Symplectic fillings of the link of simple elliptic singularities *

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Abstract: The diffeomorphism types of the minimal strong symplectic fillings of the link of a simple elliptic singularity are completely determined.

§1. Introduction.

The link of an isolated singularity carries a contact structure given by the maximal complex tangencies. In this paper, we consider the link as a contact manifold with the contact structure mentioned above. From the point of view of symplectic and contact geometry, the minimal resolution of an isolated singularity in a complex surface gives a typical example of a symplectic filling of the link of the singularity. Furthermore, if the singularity is realized as complete intersections, its Milnor fiber gives another typical example of a symplectic filling of the link. In a previous work [O-O3], we showed that the diffeomorphism type of any minimal symplectic filling of

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the link of a simple singularity on a complex surface is unique, in particular, it is diffeomorphic to its Milnor fiber. This reflects a special feature of simple singularities established by Brieskorn. We cannot expect to extend uniqueness result to other isolated surface singularities, since, for example, the Milnor fiber and the minimal resolution may have different signatures in general. In this note, we study the case of *simple elliptic singularities*, i.e., a class of isolated singularities, the exceptional divisor of whose minimal resolution consists of a non-singular elliptic curve of negative self intersection number $-k < 0$. We call the number $k > 0$ the *degree* of the simple elliptic singularity. Topologically, the link of the simple elliptic singularity of degree k is the boundary of the total space of the D^2 bundle over T^2 with the first Chern class $-k$. If such a singularity is realized as a complete intersection singularity, the degree k is quite restricted. In fact, Saito [Sai] showed that k must be 1, 2 or 3, when the simple elliptic singularity is realized as a hypersurface singularity. The degree 4 appears when the singularity is realized as a complete intersection singularity. See the table in Section 2. According to Pinkham [P], if the singularity is "smoothed by deformation" if and only if $0 < k \leq 9$.

We interpret these results from symplectic and contact viewpoint. Firstly, we study the diffeomorphism types of the strong symplectic fillings of the simple elliptic singularity, which contains the case of the Milnor fiber. For the definition of strong symplectic fillings, see Section 2. Note also that the minimal resolution and smoothings, if there exist, are in fact strong symplectic fillings. Namely we have the following

Theorem 1. (1) *If the link of a simple elliptic singularity of the degree k carries a strong symplectic filling X with $c_1(X) = 0$, then we have $0 < k \leq 9$.*
(2) *Moreover, the diffeomorphism type of X is unique, unless $k = 8$. If $k = 8$, there are exactly two diffeomorphism types. More explicitly, when $k \neq 8$, X is diffeomorphic to the complement of a holomorphically embedded 2-torus in the $(9-k)$-point blow-up of $\mathbf{C}P^2$, which represents the anti-canonical divisor. When $k = 8$, X is diffeomorphic to the complement of a holomorphically embedded 2-torus either in $\mathbf{C}P^1 \times \mathbf{C}P^1$, or in the one-point blow-up of $\mathbf{C}P^2$, which represents the anti-canonical divisor in both cases.*

In the proof, the diffeomorphism type of X is described not only as the complement of the anti-canonical divisor, but also as following: We consider the blow-up of $\mathbf{C}P^2$ at $(9-k)$-points on a non-singular cubic curve. For the case $k \neq 8$, X is diffeomorphic to the complement of the proper transform of the cubic curve. When $k = 8$, X is diffeomorphic to either the complement of a holomorphically embedded 2-torus in $\mathbf{C}P^1 \times \mathbf{C}P^1$ which represents the homology class $2([\mathbf{C}P^1 \times pt] + [pt \times \mathbf{C}P^1])$, or the complement of the proper transform of the cubic curve. In the former case of $k = 8$, we blow-up $\mathbf{C}P^1 \times \mathbf{C}P^1$ at one point on the holomorphically embedded 2-torus and contract two (-1)-curves which are proper transforms of $\mathbf{C}P^1 \times pt$ and $pt \times \mathbf{C}P^1$. Then we have a non-singular cubic curve in $\mathbf{C}P^2$. We note that it is known, when $k = 8$, that there exist five components in the deformation space of the simple elliptic singularity and two diffeomorphism types of smoothings [L-W] [1]. The two diffeomorphism types appearing in Theorem 1 correspond to this fact. In this note we call each of them *its smoothing.* In this sense, we can restate Theorem 1 (2) as following.

Theorem 1'. *Any strong symplectic filling X of the link of a simple elliptic singularity is diffeomorphic to its smoothing, if $c_1(X) = 0$.*

We say that a symplectic 4-manifold is *minimal* if it does not contain any symplectically embedded 2-sphere of self-intersection number -1. We note that any strong symplectic filling X of the link of a simple elliptic singularity is automatically minimal, provided that $c_1(X) = 0$. See Section 3.2.

On the other hand, the disk bundle of the line bundle over an elliptic curve with negative first Chern class gives an example of a strong symplectic filling of the link of the corresponding simple elliptic singularity. The total space of the line bundle gives the minimal resolution of the simple elliptic singularity. The zero section is the exceptional divisor. Note that in this case we can not expect the property $c_1(X) = 0$ as in Theorem 1. However, we can also show the following:

Theorem 2. *Any minimal strong symplectic filling of the link of a simple elliptic singularity is diffeomorphic to either one of its smoothings or the minimal resolution. More precisely, if the degree $k \geq 10$, any minimal strong symplectic filling is diffeomorphic to the minimal resolution. If $0 < k \leq 9$, the symplectic filling is diffeomorphic to either its smoothing or the minimal resolution.*

[1] We learnt from Jonathan Wahl that smoothings in four components of the deformation space are diffeomorphic.

A more explicit description of the minimal strong symplectic filling is given in Theorem 7.3.

§2. Review on basic facts.

Let $\pi_L : L \to T^2$ be a complex hermitian line bundle of degree $-k < 0$ over a 2-torus. Then there is a connection on L with the curvature $-2k\sigma$, where σ is an area form on T^2 with the total area 2π. Write ξ the horizontal distribution on the unit circle bundle $S(L)$. Then ξ is a contact distribution. Denote by α the connection form on $S(L)$. Then we have a symplectic form on L defined by $d(r^2\alpha)/2 + \pi_L^*\sigma$, where r is the radial fiber coordinate on L. Note that the restriction of this symplectic form to the unit circle bundle $S(L)$ is exact and written as $(k+2)d\alpha/(2k)$. Note that the contact structure is unique up to isotopy, when we vary the connection. This follows from Gray's stability theorem. We can also interpret this symplectic structure as follows. Let θ be a contact form for $(S(L), \xi)$. The symplectization of it is described as $(S(L) \times \mathbf{R}_{>0}, d(t \cdot \theta))$, where $\mathbf{R}_{>0}$ is the positive part of the real line and t is the coordinate on the real line. For a suitable $T_1 > 0$, the symplectic cutting construction along the hypersurface $t = T_1$ gives the symplectic structure above.

For any complex structure on T^2 and any holomorphic structure on L, the zero section is a non-singular divisor of negative self-intersection number $-k$. Then we can contract the zero section and get an isolated singularity. Conversely, the total space of the line bundle over the T^2 with degree $-k$ gives the minimal resolution of the simple elliptic singularity of degree k. If we choose the connection compatible with the holomorphic structure on L, the contact structure ξ above is the one on the link of the isolated singularity, the maximal complex tangency on the link.

Here are realization of some of these singularities as hypersurface or complete intersection singularities. See [Sai] and [D], for example.

Notation	Degree k	Defining equation
$\widetilde{E_8}$	1	$x^6 + y^3 + z^2 + \lambda xyz$
$\widetilde{E_7}$	2	$x^4 + y^4 + z^2 + \lambda xyz$
$\widetilde{E_6}$	3	$x^3 + y^3 + z^3 + \lambda xyz$
$\widetilde{D_5}$	4	$\begin{cases} x^2 + y^2 + \lambda zw \\ xy + z^2 + w^2 \end{cases}$

In these cases, we have the Milnor fiber of the isolated singularity, which is an example of symplectic fillings of its link. Milnor fibers for the isolated complete intersection singularity enjoy some special properties.

(2.1.1) The tangent bundle of the Milnor fiber is stably trivial. (This is because it is the inverse image of a regular value of a smooth function.)

(2.1.2) The homotopy type of the Milnor fiber is the same as that of the bouquet of spheres of the middle dimension. In particular, it is simply connected.

We will see later that one of symplectic fillings with vanishing Chern class in the case of $k = 8$ is not simply connected. Meanwhile, as we mentioned above, such a singularity cannot be realized as a complete intersection singularity.

Let (M, ξ) be a closed contact manifold. A compact symplectic manifold (X, ω), which bounds M, is called a *strong symplectic filling* of (M, ξ), if there exists a 1-form λ on a collar neighborhood U of $M = \partial X$ such that

(2.2.1) $\omega = d\lambda$ on U,

(2.2.2) $\xi = \ker \lambda$, i.e., $\lambda|_M$ is a contact form of ξ,

(2.2.3) The orientation of M given by $\lambda \wedge \omega$ coincides with the orientation as the boundary of the symplectic manifold X.

From now on, we consider the case that (M, ξ) is the link of a simple elliptic singularity with the natural contact structure, i.e., $(S(L), \xi)$ above. Note that the unit disk bundle $D(L)$ of L is an example of its strong symplectic filling.

Take the projective bundle $\mathbf{P}(L \oplus \underline{\mathbf{C}}) \to T^2$. It is a compactification of L and is equipped with a symplectic structure, which is an extension of the one on the unit disk bundle $D(L)$. This is also represented by the symplectic cutting construction applied to $(S(L) \times \mathbf{R}_{>0}, d(t \cdot \theta))$ along the hypersurfaces $t = T_1, T_2$, where $T_2 > T_1 > 0$.

We may suppose that the unit disk bundle $D(L)$ is symplectically embedded as a tubular neighborhood of the zero section $O_{T^2} = \mathbf{P}(O \oplus \{1\})$. Denote by Y the complement of $D(L)$ in $\mathbf{P}(L \oplus \underline{\mathbf{C}})$ and by E_∞ the section at infinity, the compactifying divisor. In a collar neighborhood of the boundary $S(L) = \partial Y$, there exists a 1-form λ satisfying the conditions (2.2.1) and (2.2.2) above and

(2.2.4) The orientation of $S(L)$ as a contact manifold is opposite to the orientation as the boundary of the symplectic manifold Y.

This allows us to glue a strong symplectic filling X and Y to get a closed symplectic 4-manifold (Z, Ω), which we call a *compactification* of the filling X. We note that E_∞ is symplectically embedded in Z with the self-intersection number $k > 0$.

In the next section, we prove that Z is a rational symplectic 4-manifold in the sense of [M1], when $c_1(X) = 0$.

§3. Proof of Theorem 1.

Firstly, we describe the compactification Z of a strong symplectic filling X. Namely, we show the following.

Proposition 3.1. *Let X be a strong symplectic filling of a simple elliptic singularity of degree $k > 0$. Suppose that $c_1(X) = 0$. Then the compactification Z of X constructed in Section 2 is diffeomorphic to either the $(9-k)$-point blow-up of $\mathbf{C}P^2$, or $\mathbf{C}P^1 \times \mathbf{C}P^1$. More precisely, when $k \neq 8$, Z is diffeomorphic to the $(9-k)$-point blow-up of $\mathbf{C}P^2$, and when $k = 8$, Z is diffeomorphic to either the one-point blow-up of $\mathbf{C}P^2$, or $\mathbf{C}P^1 \times \mathbf{C}P^1$.*

In particular, we have $k \leq 9$, which is the conclusion of Theorem 1 (1). Then the symplectic filling X is diffeomorphic to the complement of a tubular neighborhood of the proper transform of the compactifying divisor. Secondly, we note that a theorem in [Sik] implies that the isotopy type of the embedding of the compactifying divisor is unique. Combining these, we get classification of the diffeomorphism types of the strong symplectic fillings with $c_1 = 0$. There are two diffeomorphism types of minimal rational symplectic 4-manifolds, i.e., $\mathbf{C}P^2$ and $\mathbf{C}P^1 \times \mathbf{C}P^1$. Hence, there are two diffeomorphism types of rational symplectic 4-manifolds with $b_2 = 2$, meanwhile the diffeomorphism type is unique unless $b_2 = 2$. This fact reflects the conclusion that the diffeomorphism type of the strong symplectic filling with $c_1 = 0$ is unique, unless the degree $k = 8$ and that there are two diffeomorphism types when the degree $k = 8$. Smoothings are examples of strong symplectic fillings with $c_1 = 0$. We will see that they exhaust all possible diffeomorphism types. Thus X must be diffeomorphic to its smoothing, which is the conclusion of Theorem 1 (2).

3.1. Proof of Proposition 3.1.

We recall the following

Theorem 3.2. [O-O1] *If a closed symplectic 4-manifold (M, ω) satisfies*

$$\int_M c_1(M) \wedge \omega > 0,$$

then (M, ω) is a rational or ruled symplectic 4-manifold up to blow-up and down.

Since $c_1(X) = 0$, the support of the Poincaré dual of $c_1(Z)$ (over $\mathbf{Q}$) is contained in Y. Thus we can put the Poincaré dual of $c_1(Z) = aE_\infty$ for some number a. Then the adjunction formula for the symplectically embedded 2-torus E_∞ shows that $a = 1$. Thus the Poincaré dual of $c_1(Z)$ is represented by E_∞. Hence we have

$$\int_Z c_1(Z) \wedge \omega = \int_{E_\infty} \omega > 0.$$

Theorem 3.2 implies that (Z, ω) is a rational or ruled symplectic 4-manifold up to blow-up and down. In the case that Z is not minimal, we blow down (-1) curves to obtain a minimal symplectic 4-manifold Z_{min}. We denote by E the exceptional divisor of $\pi : Z \to Z_{min}$. Since we have

$$c_1^2(Z_{min}) = c_1^2(Z) - E \cdot E = E_\infty \cdot E_\infty - E \cdot E = k - E \cdot E > 0,$$

which yields that Z_{min} is not an irrational ruled symplectic 4-manifold. Therefore Z is a rational symplectic 4-manifold up to blow up and down.

Now we compute the second Betti number $b_2(Z)$ of Z. Since Z is a rational symplectic 4-manifold, we have $b_1(Z) = 0$ and $b_2^+(Z) = 1$, where $b_2^\pm(Z)$ is the maximal dimension of $\pm$-definite subspace in $H_2(Z; \mathbf{Q})$ with respect to the intersection pairing. We have $b_2^- + 3 = c_2(Z)[Z]$, since $c_2(Z)[Z]$ is the Euler number of Z, and $3(1 - b_2^-(Z)) = c_1(Z)^2 - 2c_2(Z)$ (the signature formula). Thus $b_2^-(Z)$ equals $9 - c_1(Z)^2 = 9 - k$, which implies that $k \leq 9$. Moreover, we can easily see that when $k = 8$, Z is either the one point blow-up of $\mathbf{C}P^2$ or $\mathbf{C}P^1 \times \mathbf{C}P^1$, and when $k \neq 8$, Z is the $(9-k)$-point blow-up of $\mathbf{C}P^2$. ∎

Remark 3.3. The assumption that $c_1(X) = 0$ is not necessary to prove that Z is a rational or ruled symplectic 4-manifold up to blow up down. See Section 4 and Section 7. But it is needed for the proof that it is not an irrational ruled symplectic 4-manifold.

3.2. The isotopy type of the embedding of E_∞.

By our construction, Z contains a symplectically embedded 2-torus E_∞. We pick a compatible almost complex structure J , which is generic among those making E_∞ pseudo-holomorphic. Since E_∞ represents the Poincaré dual of the first Chern class, i.e., "anti-canonical divisor", the adjunction formula implies that any J-holomorphic (-1)-curve S must intersect E_∞ once transversally. In particular, we note that the complement of a tubular neighborhood of E_∞ in Z, which is nothing but the filling X, is minimal. Thus, after blowing down S, E_∞ becomes an embedded pseudo-holomorphic torus with respect to an appropriate almost complex structure. It is also easy to see that this embedded pseudo-holomorphic torus represents the Poincaré dual of the first Chern class of the blown-down manifold. After successive blow-down, we arrive at a minimal rational symplectic 4-manifold Z_{min}, which is either the complex projective plane $\mathbf{C}P^2$ or the product of two copies of the 2-sphere $\mathbf{C}P^1 \times \mathbf{C}P^1$. [2]

If Z_{min} is $\mathbf{C}P^1 \times \mathbf{C}P^1$, then E' is a symplectically embedded 2-torus of the self-intersection number 8 representing the homology class $2[\mathbf{C}P^1 \times pt + pt \times \mathbf{C}P^1]$. If Z_{min} is the complex projective plane, E_∞ is a symplectically embedded 2-torus E' with self-intersection number 9. Hence E' represents the homology class $3[\mathbf{C}P^1]$. We apply the following theorem due to Sikorav.

Theorem 3.4.[Sik] *Any symplectically embedded curve in $\mathbf{C}P^2$ representing $3[\mathbf{C}P^1]$ is symplectically isotopic to a non-singular cubic curve with respect to the standard complex structure.*

Thus the isotopy type of the pair $(\mathbf{C}P^2, E')$ is uniquely determined. Following the blowing-up procedure conversely, we get uniqueness of the isotopy type of the pair (Z, E_∞) if $Z_{min} = \mathbf{C}P^2$. In this case, the diffeomorphism type of X is realized by the complement of a tubular neighborhood of E_∞ in Z.

If $Z_{min} = \mathbf{C}P^1 \times \mathbf{C}P^1$, we blow up it at a point p on E'. Then we blow-down the proper transform of two symplectically embedded 2-spheres, passing through p, of self-intersection number 0 in Z_{min}, which represent the homology classes $[\mathbf{C}P^1 \times pt]$ and $[pt \times \mathbf{C}P^1]$. Then we get $\mathbf{C}P^2$ and E' becomes

[2] In complex geometry, minimal surfaces are the complex projective plane and Hirzebruch surfaces F_k with the degree $k \neq 1$. But F_k with k odd contains a symplectically embedded 2-sphere of self-intersection number -1, which can be symplectically blown-down, hence it is not minimal in symplectic category. F_k with k even is symplectomorphic to $\mathbf{C}P^1 \times \mathbf{C}P^1$ with a suitable split symplectic form.

a symplectically embedded 2-torus representing the class $3[\mathbf{C}P^1]$, whose isotopy type is uniquely determined by Sikorav's theorem again. Going back the blowing up and down procedure, we get uniqueness of the diffeomorphism type of the pair (Z_{min}, E'). Note that the complement of E' in this case has non-trivial torsion in the first homology group. On the other hand, the complement of the proper transform of the non-singular cubic curve in a one point blow-up of the complex projective plane has the vanishing first homology group. Hence, when $k = 8$, we have exactly two diffeomorphism types of minimal symplectic fillings with vanishing the first Chern class. We have finished the proof of Theorem 1. ∎

§4. Hartshorne's theorems and idea of proof of Theorem 2.

Our strategy of the proof of Theorem 2 is similar to the one in Section 3. Let X be a strong symplectic filling of the link of a simple elliptic singularity of degree $k > 0$. Denote by Z the symplectic 4-manifold obtained by gluing X and Y. Note that E_∞ is an embedded 2-torus with a positive self-intersection number k. The adjunction formula for E_∞ tells us that $c_1(Z)[E_\infty] = k > 0$. By the inflation argument in [L-M], we can get a symplectic form which is arbitrary close to the Poincaré dual of $[E_\infty]$. Applying Theorem 3.2 to Z with such a symplectic form, it turns out that Z is a rational or ruled symplectic 4-manifold up to blow-up and down. In this case, we can not exclude the possibility of a ruled symplectic 4-manifold, because we do not assume $c_1(X) = 0$ in Theorem 2. To describe the structure of the pair of Z and E_∞ in more details, we recall and explain Hartshorne's theorems [H] in algebraic geometry, because it gives a guide line of our proof. In fact, we will give the symplectic analogs of Hartshorne's theorems in Section 6. We remark that the results for the case $g = 1$ are due to Nagata [N].

For an algebraic varieties X_1 and X_2, we say that two embeddings $C \to X_1$ and $C \to X_2$ are *equivalent* if there is a birational map $\varphi : X_1 \to X_2$ which is an isomorphism on an open neighborhood of C in X_1, and induces the identity on C. Hartshorne proved the following.

Theorem 4.1.[H] *Let F be a rational surface and C a non-singular curve of genus $g \geq 0$ on F. Then either*

a) *$C \cong \mathbf{C}P^1$, and the embedding $C \to F$ is equivalent to a section of a Hirzebruch surface, or*

b) *$g = 1$ and the embedding $C \to F$ is equivalent to a non-singular cubic* *curve in* $\mathbf{C}P^2$*, or*

c) $C^2 \leq 4g+4$.

In the case that $g=1$ and $C^2>0$, Hartshorne's argument implies more information. Namely, we can see that the case a) does not happen and it is equivalent to one of the following:

b) a non-singular cubic in $\mathbf{C}P^2$,

c) the proper transform of an embedded elliptic curve of bidegree $(2,2)$ in $\mathbf{C}P^1 \times \mathbf{C}P^1$ under blowing up at some points on the elliptic curve.

The symplectic analog of this result is shown in Section 6 (Theorem 6.10) and the proof of Theorem 2 in Section 7.

On the other hand, a curve in a non-rational ruled surface is described as following.

Theorem 4.2.[H] *Let C be a non-singular curve of genus $g>0$ on a non-rational ruled surface F with the base B. Then either*

a) *$C \cong B$ and the embedding $C \to F$ is equivalent to a section of a ruled surface over B, or*

b) $C^2 \leq 4g-4$.

If we only consider an embedded elliptic curve of positive self-intersection number, the case b) is excluded.

These results work in birational category. In fact, we do not have notions of function field, birational map and fundamental points etc, directly in our symplectic setting. But we can establish symplectic analogs of these results. Most of the argument in [H] only uses intersection of curves and elementary transformations, which are also available in symplectic setting. E.g., the positivity of intersection for pseudo-holomorphic curves, symplectic blow-up and blow-down. Moreover, we should note that a basic ingredient, when we carry out the symplectic analogs below, is the flexibility of symplectic structures. In complex geometry, the biholomorphic type of the blown-up manifolds depend on the points, which are blown-up. In particular, the manifold obtained by successive blowing-ups may not be obtained by blowing-up simultaneously some points on the initial manifold. In contrast, the symplectic deformation type of the blown-up manifolds does not depend on the points, since the symplectomorphism group action is n-transitive for any positive integer n. We will explain this point in Section 6.1 in more detail.

Once we establish the analogs, the proof of Theorem 2 goes as follows. We introduce the notion of *s-equivalence* (see Definition 5.3) and classify minimal symplectic fillings up to *s-equivalence*. Then we investigate the situation carefully (Propositions 7.2, 7.3) and establish the classification up to diffeomorphisms. In the case that Z is rational, X is described exactly as in Theorem 1. If Z is not rational, then symplectic analog of Theorem 4.2 implies that X is *s-equivalent* to the complement of a section in a ruled symplectic 4-manifold over an elliptic curve, which is diffeomorphic to the disk bundle over a 2-torus. Hence we can classify the s-equivalent classes of the filling X. Finally we will prove Proposition 7.1 and Proposition 7.2 which make us obtain the classification of the diffeomorphism types of the filling X, in stead of the classification of the s-equivalent classes.

We will define and clarify the notion in symplectic category corresponding to the birational equivalence and prove the symplectic analogs of these results in the following two sections.

§5. Symplectic blow-up at infinity.

In this section, we explain how the complement of a symplectically embedded surface changes under the blow-up at a point on the surface. Let M be a symplectic 4-manifold and S a symplectically embedded surface in M. Choose a Darboux ball B centered at the point p on S. Here a Darboux ball means an embedded ball, which is symplectomorphic to a round ball in a symplectic vector space with respect to a compatible metric. We may assume that S is a "flat" symplectic surface in a neighborhood of B so that $S \cap B$ bounds a characteristic orbit on the boundary of the Darboux ball. The symplectic blow-up of M at p is obtained by gluing $M - B$ and a disk bundle over the complex projective line with the first Chern number -1 along their boundaries. We denote by $\widehat{M}$ the blow-up manifold. In this process, characteristic orbits on ∂B are identified with fibers of the unit circle bundle. Since the surface S is "flat" near ∂B, we can glue $S - B$ with a fiber disk to get a symplectically embedded surface S' in $\widehat{M}$. Certainly, we have $S \cdot S = S' \cdot S' + 1$. Let $U(S)$ and $U(S')$ be tubular neighborhoods of S and S' respectively. Note that $U(S')$ may not have concave boundary even though $U(S)$ has concave boundary. (If $U(S)$ has convex boundary, we may choose $U(S')$ having convex boundary.) We describe topological transition from $M - U(S)$ to $\widehat{M} - U(S')$ and call this operation as symplectic blow-up at infinity.

Let $W \cong D^2_b \times D^2_f$ be a trivialization of $U(S)$, as a disk bundle over S, around $p \in S$. Here the first factor D^2_b is the base disk and the second factor D^2_f is the fiber disk. Define $\varphi : D^2_b \times \partial D^2_f \to \partial D^2 \times D^2$ by $\varphi(z_1, z_2) = (z_2, z_2^{-1} z_1)$. Then the above description of symplectic blow-up yields the following:

Proposition 5.1. *$\widehat{M} - U(S')$ is diffeomorphic to the manifold $(M - U(S)) \cup_\varphi D^2 \times D^2$ with its corner smoothen.*

The converse operation is called symplectic blow-down at infinity. More precisely, let (M, ω) be a symplectic 4-manifold M with boundary. Suppose that ∂M contains a loop l, which is an integrable curve of the characteristic distribution $\ker \omega|_{\partial M}$ such that

(5.2.1) In a neighborhood U of l, integral curves of the characteristic distribution are described as fibers of a S^1-fibration $U \to D^2$.

(5.2.2) The loop l bounds a symplectically embedded disk D with self-intersection number -1 with respect to the trivialization of the normal bundle given by the fibers in (5.2.1).

Symplectic blow-down at infinity is topologically described as follows. Denote by $N(D)$ a tubular neighborhood of D in M. We may assume that the closure of $\partial N(D) - \partial M$ is a S^1-bundle P over D with the first Chern class -1 with respect to the trivialization on ∂D above. The blown-down manifold is obtained from $M - N(D)$ by gluing $D^3 \times S^1$ identifying P and $D^2_+ \times S^1$, where D^2_+ is a hemisphere in the boundary ∂D^3 of the first factor, so that the fibers of P are matched with the fibers of $D^2_+ \times S^1 \to D^2_+$.

Definition 5.3. Under the situation above, if $M_1 - U(S_1)$ is obtained from $M_2 - U(S_2)$ by a sequence of blow-up and blow-down at interior points or infinity while keeping convex boundary condition, then they are said to be *s-equivalent*. We call these operations *"symplectic birational transformations"*. For two symplectic embeddings $S \to M_1$ and $S \to M_2$ in symplectic 4-manifolds, we say they are *s-equivalent* if there is a composition of "symplectic birational transformations" $\varphi : M_1 \to M_2$ which is an isomorphism on S.

§6. Symplectic analogs of Hartshorne's theorems.

In this section, we give the symplectic analogs of Hartshorne's theorems and the proofs for the sake of completeness.

In [H], a curve is not assumed to be an immersion of a non-singular Riemann surface. In this section, however, we assume that a symplectic surface is the image of a pseudo-holomorphic immersion with respect to some almost complex structure. In the course of the proof of Theorem 2, we apply results in this section to the image of the *compactifying divisor* E_∞ under the blowing down operation. In such a situation, the condition above is guaranteed by choosing a compatible almost complex structure on Z generically so that exceptional (-1)-curves intersects the (image of) compactifying divisor transversally. In this situation, we can find that near the intersection point, the image of the compactifying divisor is "flat" in the sense of Section 5. We can also assume that each exceptional (-1)-curve does not pass through any multiple points of the image of the compactifying divisor, which is caused by some previous blow-down operation. The local structure of singular points of pseudo-holomorphic curves are fairly understood in [M2] and the multiplicity at a singular point also makes sense. In our simple situation, it is just the number of local branches. The adjunction formula and positivity of intersection for pseudo-holomorphic curves are also established by McDuff. They are principal tools in the argument below.

6.1. Elementary transformations.

First of all, we review elementary transformations used in [H] and note that all of them have symplectic analogs as follows. A basic operation is the blow-up at p denoted by $\mathbf{dil}_p$. We regard $\mathbf{dil}_p$ as an operation on symplectic deformation types. Its converse operation is the blow-down of an exceptional curve E denoted by $\mathbf{cont}_E$. Other three operations are defined for ruled surfaces. Let $F = \mathbf{P}(\mathcal{E})$ be a ruled surface over a Riemann surface C. Denote by D a section and by f a fiber. In the case that $C = \mathbf{C}P^1$, they are one of Hirzebruch surfaces $F_e = \mathbf{P}(\mathcal{O}(-e) \oplus \underline{\mathbf{C}})$ with $e \geq 0$. We take D as a section with self-intersection number e and denote by D_e the zero section with self-intersection number $-e$. The operation $\mathbf{elm}_p : F \to F$ is defined as the composition $\mathbf{cont}_{\hat{f}_p} \circ \mathbf{dil}_p$, where f_p is the fiber, through p, of the ruled surface and $\hat{f}_p$ is its proper transform under $\mathbf{dil}_p$. For the rational surface F_0, the operation $\mathbf{refl}$ is the operation exchanging two factors ($F_0 = \mathbf{C}P^1 \times \mathbf{C}P^1$). For the rational surface F_1, the operation $\mathbf{int}_p$, for $p \notin D_1$, is defined by $\mathbf{cont}_{D_1} \circ \mathbf{dil}_p$. There are analogs of blow-up and blow-down in symplectic category. Hence these operations except $\mathbf{refl}$ makes sense in symplectic category. The operation $\mathbf{refl}$ just exchanges two factors and it is also realized as symplectic operation when we allow changing symplectic structure. As in the case of $\mathbf{dil}_p$, we also regard $\mathbf{refl}$ as an operation on symplectic deformation types.

In Section 4, we note that the symplectic deformation type after successive blowing-up does not depend on the points, which are blown-up. From an opposite viewpoint, the symplectic deformation type after successive blowing-down can be obtained by blowing-down some disjoint symplectic (-1)-curves simultaneously. During the arguemnt in 6.3 and 6.4, we adopt simultaneous blowing down operations (blowing down of disjoint (-1)-curves). Some statements are proven by induction and we regard the simultaneous resolution as iterative process of blowing downs of disjoint symplectic (-1)-curves. We also make these disjoint symplectic (-1)-curves are pseudo-holomorphic with respect to a compatible almost complex structure. In our later purpose, we have to keep track of the compactifying divisor E_∞ under blowing-down. To do this, we choose a compatible almost complex structure, mentioned above, such that E_∞ is also pseudo-holomorphic, and the pseudo-holomorphic representatives of the symplectic (-1)-curves intersect E_∞ transversally. This is possible because the self intersection number of the symplectically embedded 2-torus E_∞ is positive, hence the genericity argument works.

6.2. Preliminary discussion on symplectic surfaces in ruled symplectic 4-manifolds.

Let $F \to C$ be a ruled symplectic 4-manifold, D a symplectic section with self-intersection number $e \geq 0$, and f a fiber, which is also symplectically embedded. Denote by Y a symplectically immersed surface represented by a pseudo-holomorphic curve with respect to some almost complex structure compatible with the ruled symplectic structure. The homology class of Y equals to $mD + nf$ for some integers m and n. Then we have

$$Y \cdot Y = m^2 e + 2mn, \tag{6.1}$$

$$2p_a(Y) - 2 = m^2 e + 2mn - me - 2n + m(2\gamma - 2), \tag{6.2}$$

where γ is the genus of the base C and $p_a(Y)$ is the virtual genus (cf. [M1,2]). Because of positivity of intersection of pseudo-holomorphic curves, we have

(6.3.1) m must be non-negative,
and
(6.3.2) when $m = 0$, $n > 0$.

If F is a rational ruled symplectic 4-manifold, i.e., $C = \mathbf{C}P^1$, we put $D_e = D - ef$. There are only two symplectic deformation types of rational ruled symplectic 4-manifolds, i.e., $F_0 = \mathbf{C}P^1 \times \mathbf{C}P^1$ and F_1. The homology class D_1 is represented by a pseudo-holomorphic sphere, which is unique when fixing an almost complex structure.

Let M be a symplectic 4-manifold and S the image of a pseudo-holomorphic immersion such that, near singular points, any two branches intersect transversally. Let p be a singular point (or a multiple point) of S. Denote by r the multiplicity of S at p, i.e., the number of local branches passing through p. After the blow-up $\mathbf{dil}_p$ at p, we have

$$\tilde{S} \cdot \tilde{S} = S \cdot S - r^2, \tag{6.4}$$

and

$$p_a(\tilde{S}) = p_a(S) - \frac{1}{2}r(r-1), \tag{6.5}$$

where $\tilde{S}$ is the proper transform of S.

6.3. Symplectic surfaces on irrational ruled symplectic 4-manifolds.

Let $\widehat{F}$ be a symplectic 4-manifold obtained by successive blown-up of points of an irrational ruled symplectic 4-manifold F over a Riemann surface C of genus $\gamma \geq 0$. Forgetting the symplectic structure, we have a C^∞-mapping $h : \widehat{F} \to F$. Composing h with the projection $\pi : F \to C$, we get a C^∞ mapping $\pi \circ h : \widehat{F} \to C$.

Theorem 6.6. *Let S be a symplectically immersed surface, which is represented by a pseudo-holomorphic immersion of a Riemann surface, in $\widehat{F}$ and m the degree of $\pi \circ h|_S : S \to C$. If $m > 1$, then we have*

$$S \cdot S \leq \frac{2m}{m-1}(p_a(S) - 1).$$

Proof. Firstly, we consider the case that $\widehat{F}$ is minimal, i.e., it does not contain any symplectic (-1)-curve. Note that $p_a(S)$ is not less than the genus of the domain of the immersion. Hence, if $p_a(S) \leq 1$, the condition that $m > 1$ excludes the case that $p_a(S) = 0$ and implies that $p_a(S) = 1 = \gamma$. Then the same computation as in the proof of Theorem (2.3) in [H] works and we get the conclusion. When $p_a(S) > 1$, the proof is similar to that in [H].

General case is proven by induction on the number of symplectic (-1)-curves blown-down to a minimal ruled symplectic 4-manifold. Note that the positivity of intersection applied to S and a fiber implies that the multiplicity of p on S is bounded by m. Then the computation in [H], using (6.4) and (6,5), derives our conclusion. ∎

Corollary 6.7. *Let S be a symplectically embedded surface of genus $g > 0$ on a symplectic 4-manifold M obtained by successive blowing-up of an irrational ruled symplectic 4-manifold over a Riemann surface C. Then either*

a) *$S \cong C$, and the embedding $S \to M$ is s-equivalent to a section of an irrational ruled symplectic 4-manifold, or*

b) *$S \cdot S \leq 4g - 4$.*

Proof. The proof of Corollary (2.4) in [H] works after replacing transformations in birational geometry by symplectic operation. Let F be an irrational ruled symplectic 4-manifold obtained from M by blowing-down disjoint symplectic (-1)-curves in M. The integer m above equals the intersection number of S and a fiber in F away from blow-up and down locus. If $m = 0$ or $m > 1$, the computation in [H] works and we get our conclusion b). If $m = 1$, the image $\overline{S}$ of S in F has intersection number 1 with the fiber class $[f]$. Therefore $\overline{S}$ is considered as a cross section with respect to the ruled structure $F \to C$. We always have $\overline{S} \cdot \overline{S} \geq S \cdot S$. If $\overline{S} \cdot \overline{S} > S \cdot S$, we apply the symplectic version of $\mathbf{elm}_p$ for $p \in \overline{S}$, to which a certain symplectic (-1)-curve is contracted. Note that the composition of the original blowing-down operation with $\mathbf{elm}_p$ is also an operation of blowing-down disjoint symplectic (-1)-curves. Then the self-intersection number $\overline{S} \cdot \overline{S}$ decreases by 1. After finitely many operations, we may arrive at $\overline{S} \cdot \overline{S} = S \cdot S$. In other words, the normal bundles of S and $\overline{S}$ are isomorphic. This finishes the proof of the corollary. ∎

6.4. Symplectic surfaces in rational ruled symplectic 4-manifolds.

We start an analog of Proposition (3.1) in [H]. If the ambient rational symplectic 4-manifold is $\mathbf{C}P^2$, the adjunction formula gives a relation between the genus of a symplectically embedded surface and the degree of the surface, i.e., the homology class (see the proof of Theorem 6.10 below). In our application, we consider a 2-torus, which appears only in the case of degree 3. In this case, Sikorav's theorem (Theorem 3.4) tells us the isotopy class of the embedding.

Proposition 6.8. *Let M be a rational symplectic 4-manifold and S a symplectically embedded surface in M. Suppose that M is not $\mathbf{C}P^2$. After blowing-down disjoint symplectic (-1)-curves, we arrive at a rational ruled symplectic manifold F_e and S' the image of S with $S' = mD + nf$ such that*

a) *every multiple point of S' has multiplicity $\leq m/2$;*
b) *if $e = 0$, then $m \leq n$;*
c) *if $e = 1$, then every multiple point of S' has multiplicity $\leq n$;*
d) *if $m = 1$, then $S \cdot S = S' \cdot S'$;*
e) *if $e = 1$, $m = 2$, and $n = 0$, then $S \cdot S = S' \cdot S'$.*

Proof. We consider all possible symplectic operations obtained by blowing-down disjoint symplectic (-1)-curves applied to M such that S is not transformed to a point. Among these operations, we choose one which first minimizes m, and secondly minimizes $S' \cdot S'$. We denote this operation by T. This is possible, since m is always non-negative (see (6.3)) and the self-intersection number does not decrease under blow-down operation.

a) If S' has a multiple point p of multiplicity $r > m/2$, we apply the symplectic operation $\mathbf{elm}_p$. Since the original S is non-singular, the multiple point is made under some blow-down process, which is converse to $\mathbf{dil}_p$. Composing $\mathbf{elm}_p$ with T, we get another operation T' obtained by blowing-down disjoint (-1)-curves. Exactly same as [H], the image $\overline{S}$ of S under the operation T' has a new multiple point of multiplicity $r' = m - r$ instead. (Here $m - r$ is the sum of intersection indices of S' and the fiber f_p through p without counting the index at p). Since m equals the intersection number of S' with a general fiber f, the number m', which is similarly defined for $\overline{S}$, equals m. We also have $\overline{S} \cdot \overline{S} = S' \cdot S' - r^2 + r'^2 < S' \cdot S'$, which contradicts the minimality of $S' \cdot S'$.

b) The proof of Proposition (3.1) b) in [H] also works here.

c) If p is a multiple point of S' of multiplicity r. If $p \in D_1 = D - f$, the argument in [H] works. Suppose that $p \notin D_1$. Since the original S is non-singular, the multiple point p is made by some blow-down operation, which is converse to $\mathbf{dil}_p$. Composing $\mathbf{int}_p$ with T, we get another blowing-down operation, denoted by T'. The rest of the argument is similar to the one in [H].

d) If $m = 1$ and $S' \cdot S' < S \cdot S$, S is effected by some blow-down operation.

So forget the last blow-down operation, which effects S, i.e., changes the self-intersection number of S, and apply $\mathbf{cont}_{\hat{f}}$. This is the symplectic analog of $\mathbf{elm}_p$. The rest of the argument is similar to the one in [H].

e) After modifying the argument in [H] as in d) above, we get a conclusion.

∎

The following proposition is proven in the same way as in Proposition (3.2) in [H].

Proposition 6.9. *Let S be a symplectically immersed surface, which is represented by a pseudo-holomorphical immersion, on a rational symplectic 4-manifold M, and suppose that after blowing-down disjoint symplectic (-1)-curves and deforming the symplectic structure M becomes a rational ruled symplectic 4-manifold F_e, $e = 0, 1$, and satisfies the properties of a), b), c), d), e) of Proposition 6.8. Then*

(i) *if $m = 2$, then S is non-singular and $S \cdot S \leq 4g + 4$;*

(ii) *if $m = 3$, then S is non-singular and $S \cdot S \leq 3g + 6$;*

(iii) *if $m \geq 4$ and either $e \neq 1$, or $e = 1$ and $n \geq m/2$, then*

$$S \cdot S \leq \frac{2m}{m-2}(p_a(S) - 1);$$

(iv) *if $m \geq 4$ and $e = 1$ and $n < m/2$, then*

$$S \cdot S < \frac{2j}{j-3}(p_a(S) - 1),$$

where $j = m + n$.

The proof of the following theorem is almost identical with the proof of Theorem (3.5) in [H], but we give it for deriving more precise conclusion in the case of $g = 1$. See Section 7.

Theorem 6.10. *Let S be a symplectically embedded surface of genus $g \geq 0$ in a rational symplectic 4-manifold M. Then either,*

A) *$S \cong \mathbf{C}P^1$, and the embedding is s-equivalent to a section of a rational ruled symplectic 4-manifold $F_e \to \mathbf{C}P^1$, or*

B) *$g = 1$, and the embedding is s-equivalent to a non-singular cubic complex curve in $\mathbf{C}P^2$, in which case $S \cdot S = 9$, or*

C) *$S \cdot S \leq 4g + 4$.*

Proof. If $M = \mathbf{C}P^2$, the adjunction formula for an embedded pseudo-holomorphic curve tells us $S \cdot S = 4g + 4 - (m-2)(m-4)$, where m is the degree of S, which is non-negative.

If $m \neq 3$, it implies $S \cdot S \leq 4g+4$. If $m = 3$, Theorem 3.4, due to Sikorav, implies the assertion B).

Suppose that $M \neq \mathbf{C}P^2$. From Proposition 6.8, after the blow-down operation T, we arrive at the situation satisfying the conditions a), b), c), d), e) in the proposition.

If $m = 0$, we get $S \cdot S \leq 0$, so C) holds.

If $m = 1$, then d) implies that the self intersection number does not change under the operation T. Hence their tubular neighborhoods are isomorphic and T gives an s-equivalence between the embedding of S to M and a cross section of a rational ruled symplectic 4-manifold.

If $m = 2$, (i) in Proposition 6.9 implies that $S \cdot S \leq 4g + 4$.

If $m = 3$, (ii) in Proposition 6.9 implies that $S \cdot S \leq 3g + 6$, which implies the condition C) unless $g = 0, 1$. The case that $g = 0$ is excluded by the same reason in [H]: in this case, the formula (6.2) shows $6e + 4n = 4$, so $e = 0$, $n = 1$, which is impossible by b) in Proposition 6.8. If $g = 1$, we have $6e + 4n = 6$, so $e = 1$, $n = 0$ and $S' \cdot S' = 9$. Since we have $S' \cdot D_1 = S' \cdot (D - f) = 0$ in this case, we can blow-down the symplectic (-1)-curve D_1 without changing a neighborhood of S'. So the neighborhood of S' is isomorphic to a neighborhood of a non-singular cubic complex curve in $\mathbf{C}P^2$. Note that $S \cdot S \leq S' \cdot S'$. Hence if $S \cdot S = 9$, we have the case B). If $S \cdot S < S' \cdot S'$, C) holds.

If $m \geq 4$ and the case (iii) applies, we get $S \cdot S \leq 4g - 4$, hence C) holds.

If $m \geq 4$ and the case (iv) applies, we have $S \cdot S \leq 2j(g-1)/(j-3)$. Hence, if $j \geq 6$ in addition, C) holds. Recall that $j = m + n$. Hence the remaining cases are $j = 4, 5$, which implies that $n = 0, 1$. Every multiple point has the multiplicity at most n by c) in Proposition 6.8. This implies that S' is non-singular. If $j = 4$, i.e., $m = 4$ and $n = 0$. The adjunction formula and computation of the self-intersection number imply that $g = 3$ and $S' \cdot S' = 16$. Hence $S \cdot S \leq S' \cdot S' = 4g+4$. If $j = 5$, either $m = 4$, $n = 1$, $g = 6$, $S' \cdot S' = 24$, or $m = 5$, $n = 0$, $g = 6$, $S' \cdot S' = 25$. In either case, we have $S \cdot S \leq S' \cdot S' < 4g + 4$. ∎

§7. Proof of Theorem 2.

Let X be a strong symplectic filling of the link of a simple elliptic singularity of degree $k > 0$. As in Section 2, we write $Y = \mathbf{P}(L \oplus \underline{\mathbf{C}}) \setminus D(L)$, which contains the section at infinity E_∞ and we have the compactification $Z = X \cup Y$ of X. Note that the symplectically embedded 2-torus E_∞ in Z has a positive self-intersection number k. The adjunction formula for E_∞ tells us that $c_1(Z)[E_\infty] = k > 0$. By the inflation argument, we can get a symplectic form which is arbitrary close to the Poincaré dual of $[E_\infty]$. Applying Theorem 3.2 to Z with such a symplectic form, it turns out that Z is a rational or ruled symplectic 4-manifold up to blow-up and down. Firstly, we show a weaker statement than Theorem 2. Namely, we will show that a minimal strong symplectic filling X is s-equivalent to either minimal resolution or a smoothing.

Case (1). $k \geq 10$.

In this case $E_\infty \cdot E_\infty \geq 10$. Suppose that Z is obtained by successive blowing-up of the complex projective plane or a rational ruled symplectic 4-manifold Z'. We denote by $E'_\infty \subset Z'$ the image of E_∞ of the blow down operations. By Theorem 6.10 only the case A) would occur. But in this case the genus of E'_∞ is zero, which is less than that of E_∞. This is contradiction. Therefore Z is obtained by successive blowing-up of an irrational ruled symplectic 4-manifold. Since $E_\infty \cdot E_\infty > 0$, Corollary 6.7 implies that E_∞ is s-equivalent to a section of the irrational ruled surface of degree $-k \leq -10$ over a 2-torus. Thus the minimal symplectic filling X is s-equivalent to the minimal resolution.

Case (2). $k = 9$.

When Z is obtained by successive blowing-up of the complex projective plane or a rational ruled symplectic 4-manifold, this is just the case B) in Theorem 6.10.

When Z is obtained by successive blowing-up of an irrational ruled symplectic 4-manifold, E_∞ is s-equivalent to a section of the irrational ruled symplectic 4-manifold of degree -9 over a 2-torus by Corollary 6.7. Then the minimal symplectic filling X is s-equivalent to the minimal resolution. Thus we have two kinds of the symplectic fillings for this case depending on Z.

Case (3). $0 < k \leq 8$.

When Z is obtained by successive blowing-up of an irrational ruled symplectic 4-manifold, Corollary 6.7 again tells us that the minimal symplectic filling X is s-equivalent to the minimal resolution.

When Z is $\mathbf{C}P^2$, by the proof of Theorem 6.10, E_∞ must be a non-singular cubic curve, whose self intersection number is 9. So this case never happens.

When Z is obtained by successive blowing-up of a rational ruled symplectic 4-manifold, the case A) in Theorem 6.10 does not occur because of the same reason as in Case (1). By Proposition 6.9 (iii) and (iv), if $m \geq 4$, we have $E_\infty \cdot E_\infty \leq 0$ which contradicts our assumption. Thus $m \leq 3$. When $m = 0$, $E_\infty \cdot E_\infty \leq 0$. So this is not our case. When $m = 1$, by the proof of Theorem 6.10, E_∞ is s-equivalent to a section of a rational ruled symplectic 4-manifold. Thus this does not happen. When $m = 3$, by the proof of Theorem 6.10 this is the case of the non-singular cubic curve. But we are considering the case $E_\infty \cdot E_\infty \leq 8$, so this is not the case.

Thus, the remaining cases which we should classify are $m = 2$ with $0 < E_\infty \cdot E_\infty \leq 8$. We denote by E'_∞ the image of E_∞ after symplectic blowing-down disjoint symplectic (-1)-curves in Z to a rational ruled symplectic 4-manifold satisfying Proposition 6.8. Note that the property a) in Proposition 6.8 implies that E'_∞ is non-singular and has the same genus as E_∞. Let $e = 0$, i.e., $F_0 = \mathbf{C}P^1 \times \mathbf{C}P^1$ and $[E'_\infty] = 2[D] + n[f]$. The adjunction formula for embedded curves implies that the genus of E'_∞ is $n - 1$. The genus of E'_∞, hence of E_∞, equals 1, if and only if $n = 2$. This is the case discussed in Section 3 and they are blow-up, of $\mathbf{C}P^1 \times \mathbf{C}P^1$, at l, $0 \leq l < 8$, points on the symplectically embedded 2-torus and some points outside of it. As we note, after the blow-up operation at a point on the 2-torus, we can blow-down two symplectic (-1)-curves, which are proper transforms of rulings passing through the point, and get $\mathbf{C}P^2$ with a cubic curve. Let $e = 1$, i.e., F_1 is the one-point blow-up of $\mathbf{C}P^2$ and $[E'_\infty] = 2[D] + n[f]$. The adjunction formula for embedded curves implies that the genus of E'_∞ is n. In this case, the intersection number of E'_∞ and the symplectic exceptional (-1)-curve D_1, which is homologous to $D - f$, is 1. Hence it is the proper transform of a non-singular complex cubic curve by the blow-up at a point on the cubic.

Therefore there appear the following cases only.

1) a non-singular complex cubic curve in $\mathbf{C}P^2$ and its blow-up at l-points, $1 \leq l < 9$, on the cubic curve and some points outside of it,

2) a symplectically embedded 2-torus representing the class $(2, 2)$ in $\mathbf{C}P^1 \times \mathbf{C}P^1$ and its blow-up at some points outside of the 2-torus,

3) a section of an irrational ruled symplectic 4-manifold with positive self-intersection number.

Up to now, we classified the s-equivalent classes of minimal strong symplectic fillings. To finish the proof of Theorem 2, we need to replace "s-equivalence" by "diffeomorphism". This will be done in Propositions 7.1 and 7.2 below. First of all, we note that any composite of elementary transformations can be described as the composite of a simultaneous blowing-up and a simultaneous blowing-down and a diffeomorphism preserving the symplectic deformation type. This fact follows from two observations. One is that the operation $\mathbf{dil}_p \circ \mathbf{cont}_E$ is also written as the operation $\mathbf{cont}_{E'} \circ \mathbf{dil}_{p'}$ for some p' and E'. A similar fact is well-known in algebraic geometry (see Theorem 4 in Chapter IV in [Sh]), but the situation is much simpler in symplectic geometry. Because we can slightly change the position of p by a symplectic isotopy so that p is not contained in E. The other is, as we noticed in §6.1, a successive blowing-up can be considered as a simultaneous blowing-up in symplectic setting. It is also true that a successive blowing-down can be considered as a simultaneous blowing-down.

Proposition 7.1. *If the compactification Z of the minimal strong filling X is a rational symplectic 4-manifold, then we have $c_1(X) = 0$. Therefore the diffeomorphism type of X is described by Theorem 1.*

Proof. We have shown that the pair (Z, E_∞) is s-equivalent to the pair of a rational symplectic 4-manifold Z' and a symplectically embedded torus E' , which represents the anti canonical class. Therefore there exists a symplectic 4-manifold $\widehat{Z}$, which can be commonly obtained by blowing-up from Z as well as Z'. Moreover, the symplectic submanifolds E_∞ and E' are in correspondence under these blowing-up and down processes. We denote by $\widehat{E}$ the proper transform of E_∞, hence also of E'.

$$\begin{tikzcd}
 & (\widehat{Z}, \widehat{E}) \arrow[dl] \arrow[dr] & \\
(Z, E_\infty) & & (Z', E')
\end{tikzcd}$$

Figure 7.1

To get $\widehat{Z}$ from Z', we blow-up k points on E' and l points outside of E'. We denote by $\{e_i\}$ the set of symplectic (-1)-curves corresponding to the points outside of E'. Since E' is the anti canonical class of Z', the anti canonical class of $\widehat{Z}$ is represented by $\widehat{E} - \sum e_i$. To get Z from $\widehat{Z}$, we blow-down a number of disjoint symplectic (-1)-curves $\{\epsilon_k\}$ simultaneously. Since Z is assumed to be minimal, any e_i must intersect at least one of ϵ_j. Suppose that $\epsilon_j \neq e_i$. Since e_i and $e_{i'}$ are disjoint, whenever $i \neq i'$, ϵ_j does not belong to $\{e_i\}$. Therefore the intersection number of ϵ_j and $\sum e_i$ is at least 1. On the other hand, since ϵ_j is a symplectic (-1)-curve, $c_1(\widehat{Z})[\epsilon_j] = 1$, which implies that $\widehat{E} \cdot \epsilon_j \geq 2$. Hence after blowing-down ϵ_j, the image of $\widehat{E}$, i.e., E_∞ is no longer symplectically embedded. This contradicts to the fact that E_∞ is symplectically embedded. Therefore, $\epsilon_j = e_i$. This argument tells us that ϵ_k is either one of e_i or disjoint from any e_i. In particular, all the e_i are blown-down in Z. The adjunction formula again implies that other ϵ_k must intersects $\widehat{E}$ exactly at one point transversally. Hence E_∞ is the anti-canonical class of Z. Hence we have $c_1(X) = 0$. ∎

Proposition 7.2. *If Z is an irrational ruled symplectic 4-manifold, the pair (Z, E_∞) is diffeomorphic to the pair of Z' and a section with self intersection number k. Hence in this case, X is diffeomorphic to the minimal resolution.*

Proof. We have shown that (Z, E_∞) is s-equivalent to the pair of an irrational ruled symplectic 4-manifold Z' and a section E' with the self intersection number k. Then we have a symplectic 4-manifold $\widehat{Z}$, which is commonly obtained by blowing-up from Z as well as Z'. Moreover, E_∞ and E' are in correspondence under these blowing-up and down processes. To get $\widehat{Z}$ from Z, we blow-up k_1 points on E_∞ and k_2 points outside of E_∞. Similarly, we blow-up l_1-points on E' and l_2 points outside of E' to get $\widehat{Z}$. Since Z is an irrational ruled symplectic 4-manifold, any symplectic (-1)-curve in $\widehat{Z}$ can be isotoped into the total transform of some fiber of the ruling structure. Note that the self intersection number of E_∞ and E' are equal from the definition of s-equivalence. Therefore $k_1 = l_1$. Since the second Betti numbers of Z and Z' are 2. Their sections have the same intersection number, in particular, the same parity. Hence Z and Z' are diffeomorphic. If we identify them by a diffeomorphism, E_∞ and E' are sections with the same intersection numbers. This implies that they are isotopic through sections, which implies the conclusion. ∎

Therefore, combing the classification of s-equivalence classes of the minimal strong symplectic fillings with Proposition 7.1, Proposition 7.2 and Theorem 1, we have obtained the following.

Theorem 7.3. *Let X be a minimal strong symplectic filling of the link of a simple elliptic singularity of degree $k > 0$. Then we have the following:*

(1) *When $k \geq 10$, X is diffeomorphic to the minimal resolution.*

(2) *When $k \leq 9$, X is diffeomorphic to either its smoothing or the minimal resolution. More precisely, when $k \neq 8$, X is diffeomorphic to either its smoothing or the minimal resolution. In this case, the diffeomorphism type of smoothings is unique and same as the complement of a holomorphically embedded 2-torus in $(9-k)$-point blow up of the projective plane $\mathbf{C}P^2\#\overline{\mathbf{C}P^2}$ which represents the homology class $3[\mathbf{C}P^1] - \sum_{i=1}^{9-k} e_i$ (where $\{e_i\}$ is the set of exceptional divisors). When $k = 8$, the simple elliptic singularity of degree 8 has two diffeomorphism type of smoothings, which are described in (i) and (ii) below. In this case, X is diffeomorphic to one of (i) the complement of a holomorphically embedded 2-torus in $\mathbf{C}P^1 \times \mathbf{C}P^1$ which represents the homology class $2([\mathbf{C}P^1 \times pt] + [pt \times \mathbf{C}P^1])$, (ii) the complement of a holomorphically embedded 2-torus in one point blow up of the projective plane $\mathbf{C}P^2\#\overline{\mathbf{C}P^2}$ which represents the homology class $3[\mathbf{C}P^1] - e$ (where e is the exceptional divisor), or (iii) the minimal resolution. Furthermore we note that (i) and (ii) are s-equivalent.*

§8. Remarks on higher genus cases.

Our proofs of Theorem 1 and Theorem 2 can be divided into two steps. The first step is the description of the compactification Z of the symplectic filling X, and the second one is the description of the pair of the compactification and compactified divisor E_∞ and the symplectic isotopy problem of the pair. In the higher genus cases, the isotopy problem is more subtle and we cannot expect uniqueness result in general. But we have some results on the pair (Z, E_∞) in Section 6 for higher genus cases as well, provided that Z is a rational or ruled symplectic 4-manifold. If Z is an irrational ruled symplectic 4-manifold and E_∞ is a section, then the diffeomorphism type of the pair (Z, E_∞) is uniquely determined by the diffeomorphism type of the ambient space Z and the self-intersection number of E_∞. In this section, we like to discuss with what we can show for the higher genus cases at this stage.

Let $L_{-k,g} \to \Sigma_g$ be a complex hermitian line bundle of degree $-k < 0$ over a Riemann surface Σ_g of genus $g \geq 2$. We denote by $D(L_{-k,g}) \to \Sigma_g$ the unit disk bundle of $L_{-k,g}$ and by $S(L_{-k,g}) \to \Sigma_g$ the unit circle bundle. Then as in Section 2, the horizontal distribution ξ on $S(L_{-k,g})$ defines a contact structure and $D(L_{-k,g})$ gives a strong symplectic filling of $(S(L_{-k,g}), \xi)$. Take the projective bundle $\mathbf{P}(L_{-k,g} \oplus \underline{\mathbf{C}}) \to \Sigma_g$. Denote by Y the complement of $D(L_{-k,g})$ in $\mathbf{P}(L_{-k.g} \oplus \underline{\mathbf{C}})$ and by E_∞ the section at infinity, the compactifying divisor.

Now let X be a strong symplectic filling of $(S(L_{-k,g}), \xi)$. Then as in Section 2, we can glue X and Y to obtain a closed symplectic 4-manifold Z, which is our compactification of X. The compactified divisor E_∞ is symplectically embedded in Z with the self-intersection number $k > 0$. Then we can show the following.

Proposition 8.1. *Under the situation above, if $k > 2g - 2$, then the compactification Z of the strong symplectic filling X of $(S(L_{-k,g}), \xi)$ is diffeomorphic to a rational or ruled symplectic 4-manifold up to blow-up and down.*

Proof. If $k > 2g - 2$, the adjunction formula implies that $c_1(Z)[E_\infty] > 0$. Then by the inflation argument similar to Section 4, we can find that there exists a symplectic structure ω on Z such that $c_1(Z)[\omega] > 0$. Thanks to Theorem 3.2 again, we get the conclusion. ∎

Proposition 8.2. *If $k > 4g + 4$, then X is diffeomorphic to the complement of a section of an irrational ruled symplectic 4-manifold over Σ_g.*

Proof. By our assumption, Theorem 6.10 implies that Z is not rational, but an irrational ruled symplectic 4-manifold. Since $E_\infty \cdot E_\infty = k > 4g + 4 > 4g - 4$, Corollary 6.7 and Proposition 7.2 imply the conclusion. ∎

Remark 8.3. Hartshorne pointed out in [H] that the bound in Corollary 6.7 (b) are best possible for curves on a ruled surface of genus one, but we could get better bounds for higher genus cases.

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