NONCOMMUTATIVE GEOMETRY ON TREES AND BUILDINGS

GUNTHER CORNELISSEN, MATILDE MARCOLLI, KAMRAN REIHANI, AND ALINA VDOVINA

1. Introduction

The notion of a spectral triple, introduced by Connes (*cf.* [9], [7], [10]), provides a powerful generalization of Riemannian geometry to noncommutative spaces. It originates from the observation that, on a smooth compact spin manifold, the infinitesimal line element ds can be expressed in terms of the inverse of the classical Dirac operator $\mathcal{D}$, so that the Riemannian geometry is entirely encoded by the data $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of the algebra of smooth functions (a dense subalgebra of the C^*-algebra of continuous functions), the Hilbert space of square integrable spinor sections, and the Dirac operator. For a noncommutative space, the geometry is then defined in terms of a similar triple of data $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, where $\mathcal{A}$ is a C^*-algebra, $\mathcal{H}$ is a Hilbert space on which $\mathcal{A}$ acts by bounded operators, and $\mathcal{D}$ is an unbounded self adjoint operator on $\mathcal{H}$ with compact resolvent $(\mathcal{D}-z)^{-1}$ for $z \notin \mathbb{R}$, and such that the commutators $[D, a]$ are bounded operators for all a in a dense subalgebra of $\mathcal{A}$.

Consani and Marcolli constructed in [11] a noncommutative space describing the geometry of the special fibers at the archimedean places of an arithmetic surface, in the form of a spectral triple for the action of a Kleinian Schottky group on its limit set. The motivation for the construction was a result of Yuri Manin [27], computing the Arakelov Green function of a compact Riemann surface in terms of a Schottky uniformization, and the proposed interpretation of the "dual graph" of the fiber at arithmetic infinity of an arithmetic surface in terms of the tangle of bounded geodesics in a hyperbolic handlebody having the Riemann surface as its conformal boundary at infinity ([27], *cf.* also [11]).

There is a well known analogy between archimedean places of an arithmetic surface and non-archimedean places with maximally degenerate fiber. The fibers over such non-archimedean places also admit a Schottky uniformization, by a p-adic Schottky group. This analogy was considered in [12], where the construction of [11] was generalized to the case of p-adic Schottky groups and Mumford curves. The use of Schottky uniformization in [27] implies that the results of Manin on ∞-adic Arakelov geometry and hyperbolic geometry appear to be confined to the 2-dimensional case. Some results in higher dimension were obtained, in the case of linear cycles in projective spaces, by Annette Werner [43], in terms of the geometry of the Bruhat-Tits building for $\mathrm{PGL}(n)$.

Motivated by this circle of ideas, we pursue two main directions in this paper. One is a refinement of the construction of spectral triples for Mumford curves. The main result is that we can improve the theta summable construction of [11] to a *finitely summable* construction, upon passing to the stabilization of the graph C^*-algebra of the dual graph of the special fiber of a Mumford curve. The main advantage of a finitely summable spectral triple is that it makes it possible to extract invariants of the geometry through zeta functions and through the Connes–Moscovici local index formula of [10]. The construction of the finitely summable triple is based on a modification for the non-unital case of a construction of Antonescu and Christensen [1] of spectral triples for AF algebras.

The other main direction we consider in the paper is that of generalizations of the spectral triples from the case of Mumford curves and trees to some classes of Euclidean and hyperbolic 2-dimensional buildings.

2. Theta summable spectral triples

We begin by recalling briefly the construction of the theta summable spectral triple for the Kleinian Schottky case of [11] and give a general formulation for group actions on trees, which also includes the case of Mumford curves of [12].

2.1. **Kleinian Schottky groups.**

In the construction of [11], one considers a Kleinian Schottky group $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ acting by isometries on real hyperbolic 3-space $\mathbb{H}^3$. This extends to an action on $\mathbb{P}^1(\mathbb{C}) = \partial\mathbb{H}^3$ by fractional linear transformations. A Kleinian Schottky group is a finitely generated discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$, which is isomorphic to a free group in g generators and such that all elements are hyperbolic. The limit set $\Lambda_\Gamma \subset \mathbb{P}^1(\mathbb{C})$ is the set of accumulation points of orbits of the Γ-action. The quotient $X = \Omega_\Gamma/\Gamma$, for $\Omega_\Gamma = \mathbb{P}^1(\mathbb{C}) \smallsetminus \Lambda_\Gamma$, is a compact Riemann surface of genus g, which is the conformal boundary at infinity of a hyperbolic handlebody of infinite volume obtained as the quotient $\mathbb{H}^3/\Gamma$. The convex core $H(\Lambda_\Gamma)/\Gamma$, where $H(\Lambda_\Gamma) \subset \mathbb{H}^3$ is the convex hull of the limit set, is a region of finite volume and a deformation retract of $\mathbb{H}^3/\Gamma$. For genus $g \geq 2$, the group Γ is non-elementary, namely the limit set Λ_Γ consists of more than two points (it is in fact a totally disconnected compact Hausdorff space – a fractal in $\mathbb{P}^1(\mathbb{C})$).

Consider the crossed product algebra $\mathcal{A} = C(\Lambda_\Gamma) \rtimes \Gamma$ of the action of the Schottky group on its limit set. This can be identified with a Cuntz–Krieger algebra O_A (*cf.* [18]), where A is the $2g \times 2g$ matrix with entries in $\{0,1\}$ associated to the subshift of finite type $\mathcal{S}$ given by all admissible doubly infinite sequences in the generators of the group Γ and their inverses.

The invertible shift T on $\mathcal{S}$ defines a noncommutative space $C(\mathcal{S}) \rtimes_T \mathbb{Z}$. The corresponding homotopy quotient in the sense of Baum–Connes is given by the quotient $\mathcal{S}_T = (\mathcal{S} \times \mathbb{R})/\mathbb{Z}$. The cohomology $H^1(\mathcal{S}_T, \mathbb{C})$ is computed by an exact sequence

$$0 \to \mathbb{C} \to \mathcal{V} \xrightarrow{\delta} \mathcal{V} \to H^1(\mathcal{S}_T, \mathbb{C}) \to 0, \tag{2.1}$$

where $\mathcal{V}$ is the infinite dimensional vector space $\mathcal{V} = C(\Lambda_\Gamma, \mathbb{Z}) \otimes \mathbb{C}$ of locally constant functions on the limit set Λ_Γ. The coboundary δ is given by $\delta f = f - f \circ T$.

The space $\mathcal{V}$ has a natural filtration by finite dimensional vector spaces, where $\mathcal{V}_n \subset \mathcal{V}$ is the space of locally constant functions that only depend on the first $(n+1)$ coordinates. That is, for $\gamma \in \Gamma$ we let $\Lambda_\Gamma(\gamma) \subset \Lambda_\Gamma$ denote the set of admissible infinite sequences $a_0, a_1, \ldots, a_n \ldots$ in the generators and their inverses beginning with the word γ. Then $\mathcal{V}_n$ consists of continuous functions on Λ_Γ that are constant on each $\Lambda_\Gamma(\gamma)$ with γ of length $|\gamma| = n+1$ as an admissible word in the generators and their inverses. The filtration of $\mathcal{V}$ by the $\mathcal{V}_n$ induces a filtration on the cohomology

$$H^1(\mathcal{S}_T, \mathbb{C}) = \varinjlim_n \mathcal{V}_n/\delta\mathcal{V}_{n-1}.$$

In [11] this cohomology was interpreted as a model for the cohomology of the “dual graph” of the special fiber at infinity of an arithmetic surface.

The filtered vector space $\mathcal{V}$ is a dense subspace of $L^2(\Lambda_\Gamma, d\mu)$, where $d\mu$ is the Patterson–Sullivan measure on the limit set Λ_Γ (*cf.* [39]), which satisfies the scaling property

$$(\gamma^* d\mu)(x) = |\gamma'(x)|^{\delta_H}\, d\mu(x), \quad \forall \gamma \in \Gamma, \tag{2.2}$$

with δ_H the Hausdorff dimension of Λ_Γ.

The grading operator $D = \sum_n n\,\hat{\Pi}_n$, where Π_n is the orthogonal projection onto $\mathcal{V}_n$ and $\hat{\Pi}_n = \Pi_n - \Pi_{n-1}$, is a densely defined self adjoint operator on $L^2(\Lambda_\Gamma, d\mu)$ with compact resolvent.

The Cuntz–Krieger algebra O_A is the universal C^*-algebra generated by $2g$ partial isometries S_i subject to the relations

$$\sum_j S_j S_j^* = I \quad \text{and} \quad S_i^* S_i = \sum_j A_{ij}\, S_j S_j^*, \tag{2.3}$$

where $A = (A_{ij})$ has entries $A_{ij} = 1$ for $|i-j| \neq g$, and $A_{ij} = 0$ otherwise. There is a faithful representation of the algebra O_A on the algebra of bounded operators $\mathcal{B}(\mathcal{L})$ on the Hilbert space $\mathcal{L} = L^2(\Lambda_\Gamma, d\mu)$, by setting

$$(T_{\gamma^{-1}} f)(x) := |\gamma'(x)|^{\delta_H/2} f(\gamma x), \quad \text{and} \quad (P_\gamma f)(x) := \chi_\gamma(x) f(x), \tag{2.4}$$

with χ_γ the characteristic function of the set $\Lambda_\Gamma(\gamma)$, so that the $S_i = \sum_j A_{ij} T^*_{\gamma_i} P_{\gamma_j}$ are partial isometries satisfying (2.3).

We give here a slightly modified version of the construction of [11]. Given an automorphism U of the algebra $\mathcal{A}$, and a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{L})$ in the algebra of bounded operators of a Hilbert space $\mathcal{L}$, we consider the representation of $\mathcal{A}$ on $\mathcal{H} = \mathcal{L} \oplus \mathcal{L}$

$$\pi_U(a)\ (\xi, \zeta) := (\pi(a)\,\xi,\ \pi(U(a))\,\zeta)\,. \tag{2.5}$$

Typically, for a Cuntz–Krieger algebra $\mathcal{A} = O_A$, automorphisms of the algebra can be obtained from automorphisms of the corresponding space $\mathcal{S}_A^+$ of admissible right infinite sequences associated to the subshift of finite type with matrix A. In fact, by [29], there is a homomorphism $\mathrm{Aut}(\mathcal{S}_A^+) \to \mathrm{Aut}(O_A)$ that to an automorphism u of $\mathcal{S}_A^+$ assigns an automorphism U of O_A which restricts to $u^*(f) = f \circ u^{-1}$ on the maximal commutative subalgebra $C(\mathcal{S}_A^+)$. By an automorphism of $\mathcal{S}_A^+$ one denotes a homeomorphism u of $\mathcal{S}_A^+$ such that $T \circ u \circ T^{-1} = u$, where T is the one-sided shift on $\mathcal{S}_A^+$. In the case we are considering, we can identify $\mathcal{S}_A^+$ with the limit set Λ_Γ.

A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for Kleinian Schottky groups is then obtained as follows (*cf.* [11]).

Proposition 2.1. *Let $\mathcal{A} = O_A$, let π be the representation* (2.4) *on $\mathcal{L} = L^2(\Lambda_\Gamma, d\mu)$. Let u be an automorphism of Λ_Γ and let π_U be the representation* (2.5) *of $\mathcal{A}$ on $\mathcal{H} = \mathcal{L} \oplus \mathcal{L}$, for the induced automorphism U of $\mathcal{A}$. Let F be the linear involution that exchanges the two copies of $\mathcal{L}$ and $\mathcal{D} = FD$, with $D = \sum_n n\hat{\Pi}_n$. Then, for $\delta_H < 1$, the data $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ define a spectral triple.*

Proof. The result follows by showing that the commutators $[\mathcal{D}, a]$, for a in a dense subalgebra of O_A, are bounded operators on $\mathcal{H}$. For that it is sufficient to estimate the norm of the commutators $[D, S_i]$ and $[D, S_i^*]$ on $\mathcal{L}$. An estimate is given in [11] in terms of the Poincare' series of the Schottky group (hence the $\delta_H < 1$ condition). This takes care also of the commutators with $\tilde{S}_i = U S_i$ and their adjoints. In fact, arguing as in [29] we see that $\tilde{S}_i = \sum_j S_j P_{\chi_{ij}}$, where $P_{\chi_{ij}}$ denotes the orthogonal projection associated to the characteristic function χ_{ij} of the set E_{ij} of points $\omega \in \Lambda_\Gamma$ such that the infinite word $u\gamma_i u^{-1}\omega$ is admissible (*i.e.* it defines a point in Λ_Γ) and has γ_j as first letter. In particular, this implies that the automorphism U preserves the dense subalgebra of $\mathcal{A} = O_A$ generated algebraically by the partial isometries S_i and their adjoints S_i^*, so that we still have $[D, \tilde{S}_i]$ and $[D, \tilde{S}_i^*]$ bounded. □

The spectral triple of Proposition 2.1 is θ-summable since the dimension of the eigenspaces of D grows like $2g(2g-2)^{n-1}(2g-2)$ so that we have

$$\mathrm{Tr}(e^{-tD^2}) < \infty \quad \forall t > 0. \tag{2.6}$$

The fact that this spectral triple is not finitely summable falls under a general result of Connes [7], which shows that non-amenable discrete groups (like Schottky groups) do not admit finitely summable spectral triples.

A reason for introducing the choice of an automorphism U of the algebra is in order to allow for a non-trivial K-homology class of the spectral triple. In specific cases of geometric interest one often has a particular choice of an automorphism available as part of the data.

2.2. **Theta summable spectral triples for actions on trees.**

We now present a similar construction of spectral triples in the case of actions on trees, which refines the construction given in [12] for the case of Mumford curves. We show here that the construction indeed follows very closely the case of Kleinian Schottky groups.

The results of Lubotzky [26] show that the geometry of actions on trees by finitely generated, torsion free, discrete subgroups of automorphisms is in many ways analogous to that of Kleinian Schottky groups. In the same philosophy, we shall see that the construction of the spectral triple recalled in Section 2.1 also extends to this case. (This in particular includes the case of Mumford curves.) The main results that we need for this constructions are provided in [14], [19], and [26].

Let $\mathcal{T}$ be a locally finite tree, with $\mathcal{T}^0$ the set of vertices and $\mathcal{T}^1$ the set of edges, and let $\Gamma \subset \mathrm{Aut}(\mathcal{T})$ be a finitely generated discrete subgroup. A path in $\mathcal{T}$ is a sequence $v_0, v_1, \dots, v_n \dots$ with $v_i \in \mathcal{T}^0$, where v_i and v_{i+1} are adjacent and there are no cancellations (namely $v_i \neq v_{i+2}$). The set of ends $\partial\mathcal{T}$ is the set of equivalence classes of paths in $\mathcal{T}$, where equivalent means having infinitely many v_i's in common. A geodesic in $\mathcal{T}$ is a doubly infinite path, namely a sequence $\dots v_{-m}, \dots, v_{-1}, v_0, v_1, \dots, v_n \dots$ with v_i and v_{i+1} adjacent and $v_i \neq v_{i+2}$. A distance function on the tree is obtained by assigning distance one to any pair of adjacent vertices.

The action of $\Gamma \subset \mathrm{Aut}(\mathcal{T})$ extends to an action on $\overline{\mathcal{T}} = \mathcal{T} \cup \partial\mathcal{T}$. The limit set $\Lambda_\Gamma \subset \partial\mathcal{T}$ is the set of accumulation points of Γ-orbits on $\mathcal{T}$. Let $H(\Lambda_\Gamma)$ be the geodesic hull of the limit set, namely the set of geodesics in $\mathcal{T}$ with both ends on Λ_Γ. It is a closed Γ-invariant subset of $\mathcal{T}$, and $H(\Lambda_\Gamma)/\Gamma$ is the convex core of $\mathcal{T}/\Gamma$.

Lubotzky showed in [26] that, if a finitely generated discrete subgroup $\Gamma \subset \mathrm{Aut}(\mathcal{T})$ is torsion free, then it is a Schottky group, in the sense that Γ is isomorphic to a free group and every element is hyperbolic. In this case, the convex core $H(\Lambda_\Gamma)/\Gamma$ is a finite graph.

We first show how to extend naturally the construction of the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of Proposition 2.1 to actions on trees.

We will use essentially the fact that, by the result of Coornaert [14], there is an analog for $\Gamma \subset \mathrm{Aut}(\mathcal{T})$ of the Patterson–Sullivan measure on the limit set of a Kleinian Schottky group. We follow the notation of [19] and assign to a hyperbolic element $\gamma \in \mathrm{Aut}(\mathcal{T})$, the expression

$$(v, \gamma^{-1}v, x) = d(v, u) - d(\gamma^{-1}v, u), \tag{2.7}$$

where $v \in \mathcal{T}^0$ is a base point, x is a point on the boundary $\partial\mathcal{T}$, and u is any vertex in the intersection of the two paths from v to x and from $\gamma^{-1}v$ to x. The distance $d(v, w)$ is the length of the geodesic arc in $\mathcal{T}$ connecting $v, w \in \mathcal{T}^0$. The horospheric distance (2.7) does not depend on u and one defines

$$\gamma_v'(x) = e^{(v,\gamma^{-1}v,x)}, \quad \text{for } x \in \partial\mathcal{T}. \tag{2.8}$$

Then by [14], for δ_H the critical exponent of the Poincaré series, there exists a normalized measure on $\partial\mathcal{T}$ with support on Λ_Γ, satisfying

$$(\gamma^* d\mu_v)(x) = (\gamma_v'(x))^{\delta_H} \, d\mu_v(x), \quad \forall \gamma \in \Gamma. \tag{2.9}$$

The Hausdorff dimension of Λ_Γ is equal to the critical exponent δ_H.

As in the case of the Kleinian Schottky group, we can then consider the Hilbert space $\mathcal{L} = L^2(\Lambda_\Gamma, d\mu)$, with respect to the Γ-conformal measure (2.9). The dense subspace of locally constant functions $\mathcal{V} = C(\Lambda_\Gamma, \mathbb{Z}) \otimes \mathbb{C}$ has a filtration by $\mathcal{V}_n$ defined as in the case of the Kleinian Schottky group, by taking functions that depend only on the first $(n+1)$-coordinates. Here we use an identification of Λ_Γ with admissible infinite sequences $a_0 a_1 \dots a_n \dots$ in the generators of Γ and their inverses. Such identification is determined by the choice of the base point $v \in \mathcal{T}^0$ and the identification $\Lambda_\Gamma = \overline{\Gamma v} \cap \partial\mathcal{T}$.

Proposition 2.2. *Let $\mathcal{T}$ be a locally finite tree and $\Gamma \subset \mathrm{Aut}(\mathcal{T})$ be a torsion free finitely generated discrete subgroup with $\Lambda_\Gamma \subset \partial\mathcal{T}$ its limit set. Let u an automorphism of Λ_Γ and U the induced automorphism of the C^*-algebra $\mathcal{A} = C(\Lambda_\Gamma) \rtimes \Gamma$. Then the data $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ as in Proposition 2.1 define a θ-summable spectral triple.*

Proof. For $\Gamma \subset \mathrm{Aut}(\mathcal{T})$ a torsion free finitely generated discrete subgroup (hence a Schottky group by [26]), let $\{\gamma_i\}_{i=1}^g$ be a set of generators and let $\gamma_{i+g} = \gamma_i^{-1}$.

The representation of the algebra $\mathcal{A} = C(\Lambda_\Gamma) \rtimes \Gamma$ on the Hilbert space $\mathcal{L} = L^2(\Lambda_\Gamma, d\mu)$ is then given as in (2.5), with U induced by an automorphism u of Λ_Γ and the representation π on $\mathcal{B}(\mathcal{L})$ defined as in (2.4) by setting

$$(T_{\gamma^{-1}} f)(x) := (\gamma_v'(x))^{\delta_H/2} \, f(\gamma x), \quad \text{and} \quad (P_\gamma f)(x) := \chi_\gamma(x) f(x), \tag{2.10}$$

where χ_γ is the characteristic function of the subset $\Lambda_\Gamma(\gamma) \subset \Lambda_\Gamma$.

As in the proof given in [11] of the result for Kleinian Schottky groups of Proposition 2.1, it is then sufficient to prove that, for $i = 1, \dots, 2g$, the commutators $[D, S_i]$ and $[D, S_i^*]$ are bounded. Here $S_i = \sum_j A_{ij} T^*_{\gamma_i} P_{\gamma_j}$ is the operator $(S_i \, f)(x) = (1 - \chi_{\gamma_i^{-1}}(x))(\gamma_i'(x))^{\delta_H/2} \, f(\gamma_i x)$, while the adjoint S_i^* acts as $(S_i^* f)(x) = \chi_{\gamma_i}(x)(\gamma_i'(Tx))^{-\delta_H/2} f(Tx)$, where T is the one sided shift on Λ_Γ satisfying $T|_{\Lambda_\Gamma(\gamma_i)}(x) = \gamma_i^{-1}(x)$.

We use the fact ([14], [19]) that the function $\gamma_v'(x)$ is locally constant on $\partial\mathcal{T}$, for any chosen $v \in \mathcal{T}^0$. This means that

$$\Pi_{k_i}(\gamma_i') = \gamma_i',$$

for some $k_i > 0$. This implies that we have

$$S_i : \mathcal{V}_{n+1} \to \mathcal{V}_n, \quad S_i^* : \mathcal{V}_n \to \mathcal{V}_{n+1}, \quad \forall n \geq k_i.$$

Thus the commutator $[D, S_i]$ can be written as

$$[D, S_i] = -S_i(1 - \Pi_0) + \sum_{k=0}^{k_i - 1} (S_i \Pi_{k+1} - \Pi_k S_i), \tag{2.11}$$

since $S_i \Pi_{k+1} = \Pi_k S_i$ for $k \geq k_i$. Thus $[D, S_i]$ is a bounded operator. The argument for $[D, S_i^*]$ is analogous. Thus the commutators $[D, a]$ are bounded for all elements a in the dense involutive subalgebra of $C(\Lambda_\Gamma) \rtimes \Gamma$ generated algebraically by the S_i.

We use again, as in Proposition 2.1 above, the fact that the automorphism U of $\mathcal{A}$ induced by the automorphism u of Λ_Γ preserves the subalgebra generated algebraically by the S_i and S_i^* so that we have bounded commutators with elements of this dense subalgebra in the representation twisted by U. The rest of the argument is then analogous to [11] and shows that we obtain a spectral triple.

□

This construction can be refined by working with the graph $H(\Lambda_\Gamma)/\Gamma$, instead of directly with the limit set Λ_Γ. The hull $H(\Lambda_\Gamma)$ consists of all the axes $L(\gamma)$ of $\gamma \in \Gamma$. These are the geodesics in $\mathcal{T}$ connecting the fixed points $z^-(\gamma)$ and $z^+(\gamma)$ in Λ_Γ. Let A be the *directed edge matrix* of the finite graph $\mathcal{G} = H(\Lambda_\Gamma)/\Gamma$. This is a $2\#\mathcal{G}^1 \times 2\#\mathcal{G}^1$ matrix with entries in $\{0,1\}$, such that $A_{ee'} = 1$ is ee' is an admissible path in $\mathcal{G}$ where e, e' are edges with either possible orientation, and $A_{ee'} = 0$ otherwise. The set $\mathcal{S}_A^+$ of infinite admissible words, with the admissibility condition specified by the matrix A, describes the set of infinite walks on the tree $H(\Lambda_\Gamma)$ starting at any vertex of a given fundamental domain for the action of Γ. This gives an identification of $\mathcal{S}_A^+$ with a union of n copies of Λ_Γ, for $n = \#\mathcal{G}^0$, each corresponding to walks starting at a given vertex in $\mathcal{G}$. Using this identification, we can define the Hilbert space $\mathcal{L}_A = L^2(\mathcal{S}_A^+, d\mu)$, with the measure induced by the Patterson–Sullivan measure (2.9) on Λ_Γ. This identification also induces on $\mathcal{L}_A$ a filtration, for which we can consider the associated grading D. As in the previous cases, we have a representation of the algebra $\mathcal{A} = C(\mathcal{S}_A^+) \rtimes \Gamma$ on the Hilbert space $\mathcal{L}_A$. The algebra $\mathcal{A}$ is Morita equivalent to a graph C^*-algebra of $H(\Lambda_\Gamma)/\Gamma$. As before, we can consider an automorphism of the Cantor set $\mathcal{S}_A^+$ and a corresponding automorphism U of the algebra $\mathcal{A}$ and a representation on $\mathcal{H}_A = \mathcal{L}_A \oplus \mathcal{L}_A$, where the $[F, \pi_U(a)]$ are compacts (*cf.* Lemma 4.6, [29]). We can then extend the result of Proposition 2.2 to θ-summable spectral triples of the form $(C(\mathcal{S}_A^+) \rtimes \Gamma, \mathcal{H}_A, \mathcal{D} = FD)$.

The original spectral triple of Proposition 2.2 corresponds to considering the graph $\mathcal{T}_\Gamma/\Gamma$ instead of $H(\Lambda_\Gamma)/\Gamma$, where $\mathcal{T}_\Gamma$ is the Cayley graph of Γ and $H(\Lambda_\Gamma)$ is the smallest subtree of $\mathcal{T}$ containing all the axes of the elements of Γ, *i.e.* the infinite geodesics in $\mathcal{T}$ with endpoints $z_\gamma^\pm \in \Lambda_\Gamma$ the fixed points of γ. In the case of Mumford curves, where $\mathcal{T}$ is the Bruhat–Tits tree of $\mathrm{PGL}(2, K)$, with K a finite extension of $\mathbb{Q}_p$, the graph $\mathcal{T}_\Gamma/\Gamma$ gives the dual graph of the specialization over the ring of integers $\mathcal{O} \subset K$ of the algebraic curve C holomorphically isomorphic to $X = \Omega_\Gamma/\Gamma$, while $H(\Lambda)/\Gamma$ is the dual graph of the minimal smooth model of C over $\mathcal{O}$, *cf.* [30].

It is known by [21] p.124 that all finite graphs that are connected and such that every vertex that is not connected by an edge to itself is the source of at least three edges can occur as the graph $H(\Lambda_\Gamma)/\Gamma$ of a Mumford curve (*i.e.* for Γ a p-adic Schottky group acting on a Bruhat–Tits tree). In the recent work [16], we consider the graph C^*-algebras associated to such graphs $H(\Lambda_\Gamma)/\Gamma$ and classified them via a K-theory computation obtained using the Bass–Hashimoto edge incidence operator on graphs.

The difference between the graphs $\mathcal{T}_\Gamma/\Gamma$ and $H(\Lambda)/\Gamma$ may be used to define a "code building" procedure of the type considered in §5 of [25] to produce codes from Mumford curves. We shall not deal with this aspect in the present paper.

The above construction will also work if the group Γ is replaced by a general finitely generated discrete subgroup N of $\mathrm{PGL}(2, K)$. On the side of Mumford curves this corresponds to orbifold uniformization, or Mumford curves with automorphisms, cf. [15]. Such N always has a finite index normal free subgroup. Conversely, given such a free group Γ, a finitely generated discrete subgroup $N \subset \mathrm{PGL}(2, K)$ contained in the normalizer $N(\Gamma)$ of Γ in $\mathrm{PGL}(2, K)$

determines a finite group $G = N/\Gamma \hookrightarrow \mathrm{Aut}(X)$ of automorphisms of the Mumford curve $X = \Omega_\Gamma/\Gamma$, since $\mathrm{Aut}(X) = N(\Gamma)/\Gamma$. Thus, it becomes relevant to study the equivariant deformation problem, of how these data can be deformed to another curve of the same genus with an action of the same group (*cf.* [15]). For $\Gamma \subset N$ as above, the finite group $G = N/\Gamma$ acts on $\mathcal{G} = H(\Lambda_\Gamma)/\Gamma$ with quotient the finite graph $\mathcal{G}_N = H(\Lambda_\Gamma)/N$. Let $\rho_0 : N \hookrightarrow \mathrm{Aut}(\mathcal{T})$ denote the inclusion of $N \subset N(\Gamma) \subset \mathrm{Aut}(\mathcal{T})$. Let $\mathrm{Hom}^*(N, \mathrm{Aut}(\mathcal{T}))$ denote the subset of $\mathrm{Hom}(N, \mathrm{Aut}(\mathcal{T}))$ of injective homomorphisms with discrete image. This governs the equivariant deformations. There is an open neighborhood of ρ_0 in the space $\mathrm{Hom}^*(N, \mathrm{Aut}(\mathcal{T}))$ of the form

$$\mathcal{U}(\rho_0) = \{\rho \in \mathrm{Hom}^*(N, \mathrm{Aut}(\mathcal{T}))|\ \rho(\gamma_i)(x_i) = y_i,\ \rho(\gamma_i)(y_i) = \gamma_i(y_i)\}. \tag{2.12}$$

Here $\{\gamma_i\}_{i=1}^g$ is a set of generators for the Schottky group $\Gamma \subset N$ and x_i, y_i are points on the axes $L(\gamma_i)$ that specify the Schottky data. As shown in [19], the neighborhood $\mathcal{U}(\rho_0)$ has the property that, for all $\rho \in \mathcal{U}(\rho_0)$, the group $\rho(\Gamma)$ is a Schottky group, of finite index in $\rho(N)$, with the same Schottky data as the original $\rho_0(\Gamma)$. In particular, in our setting, this shows that, by themselves, the spectral triples introduced in the previous section will not distinguish Mumford curves in the family $\rho(\Gamma) \subset \rho(N)$, with $\rho \in \mathcal{U}(\rho_0)$. However, one can implement in the construction the induced action of the group G on the C^*-algebra of the graph $\mathcal{G} = H(\Lambda_\Gamma)/\Gamma$, or consider G-equivariant spectral geometries for the Morita equivalent algebra $C(\mathcal{S}_A^+ \times G) \rtimes N$, with N acting on the left on $G = N/\Gamma$.

In [12] the local L-factor $L_v(H^1(X), s) = \det(1 - Fr_v^* N(v)^{-s} | H^1(\bar{X}, \mathbb{Q}_\ell)^{I_v})^{-1}$ of a Mumford curve was recovered from the data $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of a theta-summable spectral triple as described above. Another possible direction in which the construction of such spectral triples may be of arithmetic significance is by associating to a Mumford curve some cocycles in the entire cyclic cohomology of a smooth subalgebra of $\mathcal{A}$.

Cyclic cohomology was introduced by Connes as a natural receptacle for the characters of finitely summable Fredholm modules. Similarly, in the theta summable case that corresponds to "infinite dimensional geometries", one can also define characters through the JLO cocycle $\varphi = (\varphi_{2n})$ (*cf.* IV.8.ϵ [8]) of the form

$$\varphi_{2n}(a^0, \ldots, a^n) = \int_{\substack{s_i \geq 0 \\ \sum s_i = 1}} \mathrm{sTr}\left(a^0 e^{-s_0 \mathcal{D}^2} [\mathcal{D}, a^1] e^{-s_1 \mathcal{D}^2} \cdots [\mathcal{D}, a^{2n}] e^{-s_{2n} \mathcal{D}^2}\right)\, ds_0 \cdots ds_{2n}, \tag{2.13}$$

where sTr denotes the supertrace. These live naturally in the entire cyclic cohomology, as shown in §IV.7 of [8].

3. Finitely summable spectral triples

It is clear that the θ-summable condition imposes a strong limitation on how one may be able to apply tools from noncommutative geometry to the arithmetic context, most notably the local index formula of [10], which requires the finitely summable setting. There were already strong indications from the original construction (*cf.* [13]) that it should be possible to obtain finitely summable spectral triples associated to Mumford curves.

In fact the Cuntz–Krieger algebra O_A has up to stabilization (*i.e.* tensoring with compact operators) a second description as a crossed product (*cf.* [18]). Namely, one has an identification

$$\overline{O_A} \cong \overline{\mathcal{F}}_A \rtimes_T \mathbb{Z}, \tag{3.1}$$

where $\mathcal{F}_A$ is an approximately finite dimensional (AF) algebra, *i.e.* a direct limit of finite dimensional algebras. Here for a unital C^*-algebra $\mathcal{A}$ we use the notation $\overline{\mathcal{A}} = \mathcal{A} \otimes \mathcal{K}$ with $\mathcal{K}$ the algebra of compact operators. The algebra $\overline{\mathcal{A}}$ is no longer unital.

In the case of the algebra $O_A \cong C(\Lambda_\Gamma) \rtimes \Gamma$, the hyperbolic growth of the Schottky group Γ prevents one from constructing a finitely summable Dirac operator. This is no longer the case for the algebra $\mathcal{A}_\Gamma := \overline{\mathcal{F}_A} \rtimes \mathbb{Z}$. The fact that this algebra can be written as a crossed product by the integers implies that, by Connes' result on hyperfiniteness [7], it may carry a finitely summable spectral triple.

However, the fact of working with non-unital algebras forces one to relax the axioms of finitely summable spectral triple to a suitable "local" version, as discussed in [20] in the important example of Moyal planes.

Definition 3.1. *Let $\mathcal{A}$ be a non-unital C^*-algebra. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ consists of the data of a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of the algebra as bounded operators on a separable Hilbert space $\mathcal{H}$, together with an unbounded self-adjoint operator $\mathcal{D}$ on $\mathcal{H}$ such that the subalgebra*

$$\mathcal{A}_\infty := \{a \in \mathcal{A} \mid a\mathrm{Dom}\mathcal{D} \subseteq \mathrm{Dom}\mathcal{D},\ [\mathcal{D}, a] \in \mathcal{B}(\mathcal{H}),\ a(1+\mathcal{D}^2)^{-1} \in \mathcal{K}(\mathcal{H})\} \tag{3.2}$$

is dense in $\mathcal{A}$.

In (3.2) $\mathcal{K}(\mathcal{H})$ denotes the ideal of compact operators. In particular, if $\mathcal{D}$ has compact resolvent, then the last property of $\mathcal{A}_\infty$ is automatically fulfilled.

In the case we are interested in, the AF algebra $\mathcal{F}_A$ can be described in terms of a groupoid C^*-algebra associated to the "unstable manifold" in the Smale space $(\mathcal{S}, T)$. In fact, consider the algebra $\mathcal{O}_A^{alg}$ generated algebraically by the S_i and S_i^* subject to the Cuntz–Krieger relations (2.3). Elements in $\mathcal{O}_A^{alg}$ are linear combinations of monomials $S_\mu S_\nu^*$, for multi-indices μ, ν, *cf.* [18]. The AF algebra $\mathcal{F}_A$ is generated by elements $S_\mu S_\nu^*$ with $|\mu| = |\nu|$, and is filtered by finite dimensional algebras $\mathcal{F}_{A,n}$ generated by elements of the form $S_\mu P_i S_\nu^*$ with $|\mu| = |\nu| = n$ and $P_i = S_i S_i^*$ the range projections, and embeddings determined by the matrix A. The commutative algebra $\mathrm{C}(\Lambda_\Gamma)$ sits as a subalgebra of $\mathcal{F}_A$ generated by all range projections $S_\mu S_\mu^*$. The embedding is compatible with the filtration and with the action of the shift T, which is implemented on $\mathcal{F}_A$ by the transformation $a \mapsto \sum_i S_i\, a\, S_i^*$. (*cf.* [18].) The stabilization $\overline{\mathcal{F}_A}$ is a non-unital AF algebra.

We use the description (3.1) as the starting point for the construction of finitely summable spectral triples.

The following result provides a modification of Theorem 2.1 of [1], where the AF-algebra is now not necessarily unital, the sequence of eigenvalues need not be positive, and the growth condition required for the sequence of eigenvalues is finer. This will suffice for our purpose of defining a finitely summable spectral triple.

Theorem 3.2. *Let $\mathcal{A}$ be a (not necessarily unital) AF-algebra, and let p be a positive real number. Then there exists an unbounded Fredholm module $(\mathcal{H}, \mathcal{D})$ over $\mathcal{A}$ such that $(1 + \mathcal{D}^2)^{-p/2} \in \mathcal{L}^1(\mathcal{H})$. In particular, $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a p-summable odd spectral triple.*

Proof. We can write $\mathcal{A} = \overline{\cup_{n=1}^\infty \mathcal{A}_n}$, where each $\mathcal{A}_n$ is a finite dimensional C^*-algebra.

Let τ be a state on $\mathcal{A}$, that is, a continuous linear functional $\tau : \mathcal{A} \to \mathbb{C}$ of norm one, such that $\tau(a^*a) \geq 0$ for all $a \in \mathcal{A}$. The norm of a positive linear functional is the limit of its evaluation at any approximate identity, so that, in the unital case a state is just a positive linear functional with $\tau(1) = 1$. We denote by $\mathcal{H} = L^2(\mathcal{A}, \tau)$ the Hilbert space of the GNS

representation defined by the state τ. Namely, $\mathcal{H}$ is the Hilbert space completion of the quotient $\mathcal{A}/\mathcal{N}_\tau$ with respect to the inner product induced by $\tau(b^*a)$, where $\mathcal{N}_\tau$ is the closed left ideal $\mathcal{N}_\tau = \{x \in \mathcal{A} : \tau(x^*x) = 0\}$. In this non-unital case the cyclic vector ξ in the GNS representation is obtained as the limit in the norm of $\mathcal{H}$ of the classes of a given approximate identity for $\mathcal{A}$.
Let η denote the quotient map $\eta : \mathcal{A} \to \mathcal{H}$. We set $\mathcal{H}_n = \eta(\mathcal{A}_n)$. These are finite dimensional subspaces of $\mathcal{H}$ with $\dim \mathcal{H}_n \leq \dim \mathcal{A}_n$ and $\mathcal{H}_n \subset \mathcal{H}_{n+1}$. We assume that the $\mathcal{H}_n$ give a filtration of $\mathcal{H}$. Let $\Pi_n : \mathcal{H} \to \mathcal{H}_n$ be the orthogonal projection onto $\mathcal{H}_n$ and put $\hat{\Pi}_n = \Pi_n - \Pi_{n-1}$, for $n \geq 2$, and $\hat{\Pi}_1 = \Pi_1$.
Following the construction of [1], we now show that we can choose a sequence (λ_n) of *real* numbers such that the unbounded operator $\mathcal{D} = \sum_{n=1}^\infty \lambda_n \hat{\Pi}_n$ defined on the dense subspace $\cup_{n=1}^\infty \mathcal{H}_n$ of $\mathcal{H}$ satisfies the p-summability condition.
In fact, for any n we may assume that $\mathcal{H}_n \subsetneq \mathcal{H}_{n+1}$, since otherwise we would have $\hat{\Pi}_{n+1} = 0$, and $\mathcal{A}_n \subsetneq \mathcal{A}_{n+1}$, so that we have $\dim \mathcal{A}_{n+1} > \dim \mathcal{A}_n$. This gives $\dim \mathcal{A}_n \geq n$, since $\dim \mathcal{A}_1 \geq 1$. Now, if we choose the eigenvalues λ_n so that $|\lambda_n| \geq (\dim \mathcal{A}_n)^q$, for some $q > 2/p$, we obtain an estimate of the form

$$\begin{aligned}
\mathrm{Tr}(1+\mathcal{D}^2)^{-p/2} &= \sum_{n=1}^\infty (1+|\lambda_n|^2)^{-p/2} \dim E_{\lambda_n} \\
&= \sum_{n=1}^\infty (1+|\lambda_n|^2)^{-p/2} (\dim \mathcal{H}_n - \dim \mathcal{H}_{n-1}) \\
&\leq \sum_{n=1}^\infty |\lambda_n|^{-p} (\dim \mathcal{H}_n) \leq \sum_{n=1}^\infty |\lambda_n|^{-p} (\dim \mathcal{A}_n) \\
&\leq \sum_{n=1}^\infty (\dim \mathcal{A}_n)^{-pq} (\dim \mathcal{A}_n) \leq \sum_{n=1}^\infty \frac{1}{n^{pq-1}} < \infty \ ,
\end{aligned}$$

where we set $\mathcal{A}_0 = \{0\}$. Also notice that, for $m \geq n$ we have $\mathcal{A}_m(\mathcal{H}_n) \subset \mathcal{H}_n$, so that for $n > m$ and $a \in \mathcal{A}_m$ we have $[\hat{\Pi}_n, a] = 0$. Thus, we obtain $[\mathcal{D}, a] = \sum_{n=1}^m \lambda_n [\hat{\Pi}_n, a]$. This shows that $[\mathcal{D}, a]$ has a bounded closure on $\mathcal{H}$. Moreover, if $a \in \mathcal{A}_m$ then $a\mathrm{Dom}\mathcal{D} \subseteq \mathrm{Dom}\mathcal{D}$. Thus, the subalgebra

$$\mathcal{A}_\infty := \{a \in \mathcal{A} \mid a\mathrm{Dom}\mathcal{D} \subseteq \mathrm{Dom}\mathcal{D} \text{ and } [\mathcal{D}, a] \text{ admits a bounded closure}\}$$

contains $\cup_{n=1}^\infty \mathcal{A}_n$, hence it is dense in $\mathcal{A}$. □

In fact, one can strenghten the result of the theorem as in Corollary 3.3 below. However, we are interested in allowing for possibly non-faithful states, since we are interested in a state that encodes the data of the uniformization through the Patterson–Sullivan measure on the limit set as in the theta summable construction.

Corollary 3.3. *In the construction of Theorem 3.2 one can always choose the state τ to be faithful.*

Proof. It is enough to show that every separable C^*-algebra admits a faithful state. Let $\mathcal{A}$ be a separable C^*-algebra and let $\mathcal{A}_+$ be its positive cone. Choose a sequence (a_m) which is dense in $\mathcal{A}_+$, and choose a bounded approximate identity (u_n) for $\mathcal{A}$ in $\mathcal{B}_1(\mathcal{A}_+)$. Set $w_{n,m} := u_n a_m u_n$ in $\mathcal{A}_+$, and choose a state $\tau_{n,m}$ on $\mathcal{A}$ such that $\tau(w_{n,m}) = \|w_{n,m}\|$. Now define

$$\tau := \sum_{n,m\geq 1}^{\infty} \frac{\tau_{n,m}}{2^{n+m}}.$$

One can see that τ is a faithful state on $\mathcal{A}$. $\square$

In the case of a faithful state, the Hilbert space $\mathcal{H}$ is the closure of $\mathcal{A}$ in the inner product $\langle a,b\rangle = \tau(b^*a)$ and the filtration of $\mathcal{H}$ is given by $\mathcal{H}_n = \eta(\mathcal{A}_n) = \mathcal{A}_n$.

We then obtain a finitely summable spectral triple for the algebra (3.1) through the following construction.

Theorem 3.4. *Let $(\mathcal{A}, H, D)$ be an odd spectral triple for the (not necessarily unital) C^*-algebra $\mathcal{A}$, and assume that $\mathcal{D}$ has compact resolvent. Consider the crossed product $\mathcal{A} \rtimes_\alpha \mathbb{Z}$, and assume that the dense subalgebra $\mathcal{A}_\infty$ of the spectral triple contains a dense α-invariant subalgebra $\mathcal{A}'_\infty$ such that, for any fixed $a \in \mathcal{A}'_\infty$ the sequence of bounded operator $\{[\mathcal{D}, \alpha^n(a)]\}_{n\in\mathbb{Z}}$ is uniformly bounded in the operator norm. Let $(C(S^1), \ell^2(\mathbb{Z}), \partial)$ be the standard spectral triple on the circle. Consider then the data*

$$(\mathcal{A} \rtimes_\alpha \mathbb{Z}, \mathcal{H}, \mathcal{D}) \tag{3.3}$$

where $\mathcal{H} = \ell^2(\mathbb{Z}, H) \oplus \ell^2(\mathbb{Z}, H)$ is the direct sum of two copies of the Hilbert space for the regular representation of $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ (as a reduced crossed product) and

$$\mathcal{D} = \begin{pmatrix} 0 & D_0^* \\ D_0 & 0 \end{pmatrix},$$

for $D_0 = D \otimes 1 + i \otimes \partial$. The data (3.3) define an even spectral triple for the algebra $\mathcal{A} \rtimes_\alpha \mathbb{Z}$, with respect to the grading on $\mathcal{H}$ given by

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If $(\mathcal{A}, H, D)$ is (p,∞)-summable, then $(\mathcal{A} \rtimes_\alpha \mathbb{Z}, \mathcal{H}, \mathcal{D})$ is $(p+1,\infty)$-summable.

Proof. Notice $\mathcal{D}$ is in fact the tensor product of D and ∂ in the Baaj-Julg's picture [2] of Kasparov's external product in K-homology

$$K^1(\mathcal{A}) \times K^1(C(S^1)) \to K^1(\mathcal{A} \otimes C(S^1)).$$

In particular, the facts that $\mathcal{D}$ is a selfadjoint operator with compact resolvent and is finitely summable is already well known.

Let V denote the regular representation of $C(S^1) = C^*(\mathbb{Z})$ in $\ell^2(\mathbb{Z})$, as part of the data $(C(S^1), \ell^2(\mathbb{Z}), \partial)$. Put $\mathcal{L} = \ell^2(\mathbb{Z}, H)$, so that $\mathcal{H} = \mathcal{L} \oplus \mathcal{L}$ and $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ acts diagonally on $\mathcal{H}$. Then, for all $\xi \in \mathcal{L}$, one has

$$(\mathcal{D}_0\xi)(k) = D(\xi(k)) - ik\xi(k), \quad \forall k \in \mathbb{Z}. \tag{3.4}$$

An element $a \otimes V^n \in C_c(\mathbb{Z}, \mathcal{A}) \subset \mathcal{A} \rtimes_\alpha \mathbb{Z}$, for $a \in \mathcal{A}'_\infty$ and $n \in \mathbb{Z}$, is represented in $\mathcal{L}$ by

$$((a \otimes V^n)\xi)(k) = \alpha^k(a)(\xi(n+k)), \quad \forall k \in \mathbb{Z}.$$

Using (3.4) one obtains

$$[\mathcal{D}, a \otimes V^n] = \begin{pmatrix} 0 & T_1 + inT_2 \\ T_1 - inT_2 & 0 \end{pmatrix},$$

where $T_i : \mathcal{L} \to \mathcal{L}$ are (unbounded) operators for $i = 1, 2$ given by $(T_1\xi)(k) = [D, \alpha^k(a)](\xi(n+k))$ and $(T_2\xi)(k) = \alpha^k(a)(\xi(n+k))$, for $k \in \mathbb{Z}$. These satisfy the estimates $\|T_1\| \leq$

$\sup_k \|[D,\alpha^k(a)]\| < \infty$, and $\|T_2\| \le \|a\| < \infty$. Therefore we see that $[\mathcal{D}, a\otimes V^n]$ admits a bounded closure in $\mathcal{B}(\mathcal{H})$. □

From an index theoretic perspective, it may be preferable to work with a finitely summable even spectral triples on the AF-algebra, since it is the K_0-group of an AF algebra that carries all the interesting information, while the K_1-group is trivial. It is easy to modify the previous construction to accommodate this case. As before, let $\mathcal{A} = \overline{\cup_{n=1}^\infty \mathcal{A}_n}$ be an AF-algebra and let τ be a state on $\mathcal{A}$. Consider the Hilbert space $H = L^2(\mathcal{A},\tau)\oplus L^2(\mathcal{A},\tau)$. One can take the diagonal action of $\mathcal{A}$ on H in the GNS representation, although it is better to proceed as in the θ-summable case and introduce a twisting on one of the copies of $L^2(\mathcal{A},\tau)$ by a nontrivial automorphism of the algebra. Now one can choose a sequence of *complex* (not necessarily real) numbers λ_n satisfying a suitable growth condition (*e.g.* $|\lambda_n| \ge (\dim \mathcal{A}_n)^q$, where $q > 2/p$) as the eigenvalues of an operator $D_0 = \sum_{n=1}^\infty \lambda_n \hat{\Pi}_n$ on $L^2(\mathcal{A},\tau)$. One then consider the operator on $\mathcal{H}$ defined by

$$D = \begin{pmatrix} 0 & D_0^* \\ D_0 & 0 \end{pmatrix}.$$

Then $(\mathcal{A}, H, D)$ is a p-summable even spectral triple with respect to the grading on H given by

$$\gamma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The construction of Theorem 3.4 is correspondingly modified to yield an *odd* spectral triple $(\mathcal{A}\rtimes_\alpha\mathbb{Z}, \mathcal{H}, \mathcal{D})$ with $\mathcal{H} = \ell^2(\mathbb{Z}, H) = H\otimes\ell^2(\mathbb{Z})$, the Hilbert space for the regular representation of $\mathcal{A}\rtimes_\alpha\mathbb{Z}$ as before, and with the Dirac operator given by $\mathcal{D} := D\otimes 1 + \gamma\otimes\partial$. Again, if $(\mathcal{A}, H, D)$ is (p,∞)-summable then $(\mathcal{A}\rtimes_\alpha\mathbb{Z}, \mathcal{H}, \mathcal{D})$ is $(p+1,\infty)$-summable.

4. Some motivating examples

We look here at some simple examples that give some motivation for introducing spectral triples associated to Mumford curves and justify why it may be interesting to derive invariants from these spectral geometries that are more refined than the C^*-algebra O_A itself.

We first look at the case of genus two Mumford curves discussed in [12], [13]. In this case we are considering a Schottky group Γ of rank two in $\mathrm{PGL}(2,K)$ where K is a finite extension of $\mathbb{Q}_p$. The combinatorially different forms of the graph $\mathcal{T}_\Gamma/\Gamma$ of the special fiber are illustrated in Figure 1.

In terms of corresponding Cuntz–Krieger C^*-algebras, we are considering in this case the algebras O_{A_i}, $i = 1,2,3$ with directed edge matrices A_i of the form

$$A_1 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

in the first case. In the second case (*cf.* Figure 1) we label by $a = e_1$, $b = e_2$ and $c = e_3$ the oriented edges in the graph $\mathcal{T}_\Gamma/\Gamma$, so that we have a corresponding set of labels $E = \{a, b, c, \bar{a}, \bar{b}, \bar{c}\}$ for the edges in the covering tree $\mathcal{T}_\Gamma$. A choice of generators for the group $\Gamma \simeq \mathbb{Z} * \mathbb{Z}$ acting on $\mathcal{T}_\Gamma$ is obtained by identifying the generators g_1 and g_2 of Γ with the

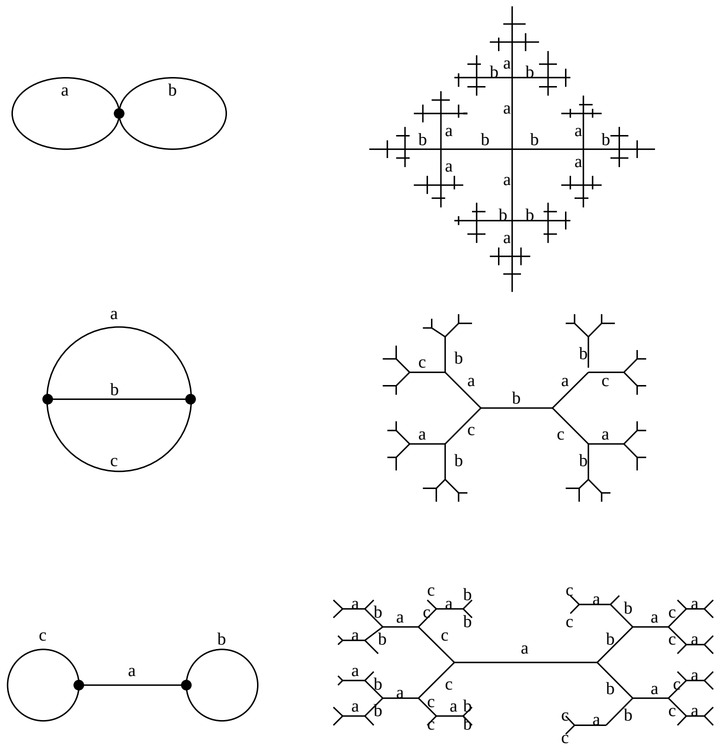

FIGURE 1. The graphs $\mathcal{T}_\Gamma/\Gamma$ for genus $g=2$, and the corresponding trees $\mathcal{T}_\Gamma$.

chains of edges $a\bar{b}$ and $a\bar{c}$. The directed edge matrix is then of the form

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The third case in Figure 1 is analogous. A choice of generators for the group $\Gamma \simeq \mathbb{Z}*\mathbb{Z}$ acting on Δ_Γ is given by $ab\bar{a}$ and c. The directed edge matrix is then

$$A_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Recall that, for a Cuntz–Krieger algebra the K-theory is computed in terms of the $n \times n$ matrix A in the form (*cf.* [18])

$$K_0(\overline{O_A}) = \mathbb{Z}^n/(1-A^t)\mathbb{Z}^n \qquad K_1(\overline{O_A}) = \mathrm{Ker}(1-A^t) \subset \mathbb{Z}^n. \tag{4.1}$$

In general, combinatorially different graphs $\mathcal{T}_\Gamma/\Gamma$ are not distinguished by the associated graph C^*-algebras alone. We can see this in the genus two case as follows.

Lemma 4.1. *In the case of genus $g = 2$, the C^*-algebras $\overline{O_{A_i}}$, $i = 1, 2, 3$, associated to the graphs of Figure 1 are isomorphic.*

Proof. First an explicit calculation shows that the K-groups are of the form

$$K_j(O_{A_i}) \cong \mathbb{Z}^2, \tag{4.2}$$

for $j = 0, 1$ and $i = 1, 2, 3$. Moreover, by [37], for simple Cuntz–Krieger algebras (*i.e.* algebras O_A where the matrix A is irreducible and not a permutation matrix) the condition that the groups $K_0(O_{A_i})$ are isomorphic implies that the algebras $\overline{O_{A_i}}$ are isomorphic. □

This can be seen as a special case of the more general result of [16].

Thus, a first question is whether more refined invariants coming from a spectral triple may be able to distinguish combinatorially different geometries. There is a more subtle kind of question of a similar nature.

As we discussed in §2.2 above, while the finite graph $\mathcal{T}_\Gamma/\Gamma$ only carries the combinatorial information on the special fiber of the Mumford curve, one can consider the finite graph $H(\Lambda_\Gamma)/\Gamma$, where $H(\Lambda_\Gamma)$ is the smallest subtree in the Bruhat–Tits tree $\mathcal{T}$ of the field K that contains axes of all elements of Γ. When one considers the tree $H(\Lambda_\Gamma)$ instead of $\mathcal{T}_\Gamma$ one is typically adding extra vertices. The way the tree $H(\Lambda_\Gamma)$ sits inside the Bruhat–Tits tree $\mathcal{T}$ depends on where the Schottky group Γ lies in $\mathrm{PGL}(2, K)$, unlike the information on the graph $\mathcal{T}_\Gamma/\Gamma$ which is purely combinatorial (*cf. e.g.* [28]).

We can consider a specific geometric example, again in the genus two case, by looking at a 1-parameter family of Mumford curves considered by Fumiharu Kato in [22]. There one considers the free amalgamated product $N = \mathbb{Z}_m * \mathbb{Z}_n$ of two cyclic groups. For k a (discretely) non-archimedean valued field of characteristic coprime to m and n, one considers the discrete embedding of N in $\mathrm{PGL}(2, k)$ given by

$$\langle \begin{pmatrix} \zeta^m t & 0 \\ \zeta^m - 1 & t \end{pmatrix}, \begin{pmatrix} \zeta^n & -(\zeta^n - 1)t^{-1} \\ 0 & 1 \end{pmatrix} \rangle. \tag{4.3}$$

One then considers the free subgroup Γ of N generated by commutators

$$\Gamma := [\mathbb{Z}_m, \mathbb{Z}_n]. \tag{4.4}$$

This group Γ is a maximal free subgroup of N of free rank $g := (m-1)(n-1)$. It is the Schottky group of a curve X_Γ. If one looks at the particularly simple case with $m = 2$ and $n = 3$ one finds a curve of genus two with $\mathbb{Z}_6$-symmetry. It is not hard to see that the graph of the special fiber is the second graph in Figure 1. Moreover, one can see that, if π denotes a uniformizer and we take $t = \pi^r$ then the graph $H(\Lambda_\Gamma)/\Gamma$ is again topologically of the same form but with $2r + 1$ vertices inserted on each of the three lines, namely it looks like the graph in Figure 2.

A direct calculation shows that in this case again the corresponding algebras O_{A_r} associated to these graphs are stably isomorphic. In fact, one can show as in the previous example that they have the same K-theory groups. (Again this follows from the general result of [16].)

This again raises the question of whether more refined invariants, such as the spectral triple which introduces a smooth subalgebra and a Dirac operator, can capture more interesting information on the geometry and the uniformization parameters. We hope to return to this general question in future work by looking more closely at such invariants. For the moment we can give a heuristic justification of why we think this may be possible.

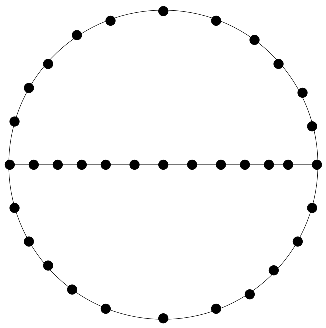

FIGURE 2. The graph $H(\Lambda_\Gamma)/\Gamma$ for $t = \pi^5$ in Kato's family.

In the case of the θ-summable case the information on the uniformization is stored in the Patterson–Sullivan measure which is used to define both the representation of the algebra on the Hilbert space and the Dirac operator. In the case of the finitely summable construction the dependence on the uniformization will enter again through a choice of a state τ on the AF algebra induced by the the Patterson–Sullivan measure.

The reason why the Patterson–Sullivan measure on the limit set is especially good in order to detect geometric properties that depend on the Schottky uniformization lies in a very general type of rigidity result (see [44]). This type of result implies, in our case, given two Schottky uniformizations of a fixed genus, if the abstract isomorphism of the Schottky groups Γ induces a homeomorphism of the limit sets which is absolutely continuous with respect to the Patterson–Sullivan measure, then the abstract isomorphism comes from an automorphism of the ambient group $\mathrm{PGL}(2, K)$. One knows that these are either inner or they come from automorphisms of the field K, so that one can in fact recover much of the information on the Schottky uniformization (the Schottky group up to conjugation) from information on the Patterson–Sullivan measure.

Ongoing work in this direction [17] shows that the zeta functions of a finitely summable spectral triple on a Mumford curve, for a choice of the state τ that encodes the Patterson–Sullivan measure, are enough to completely reconstruct the isomorphism class of the Mumford curve.

5. Higher rank cases

We now proceed to consider some higher dimensional classes of buildings. The case of rank two is the most interesting because the classification problem is especially difficult for rank two and the construction of new invariants can be useful to that purpose. We first consider the simplest case that can be reduced to the construction for trees and some simple generalizations. We then discuss some more general classes in the hyperbolic case.

5.1. **Products of trees.**

The first case of 2-dimensional buildings to which the construction given for trees can be extended is the case considered in [24]. This deals with affine buildings Δ whose 2-cells are euclidean squares and whose 1-skeleton is the product of two trees. On such buildings one considers the action of Burger–Mozes (BM) groups (*cf.* [5]), which are discrete torsion free subgroups $\Gamma \subset \mathrm{Aut}(\mathcal{T}_1) \times \mathrm{Aut}(\mathcal{T}_2)$ acting freely and transitively on the set of vertices. If the trees $\mathcal{T}_i$ have even valences n_i, then the quotient Δ/Γ is a polyhedron with one vertex and $n_1 n_2/4$ square faces, with fundamental group Γ. The link of the vertex is a complete bipartite graph with $n_1 + n_2$ vertices and edges between each of the first set of n_1 vertices and each of the second set of n_2 vertices.

The boundary $\partial\Delta$ is defined by an equivalence relation on sectors in apartments of Δ, by which two sectors are equivalent if their intersection contains a sector (this notion extends the usual shift-tail equivalence of paths in a tree). As in the case of trees, the choice of a vertex determines a choice of a representative in each class of sectors. In the case we are considering, this gives a non-canonical identification $\partial\Delta \simeq \partial\mathcal{T}_1 \times \partial\mathcal{T}_2$.

The action of Γ on Δ extends to an action on the boundary, hence one can consider the C^*-algebra $C(\partial\Delta) \rtimes \Gamma$. It is shown in [24] that this is isomorphic to a rank 2 Cuntz–Krieger algebra associated to two subshifts of finite type in the alphabet given by Γ-equivalence classes of oriented chambers in Δ. More precisely, the alphabet $\mathcal{R}$ is given by letters of the form $r = (a, b, b', a')$ with $ab = b'a'$, and the horizontal and vertical transition matrices A_1, A_2 are commuting $n_1n_2 \times n_1n_2$ matrices with entries in $\{0, 1\}$. They give the admissibility condition for adjacent squares, in the horizontal and vertical direction, respectively, *cf.* [24]. (The notion is well defined, due to the special class of groups Γ considered.)

As in the case of trees, we consider functions on the boundary. Namely, we consider the Hilbert space $\mathcal{L} = L^2(\partial\Delta, d\mu)$, where (after a choice of a base vertex in Δ) the measure can be identified with a product $d\mu(x) = d\mu_1(x_1) \times d\mu_2(x_2)$ of Patterson–Sullivan measures on $\partial\mathcal{T}_1$ and $\partial\mathcal{T}_2$.

Moreover, the subshifts of finite type in the horizontal and vertical directions associated to the matrices A_i determine a filtration on functions on the boundary, analogous to the one considered in the case of a single tree. In fact, for a fixed base vertex v_0 of Δ, a point $x \in \partial\Delta$ is identified with a sector originating at v_0. Under the identification $\partial\Delta \simeq \partial\mathcal{T}_1 \times \partial\mathcal{T}_2$ determined by v_0, we can write, for $f \in C(\partial\Delta, \mathbb{Z}) \otimes \mathbb{C}$,

$$f(x) = f(x_1, x_2) = f(a_0a_1a_2 \dots, b_0b_1b_2 \dots) = f(a_0 \dots a_\ell, b_0 \dots b_k),$$

where $a_0a_1a_2 \dots$ is a path in $\mathcal{T}_1$ and $b_0b_1b_2 \dots$ is a path in $\mathcal{T}_2$, such that the corresponding infinite 2-dimensional word is admissible according to the conditions given by the matrices A_i. This means that the space $C(\partial\Delta, \mathbb{Z}) \otimes \mathbb{C}$ has a filtration by finite dimensional linear subspaces $\mathcal{V}_{\ell,k}$ of functions of finite admissible 2-dimensional words in the alphabet $\mathcal{R}$.

We associate a grading operator to this filtration as follows. We denote by $\Pi_{\ell,k}$ the orthogonal projection of $\mathcal{L}$ onto the finite dimensional subspace $\mathcal{V}_{\ell,k}$. We consider the densely defined unbounded self-adjoint grading operator on $\mathcal{L}$ of the form

$$D = \sum_{m=0}^{\infty} m \sum_{k=0}^{m} \hat{\Pi}_{m-k,k}, \tag{5.1}$$

where $\hat{\Pi}_{\ell,k}$ is defined by the inclusion-exclusion $\hat{\Pi}_{\ell,k} = \Pi_{\ell,k} - \Pi_{\ell-1,k} - \Pi_{\ell,k-1} + \Pi_{\ell-1,k-1}$.

Proposition 5.1. *Let the data $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be given by the algebra $\mathcal{A} = C(\partial\Delta) \rtimes \Gamma$ acting on the Hilbert space $\mathcal{H} = \mathcal{L} \oplus \mathcal{L}$ through a representation* (2.5)*, and the operator $\mathcal{D} = FD$, with D as in* (5.1) *and F the involution exchanging the two copies of $\mathcal{L}$. The data $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ define a θ-summable spectral triple.*

Proof. The setting is very similar to the previous cases. We obtain a representation of the algebra $\mathcal{A}$ on the Hilbert space $\mathcal{H}$ by letting functions in $C(\partial\Delta)$ act on $\mathcal{L}$ as multiplication operators and elements in the group Γ by unitary operators

$$(T_{\gamma^{-1}} f)(x) = \gamma_1'(x_1)^{\delta_1/2} \gamma_2'(x_2)^{\delta_2/2} f(\gamma(x)), \tag{5.2}$$

where $x = (x_1, x_2) \in \partial\Delta \simeq \partial\mathcal{T}_1 \times \partial\mathcal{T}_2$ and $\gamma = (\gamma_1, \gamma_2) \in \Gamma \subset \mathrm{Aut}(\mathcal{T}_1) \times \mathrm{Aut}(\mathcal{T}_2)$.

In order to show that there exists a dense involutive subalgebra $\mathcal{A}_0 \subset \mathcal{A}$, such that the commutators $[\mathcal{D}, a]$ are bounded operators on $\mathcal{H}$ for all $a \in \mathcal{A}_0$, it is sufficient to prove that the commutators of D with functions in $C(\partial\Delta, \mathbb{Z}) \otimes \mathbb{C}$ and with operators T_γ for $\gamma \in \Gamma$ are bounded. The first case is clear since a function $h \in C(\partial\Delta, \mathbb{Z}) \otimes \mathbb{C}$ is contained in some $\mathcal{V}_{u,v}$, hence the corresponding multiplication operator maps $\mathcal{V}_{\ell,k}$ to itself whenever $\mathcal{V}_{\ell,k} \supset \mathcal{V}_{u,v}$. Since the γ_i' are locally constant on $\partial\mathcal{T}_i$, we also obtain that there exists some integer $n = n(\gamma)$ such that $T_\gamma : \mathcal{V}_{\ell,k} \to \mathcal{V}_{\ell+n,k+n}$ whenever $\mathcal{V}_{\ell,k} \supset \mathcal{V}_{u,v}$, for $\gamma_i' \in \mathcal{V}_{u,v}$. $\square$

It is easy to see that, even if the case considered here is euclidean and not hyperbolic, still the spectral triple constructed in this way will not be finitely summable. For example, if the group acting is a product $\Gamma \times \Gamma$ of two copies of a Schottky group of genus g and $\mathcal{T}_1 = \mathcal{T}_2$ is the Cayley graph of Γ, and $A_1 = A_2$ is the directed edge matrix of the quotient group, namely the matrix with $A_{ij} = 1$ unless $|i - j| = g$, then the dimensions of the eigenspaces of the operator (5.1) are $\dim E_m = (m+1)2g(2g-1)^{m-1}(2g-2)^2$ for $m \geq 2$, $4g(2g-1)(2g-2)$ for $m = 1$ and $2g(2g-1)$ for $m = 0$.

Noncommutative spaces associated to Euclidean buildings, in the form of crossed product C^*-algebras $C(\partial\Delta) \rtimes \Gamma$ were also considered in the case of buildings with triangular presentations. Of particular arithmetic interest is the case of the "fake projective planes". These are algebraic surfaces with ample canonical class and the same numbers $p_g = q = 0$ and $c_1^2 = 3c_2 = 9$ as the projective plane $\mathbb{P}^2$. The first such example was constructed by Mumford [31] using p-adic uniformization by a discrete cocompact subgroup $\Gamma \subset \mathrm{PGL}(3, \mathbb{Q}_2)$. More recent examples (known as CMSZ fake projective planes) were obtained, again using actions on the Bruhat–Tits building of $\mathrm{PGL}(3, \mathbb{Q}_2)$ (see [6], [23]). The formal model of the CMSZ fake projective planes is obtained by the action on the Bruhat–Tits building Δ of $\mathrm{PGL}(3, \mathbb{Q}_2)$ of two subgroups Γ_1, Γ_2 of index 3. The quotients $\Gamma_i\backslash\Omega$, where Ω is Drinfeld's symmetric space of dimension 2 over $\mathbb{Q}_2$, are non-isomorphic fake projective planes. All these examples admit a description as Shimura varieties (*cf.* [23]). The topology of the corresponding noncommutative spaces $\mathcal{A}_i = C(\partial\Delta) \rtimes \Gamma_i$ distinguishes between the fake projective planes. In fact, the results of [36] show for instance that, for the CMSZ cases, the K-theory of $\mathcal{A}_i$ is $K_0(\mathcal{A}_1) = \mathbb{Z}/3$ while $K_0(\mathcal{A}_2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$. Recently, Gopal Prasad and Sai-Kee Yeung identified in [32] the complete list of fake projective planes (see also [33] for a higher dimensional case). It would be interesting to see if these can be studied from the operator theoretic point of view and whether the K-theory of the relevant C^*-algebras distinguishes them or whether more refined invariants of spectral geometry can be used to that purpose.

As we discussed in §4, one can consider higher dimensional cases of combinatorially different actions of the same group Γ on a building Δ, such as two combinatorially non-equivalent presentations of the same group acting on a product of two trees of degree four (presentations 42 and 44 from [24]):

$$P_1 = \{a_1, b_1, a_2, b_2 : a_1 b_1 a_1^{-1} b_1^{-1}, a_1 b_2 a_1^{-1} b_2^{-1}, a_2 b_1 a_2^{-1} b_2, a_2 b_2 a_2^{-1} b_1\}$$

$$P_2 = \{a_1, b_1, a_2, b_2 : a_1 b_1 a_1^{-1} b_2, a_1 b_2 a_1^{-1} b_1, a_2 b_1 a_2^{-1} b_2, a_2 b_2 a_2^{-1} b_1\}.$$

One can ask whether in such examples invariants coming from spectral triples may be able to distinguish the combinatorially different actions.

5.2. **Polyhedra covered by products of trees.**

In the previous section we looked at 2-dimensional buildings that are products of trees, with the action of groups of BM type, $\Gamma \subset \mathrm{Aut}(\mathcal{T}_1) \times \mathrm{Aut}(\mathcal{T}_2)$. In this section we show that the results of the previous section may be applied more generally. Namely, we show the existence of an infinite family of examples that are not of BM type, but which can be reduced to BM type by passing to a subgroup of index four.

A *polyhedron* is a two-dimensional complex obtained from several oriented p-gons by identification of corresponding sides. Consider a vertex of the polyhedron and take a sphere of a small radius at this point. The intersection of the sphere with the polyhedron is a graph, which is called the *link* at this vertex.

Recall that a graph is *bipartite* if its set of vertices can be partitioned into two disjoint subsets P and L such that no two vertices in the same subset lie on a common edge. It is known by the result of [3] that the universal cover of a polyhedron with square faces and complete bipartite graphs as links is a 2-dimensional Euclidean building which is a product of two trees $\mathcal{T}_1 \times \mathcal{T}_2$. This means that an efficient method to construct Euclidean buildings with compact quotients is by constructing finite polyhedra with appropriate links.

We recall the definition of *polygonal presentation* given in [40].

Definition 5.2. *Suppose given n disjoint connected bipartite graphs $G_1, G_2, \dots G_n$. Let P_i and L_i denote the sets of black and white vertices in G_i, for $i = 1, ..., n$. Let $P = \cup P_i$ and $L = \cup L_i$, with $P_i \cap P_j = \emptyset$ and $L_i \cap L_j = \emptyset$, for $i \neq j$. Let λ be a bijection $\lambda : P \to L$.*
A set $\mathcal{P}$ of k-tuples $(x_1, x_2, \dots, x_k)$, with $x_i \in P$, is be called a polygonal presentation *over P compatible with λ if the following properties are satisfied.*

(1) *If $(x_1, x_2, x_3, \dots, x_k) \in \mathcal{P}$, then $(x_2, x_3, \dots, x_k, x_1) \in \mathcal{P}$.*
(2) *Given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \dots, x_k) \in \mathcal{P}$ for some $x_3, \dots, x_k$ if and only if x_2 and $\lambda(x_1)$ are incident in some G_i.*
(3) *Given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \dots, x_k) \in \mathcal{P}$ for at most one $x_3 \in P$.*

If there exists such $\mathcal{P}$, then the corresponding λ is called a basic bijection*.*

Polygonal presentations for $n = 1$, $k = 3$, with the incidence graph of the finite projective plane of order two or three as the graph G_1, were listed in [6].

One can associate a polyhedron X with n vertices to a polygonal presentation $\mathcal{P}$ in the following way. To every cyclic k-tuple $(x_1, x_2, x_3, \dots, x_k)$ we assign an oriented k-gon, with the word $x_1 x_2 x_3 \dots x_k$ written on its boundary. The polyhedron is then obtained by identifying sides with the same labels in these polygons, preserving orientation. We say then that the polyhedron X corresponds to the polygonal presentation $\mathcal{P}$. It was shown in [40] that a polyhedron X that corresponds to a polygonal presentation $\mathcal{P}$ has the graphs $G_1, G_2, \dots, G_n$ as links.

In particular, suppose that X is a polyhedron corresponding to a polygonal presentation $\mathcal{P}$ and let s_i and t_i be, respectively, the number of vertices and of edges of the graph G_i, for $i = 1, ..., n$. Then X has n vertices (the number of vertices of X is equal to the number of graphs), $k \sum_{i=1}^{n} s_i$ edges and $\sum_{i=1}^{n} t_i$ faces. All the faces are polygons with k sides.

We use the procedure illustrated above to construct certain compact polyhedra with square faces whose links are complete bipartite graphs.

Definition 5.3. *We say that the sets $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k$ of connected bipartite graphs are* compatible*, if all graphs in every set have the same number n of white vertices and, for every* *pair $\mathcal{G}_1$, $\mathcal{G}_j$, $j = 2, \dots, k$, there is a corresponding bijection between sets of white vertices that preserves the degrees of the vertices.*

Proposition 5.4. [42] *Let $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k$ be compatible sets of connected bipartite graphs, $k \geq 1$. Then there exists a family of finite polyhedra with $2k$-gonal faces, whose links at the vertices are isomorphic to the graphs from $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k$.*

We give now an explicit construction of a particular case of this theorem, when $k = 2$ and each of the families $\mathcal{G}_i$, $i = 1, 2$ contains exactly one complete bipartite graph G_i, with n white and r black vertices.

By [40], to construct the polyhedron with given links, it is sufficient to construct a corresponding polygonal presentation. By the definition of compatible sets of bipartite graphs, there is a bijection α from the set of white vertices of G_1 to the set of white vertices of G_2, preserving the degrees of the white vertices. We mark the white vertices of G_i, $i = 1, 2$ by letters of an alphabet $\mathcal{A}_i = \{x_1^i, \dots, x_n^i\}$, such that the bijection α is induced by the indices of the letters, *i.e.* $\alpha(x_m^1) = x_m$. We mark the black vertices of G_i, $i = 1, 2$ by letters of an alphabet $\mathcal{B}_i = \{y_1^i, y_2^i, \dots, y_r^i\}$. Thus, every edge of $G_i, i = 1, 2$ can be presented in a form $(x_m^i y_l^i)$, for $m = 1, \dots, n$ and $l = 1, \dots, r$.

Having such a bijection α of white vertices we can choose a bijection β of the set of edges of G_1 to the set of edges of G_2, which preserves α. Let $\beta_j(x_m^1 y_l^1) = x_m^2 y_j^2$. We let the cyclic word $(x_m^1, y_l^1, x_m^2, y_j^2)$, for $m = 1, \dots, n$ and $l, j = 1, \dots, r$ belong to the set $\mathcal{P}$. It is shown in [42] (in more general form) that $\mathcal{P}$ is a polygonal presentation. Denote then by X the polyhedron corresponding to this polygonal presentation $\mathcal{P}$.

Definition 5.5. *A polygonal presentation $\mathcal{P}$ satisfies the* stable pairs condition *if any word $(x_m^1, y_l^1, x_m^2, y_j^2) \in \mathcal{P}$ if and only if every word in $\mathcal{P}$ containing x_m^1 or x_m^2 has the form $(x_m^1, y_s^1, x_m^2, y_t^2)$ and every word in $\mathcal{P}$ containing y_l^1 or y_j^2 has the form $(x_p^1, y_l^1, x_p^2, y_j^2)$.*

With the condition of Definition 5.5, we obtain the following result.

Lemma 5.6. *If X is a polyhedron X corresponding to a polygonal presentation that satsifies the stable pairs condition, then the fundamental group of X is of BM type.*

Proof. Consider a polyhedron X with polygonal presentation $\mathcal{P}$=$(x_m^1, y_l^1, x_m^2, y_j^2)$, for $m = 1, \dots, n$ and $l, j = 1, \dots, r$, which satisfies the stable pair condition. This polyhedron has four vertices and nr faces with words from $\mathcal{P}$ on their boundary. To compute its fundamental group we need letters from one word of $\mathcal{P}$, say $(x_m^1, y_l^1, x_m^2, y_j^2)$, to be trivial. Because of the stable pair condition, $y_s^1 y_t^2 = 1$, for $s, t = 1, \dots, r$, and $x_p^1 x_p^2 = 1$. All relations of the fundamental group of X have the form $(x_p^1, y_s^1, (x_p^1)^{-1}, (y_s^1)^{-1})$, for $p = 1, \dots, m-1, m+1, \dots, n$ and $s = 1, \dots, t-1, t+1, \dots, r$. It is then a BM group by definition and it acts on a product of trees of valences $2(n-1)$ and $2(r-1)$. □

We now consider the following infinite family of examples. Let G be a complete bipartite graph on $8q$ vertices, $4q$ black vertices and $4q$ white ones. Let $\mathcal{A}$ and $\mathcal{B}$ be two alphabets on $4q$ letters, $\mathcal{A} = \{x_1, x_2, \dots, x_{4q}\}$ and $\mathcal{B} = \{y_1, y_2, \dots, y_{4q}\}$. We mark every black vertex with an element from $\mathcal{A}$ and every white vertex with an element from $\mathcal{B}$.

We define a polygonal presentation $\mathcal{P}$ as the following set of cyclic words

$$\begin{aligned} &(x_{1+4i}, x_{2+4j}, x_{4+4i}, x_{3+4j}), \quad (x_{1+4i}, x_{1+4j}, x_{4+4i}, x_{4+4j}), \\ &(x_{1+4i}, x_{3+4j}, x_{4+4i}, x_{2+4j}), \quad (x_{2+4i}, x_{2+4j}, x_{3+4i}, x_{3+4j}), \end{aligned} \tag{5.3}$$

where $i, j = 0, 1, \ldots, q-1$. The basic bijection λ is given by $\lambda(x_l) = y_l$.

The polyhedron X that corresponds to $\mathcal{P}$ has square faces and one vertex whose link is naturally isomorphic to a complete bipartite graph. By [3] the universal covering Δ of X is the direct product of two trees.

For example, in the case $q = 1$ the polygonal presentation $\mathcal{P}$ contains four cyclic words (x_1, x_2, x_4, x_3), (x_1, x_1, x_4, x_4), (x_1, x_3, x_4, x_2), (x_2, x_2, x_3, x_3). The corresponding polyhedron X consists of four faces and one vertex, with the link at this vertex given by the complete bipartite graph with four vertices of each color.

In this class of examples, the fundamental group Γ of the polyhedron acts on the 2-dimensional building $\Delta \simeq \mathcal{T}_1 \times \mathcal{T}_2$ so that $X = \Delta/\Gamma$, but Γ is not a subgroup of $\mathrm{Aut}(\mathcal{T}_1) \times \mathrm{Aut}(\mathcal{T}_2)$. To reduce this case to the case of BM groups, we have to find in Γ a subgroup of finite index which is of BM type.

Lemma 5.7. *The group Γ with generators $x_k, k = 1, ..., 4q$ and relations*

$$\begin{aligned} &x_{1+4i}x_{2+4j}x_{4+4i}x_{3+4j} = 1, \quad x_{1+4i}x_{1+4j}x_{4+4i}x_{4+4j} = 1, \\ &x_{1+4i}x_{3+4j}x_{4+4i}x_{2+4j} = 1, \quad x_{2+4i}x_{2+4j}x_{3+4i}x_{3+4j} = 1, \end{aligned} \tag{5.4}$$

where $i, j = 0, 1, \ldots, q-1$, contains a subgroup of index four which is of BM type.

Proof. Consider the polyhedron Y which corresponds to the polygonal presentation (5.3). It contains one vertex and $4q$ faces and the group Γ is the fundamental group of Y. The 4-branching cover of Y is a polyhedron with four vertices of type X, which satisfies the stable pairs condition. Indeed, the polygonal presentation of Y can be obtained (5.3) by replacing each word (x_k, x_l, x_m, x_n) by four: $(x_k^1, x_l^2, x_m^3, x_n^4)$, $(x_k^4, x_l^1, x_m^2, x_n^3)$, $(x_k^3, x_l^4, x_m^1, x_n^2)$, $(x_k^2, x_l^3, x_m^4, x_n^1)$. By direct inspection we can see that this satisfies the stable pair condition. By Lemma 5.6, the fundamental group of X is of BM type, acting on the product of two trees of valence $2(4q-1)$. Thus, the group Γ contains a subgroup Γ_0 of index four of BM type. $\square$

In this case, one can still apply the construction of Proposition 5.1 to the index four subgroup of Lemma 5.7. This means that we now work with the algebra $C(\Delta) \rtimes \Gamma_0$, which is Morita equivalent to $C(\Delta \times S) \rtimes \Gamma$, for $S = \Gamma/\Gamma_0$ the coset space with the left action of Γ, instead of working with $C(\Delta) \rtimes \Gamma$.

5.3. **Hyperbolic 2-dimensional buildings.**

Finally, we give a construction of a spectral triple analogous to the one defined on trees, in the case of a class of hyperbolic 2-dimensional buildings. We will concentrate on the *right angled Fuchsian buildings.*

We recall briefly some properties of these buildings (*cf.* [4]). They are obtained as universal cover of orbihedra, according to the following construction. One begins with a regular r-gon $P \subset \mathbb{H}$ with angles $\pi/2$. With the edges labeled clockwise with $\{i\}$ $(i = 1, \ldots, r)$ and vertices correspondingly labeled $\{i, i+1\}$ and $\{r, 1\}$, for assigned labels $q_i \geq 2$, one obtains a orbihedron by assigning the trivial group to the face of P, the cyclic group $\Gamma_i = \mathbb{Z}/(q_i+1)\mathbb{Z}$ to the $\{i\}$ edge and the group $\Gamma_i \times \Gamma_{i+1}$ to the $\{i, i+1\}$ vertex. This orbihedron has universal cover Δ, with link at an $\{i\}$ labeled vertex given by the complete bipartite graph on $(q_i+1)+(q_{i+1}+1)$ vertices. The complex Δ is a hyperbolic building (in fact a Tits building),

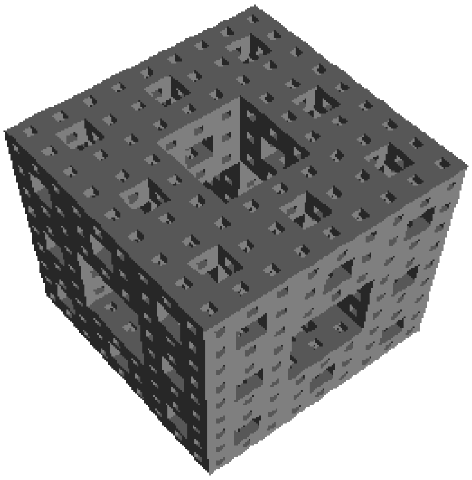

FIGURE 3. The universal Menger curve (iterative construction).

where every apartment is isomorphic to the hyperbolic plane $\mathbb{H}$ with the tessellation given by the action on P of the cocompact Fuchsian Coxeter group generated by inversions on the edges. Recall also that a wall in Δ is a doubly infinite geodesic path in the 1-skeleton of Δ. By the form of the links, all edges in such a path have the same label $\{i\}$. The equivalence relation on edges given by being in the same wall has the tree-walls as equivalence classes. A tree-wall with label $\{i\}$ divides Δ into $(q_i + 1)$ components.

One property of this type of buildings that is useful for the construction of spectral triples we outline below is the fact that the boundary can always be identified with the Menger curve. Namely, the boundary $\partial\Delta$ is defined by the set of geodesic rays from a base point in Δ. It is homeomorphic to the universal Menger curve $\mathcal{M}$. Isometries of Δ extend to homeomorphisms of the boundary.

The *universal Menger curve*, also known as the Menger sponge, is obtained by an iterative construction analogous to that of the ternary Cantor set, starting from a 3-dimensional ball (*cf.* Figure 3). It has (real) topological dimension one, and it has the universal property that any 1-dimensional separable metrizable topological space is homeomorphic to a subset of the Menger curve.

For a discrete finitely generated $\Gamma \subset \mathrm{Isom}(\Delta)$, the limit set $\Lambda_\Gamma \subset \partial\Delta$ is the set of accumulation points of orbits of Γ. The group Γ is nonelementary if Λ_Γ consists of more than two points. We can also consider, as in the case of trees, the geodesic hull $H(\Lambda_\Gamma)$, obtained by considering all infinite geodesics in Δ with endpoints on the limit set Λ_Γ. The group Γ is quasi-convex-cocompact if $H(\Lambda_\Gamma)/\Gamma$ is compact (*cf.* [14]). For instance, we can consider the case where Γ is the fundamental group of the orbihedron and $\partial\Delta = \Lambda_\Gamma$ is the Menger curve.

If $\mathcal{G}_\Delta$ denotes the dual graph of Δ, with a vertex for each chamber and an edge connecting two chambers whenever these share an edge in Δ. If such edge is of type $\{i\}$ the corresponding edge in $\mathcal{G}_\Delta$ is given length $\log q_i$. This defines a metric $d(v, w)$ on $\mathcal{G}_\Delta$. The horospherical distance is then defined as in the case of trees (2.7), (2.8), by setting $(v_1, v_2, v) = d(v_1, v) - d(v_2, v)$ which induces a well defined function

$$(v_1, v_2, x), \qquad \forall x \in \partial\Delta, \tag{5.5}$$

satisfying the cocycle relation $(v_1, v_2, x) = (v_1, v, x) + (v, v_2, x)$.
On $\partial\Delta$ one considers the combinatorial metric $\delta_v(x, y)$ (*cf.* [4]). If δ_H is the Hausdorff dimension and $d\mu_v$ the Hausdorff measure, for $\gamma \in \mathrm{Isom}(\Delta)$ one has

$$(\gamma^* d\mu_v)(x) = e^{\delta_H \tau(v, \gamma^{-1} v, x)}\, d\mu_v(x), \tag{5.6}$$

where τ is the unique positive solutions to the equation (*cf.* [4])

$$\sum_{i=1}^{r} \frac{q_i^x + q_{i+1}^x}{(1 + q_i^x)(1 + q_{i+1}^x)} = 2. \tag{5.7}$$

The idea now is to obtain a suitable measure space, constructed out of the Menger curve and the measure (5.6), on which one has a large supply of locally constant functions, as in the case of the limit sets for actions on trees that we saw before. This can be achieved by introducing a "disconnection" of the Menger curve, which produces a totally disconnected compact Hausdorff space. A similar trick was used in [38] in constructing Cuntz–Krieger type algebras associated to the action of Fuchsian groups on their limit sets, where the limit set is all of $\mathbb{P}^1(\mathbb{R})$ instead of being a totally disconnected set.

Definition 5.8. *Let $\mathcal{M}$ be a 1-dimensional compact Hausdorff space. Let W be a 0-dimensional subset of $\mathcal{M}$. Given any two points $x \neq y \in W$, and a half-open arc $c = [x, y)$ in $\mathcal{M}$, let χ_c denote the characteristic function of $c \subset \mathcal{M}$. Let $\mathcal{B}_W$ be the unital abelian C^*-algebra generated by $C(\mathcal{M})$ and the functions*

$$\{\chi_c :\ c = [x, y) \subset \mathcal{M},\ x \neq y \in W\}.$$

Since $\mathcal{B}_W$ is an abelian C^*-algebra, there is a locally compact Hausdorff space $\widehat{\mathcal{M}}_W$ such that $\mathcal{B}_W = C(\widehat{\mathcal{M}}_W)$. This is called the "disconnection" of $\mathcal{M}$ along W.

The disconnection $\widehat{\mathcal{M}}_W$ is a totally disconnected space if and only if W is dense in $\mathcal{M}$. If the space $\mathcal{M}$ is endowed with a continuous action of a discrete group Γ and the set W is Γ-invariant, then there is an induced Γ-action on $\widehat{\mathcal{M}}_W$ and one can form the crossed product algebra $C(\widehat{\mathcal{M}}_W) \rtimes \Gamma$.
In our case, we want to consider the disconnection of the Menger curve $\mathcal{M} \simeq \partial\Delta$ along the set W of endpoints of walls. We then have the following result.

Lemma 5.9. *Let $\widehat{\partial\Delta}_W$ be the disconnection of the Menger curve $\mathcal{M}$ along the endpoints of walls. The function*

$$\gamma_v'(x) := e^{\delta_H \tau(v, \gamma^{-1} v, x)} \tag{5.8}$$

is a locally constant function of $x \in \widehat{\partial\Delta}_W$.

Proof. This follows from the fact that the function (v_1, v_2, x) of (5.5) is locally constant on $\partial_{reg}\Delta$, the complement in $\partial\Delta$ of the set W of endpoints of walls of Δ (*cf.* [4]). □

Since $\partial_{reg}\Delta$ is of full μ_v-measure in $\partial\Delta$, we can identify the Hilbert spaces

$$\mathcal{L} = L^2(\partial\Delta, d\mu_v) \cong L^2(\widehat{\partial\Delta}_W, d\mu_v). \tag{5.9}$$

We now want to obtain a filtration on the Hilbert space $\mathcal{L}$. One way of doing so is by using the inductive construction of the Menger curve. To this purpose, we consider a covering $\{U_{n,i}\}_{n\geq 1, 1\leq i\leq i_n}$ of the Menger curve $\partial\Delta$, chosen in such a way that $U_{n,i} = \partial\Delta \cap C_{n,i}$, where as a cell complex each $C_{n,i}$ is a cube and the set of vertices of the $C_{n,i}$ form a Γ-invariant subset of $W = \partial\Delta \smallsetminus \partial_{reg}\Delta$. Here Γ is the fundamental group of the orbihedron. Consider the collection of arcs $c(n,\alpha)$ in $U_{n,i}$ coming from the intersection of the 1-skeleton of $C(n,i)$ with $\partial\Delta$. The characteristic functions of these arcs are locally constant functions in the C^*-algebra $C(\widehat{\partial\Delta}_W)$. In the Hilbert space $\mathcal{L} = L^2(\widehat{\partial\Delta}_W, d\mu_v)$ we can then construct a filtration where the subspace $\mathcal{V}_n$ is the span of the characteristic functions $\chi_{c(n,\alpha)}$. This choice of filtration yields a corresponding grading operator $D = \sum_n n\hat{\Pi}_n$, with Π_n the orthogonal projection onto $\mathcal{V}_n$ and $\hat{\Pi}_n = \Pi_n - \Pi_{n-1}$.

The algebra $\mathcal{B}_W = C(\widehat{\partial\Delta}_W)$ acts on $\mathcal{L}$ by multiplication operator, and for Γ as above, we have an action of the elements $\gamma \in \Gamma$ by unitaries

$$(T_{\gamma^{-1}} f)(x) = \gamma'_v(x)^{1/2} f(\gamma x), \tag{5.10}$$

with γ'_v as in (5.8). This gives a representation on $\mathcal{L}$ of the crossed product $\mathcal{A} = \mathcal{B}_W \rtimes \Gamma$.

Theorem 5.10. *Consider the data $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, where $\mathcal{A} = \mathcal{B}_W \rtimes \Gamma$, as above, $\mathcal{H} = \mathcal{L} \oplus \mathcal{L}$ and $\mathcal{D} = FD$, with F the involution exchanging the two copies of $\mathcal{L}$. Let $\mathcal{A}_0 \subset \mathcal{A}$ be the dense subalgebra generated algebraically by the characteristic functions $\chi_{c(n,\alpha)}$ and by the elements $\gamma \in \Gamma$. For U an automorphism of $\mathcal{A}$ preserving $\mathcal{A}_0$, consider the representation* (2.5) *of $\mathcal{A}$ in $\mathcal{B}_W(\mathcal{H})$. The data obtained in this way determine a θ-summable spectral triple.*

Proof. We check the condition on the commutators with $\mathcal{D}$. The dense involutive subalgebra of $\mathcal{A}_0 \subset \mathcal{A}$ is given by the algebraic crossed product $\mathcal{A}_0 = \mathcal{B}_{alg} \rtimes \Gamma$, where $\mathcal{B}_{alg}$ is the subalgebra of $\mathcal{B}$ generated algebraically by the $\chi_{c(n,\alpha)}$. If U preserves the subalgebra $\mathcal{A}_0$, it is sufficient to show that the commutators $[D,a]$ are bounded for all $a \in \mathcal{A}_0$. This is clear for $a \in \mathcal{B}_{alg}$ since such elements are in some $\mathcal{V}_k$ for some $k \geq 0$, hence the corresponding multiplication operators map $\mathcal{V}_n$ to $\mathcal{V}_n$ for all $n \geq k$. In the case of group elements, the commutator $[T_\gamma, D]$ is also bounded, since $\gamma'_v(x)^{1/2}$ is locally constant hence, for some $k = k(\gamma)$, we have $T_\gamma : \mathcal{V}_n \to \mathcal{V}_{n+k(\gamma)}$ for all sufficiently large n. □

5.4. **Finite summability in higher rank.**

There is a setting similar to the one considered in §3 in the higher rank case. In fact, in [35], Robertson and Steger considered affine buildings Δ of type $\tilde{A}_2$, whose boundary Λ is defined by an equivalence relation on sectors (just as in the case of trees it is given by an equivalence relations on geodesics). They showed that, if Γ is a group of type rotating automorphisms of Δ, then the C^*-algebra $C(\Lambda) \rtimes \Gamma$ is isomorphic to a higher rank Cuntz–Krieger algebra O_{A_1,A_2}. This is a particular (rank two) case of more general higher rank generalizations of Cuntz–Krieger algebras, associated to a finite collection of transition matrices A_j, $j = 1, \ldots, r$, with entries in $\{0,1\}$, associated to shifts in r different directions, with the transition matrices satisfying compatibility conditions (see conditions (H0)–(H3) of [35]). The matrices give admissibility conditions for r-dimensional words in an assigned alphabet. In the case of the Cuntz–Krieger algebra $O_A = C(\Lambda_\Gamma) \rtimes \Gamma$ one can choose as generators the partial isometries $S_{u,v} = T_{uv^{-1}} P_v$, for $u, v \in \Gamma$, with $t(u) = t(v)$ (same tail as edges in the Cayley graph). Similarly, in the higher rank case, one has generators that are partial isometries $S_{u,v}$, where u and v are words in the given alphabet, with $t(u) = t(v)$. These satisfy the relations

$$\begin{aligned} S^*_{u,v} &= S_{v,u} & S_{u,v}S_{v,w} &= S_{u,w} \\ S_{u,v} &= \sum S_{uw,vw} & S_{u,u}S_{v,v} &= 0, \ \forall u \neq v \end{aligned} \tag{5.11}$$

The sum here is over r-dimensional words w with source $s(w) = t(u) = t(v)$ and with shape $\sigma(w) = e_j$, for $j = 1, \ldots, r$, where e_j is the j-th standard basis vector in $\mathbb{Z}^r$ (see [34], [35] for more details). Robertson and Steger also proved ([35] §6) that the higher rank Cuntz–Krieger algebras $O_{A_1,\ldots,A_r}$ are stably isomorphic to a crossed product

$$\overline{O_{A_1,\ldots,A_r}} \cong \overline{\mathcal{F}}_{A_1,\ldots,A_r} \rtimes_T \mathbb{Z}^r, \tag{5.12}$$

where $\mathcal{F}_{A_1,\ldots,A_r}$ is the AF algebra generated by the $S_{u,v}$ with $\sigma(u) = \sigma(v)$. Again the stabilization $\overline{\mathcal{F}_A}$ is a non-unital AF algebra. One expects that a similar technique, based on the standard spectral triple of the n-torus T^n and a spectral triple for the non-unital AF algebra $\overline{\mathcal{F}}_{A_1,\ldots,A_r}$ to yield finitely summable triples for the algebra of (5.12).

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G. Cornelissen: University of Utrecht, The Netherlands
E-mail address: `cornelissen@math.uu.nl`

M. Marcolli: Max–Planck Institut für Mathematik, Bonn, Germany
E-mail address: `marcolli@mpim-bonn.mpg.de`

K. Reihani: University of Oslo, Norway
E-mail address: `kamranr@math.uio.no`

A. Vdovina: University of Newcastle, UK
E-mail address: `Alina.Vdovina@newcastle.ac.uk`