The dihedral Lie algebras and Galois symmetries of $\pi_1^{(l)}(\mathbb{P}^1\backslash\{0, \mu_N, \infty\})$

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1 Introduction

1. Summary. In this paper we study the action of the absolute Galois group on the pro-l completion $\pi_1^{(l)}(\mathbb{P}^1\backslash\{0, \mu_N, \infty\})$ of the fundamental group of $\mathbb{P}^1$ minus 0, ∞ and N-th roots of unity using the following tools.

Let G be a commutative group. In [G1] we constructed a bigraded Lie algebra $\mathbb{D}_{\bullet,\bullet}(G)$, called the dihedral Lie algebra of G, see section 2 below. Let μ_N be the group of N-th roots of unity. We relate $\mathbb{D}_{\bullet,\bullet}(\mu_N)$ to the Lie algebra of the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in $\mathrm{Aut}\pi_1^{(l)}(\mathbb{P}^1\backslash\{0, \mu_N, \infty\})$. On the other hand, in [G1] we found a mysterious connection between the structure of the Lie algebra $\mathbb{D}_{\bullet,\bullet}(\mu_N)$ and geometry of the quotient of the symmetric space $GL_m(\mathbb{R})/R^* \cdot SO_m$ by the subgroup $\Gamma_1(m;N) \subset GL_m(\mathbb{Z})$ stabilizing the vector $(1, 0, ..., 0)$ mod N. Using these results we investigate the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\pi_1^{(l)}(\mathbb{P}^1\backslash\{0, \mu_N, \infty\})$. In particular we obtain new results about the action on $\pi_1^{(l)}(\mathbb{P}^1\backslash\{0, 1, \infty\})$.

2. The action of the Galois group. Let X be a regular curve over a number field, $\overline{X}$ the corresponding projective curve, and v a tangent vector at a point $x \in \overline{X}$. Then according to Deligne ([D]) one can define the profinite fundamental group $\widehat{\pi}_1(X, v)$ based at the vector v. If X, x and v are defined over a field L then the group $\mathrm{Gal}(\overline{\mathbb{Q}}/L)$ acts by automorphisms of $\pi_1^{(l)}(X, v)$.

If $X = \mathbb{P}^1\backslash\{0, \mu_N, \infty\}$ there is a tangent vector $v_{\infty 1}$ corresponding to the inverse t^{-1} of the canonical coordinate t on $\mathbb{P}^1\backslash\{0, \mu_N, \infty\}$. So we get a map

$$\Phi_N^{(l)} : \quad Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow Aut\pi_1^{(l)}(\mathbb{P}^1\backslash\{0, \mu_N, \infty\}, v_{\infty 1}) \tag{1}$$

There are canonical isomorphisms $\mathbb{Z}_l(1) = \pi_1^{(l)}(T_0\mathbb{P}^1, v_{\infty 1}) = \pi_1^{(l)}(\mathbb{P}^1\backslash\{0, \infty\}, v_{\infty 1})$. We have a projection

$$\pi_1^{(l)}(\mathbb{P}^1\backslash\{0, \mu_N, \infty\}, v_{\infty 1}) \longrightarrow \pi_1^{(l)}(\mathbb{P}^1\backslash\{0, \infty\}, v_{\infty 1}) \tag{2}$$

The tangent vector $v_{\infty 1}$ provides a splitting of this projection:

$$I_\infty : \pi_1^{(l)}(\mathbb{P}^1\backslash\{0, \infty\}, v_{\infty 1}) \hookrightarrow \pi_1^{(l)}(\mathbb{P}^1\backslash\{0, \mu_N, \infty\}, v_{\infty 1}) \tag{3}$$

It commutes with the action of the Galois group. Let ζ_n be a primitive n's root of unity, so $\mathbb{Q}(\zeta_n)$ is the field generated by the n's roots of unity. Then it follows that the subgroup $Gal(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty}))$ preserves the elements of the subgroup

$$I_\infty(\mathbb{Z}_l(1)) \subset \pi_1^{(l)}(\mathbb{P}^1\backslash\{0, \mu_N, \infty\}) \tag{4}$$

The restriction of the map $\Phi_N^{(l)}$ to the subgroup $Gal(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N, \zeta_{l^\infty}))$ satisfies some additional constraints. Indeed, the subgroup $Gal(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))$ preserves each of the points $0, \zeta_N^a, \infty$ on $\mathbb{P}^1$ and commutes with the natural action of the group μ_N on $\mathbb{P}^1\backslash\{0, \mu_N, \infty\}$ given by $z \longmapsto \zeta_N z$.

So we conclude that $Gal(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N, \zeta_{l^\infty}))$ preserves the elements of $I_\infty(\mathbb{Z}_l(1))$, the conjugacy classes of the loops around 0 and N-th roots of unity, and commutes with the action of the group μ_N.

3. Passing to the Lie algebras. Let $\mathbb{L}(\mathbb{P}^1\backslash\{0, \mu_N, \infty\}, v_{\infty 1})$ be the pronilpotent Lie algebra over $\mathbb{Q}$ corresponding by the Maltsev theory to the pronilpotent completion of the fundamental group $\pi_1(\mathbb{P}^1\backslash\{0, \mu_N, \infty\}, v_{\infty 1})$ (see [D] ch. 9). We use a shorthand $\mathbb{L}_N$ for it. In particular we have the following.

i) Let $\mathbb{L}_N^{(l)}$ be the Lie algebra over $\mathbb{Q}_l$ corresponding to the pro-l group $\pi_1^{(l)}(\mathbb{P}^1\backslash\{0, \mu_N, \infty\}, v_{\infty 1})$. Then $\mathbb{L}_N^{(l)} = \mathbb{L}_N \otimes_{\mathbb{Q}} \mathbb{Q}_l$. Let $\mathrm{Exp}(\mathbb{L}_N)$ be the connected, simply-connected prounipotent algebraic group over $\mathbb{Q}$ with the Lie algebra $\mathbb{L}_N$; then $\pi_1^{(l)}(\mathbb{P}^1\backslash\{0, \mu_N, \infty\}, v_{\infty 1})$ is a compact open subgroup of $\mathrm{Exp}(\mathbb{L}_N)(\mathbb{Q}_l)$.

ii) $\mathbb{L}_N$ is isomorphic to the pronilpotent completion of a free Lie algebra with $N+1$ generators corresponding to the loops around 0 and all N-th roots of unity.

Similar to (2) and (3) there is a canonical projection

$$p: \mathbb{L}(\mathbb{P}^1\backslash\{0, \mu_N, \infty\}, v_{\infty 1}) \longrightarrow \mathbb{L}(\mathbb{P}^1\backslash\{0, \infty\}, v_{\infty 1}) \tag{5}$$

and the tangent vector $v_{\infty 1}$ provides its splitting:

$$i_\infty : \mathbb{L}(\mathbb{P}^1\backslash\{0, \infty\}, v_{\infty 1}) = \mathbb{L}(T_0\mathbb{P}^1, v_{\infty 1}) = \mathbb{Q}(1) \quad \hookrightarrow \quad \mathbb{L}(\mathbb{P}^1\backslash\{0, \mu_N, \infty\}, v_{\infty 1})$$

Denote by X_∞ the corresponding generator of $\mathbb{L}_N$. Choose generators X_0 and X_ζ corresponding to "loops" around 0 and $\zeta \in \mu_N$ in such a way that $X_0 + \sum_{\zeta \in \mu_N} X_\zeta + X_\infty = 0$ and the action of μ_N on $\mathbb{P}^1$ permutes the generators X_ζ.

There is a homomorphism $\varphi_N^{(l)} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow Aut\mathbb{L}_N^{(l)}$. Denote by $\mathcal{G}_N^{(l)}$ the Lie algebra of the image of the subgroup $Gal(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N, \zeta_{l^\infty}))$ under this map. It acts by derivations of the Lie algebra $\mathbb{L}_N^{(l)}$. We will describe the constraints on the derivations we get using a more general set up presented below.

4. Special equivariant derivations. Let G be a commutative group written multiplicatively. Let $\mathbb{L}(G)$ be the pronilpotent completion of the free Lie algebra with the generators Y and X_g where g runs through the group G.

Example. The Lie algebra $\mathbb{L}(\mu_N)$ is isomorphic to $\mathbb{L}_N$.

A derivation D of the Lie algebra $\mathbb{L}(G)$ is called special if there are elements $S, S_g \in \mathbb{L}(G)$ such that

$$D(X_g) = [S_g, X_g] \quad \text{for any } g \in G, \quad \mathcal{D}(Y) = [S, Y] \quad \text{and} \quad D(Y + \sum_{g\in G} X_g) = 0$$

The group G acts on the generators by $h : Y \longmapsto Y, X_g \longmapsto X_{hg}$. So it acts by automorphisms of the Lie algebra $\mathbb{L}(G)$. A derivation D of $\mathbb{L}(G)$ is called equivariant if it commutes with the action of G. Let $\mathrm{Der}^{SE}\mathbb{L}(G)$ be the Lie algebra of all special equivariant derivations of the Lie algebra $\mathbb{L}(G)$.

There are two increasing filtrations by ideals on the Lie algebra $\mathbb{L}(G)$, going from $-\infty$.

The weight filtration $\mathcal{F}^W_\bullet$. It coincides with the lower central series for $\mathbb{L}(G)$:

$$\mathbb{L}(G) = \mathcal{F}^W_{-1}\mathbb{L}(G); \quad \mathcal{F}^W_{-n-1}\mathbb{L}(G) := [\mathcal{F}^W_{-n}\mathbb{L}(G), \mathbb{L}(G)]$$

The depth filtration $\mathcal{F}^D_\bullet$. Let $\mathcal{I}(G)$ be the the kernel of the projection $p_G : \mathbb{L}(G) \longrightarrow \mathbb{Q}, \quad X_g \longmapsto 0, Y \longmapsto 1$. Its powers give the depth filtration:

$$\mathcal{F}^D_0\mathbb{L}(G) = \mathbb{L}(G), \quad \mathcal{F}^D_{-1}\mathbb{L} = \mathcal{I}(G), \quad \mathcal{F}^D_{-n-1}\mathbb{L}(G) = [\mathcal{I}(G), \mathcal{F}^D_{-n}\mathbb{L}(G)]$$

The weight (resp. depth) of a Lie word in Y, X_g is $(-1)\times$ the total number of the generators (resp. the number of X_g's).

Example. The depth filtration on $\mathbb{L}(\mu_N)$ is provided by the inclusion $\mathbb{P}^1\backslash\{0, \mu_N, \infty\} \longrightarrow \mathbb{P}^1\backslash\{0, \infty\}$ which induces the projection p_{μ_N}.

These filtrations induce two filtrations on the Lie algebra $\mathrm{Der}^{SE}\mathbb{L}(G)$. Taking the associated graded quotient with respect to these filtrations we get a Lie algebra $\mathrm{GrDer}^{SE}_{\bullet,\bullet}\mathbb{L}(G)$ bigraded by the weight $-w$ and depth $-m$.

Now let us put $G = \mu_N$ and return to the problem of description of the map $\varphi^{(l)}_N$. It follows from s. 1.2 that $\mathcal{G}^{(l)}_N$ acts by special equivariant derivations of the Lie algebra $\mathbb{L}^{(l)}_N$, i.e. $\mathcal{G}^{(l)}_N \subset \mathrm{Der}^{SE}\mathbb{L}^{(l)}_N = \mathrm{Der}^{SE}\mathbb{L}_N \otimes \mathbb{Q}_l$. The map $\varphi^{(l)}_N$ obviously respects the two filtrations. The filtrations on $\mathrm{Der}^{SE}\mathbb{L}^{(l)}_N$ induce two filtrations on $\mathcal{G}^{(l)}_N$. Let

$$\mathrm{Gr}\mathcal{G}^{(l)}_{\bullet,\bullet}(\mu_N) \subset \mathrm{GrDer}^{SE}_{\bullet,\bullet}\mathbb{L}(G) \tag{6}$$

be the associated graded quotient of the Lie algebra $\mathcal{G}^{(l)}_N$.

Remark. The weight filtration can be defined on the Lie algebra attached to the pronilpotent completion of the fundamental group of an arbitrary algebraic variety. The weights on $\mathbb{L}(\mu_N)$ are obtained by dividing by 2 the usual weights.

5. Main tools and conjectures. The goal of the present paper is to study Lie algebra (6). Our main tool is the following result proved in section 3

Theorem 1.1 *Let G be a finite commutative group. Then there exists an injective morphism of bigraded Lie algebra*

$$\xi_G : \mathbb{D}_{\bullet,\bullet}(G) \hookrightarrow \mathrm{GrDer}^{SE}_{\bullet,\bullet}\mathbb{L}(G)$$

Conjecture 1.2 *$Gr\mathcal{G}^{(l)}_{\bullet,\bullet}(\mu_N) \subset \xi_{\mu_N}(\mathbb{D}_{\bullet,\bullet}(\mu_N)) \otimes_{\mathbb{Q}} \mathbb{Q}_l$.*

This conjecture is essentially proved in this paper, see s. 4.5.

If G is a trivial group we set $\mathbb{D}_{\bullet,\bullet} := \mathbb{D}_{\bullet,\bullet}(\{e\})$, $\mathcal{G}^{(l)}_{\bullet,\bullet} := \mathcal{G}^{(l)}_{\bullet,\bullet}(1)$ and $\xi := \xi_{\{e\}}$.

Conjecture 1.3 *One has $\xi(\mathbb{D}_{\bullet,\bullet}) \otimes \mathbb{Q}_l = \mathrm{Gr}\mathcal{G}^{(l)}_{\bullet,\bullet}$.*

Since the Lie algebra $\mathbb{D}_{\bullet,\bullet}$ is defined *very* explicitely, this would give a precise description of the image of the Galois group, and so seems to be very hard. The results of Soulé [So], Ihara [Ih2] and Deligne [D] imply it in the depth -1. In this paper we proved that conjecture 1.3 holds in the depth -2 and -3 if we assume conjecture 1.2.

6. Some results about the action of the absolute Galois group on $\pi_1^{(l)}(\mathbb{P}^1\backslash\{0,1,\infty\}, v_{\infty 1})$. For every $m \geq 1$ we have a nilpotent graded Lie algebra

$$\mathrm{Gr}\mathcal{G}^{(l)}_{\bullet,\geq -m} := \mathrm{Gr}\left(\mathcal{G}^{(l)}_{\bullet,\bullet}/\mathcal{G}^{(l)}_{\bullet,<-m}\right)$$

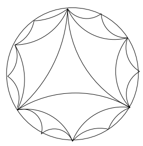
It is of the nilpotence class m. In particular $\mathrm{Gr}\mathcal{G}^{(l)}_{\bullet,\geq -1}$ is abelian. It follows from the mentioned above results of Soulé, Ihara and Deligne that

$$\mathrm{Gr}\mathcal{G}^{(l)}_{-w,-1} = \begin{cases} 0 & w \quad \text{even}, \quad \text{or } w = 1 \\ \mathbb{Q}_l & w > 1 \quad \text{odd} \end{cases} \tag{7}$$

Let us try to go further. Let $\mathcal{L}^{(l)}_{-\bullet,-k} \subset \mathrm{Gr}\mathcal{G}^{(l)}_{-\bullet,-k}$ be the subspace generated by commutators of k elements of $\mathrm{Gr}\mathcal{G}^{(l)}_{-\bullet,-1}$. Then $\mathcal{L}^{(l)}_{-\bullet,\geq -m} := \oplus_{k=1}^{m}\mathcal{L}^{(l)}_{-\bullet,-k}$ is a Lie subalgebra of $\mathrm{Gr}\mathcal{G}^{(l)}_{-\bullet,\geq -m}$. The structure of the Lie algebra $\mathcal{L}^{(l)}_{\bullet,\geq -2}$ is described by the commutator map $[,] : \Lambda^2\mathcal{L}^{(l)}_{-\bullet,-1} \longrightarrow \mathcal{L}^{(l)}_{-\bullet,-2}$. Dualizing it we get the standard cochain complex of this Lie algebra:

$$\mathcal{L}^{(l)}_{-\bullet,-2}{}^{\vee} \xrightarrow{\delta} \Lambda^2\mathcal{L}^{(l)}_{-\bullet,-1}{}^{\vee} \tag{8}$$

On the other hand, let $\mathbb{M}^*_{(2)} := \mathbb{M}^1_{(2)} \longrightarrow \mathbb{M}^2_{(2)} \longrightarrow \mathbb{M}^3_{(2)}$ be the chain complex of the classical modular triangulation of the hyperbolic plane extended by $\mathbb{P}^1(\mathbb{Q})$, placed in degrees $[1,3]$.



For example $\mathbb{M}^1_{(2)}$ is the group generated by the triangles. The group $GL_2(\mathbb{Z})$ acts naturally on the hyperbolic plane preserving the modular picture. So $\mathbb{M}^*_{(2)}$ is a complex of $GL_2(\mathbb{Z})$-modules. Let V be a GL_2-module. Then the complex $\mathbb{M}^*_{(2)} \otimes_{GL_2(\mathbb{Z})} V[1]$ computes the cuspidal cohomology $H^*_{\mathrm{cusp}}(GL_2(\mathbb{Z}), V)$. Let

$$\tau_{[1,2]}(\mathbb{M}^*_{(2)} \otimes_{GL_2(\mathbb{Z})} V) \quad :=$$

$$\mathbb{M}^1_{(2)} \otimes_{GL_2(\mathbb{Z})} V \longrightarrow \mathrm{Ker}\Big(\mathbb{M}^2_{(2)} \otimes_{GL_2(\mathbb{Z})} V \longrightarrow \mathbb{M}^3_{(2)} \otimes_{GL_2(\mathbb{Z})} V\Big) \qquad (9)$$

Theorem 1.4 *a) The weight w part of the complex (8) is canonically isomorphic to the complex $\tau_{[1,2]}\Big(\mathbb{M}^*_{(2)} \otimes_{GL_2(\mathbb{Z})} S^{w-2}V_2\Big) \otimes \mathbb{Q}_l$.*

$$b) \qquad \mathrm{dim}\mathrm{Gr}\mathcal{G}^{(l)}_{-w,-2} \geq \mathrm{dim}\mathcal{L}^{(l)}_{-w,-2} = \begin{cases} 0 & w: \quad odd \\ \left[\frac{w-2}{6}\right] & w: \quad even \end{cases} \qquad (10)$$

c) Let us assume conjecture 1.2 for $N = 1$. Then $\mathcal{L}^{(l)}_{-w,-2} = \mathrm{Gr}\mathcal{G}^{(l)}_{-w,-2}$.

The part b) has been also obtained by Ihara and Takao (unpublished, cf [M]).

In particular there is a canonical isomorphism

$$m_1^{(l)} : \mathcal{L}^{(l)}_{-w,-2}{}^{\vee} \stackrel{=}{\longrightarrow} \mathbb{M}^1_{(2)} \otimes_{GL_2(\mathbb{Z})} S^{w-2}V_2 \otimes \mathbb{Q}_l$$

Denote by Λ^2_w the weight w part of Λ^2. We define an isomorphism

$$m_2^{(l)} : \Lambda^2_w\left(\mathcal{L}^{(l)}_{\bullet,-1}\right)^{\vee} \stackrel{=}{\longrightarrow} \mathbb{M}^2_{(2)} \otimes_{GL_2(\mathbb{Z})} S^{w-2}V_2 \otimes \mathbb{Q}_l \qquad (11)$$

as follows. The stabilizer in $GL_2(\mathbb{Z})$ of the path from 0 to $i\infty$ on the upper half plane is generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore the right hand side of (11) is identified with the space of degree $w-2$ polynomials in t_1, t_2, skewsymmetric in the variables t_1, t_2 and of even degree in each of them. Let $\zeta^{(l)}_{\mathcal{M}}(n)$ be Soule's generator of $\left(\mathrm{Gr}\mathcal{G}^{(l)}_{-2n+1,-1}\right)^{\vee}$, see section 5. Then

$$m_2^{(l)} : \zeta^{(l)}_{\mathcal{M}}(2m+1) \wedge \zeta^{(l)}_{\mathcal{M}}(2n+1) \longmapsto t_1^{2m}t_2^{2n} - t_1^{2n}t_2^{2m}$$

We obtain similar results in the depth three case. The depth three part of the standard cochain complex of the Lie agebra $\mathcal{L}^{(l)}_{-\bullet,\geq -3}$ looks as follows.

$$\mathcal{L}^{(l)}_{-\bullet,-3}{}^{\vee} \longrightarrow \mathcal{L}^{(l)}_{-\bullet,-2} \otimes \mathcal{L}^{(l)}_{-\bullet,-1}{}^{\vee} \longrightarrow \Lambda^3\mathcal{L}^{(l)}_{-\bullet,-1}{}^{\vee} \qquad (12)$$

The first map is dual to the commutator map $[,] : \mathcal{L}^{(l)}_{-\bullet,-2} \otimes \mathcal{L}^{(l)}_{-\bullet,-1} \longrightarrow \mathcal{L}^{(l)}_{-\bullet,-3}$.

Theorem 1.5 *a) The complex (12) computes the cuspidal cohomology $H^i_{cusp}(GL_3(\mathbb{Z}), S^{w-3}V_3)$ at $i = 1, 2, 3$. Therefore it is acyclic.*

$$b) \qquad \mathrm{dim}\mathrm{Gr}\mathcal{G}^{(l)}_{-w,-3} \geq \mathrm{dim}\mathcal{L}^{(l)}_{-w,-3} = \begin{cases} 0 & w: \quad even \\ \left[\frac{(w-3)^2-1}{48}\right] & w: \quad odd \end{cases} \qquad (13)$$

c) Let us assume conjecture 1.2 for $N = 1$. Then $\mathrm{Gr}\mathcal{G}^{(l)}_{-w,-3} = \mathcal{L}^{(l)}_{-w,-3}$.

To prove this theorem we combine the results of section 4 with the main result of [G1] to show that *complex (12) is* naturally *isomorphic to a certain subcomplex of the Voronoi complex computing cuspidal cohomology at* $i = 1, 2, 3$ *for the local system on* $GL_3(\mathbb{Z})\backslash GL_3(\mathbb{R})/O(3)$ *associated with* $S^{w-3}V_3$.

Remarks. a) Compare theorems 1.4, 1.5 with theorems 1.4, 1.5 in [G1] where we have proved that the $\mathbb{Q}$-space of reduced multiple ζ-values of depth 2 and 3 is estimated from *above* by (10) and (13).

b) We hope to show that $\mathrm{Gr}\mathcal{G}^{(l)}_{-w,-4}/\mathcal{L}^{(l)}_{-w,-4} = H^1_{cusp}(GL_2(\mathbb{Z}), S^{w-2}V_2) \otimes \mathbb{Q}_l$ in the subsiquent paper.

One has $\mathrm{GrDer}^{SE}_{-w,-m}\mathbb{L}(G) \neq 0$ only if $w \geq m \geq 1$, and similar constraints hold for $\mathrm{Gr}\mathcal{G}^{(l)}(\mu_N)$ and $\mathbb{D}_{\bullet,\bullet}(G)$. We define the *diagonal* Lie algebras $\mathbb{D}^{diag}_{\bullet}(G)$ and $\mathrm{Gr}\mathcal{G}^{(l)}_{\bullet}(\mu_N)$ as the quotients of $\mathbb{D}_{\bullet,\bullet}(G)$ and $\mathrm{Gr}\mathcal{G}^{(l)}_{\bullet,\bullet}(\mu_N)$ by the components $\mathbb{D}_{-w,-m}(G)$ and $\mathrm{Gr}\mathcal{G}^{(l)}_{-w,-m}(\mu_N)$ with $w \neq m$. So they are graded by the weight.

Conjecture 1.6 *Let p be a prime number. Then* $\zeta_{\mu_p}(\mathbb{D}^{diag}_{\bullet}(\mu_p)) = \mathrm{Gr}\mathcal{G}^{(l)}_{\bullet}(\mu_p)$.

The results of this paper combined with [G3] prove this for the weights ≥ -2.

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2 The dihedral Lie coalgebra of a commutative group G

1. Definitions. First let us recall a definition of a bigraded Lie coalgebra $\mathcal{D}_{\bullet,\bullet}(G) = \oplus_{w \geq m \geq 1}\mathcal{D}_{w,m}(G)$ given in [G1]. The $\mathbb{Q}$-vector space $\mathcal{D}_{w,m}(G)$ is generated by the symbols

$$\{g_0 : ... : g_m\}_{n_1,...,n_m}, \qquad w = n_1 + ... + n_m, \quad n_i \geq 1 \tag{14}$$

To define the relations we introduce the generating functions

$$\{g_0 : ... : g_m | t_0 : .... : t_m\} := \sum_{n_i > 0} \{g_0 : ... : g_m\}_{n_1,...,n_m}(t_1 - t_0)^{n_1 - 1}...(t_m - t_0)^{n_m - 1} \tag{15}$$

Notice that $\{g_0 : ... : g_m | t_0 : ... : t_m\} = \{g_0 : ... : g_m | t + t_0 : ... : t + t_m\}$ (16)

We picture the generating functions (15) as $m + 1$ pairs $(g_0, t_0), ..., (g_m, t_m)$, called an *extended dihedral word*, located cyclically on an oriented circle (see the picture on page 7).

We will need also two other generating functions:

$$\{g_0 : g_1 : ... : g_m | t_0, ..., t_m\} := \{g_0 : ... : g_m | t_0 : t_0 + t_1 : ... : t_0 + ... + t_m\} \tag{17}$$

where $t_0 + ... + t_m = 0$, and

$$\{g_0, g_1, ..., g_m | t_0 : ... : t_m\} := \{g_0 : g_0 g_1 : ... : g_0 ... g_m | t_0 : t_1 : ... : t_m\}$$

where $g_0 \cdot ... \cdot g_m = 1$.

Relations. i) For any $g \in G$ one has

$$\{g \cdot g_0 : ... : g \cdot g_m | t_0 : ... : t_m\} = \{g_0 : ... : g_m | t_0 : ... : t_m\} \tag{18}$$

ii) *The dihedral symmetry relations*:

$$\{g_0 : ... : g_{m-1} : g_m | t_0 : t_1 : ... : t_m\} = \{g_1 : ... : g_m : g_0 | t_1 : ... : t_m : t_0\} \tag{19}$$

$$\{g_0 : ... : g_m | t_0 : ... : t_m\} \quad = \quad (-1)^{m+1} \{g_m : ... : g_0 | t_m : ... : t_0\} \tag{20}$$

iii) *The double shuffle relations* ($k + l = m, k \geq 1, l \geq 1$):

$$\sum_{\sigma \in \Sigma_{k,l}} \{g_0 : g_{\sigma(1)} : ... : g_{\sigma(m)} | t_0, t_{\sigma(1)}, ..., t_{\sigma(m)}\} = 0, \tag{21}$$

$$\sum_{\sigma \in \Sigma_{k,l}} \{g_0, g_{\sigma(1)}, ..., g_{\sigma(m)} | t_0 : t_{\sigma(1)} : ... : t_{\sigma(m)}\} = 0 \tag{22}$$

here $\Sigma_{k,l}$ is the set of all shuffles of the ordered sets $\{1, ..., k\}$ and $\{k+1, ..., k+l\}$.

iv) *The distribution relations* ($l \in \mathbb{Z}$ and $|l|$ divides $|G|$ if the group is finite.)

$$\{x_0^l : ... : x_m^l | t_0 : t_1 : ... : t_m\} - \sum_{y_i^l = x_i^l} \{y_0 : ... : y_m | l \cdot t_0 : ... : l \cdot t_m\} = 0$$

except the relation $\{e : e\}_1 = \sum_{y^l = e} \{y : e\}_1$, which is not supposed to hold.

v) $\{e : e\}_1 = 0$.

Remark. In [G1] we did not impose condition v).

For example the distribution relations for $l = -1$ are

$$\{x_0^{-1} : ... : x_m^{-1} | t_0 : t_1 : ... : t_m\} = \{x_0 : ... : x_m | -t_0 : ... : -t_m\} \tag{23}$$

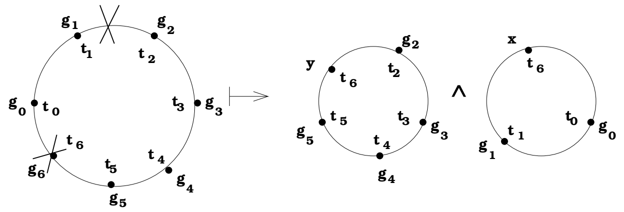
It was proved in [G1] that the dihedral symmetry and (23) follow from the double shuffle relations, see the proof of theorem 4.1 in [G1].

The cobracket $\delta : \mathcal{D}_{\bullet,\bullet}(G) \longrightarrow \mathcal{D}_{\bullet,\bullet}(G) \wedge \mathcal{D}_{\bullet,\bullet}(G)$ is given by

$$\delta\{g_0 : ... : g_m | t_0, ..., t_m\} := \tag{24}$$

$$\sum_{k=1}^{m-1} \mathrm{Cycle}_{m+1} \Big(\{g_0 : ... : g_{k-1} : g_k | t_0, ..., t_{k-1}, x_k\} \wedge \{g_k : g_{k+1} : ... : g_m | y_k, t_{k+1}, ..., t_m\} \Big)$$

where $t_0 + ... + t_{k-1} + x_k = 0$, $y_k + t_{k+1} + ... + t_m = 0$, the indices are modulo $m+1$ and $\mathrm{Cycle}_{m+1} f(x_0, ..., x_m) := \sum_{i=0}^{m} f(x_i, ..., x_{m+i})$. Each term of the formula corresponds to the following procedure: choose an arc on the circle between the two neighboring pairs (g_α, t_α) and $(g_{\alpha+1}, t_{\alpha+1})$, and in addition choose a pairs (g_β, t_β) different from the ends of the arc. Then cut the circle in the choosen arc and in the choosen point and make two oriented circles with an extended dihedral word on each of them out of the initial data.



Theorem 2.1 *δ provides a bigraded Lie coalgebra structure on $\mathcal{D}_{\bullet,\bullet}(G)$.*

We will need below an explicit description of the coproduct on the generating series $\{g_0 : ... : g_m | u_0 : ... : u_m\}$.

Lemma 2.2 *One has*

$$\delta\{g_0 : ... : g_m | u_0 : ... : u_m\} = \tag{25}$$

$$\sum_{k=1}^{m-1} \mathrm{Cycle}_{m+1}\Big(\{g_0 : ... : g_{k-1} : g_k | u_0 : ... : u_{k-1} : u_m\} \wedge \{g_k : ... : g_m | u_k : ... : u_m\}\Big)$$

Proof. To compute (2.2) we rewrite formula (24) using the transformation formula (17) and substituting $u_k := t_0 + ... + t_k$. We get

$$\delta\{g_0 : ... : g_m | u_0 : ... : u_m\} = \tag{26}$$

$$\sum_{k=1}^{m-1} \mathrm{Cycle}_{m+1}\Big(\{g_0 : ... : g_{k-1} : g_k | u_0 : ... : u_{k-1} : 0\} \wedge \{g_k : ... : g_m | u_0 : t_{k+1} : ... : t_m\}\Big)$$

Since $u_{k+i} = u_k + t_{k+1} + ... + t_{k+i}$ we have using (16)

$$\{g_k : ... : g_m | 0 : t_{k+1} : ... : t_m\} = \{g_k : ... : g_m | u_k : u_{k+1} : ... : u_m\}$$

Notice that $u_m = 0$. The lemma follows.

2. Another way to spell the definition of $\mathcal{D}_{\bullet,\bullet}(G)$. Define the elements

$$I_{n_0,..,n_m}(g_0 : ... : g_m) \in \mathcal{D}_{w,m}(G) \quad \text{where} \quad w := n_0 + ... + n_m - 1$$

by setting $I_{1,n_1,..,n_m}(g_0 : ... : g_m) = \{g_0 : ... : g_m\}_{n_1,...,n_m}$ and, more generally

$$I_{n_0,n_1,...n_m}(g_0 : ... : g_m) := \tag{27}$$

$$(-1)^{n_0-1} \cdot \sum_{i_1+...+i_m=n_0-1} \binom{n_1+i_1}{i_1} ... \binom{n_m+i_m}{i_m} \{g_0 : ... : g_m\}_{n_1+i_1,...,n_m+i_m}$$

Here $i_k \geq 0$. This definition is equivalent to the formula

$$\sum_{n_i \geq 1} I_{n_0,...,n_m}(g_0 : ... : g_m) t_0^{n_0-1} \cdot ... \cdot t_m^{n_m-1} = \{g_0 : ... : g_m | t_0 : ... : t_m\} \tag{28}$$

where the right hand side was defined in (15). In particular we have the cyclic symmetry relations

$$I_{n_0,n_1,\ldots,n_m}(g_0 : \ldots : g_m) = I_{n_1,\ldots,n_m,n_0}(g_1 : \ldots : g_m : g_0) \qquad (29)$$

Remark. For $G = \mu_N$ the properties of the generators $I_{n_0,n_1,\ldots,n_m}(a_0 : \ldots : a_m)$ reflect the properties of the iterated integral

$$\int_0^{a_0} \underbrace{\frac{dt}{t} \circ \ldots \circ \frac{dt}{t}}_{n_0-1} \circ \underbrace{\frac{dt}{a_1 - t} \circ \frac{dt}{t} \circ \ldots \circ \frac{dt}{t}}_{n_1} \circ \ldots \circ \underbrace{\frac{dt}{a_m - t} \circ \frac{dt}{t} \circ \ldots \circ \frac{dt}{t}}_{n_m} \qquad (30)$$

considered modulo the products of similar integrals and modulo the lower depth integrals. This integral is divergent if $n_0 > 1$, and so has to be regularized (see theorem 7.1 in [G1] or [G2]) . Its regularized value is given by formula (27). The shuffle relations (21) correspond precisely to the shuffle product formula for the iterated integrals (30) with $n_0 = 1$.

It follows from formula (28) and lemma 2.2 that the coproduct of these elements is given by the following formula:

$$\delta I_{n_0,n_1,\ldots,n_m}(g_0 : \ldots : g_m) = \qquad (31)$$

$$\text{Cycle}_{m+1}\Big(\sum_{j=1}^{m-1} \sum_{n'_j + n''_j = n_j + 1} I_{n_0,\ldots,n_{k-1},n'_k}(g_0 : \ldots : g_k) \wedge I_{n_k,\ldots,n_{m-1},n''_m}(g_k : \ldots : g_m)\Big)$$

Notice that the second summation is over positive integers n'_j, n''_j such that $(n'_j - 1) + (n''_j - 1) = n_j - 1$.

3. Two related bigraded Lie coalgebras: $\tilde{\mathcal{D}}_{\bullet,\bullet}(G)$ and $\mathcal{D}'_{\bullet,\bullet}(G)$.

i) The generators of $\tilde{\mathcal{D}}_{w,m}(G)$ are symbols

$$\tilde{I}_{n_0,n_1,\ldots,n_m}(g_0 : \ldots : g_m) \qquad \sum n_i = w - 1$$

The relations are the following: *the homogeneuity*

$$\tilde{I}_{n_0,n_1,\ldots,n_m}(g_0 : \ldots : g_m) = \tilde{I}_{n_0,n_1,\ldots,n_m}(hg_0 : \ldots : hg_m), \quad h \in G \qquad (32)$$

the cyclic symmetry (29), and in addition $\tilde{I}_{1,\ldots,1}(e : \ldots : e) = 0$. It is easy to se that formula (31) provides a bigraded Lie coalgebra structure on $\tilde{\mathcal{D}}_{\bullet,\bullet}(G)$.

For an associative $\mathbb{Q}$-algebra A let $\mathcal{C}(A) := A/[A, A]$ be the $\mathbb{Q}$-vector space of cyclic words in A. Denote by $\mathcal{C}(A_1A_2A_3\ldots)$ the cyclic word $A_1A_2A_3\ldots = A_2A_3\ldots A_1$.

Let $A(G)$ be the free associative algebra with the generators Y and X_g, $g \in G$. The group G acts naturally on the generators of $A(G)$ (as in s. 1.4). So it acts by automorphisms of the algebra itself. Let $\tilde{\mathcal{C}}(A(G))$ be the quotient of $\mathcal{C}(A(G))$ by the subspace generated by Y^n, X_g^n, $n \geq 0$.

Lemma 2.3 *There is a canonical isomorphism of $\mathbb{Q}$-vector spaces*

$$\eta : \tilde{\mathcal{D}}_{\bullet,\bullet}(G) \longrightarrow \tilde{\mathcal{C}}(A(G))_G$$

$$\tilde{I}_{n_0,n_1,\ldots,n_m}(g_0 : \ldots : g_m) \longmapsto \mathcal{C}\left(X_{g_0} Y^{n_0-1} X_{g_1} X^{n_1-1} \ldots X_{g_m} Y^{n_m-1}\right) \tag{33}$$

Remark. This formula reflects, in the case $G = \mu_N$, the structure of the iterated integral (30). Notice the special role of X_{g_0}: it corresponds to the integration path from 0 to g_0.

Proof. If a cyclic word in X_g, Y has a nonzero image in the quotient then it has at least one X_g. Using the cyclic symmetry we can move it to the begining of the word. Then it is in the image of η. The injectivity is clear. The lemma is proved.

ii) There is a commutative shuffle product $\circ_{Sh}$ on $A(G)$:

$$(Z_1 \ldots Z_k) \circ_{Sh} (Z_{k+1} \ldots Z_{k+l}) := \sum_{\sigma \in \Sigma_{k,l}} Z_{\sigma(1)} \ldots Z_{\sigma(k+l)}$$

We define a *shuffle relation* in $\tilde{\mathcal{D}}_{\bullet,\bullet}(G)$ as the image of an element

$$X_e \cdot \left\{ (Y^{n_0-1} X_{g_1} Y^{n_1-1} \ldots X_{g_k} Y^{n_k-1}) \circ_{Sh} (Y^{m_0-1} X_{h_1} Y^{m_1-1} \ldots X_{h_l} Y^{m_l-1}) \right\} \tag{34}$$

where $k, l \geq 0$ and $n_0, m_0 \geq 1$, under the isomorphism η^{-1}. We define the Lie coalgebra $\mathcal{D}'_{\bullet,\bullet}(G)$ as the quotient of $\tilde{\mathcal{D}}_{\bullet,\bullet}(G)$ by the subspace generated by these shuffle relations.

iii) Let $\mathcal{D}''_{w,m}(G)$ be the $\mathbb{Q}$-vector space generated by the symbols (14) subject to the relations (18),(19), (21) and v). So it has the same generators as $\mathcal{D}_{w,m}(G)$, but the relations are relaxed.

Proposition 2.4 *There is an isomorphism $i : \mathcal{D}''_{w,m}(G) \longrightarrow \mathcal{D}'_{w,m}(G)$ given by*

$$i : \{g_0 : g_1 : \ldots : g_m\}_{n_1,\ldots,n_m} \longmapsto \tilde{I}_{1,n_1,\ldots,n_m}(g_0 : g_1 : \ldots : g_m) \tag{35}$$

Proof. *Surjectivity of the map i.* Let S_1 be the subspace of $A(G)_G$ generated by elements (34) with $k = 0$ and $m_0 = 1$. We write $a \stackrel{S_1}{=} b$ if $a - b \in S_1$.

One has $\tilde{I}_n(g_0) = 0$. Indeed by the homogeneuty we need to check this only for $g_0 = e$. If $n = 1$ this is relation v). Otherwise $\binom{n}{k} X_e Y^n = X_e \left\{ Y^k \circ_{Sh} Y^{n-k} \right\} \in S_1$. Applying η^{-1} we get $\tilde{I}_n(e) = 0$.

Let us prove by induction on n_0 the following result, which allows to express $\tilde{I}_{n_0,n_1,\ldots,n_m}(g_0 : g_1 : \ldots : g_m)$ via $\tilde{I}_{1,n_1,\ldots,n_m}(g_0 : g_1 : \ldots : g_m)$:

Lemma 2.5 *a) One has the following formula in $A(G)$:*

$$X_e Y^{n_0-1} X_{g_1} Y^{n_1-1} \cdot \ldots \cdot X_{g_m} Y^{n_m-1} \stackrel{S_1}{=} \tag{36}$$

$$(-1)^{n_0-1} X_e X_{g_1} \left\{ (Y^{n_0-1}) \circ_{Sh} (Y^{n_1-1} X_{g_2} Y^{n_2-1} \cdot \ldots \cdot X^{g_m} Y^{n_m-1}) \right\}$$

b) The shuffle relations from the subspace S_1 are equivalent to formulae (36).

Remark. Applying η^{-1} to formula (36) we get relations (27).

Proof. When $n_0 = 1$ we have nothing to check. In general

$$0 \stackrel{S_1}{=} X_e \left\{ Y^k \circ_{Sh} (X_{g_1} Y^{n_1-1} \cdot ... \cdot X_{g_m} Y^{n_m-1}) \right\} = \quad (37)$$

$$\sum_{j=0}^{k} X_e Y^{k-j} X_{g_1} \left\{ Y^j \circ_{Sh} (Y^{n_1-1} \cdot ... \cdot X_{g_m} Y^{n_m-1}) \right\} \quad (38)$$

By the induction assumption this is equal modulo S_1 to

$$X_e Y^k X_{g_1} Y^{n_1-1} \cdot ... \cdot X_{g_m} Y^{n_m-1} +$$

$$(-1)^{k-j} \sum_{j=1}^{k} X_e X_{g_1} \left\{ Y^{k-j} \circ_{Sh} Y^j \circ_{Sh} (Y^{n_1-1} \cdot ... \cdot X_{g_m} Y^{n_m-1}) \right\} \quad (39)$$

Therefore $X_e Y^k X_{g_1} Y^{n_1-1} \cdot ... \cdot X_{g_m} Y^{n_m-1}$ equals modulo S_1 to

$$(-1)^{k-1} \sum_{j=1}^{k} (-1)^j X_e X_{g_1} \left\{ \binom{k}{j} Y^k \circ_{Sh} (Y^{n_1-1} \cdot ... \cdot X_{g_m} Y^{n_m-1}) \right\} \quad (40)$$

Since $\sum_{j=1}^{k} (-1)^j \binom{k}{j} = -1$ the statement follows.

Injectivity of the map i. We are dealing with the following problem. One can show that shuffle relations (21) go under the map i precisely to the space

$$\eta^{-1}(\text{the subspace of the shuffle relations (34) with } n_0 = m_0 = 1\ ) \quad (41)$$

(In fact in [G1] relations (21) appeared as a way to write the shufle relations (34) with $n_0 = m_0 = 1$). Let S be the subspace of $A(G)_G$ generated by S_1 and the shuffle relations (34) with $n_0 = m_0 = 1$. We need to show that the general shuffle relations (34) belongs to S. We use induction on $n_0 + m_0$. One has

$$Y^{n_0-1} X_{g_1} Y^{n_1-1} X_{g_2} \cdot ... \cdot X_{g_k} Y^{n_k-1} =$$

$$Y^{n_0-1} \circ_{Sh} X_{g_1} Y^{n_1-1} \cdot ... \cdot X_{g_k} Y^{n_k-1} - \sum_{i < n_0 - 1} Y^i X_{g_1} A_i \quad (42)$$

Let us take the shuffle product of (42) with the similar expression for $Y^{m_0-1} X_{h_1} \cdot ... \cdot Y^{m_l-1} X_{h_l}$. Notice that

$$Y^{n_0-1} \circ_{Sh} A \circ_{Sh} Y^{m_0-1} \circ_{Sh} B = \binom{n_0 + m_0 - 2}{n_0 - 1} Y^{n_0+m_0-2} \circ_{Sh} A \circ_{Sh} B \in S_1$$

and the other terms in the product already belong to the subcpace S by induction. The proposition is proved.

Proposition 2.6 *The formula (31) provides a well defined cobracket on* $\mathcal{D}'_{\bullet,\bullet}(G)$.

Proof. We need only to check that the shuffle relations (34) form a coideal in $\tilde{\mathcal{D}}_{w,m}(G)$. This is easy to do directly. Another proof see in the end of s. 4.

iv) The Lie coalgebra $\mathcal{D}_{\bullet,\bullet}(G)$ is a quotient of $\mathcal{D}'_{\bullet,\bullet}(G)$.

3 The Lie algebras of special derivations

1. The Lie algebra of special derivations and the depth filtration. Let $\mathbb{L}(X_0, ..., X_n)$ be the pronilpotent completion of the free Lie algebra generated by elements $X_0, ..., X_n$. Its derivation D is called special if there are $S_i \in \mathbb{L}(X_0, ..., X_n)$ such that

$$D(X_i) = [S_i, X_i], \quad \text{and} \quad D(X_0 + ... + X_n) = 0 \tag{43}$$

Denote by $\mathrm{Der}^S\mathbb{L}(X_0, ..., X_n)$ the Lie algebra of special derivations of the Lie algebra $\mathbb{L}(X_0, ..., X_n)$.

The weight filtration on the Lie algebra $\mathbb{L}(X_0, ..., X_n)$ as an increasing filtration by the lower central series. It goes from $-\infty$ to -1; the weight of X_i is -1. It provides the weight filtration on the Lie algebra $\mathrm{Der}^S\mathbb{L}(X_0, ..., X_n)$.

Let $\mathcal{I}$ be the kernel of the natural projection $\mathbb{L}(X_0, X_1, ..., X_n) \longrightarrow \mathbb{Q}$ given by $X_i \longmapsto \delta_{i0}$. Its powers $[[[\mathcal{I}, \mathcal{I}], \mathcal{I}], ..., \mathcal{I}]$ provide the depth filtration on $\mathbb{L}(X_0, ..., X_n)$. It induces the depth filtration on the Lie algebra $\mathrm{Der}^S\mathbb{L}(X_0, ..., X_n)$. Namely, a derivation (43) is of depth $-m$ if S_0 is of depth $-m$, i.e. there are at least m X_0's in S_0.

Denote by $\mathrm{GrDer}^S_{\bullet,\bullet}\mathbb{L}(X_0, ..., X_n)$ the associated graded quotient of the Lie algebra $\mathrm{Der}^S\mathbb{L}(X_0, ..., X_n)$ with respect to the depth filtration. It is a projective limit of finite dimensional Lie algebras bigraded by the weight and depth.

Let $\mathbb{A}(X_0, ..., X_n)$ be the completion of the free associative algebra generated by $X_0, ..., X_n$. By definition the Lie algebra $\mathrm{Der}^S\mathbb{A}(X_0, X_1, ..., X_n)$ of its special derivations consists of the derivations for which (43) holds with $S_i \in \mathbb{A}(X_0, ..., X_n)$. The depth filtration on $\mathbb{A}(X_0, ..., X_n)$ and $\mathrm{Der}^S\mathbb{A}(X_0, ..., X_n)$ are defined in a similar way. Let $\mathrm{GrDer}^S_{\bullet,\bullet}\mathbb{A}(X_0, ..., X_n)$ be the corresponding associated graded Lie algebra.

2. Cyclic words and special derivations ([Dr], [K]). Consider a map of $\mathbb{Q}$-linear spaces $\partial_{X_i} : \mathcal{C}(\mathbb{A}(X_0, X_1, ..., X_n)) \longrightarrow \mathbb{A}(X_0, X_1, ..., X_n)$ given on the generators by the following formula (the indices are modulo m):

$$\partial_{X_j}\mathcal{C}(X_{i_1}...X_{i_m}) := \sum_{X_{i_k}=X_j} X_{i_{k+1}}...X_{i_{k+m-1}}$$

For example $\partial_{X_1}\mathcal{C}(X_1X_2X_1X_2^2) = X_2X_1X_2^2 + X_2^2X_1X_2$. There is a map

$$\kappa : \mathcal{C}(\mathbb{A}(X_0, X_1, ..., X_n)) \longrightarrow \mathrm{Der}^S\mathbb{A}(X_0, X_1, ..., X_n)$$

$$\kappa\mathcal{C}(X_{i_1}...X_{i_m})(X_j) := \quad \left[\partial_{X_j}(X_{i_1}...X_{i_m}), X_j\right]$$

Here on the right $[A, B] = AB - BA$. We obviously get a special derivation, i.e.

$$\sum_{j=0}^{n} \left[\partial_{X_j}\mathcal{C}(X_{i_1}...X_{i_m}), X_j\right] = 0 \tag{44}$$

Indeed, for each of the $2m$ monomial terms in (44) there exists exactly one term in (44) which cancells it. For example $X_{i_1}...X_{i_m}$ appears, with the opposite signs, in $[\partial_{X_{i_1}}\mathcal{C}(X_{i_1}...X_{i_m}), X_{i_1}]$ and $[\partial_{X_{i_2}}\mathcal{C}(X_{i_1}...X_{i_m}), X_{i_2}]$

There is a Lie algebra structure on $\mathcal{C}(\mathbb{A}(X_0, X_1, ..., X_n))$ for which κ is a Lie algebra morphism. Namely, the space of cyclic words $\mathcal{C}(\mathbb{A}(X_0, ..., X_n))$ has a natural associative algebra structure given by

$$\mathcal{C}(Y_1 \cdot ... \cdot Y_k) * \mathcal{C}(Z_1 \cdot ... \cdot Z_l) := \sum_{Y_i = Z_j} \mathcal{C}(Y_{i+1} \cdot ... \cdot Y_i \cdot Z_{j+1} \cdot ... \cdot Z_{j-1}) \quad (45)$$

where Y_α, Z_β are the generators X_i, and the summation is over all pairs $Y_i = Z_j$. It is easy to check that it is associative. So there is a Lie algebra structure

$$[\mathcal{C}(Y_1 \cdot ... \cdot Y_k), \mathcal{C}(Z_1 \cdot ... \cdot Z_l)] :=$$

$$\frac{1}{2}\sum_{Y_i = Z_j}\Big(\mathcal{C}(Y_{i+1} \cdot ... \cdot Y_i \cdot Z_{j+1} \cdot ... \cdot Z_{j-1}) - \mathcal{C}(Z_{j+1} \cdot ... \cdot Z_{j-1} \cdot Z_j \cdot Y_{i+1} \cdot ... \cdot Y_{i-1})\Big)$$

(We multiply by $\frac{1}{2}$ the usual formula for $[,]$). Denote by $*_1$ the induced product on the associated graded quotient $\mathrm{Gr}\mathcal{C}(\mathbb{A}(X_0, ..., X_n))$ with respect to the depth filtration. It is given by fomula (45) where the summation is over $Y_i = Z_j$ different from X_0 only.

The centralizer of X_i in $\mathbb{A}(X_0, ..., X_n)$ is $\mathbb{Q}[[X_i]]$. So $Ker\kappa = \oplus_{i=0}^n \mathbb{Q}[[X_i]]$. Let $\tilde{\mathcal{C}}(\mathbb{A}(X_0, ..., X_n)) := \mathcal{C}(A(X_0, ..., X_n)) / \oplus_{i=0}^n \mathbb{Q}[[X_i]]$.

Proposition 3.1 $\kappa : \tilde{\mathcal{C}}(\mathbb{A}(X_0, ..., X_n)) \longrightarrow \mathrm{Der}^S(\mathbb{A}(X_0, ..., X_n))$ *is a Lie algebra isomorphism.*

Proof. We need only to show that κ is surjective. This follows from theorem 4.2 in [K]. For the convinience of the reader we reproduce the argument. Set $A := \mathbb{A}(X_0, ..., X_n)$. Let $F^1(A) := \oplus_{i=0}^n A \otimes dX_i$. There is a sequence

$$0 \longrightarrow \mathcal{C}(A) = A/[A, A] \xrightarrow{d} F^1(A) \xrightarrow{t} [A, A] \longrightarrow 0 \quad (46)$$

where $d(\mathcal{C}) := \sum \partial_{X_i}\mathcal{C} \otimes dX_i$ and $t[Z \otimes dX_i] := [Z, X_i]$. It is a complex by (44), and it is clearly exact from the left and right. There is an isomorphism of filtered (by the weight filtration) $\mathbb{Q}$-vector spaces $A \longrightarrow F^1(A), \quad X_{i_1}...X_{i_m} \longrightarrow X_{i_1}...X_{i_{m-1}} \otimes dX_{i_m}$. Thus the Euler characteristic of the weight $\geq -m$ part of (46) is zero. So the complex is exact. The proposition proved.

3. Special equivariant derivations. Let $\mathbb{A}(G)$ be the completion of the tensor algebra generated by X_g, $g \in G$, and Y. Let $\tilde{\mathcal{C}}(\mathbb{A}(G))_G$ be the space of coinvariants of G. *Let us assume that G is finite.* We define a linear map

$$\kappa_G : \tilde{\mathcal{C}}(\mathbb{A}(G))_G \longrightarrow \mathrm{Der}^S\mathbb{A}(G)^G =: \mathrm{Der}^{SE}\mathbb{A}(G)$$

as a composition of κ with the projection to G-invariants:

$$\mathcal{C}(X_{g_0}Y^{n_0-1} \cdot ... \cdot X_{g_m}Y^{n_m-1})_G \longmapsto \frac{1}{|G|}\sum_{h \in G}\kappa\Big(\mathcal{C}(X_{hg_0}Y^{n_0-1} \cdot ... \cdot X_{hg_m}Y^{n_m-1})\Big)$$

here $\mathcal{C}(...)_G$ means the image of a cyclic word $\mathcal{C}(...)$ in $\tilde{\mathcal{C}}(\mathbb{A}(G))_G$. Proposition 3.1 immediately implies

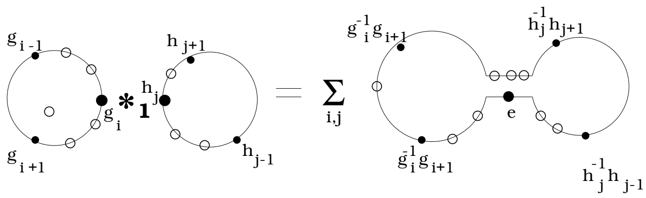
Proposition 3.2 *κ_G is a Lie algebras isomorphism.*

The algebra structure on the associate graded quotient $\mathrm{Gr}\tilde{\mathcal{C}}(\mathbb{A}(G))_G$ with respect to the depth filtration is described as follows (see the picture)

$$\mathcal{C}(X_{g_0}Y^{n_0-1}\cdot ... \cdot X_{g_k}Y^{n_k-1})_G *_1 \mathcal{C}(X_{h_0}Y^{m_0-1}\cdot ... \cdot X_{h_l}Y^{m_l-1})_G =$$

$$\sum_{i,j} \mathcal{C}(X_{g_i^{-1}g_{i+1}}Y^{n_{i+1}-1} \cdot ... \cdot X_{g_i^{-1}g_i}Y^{n_j-1}X_{h_j^{-1}h_{j+1}}Y^{m_{j+1}-1} \cdot ... \cdot X_{h_j^{-1}h_{j-1}}Y^{m_{j-1}-1})_G$$

It can be defined by the same formulae for any abelian group G.



4 The dihedral Lie algebras and special equivariant derivations

1. Assume that G is a finite group. Consider the following formal expression:

$$\xi_G = \sum_{\text{cyclic words}} \tilde{I}_{n_0,n_1,...,n_m}(g_0 : ... : g_m) \otimes \mathcal{C}(X_{g_0}Y^{n_0-1} \cdot ... \cdot X_{g_m}Y^{n_m-1})_G \quad (47)$$

where the sum is over all cyclic words in X_g, Y. (This is not the same as the sum over $n_i \geq 1$). Applying the map $Id \otimes \kappa_G$ we get a bidegree $(0,0)$ element

$$\tilde{\xi}_G \in \tilde{\mathcal{D}}_{\bullet,\bullet}(G)\widehat{\otimes}_{\mathbb{Q}}\mathrm{GrDer}^{SE}_{\bullet,\bullet}\mathbb{A}(G)$$

Notice that $\mathcal{D}_{\bullet,\bullet}(G)$ is an inductive limit of bigraded Lie coalgebras, which are finite dimensional since G is a finite group. So its dual $D_{\bullet,\bullet}(G)$ is a projective limit of bigraded Lie algebras, and $\mathbb{D}_{-w,-m}(G) = \mathcal{D}_{w,m}(G)^\vee$.

We may view the element ξ_G as a map between the bigraded $\mathbb{Q}$-vector spaces:

$$\xi_G \in Hom_{\mathbb{Q}-Vect}(\mathbb{D}_{\bullet,\bullet}(G), \mathrm{GrDer}^{SE}_{\bullet,\bullet}\mathbb{A}(G)) \quad (48)$$

Notice that $\tilde{\mathbb{D}}_{\bullet,\bullet}(G) \subset \mathbb{D}'_{\bullet,\bullet}(G) \subset \mathbb{D}_{\bullet,\bullet}(G)$.

Theorem 4.1 *a)* $\tilde{\xi}_G : \tilde{\mathbb{D}}_{\bullet,\bullet}(G) \stackrel{=}{\longrightarrow} \mathrm{GrDer}^{SE}_{\bullet,\bullet}\mathbb{A}(G)$ *is an isomorphism of the bigraded Lie algebras.*

b) Restricting $\tilde{\xi}_G$ *to* $\mathbb{D}'_{\bullet,\bullet}(G)$ *we get an isomorphism* $\xi'_G : \mathbb{D}'_{\bullet,\bullet}(G) \stackrel{=}{\longrightarrow} \mathrm{GrDer}^{SE}_{\bullet,\bullet}\mathbb{L}(G)$.

c) Restricting ξ'_G *to* $\mathbb{D}_{\bullet,\bullet}(G)$ *we get an injective Lie algebra morphism.*

Proof. a) We start with a remark from linear algebra. Let L_1, L_2 be Lie algebras and $\zeta \in Hom_{\mathbb{Q}-Vect}(L_1, L_2) = L_1^* \otimes L_2$. Denote by $\delta : L_1^* \longrightarrow \Lambda^2 L_1^*$ the Lie cobracket on L_1^* and by $[,] : \Lambda^2 L_2 \longrightarrow L_2$ the Lie bracket on L_2. Notice that $S^2(L_1^* \otimes L_2) = \Lambda^2 L_1^* \otimes \Lambda^2 L_2 \oplus S^2 L_1^* \otimes S^2 L_2$. Denote by $\zeta \circ \zeta \in \Lambda^2 L_1^* \otimes \Lambda^2 L_2$ the first component of the symmetric square of ζ. Then ζ is a morphism of Lie algebras if and only if

$$(\delta \otimes id)(\zeta) = (id \otimes [,])(\zeta \circ \zeta)$$

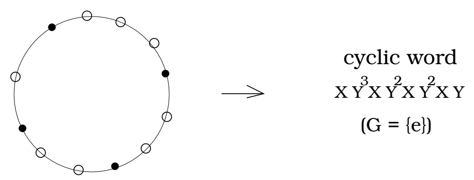
If $\zeta = \sum_i A_i \otimes B_i$ then $(id \otimes [,])(\zeta \circ \zeta) = \sum_{i,j} A_i \wedge A_j \otimes [B_i, B_j]$ and so the condition is

$$\sum_i \delta(A_i) \otimes B_i = \sum_{i,j} A_i \wedge A_j \otimes [B_i, B_j] \tag{49}$$

Let us return to our situation. To visualize a single expression

$$\delta(\tilde{I}_{n_0,n_1,\dots,n_m})(g_0 : \dots : g_m) \otimes \mathcal{C}(X_{g_0} Y^{n_0-1} \cdot \dots \cdot X_{g_m} Y^{n_m-1})_G \tag{50}$$

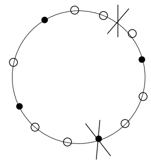
in $(\delta \otimes id)(\xi_G)$ we proceed as follows. Take an oriented circle divided on $m+1$ arcs by $m+1$ black points. Label each of the black points by an element of the set $\{X_g\}$, $g \in G$ and call it an X_g-point. The i-th arc is subdivided on n_i little arcs by $n_i - 1$ points labeled by Y (presented by little circles on the picture and called Y-points). Call such a picture a *circle with* $(n_0, \dots, n_m)$*-cyclic* G*-structure.*



It corresponds to a cyclic word $\mathcal{C}(X_{g_0} Y^{n_0-1} \cdot \dots \cdot X_{g_m} Y^{n_m-1})_G$ in an obvious way: put X_g for X_g-points and Y for Y-points and make a cyclic word.

The group G acts on circles with $(n_0, \dots, n_m)$-cyclic G-structure: an element $h \in G$ transforms X_g-points to X_{hg}-points and leaves untouched Y-points. The coinvariants are called circles with $(n_0, \dots, n_m)$-cyclic equivariant G-structure. They correspond to expressions (50) as well as to the generators $\tilde{I}_{n_0,n_1,\dots,n_m}(g_0 : \dots : g_m)$ obeying the cyclic symmetry and homogeneuty condition (32).

Let us mark such a picture by choosing in addition a little arc and a black point different from the ends of the arc containing the little arc. We call it a *marked circle with* $(n_0, \dots, n_m)$*-cyclic equivariant* G*-structure.*



a marked circle with (3,2,2,1) - cyclic structure

It follows from fromula (31) that expresion (50) is a sum of the terms which are in bijective correspondence with the marked circles with $(n_0, ..., n_m)$-cyclic equivariant G-structure. For example the marked circle with $(3, 2, 2, 1)$-cyclic structure on the picture corresponds to $I_{2,1} \wedge I_{2,1,2} \otimes \mathcal{C}(XY^2XYXY^3XY^2)$. In general we use first the marks to cut the circle on 2 oriented semicircles and make 2 new circles by gluing the endpoints of the semicircles, adding a new black point on each of the new circles instead of the marked black point on the initial circle, and using the rest of the points on each of the new circles. The new circles are getting natural cyclic structures.

On the other hand the terms of the sum

$$\sum \tilde{I}_{p_0,\ldots,p_k}(g_0 : ... : g_k) \wedge \tilde{I}_{q_0,\ldots,q_l}(h_0 : ... : h_l) \otimes$$

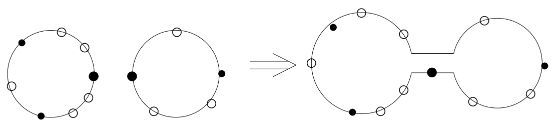
$$\mathcal{C}(X_{g_0}Y^{p_0-1} \cdot ... \cdot X_{g_k}Y^{p_k-1})_G *_1 \mathcal{C}(X_{h_0}Y^{q_0-1} \cdot ... \cdot X_{h_l}Y^{q_l-1})_G$$

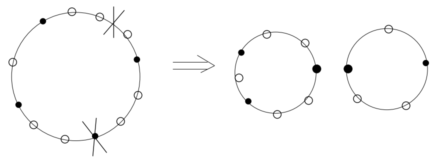
where the summation is over pars of cyclic words $\mathcal{C}(X_{g_0}Y^{p_0-1} \cdot ... \cdot X_{g_k}Y^{p_k-1})_G$ and $\mathcal{C}(X_{h_0}Y^{q_0-1} \cdot ... \cdot X_{h_l}Y^{q_l-1})_G$ and $*_1$ is the product in $\mathrm{Gr}\mathcal{C}(\mathbb{A}(G))_G$, also correspond to the marked circles with cyclic equivariant G-structure. Indeed, the product

$$\mathcal{C}(X_{g_0}Y^{p_0-1} \cdot ... \cdot X_{g_k}Y^{p_k-1})_G *_1 \mathcal{C}(X_{h_0}Y^{q_0-1} \cdot ... \cdot X_{h_l}Y^{q_l-1})_G$$

of $P := (p_0, ..., p_k)$ and $Q := (q_0, ..., q_l)$-cyclic words is a sum of $(k+1)(l+1)$ cyclic words. Let us picture P and Q-cyclic words by the corresponding circles with P and Q-cyclic G-structures. Then each of the $(k+1)(l+1)$ terms in the product corresponds to a pair $\{$*a circle with P-cyclic G-structure + a black point on it, a circle with Q-cyclic G-structure + a black point on it*$\}$.

Such pairs are in bijective correspondence with marked circles with cyclic equivariant G-structures, as demonstrated on the picture:





Namely, to get a marked circle with cyclic equivariant G-structure we proceed as follows. Make a connected sum of the P and Q-circles by connecting the choosen black points by a bridge. The orientation of the initial circles induces an orientation of their connected sum. Instead of the choosen two black points on the initial circles we put a single black point on the bottom part of the bridge. We keep the rest of the points, getting a cyclic structure on the new circle. The black point on the bottom bridge together with the top part of the bridge provide the marks on the circle with the cyclic structure we get. The distinguished point on each of the three circles is choosen to be the X_e-point. The procedure is clearly reversable, so we get the promised bijection.

We proved that $\tilde{\xi}_G$ is a morphism of bigraded Lie algebras. By lemma 2.3 it is an isomorphism. The part a) of the theorem is proved.

Lemma 4.2 *Let* $D \in \mathrm{Der}^{SE}\mathbb{A}(G)$. *Then* $D(X_e) \in \mathbb{L}(G) <=> D \in \mathrm{Der}^{SE}\mathbb{L}(G)$.

Proof. Since D is equivariant $D(X_e) \in \mathbb{L}(G)$ implies that $D(X_g) \in \mathbb{L}(G)$ for any $g \in G$. Since $D(\sum X_g + Y) = 0$ we get also $D(Y) \in \mathbb{L}(G)$.

The algebra $\mathbb{A}(X_0, ..., X_n)$ is a commutative Hopf algebra with the coproduct $\Delta(X_{i_1}...X_{i_m}) := \sum X_{i_{\alpha_1}}...X_{i_{\alpha_k}} \otimes X_{i_{\beta_1}}...X_{i_{\beta_l}}$ where the sum is over all partitions of $\{i_1, ..., i_n\}$ on two subsets $1 \leq \alpha_1 < ... < \alpha_k \leq m$ and $1 \leq \beta_1 < ... < \beta_k \leq m$. The Hopf algebra $A(X_0, ..., X_n)$ with the shufle product $\circ_{Sh}$ is its dual. Let $\tilde{\Delta}(Z) := \Delta - Id \otimes Z - Z \otimes Id$. Then for $D \in \xi'_G(\mathbb{D}_{\bullet,\bullet}(G))$ one has

$$D(X_e) \in \mathbb{L}(G) <=> (id \otimes \tilde{\Delta} \circ \partial_{X_e})(\xi'_G) \in \mathcal{D}_{\bullet,\bullet}(G)\widehat{\otimes}\Big(\mathbb{A}(G) \otimes \mathbb{A}(G)\Big) \quad \text{is zero.}$$

The element $A \otimes B \in \mathbb{A}(G) \otimes \mathbb{A}(G)$ appears in $(id \otimes \tilde{\Delta} \circ \partial_{X_e})(\xi'_G)$ with coefficient $X_e(A \circ_{Sh} B)$. Therefore shuffle relations (34) are equivalent to the condition $\tilde{\xi}_G(\mathcal{D}_{\bullet,\bullet}(G)) \subset \mathrm{GrDer}^{SE}_{\bullet,\bullet}\mathbb{L}(G)$. The lemma is proved.

Proof of proposition 2.6. We just proved that ξ'_G is an isomorphism of bigraded $\mathbb{Q}$-vector spaces. Using a) (and an obvious fact that $\mathrm{Der}^{SE}\mathbb{L}(G)$ is a Lie algebra!) we conclude that $\mathbb{D}'_{\bullet,\bullet}(G)$ is a Lie subalgebra of $\tilde{\mathbb{D}}_{\bullet,\bullet}(G)$.

So ξ'_G is a Lie algebra isomorphism. The part b) of the theorem is proved.

c) It follows from a), b) and propositions 2.4 and 2.6. The theorem is proved.

5. Towards conjecture 1.2: the distribution relations. Let $X_N := \mathbb{P}^1\backslash 0, \mu_N, \infty)$. There are maps

$$m_N : X_{NM} \longrightarrow X_M, \quad z \longmapsto z^N, \qquad\qquad i_N : X_{NM} \hookrightarrow X_M, \quad i_N : z \longmapsto z$$

They induce the homomorphisms

$$i_{N*} : \mathbb{L}(\mu_{MN}) \longrightarrow \mathbb{L}(\mu_M); \qquad Y \longrightarrow Y, X_\zeta \longrightarrow \begin{cases} 0 & \zeta \notin \mu_M \\ \zeta & \zeta \in \mu_M \end{cases}$$

$$m_{N*} : \mathbb{L}(\mu_{MN}) \longrightarrow \mathbb{L}(\mu_M); \qquad Y \longrightarrow NY, X_\zeta \longrightarrow X_{\zeta^N}$$

So they induce the maps $\tilde{m}_{N*}, \tilde{i}_{N*} : \mathrm{Der}^{SE}_{\bullet,\bullet}\mathbb{L}(\mu_{NM}) \longrightarrow \mathrm{Der}^{SE}_{\bullet,\bullet}\mathbb{L}(\mu_N)$. We claim that $\mathcal{G}^{(l)}(\mu_{NM}) \subset Ker(\tilde{i}_{N*} - \tilde{m}_{N*})$. Indeed, each of the maps i_N, m_N provides a commutative diagram

$$\begin{array}{ccc} Gal\Big(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty}, \zeta_{MN})\Big) & \stackrel{\varphi^{(l)}_{MN}}{\longrightarrow} & Aut\pi_1^{(l)}(X_{MN}) \\ \downarrow j_N & & \downarrow \\ Gal\Big(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty})\Big) & \stackrel{\varphi^{(l)}_{N}}{\longrightarrow} & Aut\pi_1^{(l)}(X_N) \end{array}$$

where j_N is the natural inclusion. So the composition $\varphi^{(l)}_N \circ j_N$ does not depend on the choise of the maps i_N, m_N. Thus an element of $\mathrm{Der}^{SE}_{\bullet,\bullet}\mathbb{L}(\mu_{MN})$ given by

$$\sum_{n_i>0} a(e : g_1 : ... : g_m)_{n_0,..,n_m} \cdot \kappa_{\mu_{MN}}\mathcal{C}(X_e Y^{n_0-1} ... X_{g_m} Y^{n_m-1})$$

(the sum is over cyclic words) is in $Ker(\tilde{i}_{N*} - \tilde{m}_{N*})$ if and only if

$$a(e : g_1 : ... : g_m)_{n_0,..,n_m} = N^{w-m} \sum_{h_i^N = g_i} a(h_0 : h_1 : ... : h_m)_{n_0,..,n_m}$$

This is precisely the distribution relations.

To complete the proof of conjecture 1.2 it remains to show that $\mathcal{G}^{(l)}(\mu_N)$ lies in the subspace of $\mathrm{Der}^{SE}_{\bullet,\bullet}\mathbb{L}(\mu_N)$ defined by the shuffle relations (22). These relations are obviously satisfied by the power series defining the multiple polylogarithms (see formula (1) in [G1]). Unfortunately I do not know any good proof of them via derivations or the iterated integrals (30). There is, however, a very indirect argument: relations (22) hold for functions => for the corresponding framed mixed Hodge structures => for the corresponding mixed Tate motives over $\mathbb{Q}(\zeta_N)$ => valid for their l-adic realization. I will publish it elswhere. It is rather amusing that the simplest relations on the level of functions are responsible for the most nontrivial constraints on the image of the Galois group!

5 Proofs of theorems 1.4 and 1.5

Applying to $\{e : e|t_0 : t_1\} = \sum_{n>0}\{e : e\}_n(t_0 - t_1)^{n-1}$ the cyclic symmetry relation we get $\mathcal{D}_{-2w,-1} = 0$. Recall that $\mathcal{D}_{-1,-1} = 0$ by definition.

Since $\mathrm{Gr}\mathcal{G}^{(l)}_{\bullet,-1}$ is abelian we may identify it with the $\mathbb{Q}_l$-points of the corresponding algebraic group. So it makes sense to talk about projection of $\varphi^{(l)}(\sigma)$ on $\mathrm{Gr}\mathcal{G}^{(l)}_{\bullet,-1}$. Let $\varphi^{(l)}_{-w,-1}(\sigma)$ be the component of this projection in $\mathrm{Gr}\mathcal{G}^{(l)}_{-w,-1}$.

In [So] Soulé constructed for each integer $m > 1$ a $Gal(\mathbb{Q}(\zeta_{l^\infty})/\mathbb{Q})$-homomorphism $\chi_m : Gal(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty}))^{ab} \longrightarrow \mathbb{Q}_l(m)$ and proved that it is zero if and only if m is even. Let $I_{2m-1}(e)^\vee$ be the generator of $\mathbb{D}_{-2m+1,-1}$ dual to $I_{2m-1}(e)$. Then it follows from the Key Lemma B in [Ih2] that

$$\varphi^{(l)}(\sigma) = \frac{(1-l^{m-1})^{-1}}{(m-1)!}\chi_m(\sigma)\xi\Big(I_{2m-1}(e)^\vee\Big) \qquad (m: odd \geq 3)$$

So $\mathbb{D}_{\bullet,-1} \otimes \mathbb{Q}_l = \mathcal{L}^{(l)}_{\bullet,-1}$. Since both $\mathbb{D}_{\bullet,\bullet} \otimes \mathbb{Q}_l$ and $\mathcal{L}^{(l)}_{\bullet,\bullet}$ are Lie subalgebras of $\mathrm{Der}^{SE}_{\bullet,\bullet}\mathbb{L}$, and in addition $\mathbb{D}_{\bullet,-1} \otimes \mathbb{Q}_l = \mathcal{L}^{(l)}_{\bullet,-1}$ we have $\mathcal{L}^{(l)}_{\bullet,\bullet} \subset \mathbb{D}_{\bullet,\bullet} \otimes \mathbb{Q}_l$. Thus there are the following commutative diagrams ($[,]$ are the commutator maps):

$$\begin{array}{ccc} \Lambda^2\mathcal{L}^{(l)}_{\bullet,-1} & \stackrel{=}{\longrightarrow} & \Lambda^2\mathbb{D}_{\bullet,-1} \otimes \mathbb{Q}_l \\ \downarrow [,] & & \downarrow [,] \\ \mathcal{L}^{(l)}_{\bullet,-2} & \hookrightarrow & \mathbb{D}_{\bullet,-2} \otimes \mathbb{Q}_l \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{L}^{(l)}_{\bullet,-2} \otimes \mathcal{L}^{(l)}_{\bullet,-1} & \stackrel{=}{\longrightarrow} & \mathbb{D}_{\bullet,-2} \otimes \mathbb{D}_{\bullet,-1} \otimes \mathbb{Q}_l \\ \downarrow [,] & & \downarrow [,] \\ \mathcal{L}^{(l)}_{\bullet,-3} & \hookrightarrow & \mathbb{D}_{\bullet,-3} \otimes \mathbb{Q}_l \end{array}$$

Theorem 5.1 *Let $H_{(w,m)}$ be the weight w, depth m part of H. Then*

$$H^i_{(w,2)}(\mathbb{D}_{\bullet,\bullet}) = H^{i-1}_{cusp}(GL_2(\mathbb{Z}), S^{w-2}V_2), \quad i = 1,2 \tag{51}$$

$$H^i_{(w,3)}(\mathbb{D}_{\bullet,\bullet}) = H^i_{cusp}(GL_3(\mathbb{Z}), S^{w-3}V_3) = 0, \quad i = 1,2,3 \tag{52}$$

Proof. Let $\mathbb{Q}_{-1,-1}$ be the one dimensional Lie algebra placed in degree $(-1,-1)$. There is an exact sequence $0 \longrightarrow \mathbb{D}_{\bullet,\bullet} \longrightarrow \widehat{\mathbb{D}}_{\bullet,\bullet} \longrightarrow \mathbb{Q}_{-1,-1} \longrightarrow 0$. It provides the follwing bicomplex where the horizontal lines are the depth $-m$ parts of the standard cochain complex for the Lie algebra $\widehat{\mathbb{D}}_{\bullet,\bullet}$:

$$\begin{array}{ccccc} \widehat{\mathbb{D}}_{\bullet,-3} & \stackrel{\delta}{\longrightarrow} & \widehat{\mathbb{D}}_{\bullet,-2} \otimes \widehat{\mathbb{D}}_{\bullet,-1} & \stackrel{\delta}{\longrightarrow} & \Lambda^3\widehat{\mathbb{D}}_{\bullet,-1} \\ & & \downarrow \partial & & \downarrow \partial \\ & & \widehat{\mathbb{D}}_{\bullet,-2} & \stackrel{\delta}{\longrightarrow} & \Lambda^2\widehat{\mathbb{D}}_{\bullet,-1} \\ & & & & \downarrow \partial \\ & & & & \widehat{\mathbb{D}}_{\bullet,\bullet} \end{array}$$

The vertical differentials have bidegree $(1,1)$ and are constracted as follows. Let V be a vector space and $f \in V^*$. Define a map $\partial : \Lambda^n V \longrightarrow \Lambda^{n-1}V$, $\partial^2 = 0$, by $v_1 \wedge ... \wedge v_n \longrightarrow \sum_{i=1}^n (-1)^{i-1} f(v_i) v_1 \wedge ... \wedge \widehat{v}_i \wedge ... \wedge v_n$. There is a canonical functional $1 \in \mathbb{Q}^\vee_{-1,-1} \subset \widehat{\mathbb{D}}_{\bullet,\bullet}$. It provides the maps $\partial : \Lambda^n\widehat{\mathbb{D}}_{\bullet,\bullet} \longrightarrow \Lambda^{n-1}\widehat{\mathbb{D}}_{\bullet,\bullet}$.

Consider the inclusion of the *subcomplex*

$$\mathbb{D}_{\bullet,-2} \longrightarrow \Lambda^2\mathbb{D}_{\bullet,-1} \quad \hookrightarrow \quad \mathbb{D}_{\bullet,-2} \longrightarrow \Lambda^2\widehat{\mathbb{D}}_{\bullet,-1} \longrightarrow \widehat{\mathbb{D}}_{\bullet,-1} \tag{53}$$

into the *complex* given by the two bottom lines of the bicomplex. The isomorphism μ from theorem 1.2 in [G1] is easyly extended to an isomorphism between the complex $\mathbb{M}^*_{(2)} \otimes_{GL_2(\mathbb{Z})} S^{\bullet-2}V_2$ and the *complex* (53) which takes $\tau_{[1,2]}\mathbb{M}^*_{(2)} \otimes_{GL_2(\mathbb{Z})} S^{\bullet-2}V_2$ just to the *subcomplex* (53). This proves the part a).

The depth -3 row of the bicomplex can be written as

$$\mathbb{D}_{\bullet,-3} \stackrel{\delta}{\longrightarrow} \mathbb{D}_{\bullet,-2} \otimes \widehat{\mathbb{D}}_{\bullet,-1} \stackrel{\delta}{\longrightarrow} \mathbb{D}_{\bullet,-1} \wedge \mathbb{D}_{\bullet,-1} \wedge \widehat{\mathbb{D}}_{\bullet,-1} \tag{54}$$

So it is maped by ∂ onto the *subcomplex* (53) and $\mathbb{D}_{\bullet,-3} \longrightarrow \mathbb{D}_{\bullet,-2} \otimes \mathbb{D}_{\bullet,-1} \longrightarrow \Lambda^2\mathbb{D}_{\bullet,-1}$ is the kernel. By theorem 1.3 and formula (36) in [G1] we have

$$H^i_{(w,3)}(\widehat{\mathbb{D}}_{\bullet,\bullet}) \quad = \quad \begin{cases} 0 & i = 1,2 \\ H^1_{\text{cusp}}(GL_2(\mathbb{Z}), S^{w-2}V_3) & i = 3 \end{cases} \tag{55}$$

According to the part a) $H^1_{cusp}(GL_2(\mathbb{Z}), S^{w-2}V_3)$ is the only nonzero cohomology group of *subcomplex* (53), the map ∂ provides a quasiisomorphism of the complex (54) and the *subcomplex* (53). Theorem 5.1 is proved.

The left vertical arrows in the commutative diagrams above are surjective by definition. Since $H^0_{cusp}(GL_2(\mathbb{Z}), S^{w-2}V_2) == 0$ if $w > 2$ the right vertical arrows there are also surjective by theorem 5.1. So the bottom horizontal maps are isomorphisms. The dimensions of $\mathbb{D}_{-w,-2}$ and $\mathbb{D}_{-w,-3}$ has been computed in theorems 1.4 and 1.5 in [G1], and are given by the right hand sides of formulas (10) and (13). This implies both theorems 1.4 and 1.5

These theorems, in particular, imply that the double shuffle relations provide a *complete* list of constraints on the Lie algebra of the image of the Galois group in $Aut\pi_1^{(l)}(\mathbb{P}^1\backslash\{0,1,\infty\})$ in the depths -2 and -3.

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