TWISTED CONJUGACY SEPARABLE GROUPS

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ABSTRACT. We define the notion of twisted conjugacy separability and study some its properties. We show that polycyclic-by-finite groups are twisted conjugacy separable. It is proved for twisted conjugacy separable groups that the Reidemeister number of an automorphism ϕ is equal to the number of finite-dimensional fixed points of $\widehat{\phi}$ on the unitary dual, if one of these numbers is finite. This theorem is a natural generalization to infinite groups of the classical Burnside-Frobenius theorem. This statement is not true for some groups with extreme properties.

CONTENTS

1. INTRODUCTION

Definition 1.1. Let G be a countable discrete group and $\phi : G \to G$ an endomorphism. Two elements $x, x' \in G$ are said to be *ϕ-conjugate* or *twisted conjugate,* iff there exists $g \in G$ with $x' = gx\phi(g^{-1})$. We shall write $\{x\}_\phi$ for the *ϕ-conjugacy* or *twisted conjugacy* class of the element $x \in G$. The number of ϕ-conjugacy classes is called the *Reidemeister number* of an endomorphism ϕ and is denoted by $R(\phi)$. If ϕ is the identity map then the ϕ-conjugacy classes are the usual conjugacy classes in the group G.

If G is a finite group, then the classical Burnside-Frobenius theorem (see e.g. [31], [23, p. 140]) says that the number of classes of irreducible representations is equal to the number of conjugacy classes of elements of G. Let $\widehat{G}$ be the *unitary dual* of G, i.e. the set of equivalence classes of unitary irreducible representations of G.

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If $\phi : G \to G$ is an automorphism, it induces a map $\widehat{\phi} : \widehat{G} \to \widehat{G}$, $\widehat{\phi}(\rho) = \rho \circ \phi$. Therefore, by the Burnside-Frobenius theorem, if ϕ is the identity automorphism of any finite group G, then we have $R(\phi) = \# \operatorname{Fix}(\widehat{\phi})$.

In [6] it was discovered that this statement remains true for any automorphism ϕ of any finite group G. Indeed, consider an automorphism ϕ of a finite group G. Then $R(\phi)$ is equal to the dimension of the space of twisted invariant functions on this group. Hence, by Peter-Weyl theorem (which asserts the existence of a two-side equivariant isomorphism $C^*(G) \cong \bigoplus_{\rho \in \widehat{G}} \operatorname{End}(H_\rho))$, $R(\phi)$ is identified with the sum of dimensions d_ρ of twisted invariant elements of $\operatorname{End}(H_\rho)$, where ρ runs over $\widehat{G}$, and the space of representation ρ is denoted by H_ρ. By Schur lemma, $d_\rho = 1$, if ρ is a fixed point of $\widehat{\phi}$, and is zero otherwise. Hence, $R(\phi)$ coincides with the number of fixed points of $\widehat{\phi}$.

The attempts to generalize this theorem to the case of non-identical automorphism and of non-finite group were inspired by the dynamical questions and were the subject of a series of papers [6, 7, 5, 10, 11, 8, 9].

In the present paper we study the following property for a countable discrete group G and its automorphism ϕ: we say that the group is *ϕ-conjugacy separable* if its Reidemeister classes can be distinguished by homomorphisms onto finite groups, and we say that it is *twisted conjugacy separable* if it is ϕ-conjugacy separable for any automorphism ϕ with $R(\phi) < \infty$ (Definitions 3.2 and 3.5). Related questions were studied in [2], [25].

After some preliminary considerations we prove **the main results** of the paper, namely

(1) **Classes of twisted conjugacy separable groups:** Polycyclic-by-finite groups are twisted conjugacy separable groups (Theorem 4.2).
(2) **Twisted conjugacy separability respects some extensions:** Suppose, there is an extension $H \to G \to G/H$, where the group H is a characteristic twisted conjugacy separable group; G/H is finitely generated FC-group (i.e., a group with finite conjugacy classes). Then G is a twisted conjugacy separable group (a reformulation of Theorem 5.1).
(3) **Twisted Burnside-Frobenius theorem for twisted conjugacy separable groups:** Let G be a twisted conjugacy separable group and ϕ its automorphism. Denote by $\widehat{G}_f$ the subset of the unitary dual $\widehat{G}$ related to finite-dimensional representations and by $S_f(\phi)$ the number of fixed points of $\widehat{\phi}_f$ on $\widehat{G}_f$. Then $R(\phi) = S_f(\phi)$, if one of them is finite (Theorem 6.2).
(4) **Counterexamples to this formulation of Twisted Burnside-Frobenius theorem:** HNN, Ivanov and Osin groups (Section 7).

A number of examples of groups and automorphisms with finite Reidemeister numbers was obtained and studied in [5, 14, 12, 11, 8].

Using the same argument as in [10] one obtains from the twisted Burnside-Frobenius theorem the following dynamical and number-theoretical consequence which, together with the twisted Burnside-Frobenius theorem itself, is very important for the realization problem of Reidemeister numbers in topological dynamics and the study of the Reidemeister zeta-function.

Let $\mu(d)$, $d \in \mathbb{N}$, be the *Möbius function*, i.e.

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1, \\ (-1)^k & \text{if } d \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{if } d \text{ is not square} - \text{free.} \end{cases}$$

Congruences for Reidemeister numbers: *Let $\phi : G \to G$ be an automorphism of a countable discrete* RP*-group G such that all numbers $R(\phi^n)$ are finite. Then one has for all n,*

$$\sum_{d|n} \mu(d) \cdot R(\phi^{n/d}) \equiv 0 \mod n.$$

These theorems were proved previously in a number of special cases in [6, 7, 10, 11, 8].

We would like to emphasize the following important remarks.

In the original formulation by Fel'shtyn and Hill [6] the conjecture about twisted Burnside-Frobenius theorem asserts an equality of $R(\phi)$ and the number of fixed points of $\widehat{\phi}$ on $\widehat{G}$. This conjecture was proved in [6, 10] for f.g. type I groups.

As it follows from a key example, which we have studied with A. Vershik in [11], an RP-group can have infinite dimensional "supplementary" fixed representations. More precisely we discuss in that paper the case of a semi-direct product of the action of $\mathbb{Z}$ on $\mathbb{Z} \oplus \mathbb{Z}$ by a hyperbolic automorphism with finite Reidemeister number (four to be precise) and the number of fixed points of $\widehat{\phi}$ on $\widehat{G}$ equal or greater than five, while the number of fixed points on $\widehat{G}_f$ is four.

This gives a counterexample to the conjecture in its original formulation and leads to the formulation using only fixed points in $\widehat{G}_f$. The new conjecture is proved in the present paper for a wide class of f.g. groups.

The origin of the phenomenon of an extra fixed point lies in bad separation properties of $\widehat{G}$ for general discrete groups. A more deep study leads to the following general theorem.

Weak twisted Burnside theorem [34]:*The number $R_*(\phi)$ of Reidemeister classes related to twisted invariant functions on G from the Fourier-Stieltjes algebra $B(G)$ is equal to the number $S_*(\phi)$ of generalized fixed points of $\widehat{\phi}$ on the Glimm spectrum of G, i. e. on the complete regularization of $\widehat{G}$, if one of $R_*(\phi)$ and $S_*(\phi)$ is finite.*

The interest in twisted conjugacy relations has its origins, in particular, in the Nielsen-Reidemeister fixed point theory (see, e.g. [21, 5]), in Selberg theory (see, eg. [33, 1]), and Algebraic Geometry (see, e.g. [17]). The congruences give some necessary conditions for the realization problem for Reidemeister numbers in topological dynamics.

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2. Preliminary considerations

The following construction relates ϕ-conjugacy classes and some conjugacy classes of another group. It was obtained in topological context by Boju Jiang and Laixiang Sun in [22]. Consider the action of $\mathbb{Z}$ on G, i.e. a homomorphism $\mathbb{Z} \to \operatorname{Aut}(G)$, $n \mapsto \phi^n$. Let Γ be a corresponding semi-direct product $\Gamma = G \rtimes \mathbb{Z}$:

$$\Gamma := < G, t \mid tgt^{-1} = \phi(g) > \tag{1}$$

in terms of generators and relations, where t is a generator of $\mathbb{Z}$. The group G is a normal subgroup of Γ. As a set, Γ has the form

$$\Gamma = \sqcup_{n\in\mathbb{Z}} G \cdot t^n, \tag{2}$$

where $G \cdot t^n$ is the coset by G containing t^n.

Remark 2.1. Any usual conjugacy class of Γ is contained in some $G \cdot t^n$. Indeed, $gg't^ng^{-1} = gg'\phi^n(g^{-1})t^n$ and $tg't^nt^{-1} = \phi(g')t^n$.

Lemma 2.2. *Two elements x, y of G are ϕ-conjugate iff xt and yt are conjugate in the usual sense in Γ. Therefore $g \mapsto g \cdot t$ is a bijection from the set of ϕ-conjugacy classes of G onto the set of conjugacy classes of Γ contained in $G \cdot t$.*

Proof. If x and y are ϕ-conjugate then there is a $g \in G$ such that $gx = y\phi(g)$. This implies $gx = ytgt^{-1}$ and therefore $g(xt) = (yt)g$ so xt and yt are conjugate in the usual sense in Γ. Conversely, suppose xt and yt are conjugate in Γ. Then there is a $gt^n \in \Gamma$ with $gt^nxt = ytgt^n$. From the relation $txt^{-1} = \phi(x)$ we obtain $g\phi^n(x)t^{n+1} = y\phi(g)t^{n+1}$ and therefore $g\phi^n(x) = y\phi(g)$. Hence, y and $\phi^n(x)$ are ϕ-conjugate. Thus, y and x are ϕ-conjugate, because x and $\phi(x)$ are always ϕ-conjugate: $\phi(x) = x^{-1}x\phi(x)$. ☐

3. Twisted conjugacy separability

We would like to give a generalization of the following well known notion.

Definition 3.1. A group G is *conjugacy separable* if any pair g, h of non-conjugate elements of G are non-conjugate in some finite quotient of G.

It was proved that polycyclic-by-finite groups are conjugacy separated ([28, 13], see also [30, Ch. 4]). Also, residually finite recursively presented Burnside p-groups constructed by R. I. Grigorchuk [16] and by N. Gupta and S. Sidki [18] are shown to be conjugacy separable when p is an odd prime in [35].

We can introduce the following notion, which coincides with the previous definition in the case $\phi = \mathrm{Id}$.

Definition 3.2. A group G is *ϕ-conjugacy separable* with respect to an automorphism $\phi : G \to G$ if any pair g, h of non-ϕ-conjugate elements of G are non-$\overline{\phi}$-conjugate in some finite quotient of G respecting ϕ.

This notion is closely related to the notion $\mathrm{RP}(\phi)$ introduced in [9].

Definition 3.3. We say that a group G has the property RP if for any automorphism ϕ with $R(\phi) < \infty$ the characteristic functions f of Reidemeister classes (hence all ϕ-central functions) are periodic in the following sense.

There exists a finite group K, its automorphism ϕ_K, and epimorphism $F : G \to K$ such that

(1) The diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ K & \xrightarrow{\phi_K} & K \end{array}$$

commutes.

(2) $f = F^* f_K$, where f_K is a characteristic function of a subset of K.

If this property holds for a concrete automorphism ϕ, we will denote this by $\mathrm{RP}(\phi)$.

One gets immediately the following statement.

Theorem 3.4. *Suppose, $R(\phi) < \infty$. Then G is ϕ-conjugacy separable if and only if G is* $\mathrm{RP}(\phi)$.

Proof. Indeed, let $F_{ij} : G \to K_{ij}$ distinguish ith and jth ϕ-conjugacy classes, where K_{ij} are finite groups, $i, j = 1, \ldots, R(\phi)$. Let $F : G \to \oplus_{i,j} K_{ij}$, $F(g) = \sum_{i,j} F_{ij}(g)$, be the diagonal mapping and K its image. Then $F : G \to K$ gives $\mathrm{RP}(\phi)$.

The opposite implication is evident. □

Definition 3.5. A group G is *twisted conjugacy separable* if it is ϕ-conjugacy separable for any ϕ with $R(\phi) < \infty$.

From Theorem 3.4 one immediately obtains

Corollary 3.6. *A group G is twisted conjugacy separable if and only if it is* RP.

Theorem 3.7. *Let $F : \Gamma \to K$ be a morphism onto a finite group K which separates two conjugacy classes of Γ in $G \cdot t$. Then the restriction $F_G := F|_G : G \to \mathrm{Im}(F|_G)$ separates the corresponding (by the bijection from Lemma* 2.2*) ϕ-conjugacy classes in G.*

Proof. First of all let us remark that $\mathrm{Ker}(F_G)$ is ϕ-invariant. Indeed, suppose $F_G(g) = F(g) = e$. Then

$$F_G(\phi(g)) = F(\phi(g)) = F(tgt^{-1}) = F(t)F(t)^{-1} = e$$

(the kernel of F is a normal subgroup).

Let gt and $\widetilde{g}t$ be some representatives of the mentioned conjugacy classes. Then

$$F((ht^n)gt(ht^n)^{-1}) \neq F(\widetilde{g}t), \qquad \forall h \in G,\ n \in \mathbb{Z},$$

$$F(ht^n gt) \neq F(\widetilde{g}tht^n), \qquad \forall h \in G,\ n \in \mathbb{Z},$$

$$F(h\phi^n(g)t^{n+1}) \neq F(\widetilde{g}\phi(h)t^{n+1}), \qquad \forall h \in G,\ n \in \mathbb{Z},$$

$$F(h\phi^n(g)) \neq F(\widetilde{g}\phi(h)), \qquad \forall h \in G,\ n \in \mathbb{Z},$$

in particular, $F(hg\phi(h^{-1})) \neq F(\widetilde{g})\ \forall h \in G$. □

Theorem 3.8. *Let some class of conjugacy separable groups be closed under taking semidirect products by $\mathbb{Z}$. Then this class consists of twisted conjugacy separable groups.*

Proof. This follows immediately from Theorem 3.7 and Theorem 3.4. □

4. First examples: polycyclic-by-finite groups

As an application we obtain another proof of the main theorem for polycyclic-by-finite groups.

Let $G' = [G, G]$ be the *commutator subgroup* or *derived group* of G, i.e. the subgroup generated by commutators. G' is invariant under any homomorphism, in particular it is normal. It is the smallest normal subgroup of G with an abelian factor group. Denoting $G^{(0)} := G$, $G^{(1)} := G'$, $G^{(n)} := (G^{(n-1)})'$, $n \geq 2$, one obtains *derived series* of G:

$$G = G^{(0)} \supset G' \supset G^{(2)} \supset \cdots \supset G^{(n)} \supset \dots \tag{3}$$

If $G^{(n)} = e$ for some n, i.e. the series (3) stabilizes by trivial group, the group G is *solvable*;

Definition 4.1. A solvable group with derived series with cyclic factors is called *polycyclic group.*

Theorem 4.2. *Any polycyclic-by-finite group is a twisted conjugacy separable group.*

Proof. The class of polycyclic-by-finite groups is closed under taking semidirect products by $\mathbb{Z}$. Indeed, let G be an polycyclic-by-finite group. Then there exists a characteristic (polycyclic) subgroup P of finite index in G. Hence, $P \rtimes \mathbb{Z}$ is a polycyclic normal group of $G \rtimes \mathbb{Z}$ of the same finite index.

Polycyclic-by-finite groups are conjugacy separable ([28, 13], see also [30, Ch. 4]). It remains to apply Theorem 3.8. □

5. Twisted conjugacy separability and extensions

It is known that conjugacy separability does not respect extensions. In particular, in [15] an example of a group G which is not conjugacy separable, but contains a subgroup H of index 2 which is conjugacy separable, is given.

For twisted conjugacy separable groups the situation is much better under some finiteness conditions. More precisely one has the following statement.

Theorem 5.1. *Suppose, there exists a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "i"] \arrow[d, "\phi'"'] & G \arrow[r, "p"] \arrow[d, "\phi"] & G/H \arrow[r] \arrow[d, "\overline{\phi}"] & 0 \\ 0 \arrow[r] & H \arrow[r, "i"] & G \arrow[r, "p"] & G/H \arrow[r] & 0, \end{tikzcd} \tag{4}$$

where H is a normal subgroup of a finitely generated group G. Suppose, $R(\phi) < \infty$, G/H is a FC *group, i.e., all conjugacy classes are finite, and H is a ϕ'-conjugacy separable group. Then G is a ϕ-conjugacy separable group.*

Proof. By Corollary 3.6 this follows from the corresponding statement for RP in [9]. □

6. Twisted conjugacy separability and twisted Burnside-Frobenius theorem

Definition 6.1. Denote by $\widehat{G}_f$ the subset of the unitary dual $\widehat{G}$ related to finite-dimensional representations. For an automorphism $\phi : G \to G$ denote by $S_f(\phi)$ the number of fixed points of $\widehat{\phi}$ on $\widehat{G}_f$.

Theorem 6.2 (Twisted Burnside-Frobenius theorem for ϕ-conjugacy separable groups)**.** *Let G be a ϕ-conjugacy separable group. Then $R(\phi) = S_f(\phi)$ if one of these numbers is finite.*

Proof. The coefficients of finite-dimensional non-equivalent irreducible representations of G are linear independent by Frobenius-Schur theorem (see [3, (27.13)]). Moreover, the coefficients of non-equivalent unitary finite-dimensional irreducible representations are orthogonal to each other as functions on the universal compact group associated with the initial group [4, 16.1.3] by the Peter-Weyl theorem. Hence, their linear combinations are orthogonal to each other as well.

It is sufficient to verify the following three statements:

1) If $R(\phi) < \infty$, than each ϕ-class function is a finite linear combination of twisted-invariant functionals being coefficients of points of $\operatorname{Fix} \widehat{\phi}_f$.

2) If $\rho \in \operatorname{Fix} \widehat{\phi}_f$, there exists one and only one (up to scaling) twisted invariant functional on $\rho(C^*(G))$ (this is a finite full matrix algebra).

3) For different ρ the corresponding ϕ-class functions are linearly independent. This follows from the remark at the beginning of the proof.

Let us remark that the property RP implies in particular that ϕ-central functions (for ϕ with $R(\phi) < \infty$) are functionals on $C^*(G)$, not only $L^1(G)$, i.e. are in the Fourier-Stieltijes algebra $B(G)$.

The statement 1) follows from the RP property. Indeed, this ϕ-class function f is a linear combination of functionals coming from some finite collection $\{\rho_i\}$ of elements of $\widehat{G}_f$ (these representations $\rho_1, \dots, \rho_s$ are in fact representations of the form $\pi_i \circ F$, where π_i are irreducible representations of the finite group K and $F : G \to K$, as in the definition of RP). So,

$$f = \sum_{i=1}^{s} f_i \circ \rho_i, \quad \rho_i : G \to \operatorname{End}(V_i), \quad f_i : \operatorname{End}(V_i) \to \mathbb{C}, \quad \rho_i \neq \rho_j,\ (i \neq j).$$

For any $g, \widetilde{g} \in G$ one has

$$\sum_{i=1}^{s} f_i(\rho_i(\widetilde{g})) = f(\widetilde{g}) = f(g\widetilde{g}\phi(g^{-1})) = \sum_{i=1}^{s} f_i(\rho_i(g\widetilde{g}\phi(g^{-1}))).$$

By the observation at the beginning of the proof concerning linear independence,

$$f_i(\rho_i(\widetilde{g})) = f_i(\rho_i(g\widetilde{g}\phi(g^{-1}))). \qquad i = 1, \dots, s,$$

i.e. f_i are twisted-invariant. For any $\rho \in \widehat{G}_f$, $\rho : G \to \operatorname{End}(V)$, any functional $\omega : \operatorname{End}(V) \to \mathbb{C}$ has the form $a \mapsto \operatorname{Tr}(ba)$ for some fixed $b \in \operatorname{End}(V)$. Twisted invariance implies twisted invariance of b (evident details can be found in [10, Sect. 3]). Hence, b is intertwining between ρ and $\rho \circ \phi$ and $\rho \in \operatorname{Fix}(\widehat{\phi}_f)$. The uniqueness of intertwining operator (up to scaling) implies 2).

It remains to prove that $R(\phi) < \infty$ if $S_f(\phi) < \infty$. By the definition of a ϕ-conjugacy separable group the Reidemeister classes of ϕ can be separated by maps to finite groups. Hence, taking representations of these finite groups and applying the twisted Burnside-Frobenius theorem to these groups we obtain that for any pair of Reidemeister classes there exists a function being a coefficient of a finite-dimensional unitary representation, which distinguish these classes. Hence, if $R(\phi) = \infty$, then there are infinitely many linear independent twisted invariant functions being coefficients of finite dimensional representations. But there are as many such functionals, as $S_f(\phi)$. $\square$

7. Examples and counterexamples

Some of examples of groups, for which the twisted Burnside-Frobenius theorem in the above formulation is true, out of the class of polycyclic-by-finite groups were obtained by F. Indukaev [20]. Namely, it is proved that wreath products $A \wr \mathbb{Z}$ are RP groups, where A is a finitely generated abelian group.

Now let us present some counterexamples to the twisted Burnside-Frobenius theorem in the above formulation for some discrete groups with extreme properties. Suppose, an infinite discrete group G has a finite number of conjugacy classes. Such examples can be found in [32] (HNN-group), [26, p. 471] (Ivanov group), and [27] (Osin group). Then evidently, the characteristic function of the unity element is not almost-periodic and the argument above is not valid. Moreover, let us show, that these groups give rise counterexamples to the above theorem.

Example 7.1. For the Osin group the Reidemeister number $R(\mathrm{Id}) = 2$, while there is only trivial (1-dimensional) finite-dimensional representation. Indeed, Osin group is an infinite finitely generated group G with exactly two conjugacy classes. All nontrivial elements of this group G are conjugate. So, the group G is simple, i.e. G has no nontrivial normal subgroup. This implies that group G is not residually finite (by definition of residually finite group). Hence, it is not linear (by Mal'cev theorem [24], [29, 15.1.6]) and has no finite-dimensional irreducible unitary representations with trivial kernel. Hence, by simplicity of G, it has no finite-dimensional irreducible unitary representation with nontrivial kernel, except of the trivial one.

Let us remark that Osin group is non-amenable, contains the free group in two generators F_2, and has exponential growth.

Example 7.2. For large enough prime numbers p, the first examples of finitely generated infinite periodic groups with exactly p conjugacy classes were constructed by Ivanov as limits of hyperbolic groups (although hyperbolicity was not used explicitly) (see [26, Theorem 41.2]). Ivanov group G is infinite periodic 2-generator group, in contrast to the Osin group, which is torsion free. The Ivanov group G is also a simple group. The proof (kindly explained to us by M. Sapir) is the following. Denote by a and b the generators of G described in [26, Theorem 41.2]. In the proof of Theorem 41.2 on [26] it was shown that each of elements of G is conjugate in G to a power of generator a of order s. Let us consider any normal subgroup N of G. Suppose $\gamma \in N$. Then $\gamma = ga^sg^{-1}$ for some $g \in G$ and some s. Hence, $a^s = g^{-1}\gamma g \in N$ and from periodicity of a, it follows that also $a \in N$ as well as $a^k \in N$ for any k, because p is prime. Then any element h of G also belongs to N being of the form $h = \widetilde{h}a^k(\widetilde{h})^{-1}$, for some k, i.e., $N = G$. Thus, the group G is simple. The discussion can be completed in the same way as in the case of Osin group.

Example 7.3. In paper [19], Theorem III and its corollary, G. Higman, B. H. Neumann, and H. Neumann proved that any locally infinite countable group G can be embedded into a countable group G^* in which all elements except the unit element are conjugate to each other (see also [32]). The discussion above related Osin group remains valid for G^* groups.

Let us remark that polycyclic-by-finite group are residually finite (see e.g. [29, 5.4.17]) while the groups from these counterexamples are not residually finite, as it is clear by definition. That is why we would like to complete this section with the following question.

Question. Suppose G is a residually finite group and ϕ is its endomorphism with finite $R(\phi)$. Does $R(\phi)$ equal $S_f(\phi)$?

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