AN EXACT GEOMETRIC MASS FORMULA

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Abstract. We show an exact geometric mass formula for superspecial points in the reduction of any quaternionic Shimura variety modulo at a good prime p.

1. Introduction

Let p be a rational prime number. Let B be a totally indefinite quaternion algebra over a totally real field F of degree d, together with a positive involution $*$. Assume that p is unramified in B. Let O_B be a maximal order stable under the involution $*$. Let (V, ψ) be a non-degenerate $\mathbb{Q}$-valued skew-Hermitian (left) B-module with dimension $2g$ over $\mathbb{Q}$. Put $m := \frac{g}{2d}$, a positive integer. A polarized abelian O_B-variety $\underline{A} = (A, \lambda, \iota)$ is a polarized abelian variety (A, λ) together with a ring monomorphism $\iota : O_B \to \mathrm{End}(A)$ such that $\lambda \circ \iota(b^*) = \iota(b)^t \circ \lambda$ for all $b \in O_B$. Let k be an algebraically closed field of characteristic p. An abelian variety over k is said to be superspecial if it is isomorphic to a product of supersingular elliptic curves. Denote by Λ_g^B the set of isomorphism classes of g-dimensional superspecial principally polarized abelian O_B-varieties over k. Define the mass of Λ_g^B to be

$$\mathrm{Mass}(\Lambda_g^B) := \sum_{\underline{A} \in \Lambda_g^B} \frac{1}{|\,\mathrm{Aut}(A, \lambda, \iota)|}. \tag{1.1}$$

The mass $\mathrm{Mass}(\Lambda_g^B)$ is studied in Ekedahl [1] (Ekedahl's result relies on an explicit volume computation in Hashimoto-Ibukiyama [4, Proposition 9, p. 568]) in the special case $B = M_2(\mathbb{Q})$. He proved

Theorem 1.1 (Ekedahl, Hashimoto-Ibukiyama)**.** *One has*

$$\mathrm{Mass}(\Lambda_g) = \frac{(-1)^{g(g+1)/2}}{2^g} \prod_{i=1}^{g} \zeta(1-2i) \cdot \prod_{i=1}^{g} p^i + (-1)^i, \tag{1.2}$$

where Λ_g is the set of isomorphism classes of g-dimensional superspecial principally polarized abelian varieties over k and $\zeta(s)$ is the Riemann zeta function.

Let $B_{p,\infty}$ be the quaternion algebra over $\mathbb{Q}$ ramified exactly at $\{p, \infty\}$. Let B' be the quaternion algebra over F such that $\mathrm{inv}_v(B') = \mathrm{inv}_v(B_{p,\infty} \otimes_{\mathbb{Q}} B)$ for all v. Let Δ' be the discriminant of B' over F.

In this paper we prove

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Theorem 1.2. *One has*

(1.3)

$$\mathrm{Mass}(\Lambda_g^B) = \frac{(-1)^{dm(m+1)/2}}{2^{md}} \prod_{i=1}^{m} \left\{ \zeta_F(1-2i) \prod_{v|\Delta'} N(v)^i + (-1)^i \prod_{v|p, v\nmid\Delta'} N(v)^i + 1 \right\},$$

where $\zeta_F(s)$ is the Dedekind zeta function.

Let $N \geq 3$ be a prime-to-p positive integer. Choose a primitive n-th root of unity $\zeta_N \in \overline{\mathbb{Q}} \subset \mathbb{C}$ and fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. Let $\mathcal{M}$ be the moduli space over $\overline{\mathbb{F}}_p$ of g-dimensional principally polarized abelian O_B-varieties with a symplectic O_B-linear level-N structure w.r.t. ζ_N. Let L_0 be a self-dual O_B-lattice of V with respect to ψ. Let G_1 be the automorphism group scheme over $\mathbb{Z}$ associated to the pair (L_0, ψ). As an immediate consequence of Theorem 1.2, we get

Theorem 1.3. *The moduli space $\mathcal{M}$ has*

(1.4)

$$|G_1(\mathbb{Z}/N\mathbb{Z})| \frac{(-1)^{dm(m+1)/2}}{2^{md}} \prod_{i=1}^{m} \left\{ \zeta_F(1-2i) \prod_{v|\Delta'} N(v)^i + (-1)^i \prod_{v|p, v\nmid\Delta'} N(v)^i + 1 \right\}$$

superspecial points.

We divide the proof of Theorem 1.2 into 4 parts; each part is treated in one section. The first part is to express the weighted sum in terms of an arithmetic mass; this is done in the author's recent work [8]. The second part is to compute the mass associated to a quaternion unitary group and a standard open compact subgroup; this is done by Shimura [7] (re-obtained by Gan and J.-K. Yu [3, 11.2, p. 522]) using the theory of Bruhat-Tits Buildings). The third part is to compare the derived arithmetic mass in Section 1 with "the" standard mass in Section 2. This reduces the problem to computing a local index at p. The last part uses Dieudonné theory to compute this local index. A crucial step is choosing a good basis for the superspecial Dieudonné module concerned; this makes the computation easier.

Notation. $\mathbb{H}$ denotes the Hamilton quaternion algebra over $\mathbb{R}$. $\mathbb{A}_f$ denotes the finite adele ring of $\mathbb{Q}$ and $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. For a number field F and a finite place v, denote by O_F the ring of integers, F_v the completion of F at v, e_v the ramification index for $F/\mathbb{Q}$, κ_v the residue field, $f_v := [\kappa_v : \mathbb{F}_p]$ and $q_v := N(v) = |\kappa_v|$. For an O_F-module A, write A_v for $A \otimes_{O_F} O_{F,v}$. For a scheme X over $\operatorname{Spec} A$ and an A-algebra B, write X_B for $X \times_{\operatorname{Spec} A} \operatorname{Spec} B$. For a linear algebraic group G over $\mathbb{Q}$ and an open compact subgroup U of $G(\mathbb{A}_f)$, denote by $\mathrm{DS}(G,U)$ the double coset space $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/U$, and write $\mathrm{Mass}(G,U) := \sum_{i=1}^h |\Gamma_i|^{-1}$ if G is $\mathbb{R}$-anisotropic, where $\Gamma_i := G(\mathbb{Q}) \cap c_i U c_i^{-1}$ and $c_1, \dots, c_h$ are complete representatives for $\mathrm{DS}(G,U)$. For a central simple algebra B over F, write $\Delta(B/F)$ for the discriminant of B over F. If B a central division algebra over a non-archimedean local field F_v, denote by O_B the maximal order of B, $\mathfrak{m}(B)$ the maximal ideal and $\kappa(B)$ the residue field. $\mathbb{Q}_{p^n}$ denotes the unramified extension of $\mathbb{Q}_p$ of degree n and write $\mathbb{Z}_{p^n} := O_{\mathbb{Q}_{p^n}}$.

2. Simple mass formulas

Let B be a finite-dimensional semi-simple algebra over $\mathbb{Q}$ with a positive involution $*$, and O_B be an order of B stable under $*$. Let k be any field.

To any polarized abelian O_B-varieties $\underline{A} = (A, \lambda, \iota)$ over k, we associate a pair (G_x, U_x), where G_x is the group scheme over $\mathbb{Z}$ representing the functor

$$R \mapsto \{h \in (\mathrm{End}_{O_B}(A_k) \otimes R)^\times \mid h'h = 1\},$$

where $h \mapsto h'$ is the Rasoti involution, and U_x is the open compact subgroup $G_x(\hat{\mathbb{Z}})$. For any prime ℓ, we write $\underline{A}(\ell)$ for the associated ℓ-divisible group with additional structures $(A[\ell^\infty], \lambda_\ell, \iota_\ell)$, where λ_ℓ is the induced quasi-polarization from $A[\ell^\infty]$ to $A^t[\ell^\infty] = A[\ell^\infty]^t$ (the Serre dual), and $\iota_\ell : O_B \otimes \mathbb{Z}_\ell \to \mathrm{End}(A[\ell^\infty])$ the induced ring monomorphism. For any two objects $\underline{A}_1$ and $\underline{A}_2$ over k, denote by Q-$\mathrm{isom}_k(\underline{A}_1, \underline{A}_2)$ the set of O_B-linear quasi-isogenies $\varphi : A_1 \to A_2$ over k such that $\varphi^*\lambda_2 = \lambda_1$, and $\mathrm{Isom}_k(\underline{A}_1(\ell), \underline{A}_2(\ell))$ the set of $O_B \otimes \mathbb{Z}_\ell$-linear isomorphisms $\varphi : A_1[\ell^\infty] \to A_2[\ell^\infty]$ over k such that $\varphi^*\lambda_2 = \lambda_1$.

Let $x := \underline{A}_0 = (A_0, \lambda_0, \iota_0)$ be a fixed polarized abelian O_B-variety over k. Denote by $\Lambda_x(k)$ the set of isomorphisms classes of polarized abelian O_B-varieties $\underline{A}$ over k such that

(I_ℓ): $\mathrm{Isom}_k(\underline{A}_0(\ell), \underline{A}(\ell)) \neq \emptyset$ for all primes ℓ.

Let $\Lambda'_x(k) \subset \Lambda_x(k)$ be the subset consisting of objects such that

(Q): Q-$\mathrm{isom}_k(\underline{A}_0, \underline{A}) \neq \emptyset$.

Let $\ker^1(\mathbb{Q}, G_x)$ denote the kernel of the local-global map $H^1(\mathbb{Q}, G_x) \to \prod_v H^1(\mathbb{Q}_v, G_x)$.

Theorem 2.1. ([8, Theorem 2.3]) *Suppose that k is a field of finite type over its prime field.*
(1) *There is a natural bijection $\Lambda'_x(k) \simeq \mathrm{DS}(G_x, U_x)$. Consequently, $\Lambda'_x(k)$ is finite.*
(2) *One has $\mathrm{Mass}(\Lambda'_x(k)) = \mathrm{Mass}(G_x, U_x)$.*

Theorem 2.2. ([8, Theorem 4.6 and Remark 4.7]) *Notation as above. If $k \supset \mathbb{F}_p$ is algebraically closed and A_0 is supersingular, then $\mathrm{Mass}(\Lambda'_x(k)) = \mathrm{Mass}(G_x, U_x)$ and $\mathrm{Mass}(\Lambda_x(k)) = |\ker^1(\mathbb{Q}, G_x)| \cdot \mathrm{Mass}(G_x, U_x)$.*

Remark 2.3. The statement of Theorem 2.2 is valid for basic abelian O_B-varieties in the sense of Kottwitz (see [6] for the definition). The present form is enough for our purpose.

3. An exact mass formula of Shimura

Let D be a totally definite quaternion division algebra over a totally real field F of degree d. Let (bar) $d \mapsto \bar{d}$ denote the canonical involution. Let (V', φ) be a D-valued totally definite quaternion Hermitian D-module of rank m. Let G^φ denote the unitary group attached to φ. This is a reductive group over F and is regarded as a group over $\mathbb{Q}$ via the Weil restriction of scalars from F to $\mathbb{Q}$. Choose a maximal order O_D of D stable under the canonical involution $\bar{\ }$. Let L be an O_D-lattice in V' which is maximal among the lattices on which φ takes its values in O_B. Let U_0 be the open compact subgroup of $G^\varphi(\mathbb{A}_f)$ which stabilizes the adelic lattice $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$.

The following is deduced from a mass formula of Shimura [7] (also see Gan - J.-K. Yu [3, 11.2, p. 522]). This form is more applicable to prove Theorem 1.2.

Theorem 3.1 (Shimura)**.** *One has*

$$\mathrm{Mass}(G^\varphi, U_0) = \frac{(-1)^{dm(m+1)/2}}{2^{md}} \prod_{i=1}^{m} \left\{ \zeta_F(1-2i) \prod_{v|\Delta(D/F)} N(v)^i + (-1)^i \right\}. \tag{3.1}$$

Deduction. In [7, Introduction, p. 68] Shimura gives the explicit formula

$$\mathrm{Mass}(G^\varphi, U_0) = |D_F|^{m^2} \prod_{i=1}^{m} D_F^{1/2} \left[(2i-1)!(2\pi)^{-2i}\right]^d \zeta_F(2i) \cdot \prod_{v|\Delta(D/F)} \prod_{i=1}^{m} N(v)^i + (-1)^i, \tag{3.2}$$

where D_F is the discriminant of F over $\mathbb{Q}$. Using the functional equation for $\zeta_F(s)$, we deduce (3.1) from (3.2).

4. Global comparison

Keep the notation as in Section 1. Fix a g-dimensional superspecial principally polarized abelian O_B-variety $x = (A_0, \lambda_0, \iota_0)$ over k. Define $\Lambda_x := \Lambda_x(k)$ as in Section 2. Let (G_x, U_x) be the pair associated to x.

Lemma 4.1. *Any two self-dual $O_B \otimes \mathbb{Z}_p$-lattices of $(V_{\mathbb{Q}_p}, \psi)$ are isomorphic.*

Proof. The proof is elementary and omitted.

Lemma 4.2. *One has (1) $\Lambda_x = \Lambda_g^B$ (2) $\ker^1(\mathbb{Q}, G_x) = \{1\}$.*

Proof. (1) The inclusion $\Lambda_x \subset \Lambda_g^B$ is clear. We show the other direction. Let $\underline{A} \in \Lambda_g^B$. It follows from Lemma 4.1 that the condition (I_ℓ) is satisfied for primes $\ell \neq p$. Let M be the covariant Dieudonné module of A. One chooses an isomorphism $O_{B,p} \simeq M_2(O_{F,p})$ so that $* : (a_{ij}) \mapsto (a_{ij})^t$. Using the Morita equivalence, it suffices to show that any two superspecial principally quasi-polarized Dieudonné modules with compatible $O_{F,p}$-action are isomorphic. This follows from Theorem 5.1.

(2) Since G_x is semi-simple and simply connected (as it is an inner form of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{Sp}_{2m,F}$), the Hasse principle for G_x holds. ∎

4.1. We compute that

(i) $G_x(\mathbb{R}) = \{h \in M_m(\mathbb{H})^d \mid \bar{h}^t h = 1\}$,

(ii) for $\ell \neq p$, we have $G_x(\mathbb{Q}_\ell) = \prod_{v|\ell} G_{x,v}$ and $U_{x,\ell} = \prod_{v|\ell} U_{x,v}$, where

$$\begin{aligned} G_{x,v} &= \begin{cases} \mathrm{Sp}_{2m}(F_v), & \text{if } v \nmid \Delta(B/F), \\ \{h \in M_m(B_v) \mid \bar{h}^t h = 1\}, & \text{otherwise,} \end{cases} \\ U_{x,v} &= \begin{cases} \mathrm{Sp}_{2m}(O_{F_v}), & \text{if } v \nmid \Delta(B/F), \\ \{h \in M_m(O_{B_v}) \mid \bar{h}^t h = 1\}, & \text{otherwise,} \end{cases} \end{aligned} \tag{4.1}$$

(iii) $G_x(\mathbb{Q}_p) = \prod_{v|p} G_{x,v}$, where

$$G_{x,v} = \begin{cases} \mathrm{Sp}_{2m}(F_v), & \text{if } v \nmid \Delta', \\ \{h \in M_m(B'_v) \mid \bar{h}^t h = 1\}, & \text{otherwise.} \end{cases} \tag{4.2}$$

Take $D = B'$ and $V' = D^{\oplus m}$ with $\varphi(\underline{x}, \underline{y}) = \sum x_i \bar{y}_i$, and take $L = O_D^{\oplus m}$. We compute that

(i)' $G^\varphi(\mathbb{R}) = \{h \in M_m(\mathbb{H})^d \mid \bar{h}^t h = 1\}$, (ii)' for any ℓ, we have $G_x(\mathbb{Q}_\ell) = \prod_{v|\ell} G_v^\varphi$ and $U_{0,\ell} = \prod_{v|\ell} U_{0,v}$, where

$$\begin{aligned} G_v^\varphi &= \begin{cases} \mathrm{Sp}_{2m}(F_v), & \text{if } v \nmid \Delta', \\ \{h \in M_m(B'_v) \,|\, \bar{h}^t h = 1\}, & \text{otherwise,} \end{cases} \\ U_{0,v} &= \begin{cases} \mathrm{Sp}_{2m}(O_{F_v}), & \text{if } v \nmid \Delta', \\ \{h \in M_m(O_{B'_v}) \,|\, \bar{h}^t h = 1\}, & \text{otherwise.} \end{cases} \end{aligned} \tag{4.3}$$

For $\ell \neq p$ and $v|\ell$, one has $B_v = B'_v$ and that $v \nmid \Delta(B/F)$ if and only if $v \nmid \Delta'$. It follows from computation above that $G_{x,\mathbb{R}} \simeq G^\varphi_{\mathbb{R}}$ and $G_{x,\mathbb{Q}_\ell} \simeq G^\varphi_{\mathbb{Q}_\ell}$ for all ℓ. Since the Hasse principle holds for the adjoint group G_x^{ad}, we get $G_x \simeq G^\varphi$ over $\mathbb{Q}$. We fix an isomorphism and write $G_x = G^\varphi$. For $\ell \neq p$ and $v|\ell$, the subgroups $U_{0,v}$ and $U_{x,v}$ are conjugate, and hence they have the same local volume.

4.2. Applying Theorem 2.2 in our setting (Section 1) and using Lemma 4.2, we get $\mathrm{Mass}(\Lambda_g^B) = \mathrm{Mass}(G_x, U_x)$. Using the result in Subsection 4.1, we get

$$\mathrm{Mass}(\Lambda_g^B) = \mathrm{Mass}(G^\varphi, U_0) \cdot \mu(U_{0,p}/U_{x,p}), \tag{4.4}$$

where $\mu(U_{0,p}/U_{x,p}) = [U_{x,p} : U_{0,p} \cap U_{x,p}]^{-1}[U_{0,p} : U_{0,p} \cap U_{x,p}]$.

5. Local index $\mu(U_{0,p}/U_{x,p})$

Let $(M', \langle\,,\rangle', \iota')$ be the covariant Dieudonné module associated to the point $x = (A_0, \lambda_0, \iota_0)$ in the previous section. Choose an isomorphism $O_B \otimes \mathbb{Z}_p \simeq M_2(O_F \otimes \mathbb{Z}_p)$ so that $*$ becomes the transpose. Let $M := eM'$, $\langle\,,\rangle := \langle\,,\rangle'|_M$ and $\iota := \iota'|_{O_F}$, where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(O_F \otimes \mathbb{Z}_p)$. The triple $(M, \langle\,,\rangle, \iota)$ is a superspecial principally quasi-polarized Dieudonné module with compatible $O_F \otimes \mathbb{Z}_p$-action of rank $g = 2dm$. Let $M = \oplus_{v|p} M_v$ be the decomposition with respect to the decomposition $O_F \otimes \mathbb{Z}_p = \oplus_{v|p} \mathcal{O}_v$; here we write $\mathcal{O}_v$ for O_{F_v}. By the Morita equivalence, we have

$$U_{x,p} = \mathrm{Aut}_{\mathrm{DM},O_B}(M', \langle\,,\rangle') = \mathrm{Aut}_{\mathrm{DM},O_F}(M, \langle\,,\rangle) = \prod_{v|p} U_{x,v}, \tag{5.1}$$

where $U_{x,v} := \mathrm{Aut}_{\mathrm{DM},\mathcal{O}_v}(M_v, \langle\,,\rangle)$.

Let $W := W(k)$ be ring of Witt vectors over k and σ the absolute Frobenius map on W. Let $\mathfrak{I} := \mathrm{Hom}(\mathcal{O}_v, W)$ be the set of embeddings; write $\mathfrak{I} = \{\sigma_i\}_{i \in \mathbb{Z}/f_v\mathbb{Z}}$ so that $\sigma\sigma_i = \sigma_{i+1}$ for all i. We identify $\mathbb{Z}/f_v\mathbb{Z}$ with $\mathfrak{I}$ through $i \mapsto \sigma_i$. Decompose $M_v = \oplus_{i \in \mathbb{Z}/f_v\mathbb{Z}} M_v^i$ into σ_i-isotypic components M_v^i. One has (1) each component M_v^i is a free W-module of rank $2m$, which is self-dual with respect to the pairing $\langle\,,\rangle$, (2) $\langle M_v^i, M_v^j \rangle = 0$ if $i \neq j$, and (3) the operations F and V shift by degree 1 and degree -1, respectively.

Theorem 5.1. *Let $(M_v, \langle\,,\rangle, \iota)$ be as above. There is a symplectic basis $\{X_j^i, Y_j^i\}_{j=1,\dots,m}$ for M_v^i such that*

(i) $Y_j^i \in VM_v^{i+1}$,

(ii) $FX_j^i = -Y_j^{i+1}$ *and* $FY_j^i = pX_j^{i+1}$,

for all $i \in \mathbb{Z}/f_v\mathbb{Z}$ and all j.

PROOF. We write f, M and q for f_v, M_v and q_v, respectively. Suppose that $f = 2c$ is even. Let $N := \{x \in M \mid F^c x = (-1)^c V^c x\}$. Since M is superspecial, we have $(*)$ $F^2 N = pN$, $\widetilde{N} \otimes_{\mathbb{Z}_q} W \simeq M$ and $N = \oplus N^i$. Since $\overline{VN^1}$ is isotropic with respect to $\langle\, ,\rangle$ in N/pN, we can choose a symplectic basis $\{X_j^0, Y_j^0\}_{j=1,\dots,m}$ for N^0 such that $Y_j^0 \in VN^1$ for all j. Define X_j^i and Y_j^i recursively for $j = 1, \cdots, j$:

$$X_{j+1}^i = p^{-1} F Y_j^i, \quad Y_{j+1}^i = -F X_j^i. \tag{5.2}$$

One has $X_{j+2}^i = \frac{-1}{p} F^2 X_j^i$ and $Y_{j+2}^i = \frac{-1}{p} F^2 Y_j^i$; hence

$$X_j^f = (-1)^c p^{-c} F^{2c} X_j^0 = X_j^0, \quad Y_j^f = (-1)^c p^{-c} F^{2c} Y_j^0 = Y_j^0,$$

for all j. It is easy to see that $\{X_j^i, Y_j^i\}_{j=1,\dots,m}$ forms a symplectic basis for N^i.

Suppose that $f = 2c+1$ is odd. Let $N := \{x \in M \mid F^{2f} x + p^f x = 0\}$. We construct a symplectic basis $\{X_j^0, Y_j^0\}_{j=1,\dots,m}$ for N^0 with the properties: $X_j^0 \notin VN^1$, $Y_j^0 \in VN^1$ and $Y_j^0 = (-1)^{c+1} p^{-c} F^f X_j^0$ for all j. We can choose $X_1^0 \in N^0 \backslash VN^1$ so that $\langle X_1^0, (-1)^{c+1} p^{-c} F^f X_1^0 \rangle \in \mathbb{Z}_{q^2}^\times$. This follows from the fact that the form $(x, y) := \langle x, p^{-c} F^f y \rangle \mod p$ is a non-degenerate Hermitian form on N^0/VN^1. Set $Y_1^0 = (-1)^{c+1} p^{-c} F^f X_1^0$ and let $\mu := \langle X_1^0, Y_1^0 \rangle$. From $\langle F^f X_1^0, .F^f Y_1^0 \rangle = \langle (-1)^{c+1} p^c Y_1^0, (-1)^c p^{c+1} X_1^0 \rangle$, we get $\mu \in \mathbb{Z}_q^\times$. Since $\mathbb{Q}_{q^2}/\mathbb{Q}_q$ is unramified, replacing X_1^0 by a suitable λX_1^0, we get $\langle X_1^0, Y_1^0 \rangle = 1$. Do the same construction for the complement of the submodule $< X_1^0, Y_1^0 >$ and use induction; we exhibit such a basis for N^0.

Define X_j^i and Y_j^i recursively for $i = 1, \dots, f$ as (5.2). We verify again that $X_j^f = X_j^0$ and $Y_j^f = Y_j^0$. It follows from the relation (5.2) that $\{X_j^i, Y_j^i\}_{j=1,\dots,m}$ forms a symplectic basis for N^i for all i. This completes the proof. ∎

Proposition 5.2. *Notation as above.*

(1) *If f_v is even, then*

$$U_{x,v} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2m}(\mathbb{Z}_{q_v}) \mid B \equiv 0 \mod p \right\}. \tag{5.3}$$

(2) *If f_v is odd, then*

$$U_{x,v} \simeq \{ h \in M_m(O_{B'_v}) \mid \bar{h}^t h = 1 \}. \tag{5.4}$$

PROOF. Let $\phi \in U_{x,v}$. Choose a symplectic basis $\mathcal{B}$ for M_v as in Theorem 5.1. Since ϕ commutes with the O_F-action, we have $\phi = (\phi_i)$, where $\phi_i \in \mathrm{Aut}(M_v^i, \langle\, ,\rangle)$. Write $\phi_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in \mathrm{Sp}_{2m}(W)$ using the basis $\mathcal{B}$. Since the map F is injective, ϕ_0 determines the remaining ϕ_i. From $\phi F^2 = F^2 \phi$, we have $\phi_{i+2} = \phi_i^{(2)}$ (as matrices). Here we write $\phi_i^{(n)}$ for $\phi_i^{\sigma^n}$. From $\phi F = F \phi$ we get $A_i^{(1)} = D_{i+1}$, $B_i^{(1)} = -pC_{i+1}$, $pC_i^{(1)} = -B_{i+1}$ and $D_i^{(1)} = A_{i+1}$.

(1) If f_v is even, then $A_0, B_0, C_0, D_0 \in \mathbb{Z}_{q_v}$ and $B_0 \equiv 0 \mod p$. This shows (5.3).

(2) Suppose f_v is odd. From $\phi_0^{(f_v+1)} = \phi_1$ we get $A_0^{(f_v)} = D_0$, $B_0^{(f_v)} = -pC_0$, $pC_0^{(f_v)} = -B_0$, $D_0^{(f_v)} = A_0$. Hence

$$U_{x,v} = \left\{ \begin{pmatrix} A & -pC^\tau \\ C & A^\tau \end{pmatrix} \in \mathrm{Sp}_{2m}(\mathbb{Z}_{q_v^2}) \right\},$$

where τ is the involution of $\mathbb{Q}_{q_v^2}$ over $\mathbb{Q}_{q_v}$. Note that $O_{B'_v} = \mathbb{Z}_{q_v^2}[\Pi]$ with $\Pi^2 = -p$ and $\Pi a = a^\tau \Pi$ for all $a \in \mathbb{Z}_{q_v^2}$. The map $A + C\Pi \mapsto \begin{pmatrix} A & -pC^\tau \\ C & A^\tau \end{pmatrix}$ gives rise to an isomorphism (5.4). This proves the proposition. ∎

Let $(V_0 = \mathbb{F}_q^{2m}, \psi_0)$ be a standard symplectic space. Let P be the stabilizer of the standard maximal isotropic subspace $\mathbb{F}_q < e_1, \dots, e_m >$.

Lemma 5.3. $|\operatorname{Sp}_{2m}(\mathbb{F}_q)/P| = \prod_{i=1}^m (q^i + 1)$.

PROOF. We have a natural bijection between the group $\operatorname{Sp}_{2m}(\mathbb{F}_q)$ and the set $\mathcal{B}(m)$ of ordered symplectic bases $\{v_1, \dots, v_{2m}\}$ for V_0. The first vector v_1 has $q^{2m} - 1$ choices. The first companion vector v_{m+1} has q^{2m-1} choices as it does not lie in the hyperplane $v_1^\perp$ and we require $\psi_0(v_1, v_{m+1}) = 1$. The remaining ordered symplectic basis can be chosen from the complement $\mathbb{F}_q < v_1, v_{m+1} >^\perp$. Therefore, we have proved the recursive formula $|\operatorname{Sp}_{2m}(\mathbb{F}_q)| = (q^{2m} - 1)q^{2m-1}|\operatorname{Sp}_{2m-2}(\mathbb{F}_q)|$. From this, we get

$$|\operatorname{Sp}_{2m}(\mathbb{F}_q)| = q^{m^2} \prod_{i=1}^m (q^{2i} - 1). \tag{5.5}$$

We have

$$P = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}; AD^t = I_m,\ BA^t = AB^t \right\}.$$

This yields

$$|P| = q^{\frac{m^2+m}{2}} |\operatorname{GL}_m(\mathbb{F}_q)| = q^{m^2} \prod_{i=1}^m (q^i - 1). \tag{5.6}$$

From (5.5) and (5.6), we prove the lemma. ∎

By Proposition 5.2 and Lemma 5.3, we get

Theorem 5.4. *One has*

$$\mu(U_{0,p}/U_{x,p}) = \prod_{v|p} \mu(U_{0,v}/U_{x,v}) = \prod_{v|p, v\nmid\Delta'} \prod_{i=1}^m (q_v^i + 1). \tag{5.7}$$

Plugging the formula (5.7) in the formula (4.4), we get the formula (1.3). The proof of Theorem 1.2 is complete.

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