}\right)$$

for a projector P_ψ and an exterior operator $\overset{3}{\hat{G}}\left(\overset{2}{B_2}, \overset{1}{A_2}\right)$. The function

$$P_{\psi(\cdot,x)} \circ \hat{G}(x,\xi)$$

will be called a *symbol* of the boundary operator $P_\psi \circ \overset{3}{\hat{G}}\left(\overset{2}{B_2}, \overset{1}{A_2}\right)$.

- D_{12} is a "*coboundary*" operator. The simplest example of such an operator is a bundle homomorphism from a finite-dimensional bundle $\mathcal{B}$ to an infinite-dimensional bundle $\mathcal{F}(C)$. For such operators we have[7]

$$e_j = \left(0,\dots,\overset{j}{1},\dots,0\right) \mapsto \psi_j(r,\omega,x),\ j=0,1,2,\dots,N_2,$$

[7]For simplicity, we consider here trivial bundles; the general case makes no difference.

where $N_2 = \dim \mathcal{B}_x$, and the operator is

$$D_{12}\begin{pmatrix} c_1 \\ \vdots \\ c_{N_2} \end{pmatrix} = c_1\psi_1(r,\omega,x) + \ldots + c_{N_2}\psi_{N_2}(r,\omega,x).$$

Such operators are called *coprojectors* and we denote them by

$$D_{12} = P^*_\psi,\ \psi = (\psi_1(r,\omega,x),\ldots,\psi_N(r,\omega,x)).$$

General *coboundary operators* are operators of the form

$$\overset{3}{\hat{F}}\left(\overset{2}{B_2},\overset{1}{A_2}\right)\circ P^*_\psi$$

for some operator $\overset{3}{\hat{F}}\left(\overset{2}{B_2},\overset{1}{A_2}\right)$ of exterior type and some coprojector P^*_ψ. The function

$$\hat{F}(x,\xi)\circ P^*_{\psi(\cdot,x)}$$

will be called a *symbol* of the coboundary operator $\overset{3}{\hat{F}}\left(\overset{2}{B_2},\overset{1}{A_2}\right)\circ P^*_\psi$.

- D_{22} is a ψDO on X in sections of a finite-dimensional bundle over X.

So, we consider matrix operators of the form

$$\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} : \begin{pmatrix} \Gamma(\mathcal{F}(C)) \\ \Gamma(\mathcal{B}) \end{pmatrix} \to \begin{pmatrix} \Gamma(\mathcal{F}(C)) \\ \Gamma(\mathcal{B}) \end{pmatrix}.$$

where by Γ we denote spaces of sections of dorresponding bundles, and

$$\begin{aligned} D_{11} &= \overset{3}{\hat{H}}\left(\overset{2}{B_2},\overset{1}{A_2}\right), \\ D_{12} &= \overset{3}{\hat{F}}\left(\overset{2}{B_2},\overset{1}{A_2}\right)\circ P^*_{\psi^{(1)}}, \\ D_{21} &= P_{\psi^{(2)}}\circ \overset{3}{\hat{G}}\left(\overset{2}{B_2},\overset{1}{A_2}\right), \\ D_{22} &= B\left(\overset{2}{x}, \overset{1}{\overline{-i\frac{\partial}{\partial x}}}\right), \end{aligned}$$

and ψ^1, ψ^2 are some tuples of function described above. Here we do not specify the smoothness properties of the considered function spaces; this will be done below.

4.2 Continuability theorems

Now we shall investigate the action of all the above operators in the function spaces $E^s_{\sigma,\gamma}$. To do this, we need the following affirmations:

Lemma 1 *Let* $\left(\psi_j^{(1)},\ j=1,\ldots,N_1\right)$ *be elements from* $E^\infty_{\sigma,\gamma}$. *Then the operator*

$$P^*_{\psi^{(1)}}\ :\ [H^s(X)]^{N_1} \to E^s_{\sigma,\gamma}$$

is continuous for any value of s.

Lemma 2 *Let* $\left(\psi_j^{(2)},\ j=1,\ldots,N_2\right)$ *be elements from* $E^\infty_{-\sigma,-\gamma}$. *Then the operator*

$$P_{\psi^{(2)}}\ :\ E^s_{\sigma,\gamma} \to [H^s(X)]^{N_2}$$

is continuous for any value of s.

The proof of both these affirmations is quite evident.

Combining these affirmations with the result on boundedness of the operators of the exterior type, we arrive at the following statement:

Theorem 9 *Let* $\psi^{(1)} \in E^\infty_{\sigma,\gamma}$ *and* $\psi^{(2)} \in E^\infty_{-\sigma,-\gamma}$. *Then the operator*

$$\mathcal{M}\ :\ \begin{pmatrix} E^s_{\sigma,\gamma} \\ [H^\sigma(X)]^{N_1} \end{pmatrix} \to \begin{pmatrix} E^{s-m}_{\sigma,\gamma} \\ [H^{\sigma-l}(X)]^{N_2} \end{pmatrix}$$

$$\mathcal{M} = \begin{pmatrix} \overset{3}{\hat{H}}\left(\overset{2}{B_2},\overset{1}{A_2}\right) & \overset{3}{\hat{F}}\left(\overset{2}{B_2},\overset{1}{A_2}\right)\circ P^*_{\psi^{(1)}} \\ P_{\psi^{(2)}}\circ\overset{3}{\hat{G}}\left(\overset{2}{B_2},\overset{1}{A_2}\right) & B\left(\overset{2}{x},\overset{1}{\overline{-i\frac{\partial}{\partial x}}}\right) \end{pmatrix}, \tag{36}$$

is continuous if

$$\operatorname{ord}\ \overset{3}{\hat{H}}\left(\overset{2}{B_2},\overset{1}{A_2}\right) = m,$$

$$\operatorname{ord}\ \overset{3}{\hat{F}}\left(\overset{2}{B_2},\overset{1}{A_2}\right) = \sigma - s + m,$$

$$\operatorname{ord}\ \overset{3}{\hat{G}}\left(\overset{2}{B_2},\overset{1}{A_2}\right) = s - \sigma + l,$$

$$\operatorname{ord} B\left(\overset{2}{x},\overset{1}{\overline{-i\frac{\partial}{\partial x}}}\right) = l.$$

4.3 Composition theorems

To construct the algebra containing operators of the form (36), we need the *pseudodifferentiality theorem.* Namely, the following affirmation is valid:

Theorem 10 (cf. [9], [10], [11], [12].) *For any exterior operator* $\overset{3}{\hat{F}}\left(\overset{2}{B_2},\overset{1}{A_2}\right)$ *the operator*

$$P_{\psi^{(2)}}\circ\overset{3}{\hat{F}}\left(\overset{2}{B_2},\overset{1}{A_2}\right)\circ P^*_{\psi^{(1)}}$$

is a pseudodifferential operator on X.

Proof. We have

$$P^*_{\psi^{(1)}}\left[c\left(x\right)\right]=\psi^{(1)}\left(r,\omega,x\right)c\left(x\right).$$

Later on,

$$\overset{3}{\hat{F}}\left(\overset{2}{B_2},\overset{1}{A_2}\right)v\left(r,\omega,x\right)=\left(\frac{1}{2\pi}\right)^{n}\int\tilde{F}\left(\tau_1,\tau_2\right)e^{i\tau_1x}e^{\tau_2\varphi(r)\partial/\partial x}v\left(r,\omega,x\right)\,d\tau_1d\tau_2$$
$$=\left(\frac{1}{2\pi}\right)^{2n}\int\left\{\int\hat{F}\left(x',\xi\right)e^{-ix'\tau_1-i\xi\tau_2}\,dx'd\xi\right\}e^{i\tau_1x}v\left(r,\omega,x+\tau_2\varphi\left(r\right)\right)\,d\tau_1d\tau_2.$$

With the change of variables

$$y=x+\tau_2\varphi\left(r\right),\tau_2=\varphi^{-1}\left(r\right)\left(y-x\right)$$

we rewrite this expression in the form

$$\left(\frac{1}{2\pi}\right)^{2n}\int\hat{F}\left(x',\xi\right)e^{-i\tau_1x'-i\varphi^{-1}(r)\xi(y-x)+i\tau_1x}v\left(r,\omega,x\right)\varphi^{-1}\left(r\right)\,dx'd\xi d\tau_1dy$$
$$=\left(\frac{1}{2\pi}\right)^{n}\int\hat{F}\left(x,\xi\right)e^{i\varphi^{-1}(r)\xi(x-y)}v\left(r,\omega,x\right)\varphi^{-1}\left(r\right)\,d\xi dy.$$

Applying this result to the function $v\left(r,\omega,x\right)=\psi^{(1)}\left(r,\omega,x\right)c\left(x\right)$, we have

$$\overset{3}{\hat{F}}\left(\overset{2}{B_2},\overset{1}{A_2}\right)P^*_{\psi^{(1)}}\left[c\left(x\right)\right]=\left(\frac{1}{2\pi}\right)^{n}\int\hat{F}\left(x,\xi\right)e^{i\varphi^{-1}(r)\xi(x-y)}\psi^{(1)}\left(r,\omega,y\right)\times c\left(y\right)\varphi^{-1}\left(r\right)\,d\xi dy.$$

Now, applying the operator $P_{\psi^{(2)}}$ to both parts of the latter relation, we arrive at the expression

$$\begin{aligned}
&P_{\psi^{(2)}} \overset{3}{\hat{F}} \left(\overset{2}{B_2}, \overset{1}{A_2} \right) P^*_{\psi^{(1)}} \left[c(x) \right] \\
&= \left(\frac{1}{2\pi} \right)^n \int \left\langle \psi^{(2)}(\cdot, x), \hat{F}(x,\xi) e^{i\varphi^{-1}(r)(x-y)} \psi^{(1)}(\cdot, y) \varphi^{-1}(r) \right\rangle c(y) \, d\xi dy \\
&= \left(\frac{1}{2\pi} \right)^n \int e^{ip(x-y)} \left\langle \psi^{(2)}(\cdot, x), \hat{F}(x, \varphi(r) p) \psi^{(1)}(\cdot, y) \right\rangle c(y) \, dp dy.
\end{aligned}$$

This is a pseudodifferential operator with the symbol

$$\left\langle \psi^{(2)}(\cdot, x), \hat{F}(x, \varphi(r) p) \psi^{(1)}(\cdot, y) \right\rangle.$$

This completes the proof.

The rest of the construction of a matrix operator algebra containing matrices of the form (36) goes in a standard way. We remark only that for such matrices to form an algebra, one must allow in the upper left and lower right cells of these matrices a finite sum of corresponding terms instead of a single term, as in (36), and add to the operator in the upper left cell of this matrices a finite sum of operators of the form

$$P^*_{\psi'} \overset{3}{\hat{F}} \left(\overset{2}{B_2}, \overset{1}{A_2} \right) P_{\psi}$$

(cf. [13], [3], [10], [11]).

Definition 4 The constructed algebra will be called an *extended local cusp algebra.* We denote it by $Op_{\text{ext}}(M)$.

4.4 Ellipticity and finiteness theorems

Let $\hat{H}(x,\xi)$ be the symbol of the operator $\overset{3}{\hat{H}} \left(\overset{2}{B_2}, \overset{1}{A_2} \right)$, that is, an operator-valued function of (x,ξ) with values of the LC-algebra. We suppose that the family $\hat{H}(x,\xi)$ is a family of Fredholm operators on the whole $T^*(\mathbf{R}^n)$. Now we shall investigate the invertibility of a symbolic matrix

$$\begin{pmatrix} \hat{H}(x,\xi) & \hat{F}(x,\xi) P^*_{\psi^{(1)}} \\ P_{\psi^{(2)}} \hat{G}(x,\xi) & B(x,p) \end{pmatrix} \tag{37}$$

of the matrix operator (36) near points where $\hat{H}(x,\xi)$ is not invertible.

To do this, we first represent the symbol $\hat{H}(x,\xi)$ of the considered operator in the splittings of the corresponding function spaces into the direct sums connected with kernel and cokernel of this symbol at some fixed point.

Second, we write down the similar decomposition for the matrix symbol (37).

Finally, we derive the conditions of invertibility of the matrix (37) and construct the inverse matrix in the explicit form.

1. We represent the operator $\hat{H}(x,\xi)$ in a special form near any point $\alpha_0 = (x_0,\xi_0)$ such that the operator

$$\hat{H}(\alpha_0) \; : \; E_1 \to E_2$$

has nontrivial kernel and cokernel.

Consider the splittings of the spaces E_1 and E_2 into the direct sums:

$$E_1 = L_1 \oplus \operatorname{Ker} \hat{H}(\alpha_0), \; E_2 = \operatorname{Im} \hat{H}(\alpha_0) \oplus L_2. \tag{38}$$

Clearly, the spaces $\operatorname{Ker} \hat{H}(\alpha_0)$ and L_2 can be identified with the complex spaces $\mathbf{C}^{N_1}$ and $\mathbf{C}^{N_2}$, where N_1 and N_2 are dimensions of the kernel and the cokernel of the operator $\hat{H}(\alpha_0)$, respectively. Then there exist natural mappings

$$\begin{array}{rcl} \pi_1 & : & E_1 \to \mathbf{C}^{N_1}, \\ i_1 & : & \mathbf{C}^{N_1} \to E_1 \end{array}$$

such that

$$\operatorname{Ker} \pi_1 = L_1, \; \operatorname{Im} i_1 = \operatorname{Ker} \hat{H}(\alpha_0), \; \pi_1 \circ i_1 = \mathbf{1}_{\mathbf{C}^{N_1}},$$

$i_1\pi_1$ is a projector to the space $\operatorname{Ker} \hat{H}(\alpha_0)$ along L_1, and $\mathbf{1} - i_1\pi_1$ is a projector to the space L_1 along $\operatorname{Ker} \hat{H}(\alpha_0)$. The decomposition of an element $u \in E_1$ into the direct sum $\mathbf{C}^{N_1} \oplus L_1$ is

$$u = \pi_1(u) \oplus (\mathbf{1} - i_1\pi_1)u \in \mathbf{C}^{N_1} \oplus L_1.$$

The identification of the space $\operatorname{Ker} \hat{H}(\alpha_0)$ with $\mathbf{C}^{N_1}$ defines a generators

$$\chi^{(1)} = {}^t\left(\chi_1^{(1)}, \ldots, \chi_{N_1}^{(1)}\right) \in \operatorname{Ker} \hat{H}(\alpha_0),$$

where t denotes the transposition. Then we have

$$i_1(c) = (c_1, \ldots, c_{N_1}) \begin{pmatrix} \chi_1^{(1)} \\ \vdots \\ \chi_{N_1}^{(1)} \end{pmatrix} = c_1\chi_1^{(1)} + \ldots + c_{N_1}\chi_{N_1}^{(1)}$$

The mapping $\pi_1 : E_1 \to \mathbf{C}^{N_1}$ defines an element

$$\chi_*^{(1)}(r,\omega) = \left(\chi_{*1}^{(1)}(r,\omega), \ldots, \chi_{*N_1}^{(1)}(r,\omega)\right)$$

of E_1^*, so that

$$\pi_1(v) = \langle \chi_*^{(1)}, v\rangle \left(\left\langle \chi_{*1}^{(1)}, v\right\rangle, \ldots, \left\langle \chi_{*N_1}^{(1)}, v\right\rangle\right).$$

The operators i_2 and π_2, and functions $\chi^{(2)}(r,\omega)$ and $\chi_*^{(2)}(r,\omega)$ are defined in the similar way by the splitting

$$E_2 = \operatorname{Im} \hat{H}(\alpha_0) \oplus L_2 \simeq \operatorname{Im} \hat{H}(\alpha_0) \oplus \mathbf{C}^{N_2}.$$

Now the matrix of the operator $\hat{H}(x,\xi)$ with respect to decompositions (38) takes the form

$$\begin{pmatrix} (\mathbf{1} - i_2\pi_2)\hat{H}(\alpha) & (\mathbf{1} - i_2\pi_2)\hat{H}(\alpha) i_1 \\ \pi_2\hat{H}(\alpha) & \pi_2\hat{H}(\alpha) i_1 \end{pmatrix}, \tag{39}$$

which equals to

$$\begin{pmatrix} (\mathbf{1} - i_2\pi_2)\hat{H}(\alpha_0) & 0 \\ 0 & 0 \end{pmatrix}$$

at α_0 since $\hat{H}(\alpha_0) i_1 = 0$ and $\pi_2\hat{H}(\alpha) = 0$.

As

$$i_1 = P^*_{\chi_*^{(1)}},\ \pi_1 = P_{\chi^{(1)}}, \tag{40}$$

$$i_2 = P^*_{\chi_*^{(2)}},\ \pi_2 = P_{\chi^{(2)}}, \tag{41}$$

the operator matrix (39) belongs to the extended local cusp algebra (note that $\pi_2\hat{H}(\alpha) i_1$ is a symbol of some pseudodifferential operator due to Theorem 10) .

2. Now let us consider the matrix of operator (37) in terms of decomposition (38). We have

$$\hat{F}(x,\xi) P^*_{\psi^{(1)}} c = (\mathbf{1} - i_2\pi_2)\hat{F}(x,\xi) P^*_{\psi^{(1)}} c \oplus \pi_2\hat{F}(x,\xi) P^*_{\psi^{(1)}} c,$$

and

$$\begin{aligned} P_{\psi^{(2)}}\hat{G}(x,\xi) u\Big|_{u\in L_1} &= P_{\psi^{(2)}}\hat{G}(x,\xi)(\mathbf{1} - i_1\pi_1) u, \\ P_{\psi^{(2)}}\hat{G}(x,\xi) u\Big|_{u=i_1(c)} &= P_{\psi^{(2)}}\hat{G}(x,\xi) i_1(c), \end{aligned}$$

which gives us the decompositions of operators $\hat{F}(x,\xi)\,P^*_{\psi^{(1)}}$ and $P_{\psi^{(2)}}\hat{G}(x,\xi)$ with respect to splittings (38), respectively. So, the matrix (37) becomes

$$\begin{pmatrix} (\mathbf{1}-i_2\pi_2)\,\hat{H}(\alpha) & (\mathbf{1}-i_2\pi_2)\,\hat{H}(\alpha)\,i_1 & (\mathbf{1}-i_2\pi_2)\,\hat{F}(\alpha)\,P^*_{\psi^{(1)}} \\ \pi_2\hat{H}(\alpha)(\mathbf{1}-i_1\pi_1) & \pi_2\hat{H}(\alpha)\,i_1 & \pi_2\hat{F}(\alpha)\,P^*_{\psi^{(1)}} \\ P_{\psi^{(2)}}\hat{G}(\alpha)(\mathbf{1}-i_1\pi_1) & P_{\psi^{(2)}}\hat{G}(\alpha)\,i_1 & B(x,p) \end{pmatrix}, \tag{42}$$

or, in view of relations (40) and (41),

$$\begin{pmatrix} (\mathbf{1}-i_2\pi_2)\,\hat{H}(\alpha) & (\mathbf{1}-i_2\pi_2)\,\hat{H}(\alpha)\,P^*_{\chi^{(1)}} & (\mathbf{1}-i_2\pi_2)\,\hat{F}(\alpha)\,P^*_{\psi^{(1)}} \\ P_{\chi^{(2)}_*}\hat{H}(\alpha)(\mathbf{1}-i_1\pi_1) & P_{\chi^{(2)}_*}\hat{H}(\alpha)\,P^*_{\chi^{(1)}} & P_{\chi^{(2)}_*}\hat{F}(\alpha)\,P^*_{\psi^{(1)}} \\ P_{\psi^{(2)}}\hat{G}(\alpha)(\mathbf{1}-i_1\pi_1) & P_{\psi^{(2)}}\hat{G}(\alpha)\,P^*_{\chi^{(1)}} & B(x,p) \end{pmatrix}.$$

As it follows from Theorem 10, all the operators in the right lower 2×2-block are pseudodifferential operators in sections of finite-dimensional bundles over X. For α close to α_0, the operator in the upper left cell of the latter matrix is invertible as an operator from L_1 to L_1. We denote this operator by $\hat{H}_0(\alpha)$.

3. Let us derive the invertibility conditions for matrix (42). Denote by (u_0,u_1) (resp., (f_0,f_1)) the components of the function u in the direct sum

$$E_1 = L_1 \oplus \operatorname{Ker}\hat{H}(\alpha_0)$$

(resp., of the function f in the direct sum

$$E_2 = \operatorname{Im}\hat{H}(\alpha_0) \oplus L_2).$$

Then to construct the inverse element to (42) one has to solve the following system of equations:

$$\mathcal{A}\begin{pmatrix} u_0 \\ u_1 \\ c \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ d \end{pmatrix},$$

where $\mathcal{A}$ is a matrix of the form

$$\begin{pmatrix} \hat{H}_0(\alpha) & (\mathbf{1}-i_2\pi_2)\,\hat{H}(\alpha)\,P^*_{\chi^{(1)}} & (\mathbf{1}-i_2\pi_2)\,\hat{F}(\alpha)\,P^*_{\psi^{(1)}} \\ P_{\chi^{(2)}_*}\hat{H}(\alpha)(\mathbf{1}-i_1\pi_1) & P_{\chi^{(2)}_*}\hat{H}(\alpha)\,P^*_{\chi^{(1)}} & P_{\chi^{(2)}_*}\hat{F}(\alpha)\,P^*_{\psi^{(1)}} \\ P_{\psi^{(2)}}\hat{G}(\alpha)(\mathbf{1}-i_1\pi_1) & P_{\psi^{(2)}}\hat{G}(\alpha)\,P^*_{\chi^{(1)}} & B(x,p) \end{pmatrix}$$

Let us derive the solution to this equation.

The first equation of the latter system is

$$\hat{H}_0(\alpha)\,u_0 + (1 - i_2\pi_2)\,\hat{H}(\alpha)\,P^*_{\chi^{(1)}}u_1 + (1 - i_2\pi_2)\,\hat{F}(\alpha)\,P^*_{\psi^{(1)}}c = f_0,$$

and we obtain

$$u_0 = \hat{H}_0^{-1}(\alpha)\,f_0 - \hat{H}_0^{-1}(\alpha)\,(1 - i_2\pi_2)\,\hat{H}(\alpha)\,P^*_{\chi^{(1)}}u_1 - \hat{H}_0^{-1}(\alpha)\,(1 - i_2\pi_2)\,\hat{F}(\alpha)\,P^*_{\psi^{(1)}}c.$$

Substituting the latter relation into the second and third equations of the system in question, we arrive to the equations

$$\begin{aligned}&\left[P_{\chi_*^{(2)}}\hat{H}(\alpha)\,P^*_{\chi^{(1)}} - P_{\chi_*^{(2)}}\hat{H}(\alpha)\,(1 - i_1\pi_1)\,\hat{H}_0^{-1}(\alpha)\,(1 - i_2\pi_2)\,\hat{H}(\alpha)\,P^*_{\chi^{(1)}}\right]u_1\\ &+ \left[P_{\chi_*^{(2)}}\hat{F}(\alpha)\,P^*_{\psi^{(1)}} - P_{\chi_*^{(2)}}\hat{H}(\alpha)\,(1 - i_1\pi_1)\,\hat{H}_0^{-1}(\alpha)\,(1 - i_2\pi_2)\,\hat{F}(\alpha)\,P^*_{\psi^{(1)}}\right]c\\ &= f_1 - P_{\chi_*^{(2)}}\hat{H}(\alpha)\,(1 - i_1\pi_1)\,\hat{H}_0^{-1}(\alpha)\,f_0,\end{aligned}$$

and

$$\begin{aligned}&\left[P_{\psi^{(2)}}\hat{G}(\alpha)\,P^*_{\chi^{(1)}} - P_{\psi^{(2)}}\hat{G}(\alpha)\,(1 - i_1\pi_1)\,\hat{H}_0^{-1}(\alpha)\,(1 - i_2\pi_2)\,\hat{H}(\alpha)\,P^*_{\chi^{(1)}}\right]u_1\\ &+ \left[B(x,p) - P_{\psi^{(2)}}\hat{G}(\alpha)\,(1 - i_1\pi_1)\,\hat{H}_0^{-1}(\alpha)\,(1 - i_2\pi_2)\,\hat{F}(\alpha)\,P^*_{\psi^{(1)}}\right]c\\ &= d - P_{\psi^{(2)}}\hat{G}(\alpha)\,(1 - i_1\pi_1)\,\hat{H}_0^{-1}(\alpha)\,f_0,\end{aligned}$$

with respect to the unknowns u_1 and c. These equations can be written in the form of one matrix equation

$$\begin{pmatrix}\Delta_{11} & \Delta_{12}\\ \Delta_{21} & \Delta_{22}\end{pmatrix}\begin{pmatrix}u_1\\ c\end{pmatrix} = \begin{pmatrix}F_1\\ D\end{pmatrix}, \tag{43}$$

where

$$\begin{aligned}\Delta_{11} &= P_{\chi_*^{(2)}}\hat{H}(\alpha)\,P^*_{\chi^{(1)}} - P_{\chi_*^{(2)}}\hat{H}(\alpha)\,(1 - i_1\pi_1)\,\hat{H}_0^{-1}(\alpha)\,(1 - i_2\pi_2)\,\hat{H}(\alpha)\,P^*_{\chi^{(1)}},\\ \Delta_{12} &= P_{\chi_*^{(2)}}\hat{F}(\alpha)\,P^*_{\psi^{(1)}} - P_{\chi_*^{(2)}}\hat{H}(\alpha)\,(1 - i_1\pi_1)\,\hat{H}_0^{-1}(\alpha)\,(1 - i_2\pi_2)\,\hat{F}(\alpha)\,P^*_{\psi^{(1)}},\\ \Delta_{21} &= P_{\psi^{(2)}}\hat{G}(\alpha)\,P^*_{\chi^{(1)}} - P_{\psi^{(2)}}\hat{G}(\alpha)\,(1 - i_1\pi_1)\,\hat{H}_0^{-1}(\alpha)\,(1 - i_2\pi_2)\,\hat{H}(\alpha)\,P^*_{\chi^{(1)}},\\ \Delta_{22} &= B(x,p) - P_{\psi^{(2)}}\hat{G}(\alpha)\,(1 - i_1\pi_1)\,\hat{H}_0^{-1}(\alpha)\,(1 - i_2\pi_2)\,\hat{F}(\alpha)\,P^*_{\psi^{(1)}}\end{aligned}$$

are pseudodifferential operators on X and

$$\begin{aligned}F_1 &= f_1 - P_{\chi_*^{(2)}}\hat{H}(\alpha)\,(1 - i_1\pi_1)\,\hat{H}_0^{-1}(\alpha)\,f_0,\\ D &= d - P_{\psi^{(2)}}\hat{G}(\alpha)\,(1 - i_1\pi_1)\,\hat{H}_0^{-1}(\alpha)\,f_0.\end{aligned}$$

The additional condition for ellipticity for the initial operator is a condition of ellipticity of the ψDO

$$\Delta = \|\Delta_{ij}\|$$

on the left in (43).

Denoting by

$$\begin{pmatrix} \Delta'_{11} & \Delta'_{12} \\ \Delta'_{21} & \Delta'_{22} \end{pmatrix}$$

a regularizer for the operator Δ, we obtain

$$\begin{aligned} u_1 &= \Delta'_{11}\left(f_1 - P_{\chi_*^{(2)}}\hat{H}(\alpha)(1 - i_1\pi_1)\hat{H}_0^{-1}(\alpha)f_0\right) \\ &\quad + \Delta'_{12}\left(d - P_{\psi^{(2)}}\hat{G}(\alpha)(1 - i_1\pi_1)\hat{H}_0^{-1}(\alpha)f_0\right) \end{aligned}$$

and

$$\begin{aligned} c &= \Delta'_{21}\left(f_1 - P_{\chi_*^{(2)}}\hat{H}(\alpha)(1 - i_1\pi_1)\hat{H}_0^{-1}(\alpha)f_0\right) \\ &\quad + \Delta'_{22}\left(d - P_{\psi^{(2)}}\hat{G}(\alpha)(1 - i_1\pi_1)\hat{H}_0^{-1}(\alpha)f_0\right). \end{aligned}$$

Hence, we have

$$\begin{aligned} u_0 &= \hat{H}_0^{-1}(\alpha)\Big\{1 + (1 - i_2\pi_2)\Big[\hat{H}(\alpha)P^*_{\chi^{(1)}}\left(\Delta'_{11}P_{\chi_*^{(2)}}\hat{H}(\alpha) + \Delta'_{12}P_{\psi^{(2)}}\hat{G}(\alpha)\right) \\ &\quad + \hat{F}(\alpha)P^*_{\psi^{(1)}}\left(\Delta'_{21}P_{\chi_*^{(2)}}\hat{H}(\alpha) + \Delta'_{22}P_{\psi^{(2)}}\hat{G}(\alpha)\right)\Big](1 - i_1\pi_1)\hat{H}_0^{-1}(\alpha)\Big\}f_0 \\ &\quad - \left[\hat{H}_0^{-1}(\alpha)(1 - i_2\pi_2)\left(\hat{H}(\alpha)P^*_{\chi^{(1)}}\Delta'_{11} + \hat{F}(\alpha)P^*_{\psi^{(1)}}\Delta'_{21}\right)\right]f_1 \\ &\quad - \left[\hat{H}_0^{-1}(\alpha)(1 - i_2\pi_2)\left(\hat{H}(\alpha)P^*_{\chi^{(1)}}\Delta'_{12} + \hat{F}(\alpha)P^*_{\psi^{(1)}}\Delta'_{22}\right)\right]d. \end{aligned}$$

Finally, we obtain the expression for the almost inverse for matrix (42) in the form

$$\begin{pmatrix} \hat{R}_{11} & \hat{R}_{12} & \hat{R}_{13} \\ \hat{R}_{21} & \hat{R}_{22} & \hat{R}_{23} \\ \hat{R}_{31} & \hat{R}_{32} & \hat{R}_{33} \end{pmatrix} = \begin{pmatrix} \hat{R}_{11} & \hat{R}_{12} & \hat{R}_{13} \\ \hat{R}_{21} & \Delta'_{11} & \Delta'_{12} \\ \hat{R}_{31} & \Delta'_{21} & \Delta'_{22} \end{pmatrix},$$

where

$$\begin{aligned} \hat{R}_{11} &= \hat{H}_0^{-1}(\alpha)\Big\{1 + (1 - i_2\pi_2)\Big[\hat{H}(\alpha)P^*_{\chi^{(1)}}\left(\Delta'_{11}P_{\chi_*^{(2)}}\hat{H}(\alpha) + \Delta'_{12}P_{\psi^{(2)}}\hat{G}(\alpha)\right) \\ &\quad + \hat{F}(\alpha)P^*_{\psi^{(1)}}\left(\Delta'_{21}P_{\chi_*^{(2)}}\hat{H}(\alpha) + \Delta'_{22}P_{\psi^{(2)}}\hat{G}(\alpha)\right)\Big](1 - i_1\pi_1)\hat{H}_0^{-1}(\alpha)\Big\}, \\ \hat{R}_{12} &= -\left[\hat{H}_0^{-1}(\alpha)(1 - i_2\pi_2)\left(\hat{H}(\alpha)P^*_{\chi^{(1)}}\Delta'_{11} + \hat{F}(\alpha)P^*_{\psi^{(1)}}\Delta'_{21}\right)\right], \end{aligned}$$

$$\hat{R}_{13} = -\left[\hat{H}_0^{-1}(\alpha)(\mathbf{1}-i_2\pi_2)\left(\hat{H}(\alpha)P^*_{\chi^{(1)}}\Delta'_{12}+\hat{F}(\alpha)P^*_{\psi^{(1)}}\Delta'_{22}\right)\right],$$

$$\hat{R}_{21} = -\left[\left(\Delta'_{11}P_{\chi^{(2)}_*}\hat{H}(\alpha)+\Delta'_{12}P_{\psi^{(2)}}\hat{G}(\alpha)\right)(\mathbf{1}-i_1\pi_1)\hat{H}_0^{-1}(\alpha)\right],$$

$$\hat{R}_{31} = -\left[\left(\Delta'_{21}P_{\chi^{(2)}_*}\hat{H}(\alpha)+\Delta'_{22}P_{\psi^{(2)}}\hat{G}(\alpha)\right)(\mathbf{1}-i_1\pi_1)\hat{H}_0^{-1}(\alpha)\right].$$

From the latter relations, one can see that the almost inverse for matrix (42) is the element of the algebra $Op_{\mathrm{ext}}(M)$.

The rest part of the construction of a regularizer for operator (36) defined by some element of $Op_{\mathrm{ext}}(M)$ including the construction of the left regularizer, does not differ from the corresponding construction in the previous subsection, and we leave it to the reader.

The globalization of the above constructions goes quite similar to that in the previous section. We remark only that the matrix operator

$$\mathcal{M} = \begin{pmatrix} \hat{H} & \mathcal{P}^* \\ \mathcal{P} & \hat{D} \end{pmatrix} : \begin{pmatrix} E^s_{\sigma,\gamma} \\ [H^\sigma(X)]^{N_1} \end{pmatrix} \to \begin{pmatrix} E^{s-m}_{\sigma,\gamma-(k+1)m} \\ [H^{\sigma'}(X)]^{N_2} \end{pmatrix}, \qquad (44)$$

where $\hat{H}$ is a differential operator on M of cusp-wedge degeneracy, $\mathcal{P}^*$ is a coboundary operator, $\mathcal{P}$ is a boundary operator, and $\hat{D}$ is a ψDO on X, is called to be *elliptic* if:

- $\hat{H}$ is elliptic on $M \setminus X$;
- The symbol family $\hat{H}(x,\xi)$ is Fredholm for all $(x,\xi) \in T^*(\mathbf{R}^n)$.
- The above described operator Δ (see formula (43)) is elliptic.

Then the following theorem is valid:

Theorem 11 *Suppose that the matrix $\mathcal{M}$ is elliptic. Then operator* (44) *possesses the Fredholm property.*

4.5 Example

To conclude this section, we present an example of matrix operator for which the conditions of the latter theorem are fulfilled. Clearly, it is sufficient to consider the operator only in a neighborhood of the edge X.

First of all, we note that there exist operators from LC-algebra on the cusp C with the nonvanishing index (the corresponding example can be found in [14]; being written for the conical case, it can be easily rewritten for the cuspidal case as well.) Let us fix such an operator $\hat{D}$.

Denote by $N_1 = \dim \operatorname{Ker} \hat{D}$ and $N_2 = \dim \operatorname{Coker} \hat{D}$. Consider a base

$$\left(\chi_1^{(1)}, \ldots, \chi_{N_1}^{(1)}\right)$$

in $\operatorname{Ker} \hat{D}$. Then

$$i_1 \begin{pmatrix} c_1 \\ \vdots \\ c_{N_1} \end{pmatrix} = c_1 \chi_1^{(1)} + \ldots + c_{N_1} \chi_{N_1}^{(1)} = P_{\chi^{(1)}}^{*} \begin{pmatrix} c_1 \\ \vdots \\ c_{N_1} \end{pmatrix}.$$

Suppose that the base $\left(\chi_1^{(1)}, \ldots, \chi_{N_1}^{(1)}\right)$ is orthonormalized with respect to the scalar product

$$(\chi, \chi') = \int \chi(r, \omega) \overline{\chi'(r, \omega)} r^{n-1} \, dr d\omega$$

on C, where the bar denotes complex conjugation. Then, choosing the orthogonal complement to $\operatorname{Ker} \hat{D}$ as the space L_1, we have

$$\pi_1(v) = \begin{pmatrix} \int v(r, \omega) \overline{\chi_1^{(1)}(r, \omega)} r^{n-1} \, dr d\omega \\ \cdots \\ \int v(r, \omega) \overline{\chi_{N_1}^{(1)}(r, \omega)} r^{n-1} \, dr d\omega \end{pmatrix} = P_{\overline{\chi^{(1)}}}(v).$$

Similar, if $\left(\chi_1^{(2)}, \ldots, \chi_{N_1}^{(2)}\right)$ is an orthonormalized base in the orthogonal complement L_2 of $\operatorname{Im} \hat{D}$, then

$$i_2 \begin{pmatrix} c_1 \\ \vdots \\ c_{N_2} \end{pmatrix} = P_{\chi^{(2)}}^{*} \begin{pmatrix} c_1 \\ \vdots \\ c_{N_2} \end{pmatrix}, \quad \pi_2(v) = P_{\overline{\chi^{(2)}}}(v).$$

The form of the operator $\hat{D}$ with respect to the above defined splittings

$$E_1 = L_1 \oplus \operatorname{Ker} \hat{D}, \quad E_2 = \operatorname{Im} \hat{D} \oplus L_2$$

is

$$\hat{D} = \begin{pmatrix} \hat{D}_0 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\hat{D}_0$ is an invertible operator from L_1 to $\operatorname{Im} \hat{D}$.

Let $\mathcal{D}$ be an operator with the symbol

$$smbl\,(\mathcal{D}) = \hat{D}\sigma\,(x,\xi)\,,$$

where $\sigma\,(x,\xi)$ is a positive smooth function on $T^*\,(X)$ equal to $|\xi|$ outside the zero section in $T^*\,(X)$ (we suppose that some Riemannian metrics is fixed on X.)

Consider the matrix operator

$$\begin{pmatrix} \mathcal{D} & P^*_{\chi^{(2)}} \\ P_{\overline{\chi^{(1)}}} & 0 \end{pmatrix}.$$

The symbol of this operator is

$$\begin{pmatrix} \hat{D}\sigma\,(x,\xi) & P^*_{\chi^{(2)}} \\ P_{\overline{\chi^{(1)}}} & 0 \end{pmatrix}.$$

Let us compute the symbol of the operator Δ corresponding to this symbol. We have

$$\Delta_{11} = P_{\overline{\chi^{(2)}}}\hat{D}\sigma\,(x,\xi)\left\{\mathbf{1} - (\mathbf{1} - i_1\pi_1)\left(\hat{D}_0\sigma\,(x,\xi)\right)^{-1}(\mathbf{1} - i_2\pi_2)\,\hat{D}\sigma\,(x,\xi)\right\}P^*_{\chi^{(1)}} = \mathbf{0}$$

since $P_{\overline{\chi^{(2)}}}\hat{D} = \pi_2\hat{D} = \mathbf{0}$, by the definition of the projector π_2;

$$\Delta_{12} = P_{\overline{\chi^{(2)}}}P^*_{\chi^{(2)}} - P_{\overline{\chi^{(2)}}}\hat{D}\sigma\,(x,\xi)\,(\mathbf{1} - i_1\pi_1)\left(\hat{D}_0\sigma\,(x,\xi)\right)^{-1}(\mathbf{1} - i_2\pi_2)\,P^*_{\chi^{(2)}} = \mathbf{1}$$

since $P_{\overline{\chi^{(2)}}}\hat{D} = 0$ and the base $\chi^{(2)}$ is orthonormalized;

$$\Delta_{21} = P_{\overline{\chi^{(1)}}}P^*_{\chi^{(1)}} - P_{\overline{\chi^{(1)}}}(\mathbf{1} - i_1\pi_1)\left(\hat{D}_0\sigma\,(x,\xi)\right)^{-1}(\mathbf{1} - i_2\pi_2)\,\hat{D}\sigma\,(x,\xi)\,P^*_{\chi^{(1)}} = \mathbf{1}$$

since the base $\chi^{(1)}$ is orthonormalized and $\hat{D}P^*_{\chi^{(1)}} = \mathbf{0}$;

$$\Delta_{22} = -P_{\overline{\chi^{(1)}}}(\mathbf{1} - i_1\pi_1)\left(\hat{D}_0\sigma\,(x,\xi)\right)^{-1}(\mathbf{1} - i_2\pi_2)\,P^*_{\chi^{(2)}}$$

(the concrete form of this operator is not of importance in what follows).

So, the symbol of the operator Δ equals

$$smbl\,\Delta = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \Delta_{22} \end{pmatrix}.$$

Hence, this operator is clearly elliptic and all the reqiurements of Theorem 11 are fulfilled.

References

[1] B.-W. Schulze, B. Sternin, and V. Shatalov. *Asymptotic Solutions to Differential Equations on Manifolds with Cusps.* Max-Planck Institut für Mathematik, Bonn, 1996. Preprint MPI 96-89.

[2] B.-W. Schulze, B. Sternin, and V. Shatalov. *An Operator Algebra on Manifolds with Cusp-Type Singularities.* Max-Planck Institut für Mathematik, Bonn, 1996. Preprint MPI 96-111.

[3] B.-W. Schulze. *Pseudodifferential Operators on Manifolds with Singularities.* North-Holland, Amsterdam, 1991.

[4] B. Sternin. The Sobolev problems for submanifolds with multidimensional singularities. *DAN SSSR*, **184**, No. 4, 1969.

[5] V.P. Maslov. *Operator Methods.* Nauka, Moscow, 1973. English transl.: "*Operational Methods*", Mir, Moscow, 1976.

[6] V. Nazaikinskii, B. Sternin, and V. Shatalov. *Methods of Noncommutative Analysis. Theory and Applications.* Mathematical Studies. Walter de Gruyter Publishers, Berlin - New York, 1995.

[7] V. G. Maz'ya and B. A. Plamenevskii. On the asymptotic of solutions of differential equations with operator coefficients. *DAN SSSR*, **196**, 1971, 512 -515. English transl. in: *Soviet Math. Dokl.* **12** (1971).

[8] V. G. Maz'ya and B. A. Plamenevskii. On the asymptotic solution of the Dirichlet problem near the isolated singularity of the boundary. *Vestnik Leningrad Univ. Math.*, , No. 13, 1977. English transl. in: *Vestnik Leningrad Univ. Math.* **10** (1982), 295 - 302.

[9] B. Sternin. *Topological aspects of the Sobolev problem.* Moscow Institute of Electronic Engineering, Moscow, 1971.

[10] B. Sternin. Relative elliptic theory and the Sobolev problem. *Soviet Math. Dokl.*, **17**, No. 5, 1976, 1306 - 1309.

[11] B. Sternin and V. Shatalov. *Relative Elliptic Theory and Sobolev Problems.* Max-Planck Institut für Mathematik, Bonn, 1994. Preprint MPI 94-114.

[12] B. Sternin and V. Shatalov. *The Sobolev Problems in Spaces with Asymptotics.* Max-Planck-Institut für Mathematik, Bonn, 1996. Preprint MPI 96-96.

[13] L. Boutet de Monvel. Boundary problems for pseudodifferential operators. *Acta Math.*, **126**, 1971, 11 - 51.

[14] B. V. Fedosov and B.-W. Schulze. On the Index of Elliptic Operators on a Cone. In *Schrödinger Operators, Markov Semigroups, Wavelet Analysis, Operator Algebras*, Advances in Partial Differential Equations, 1996, pages 347 - 372. Akademie Verlag, Berlin.

Potsdam