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Simple Finite-Dimensional Modular Noncommutative Jordan Superalgebras of degree > 1

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We classify the central simple finite-dimensional noncommutative Jordan superalgebras of degree > 1 over an algebraically closed field of characteristic $p > 2$. The case of characteristic 0 was considered by the authors in the previous paper [24]. In particular, we describe Leibniz brackets on all finite dimensional central simple Jordan superalgebras except mixed (nor vector neither Poisson) Kantor doubles of the supercommutative superalgebra $B(m, n)$.

Key Words: *noncommutative Jordan superalgebra, Jordan superalgebra, Leibniz bracket, Kantor double, simple superalgebra, Grassmann superalgebra*

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1 Introduction

The problem of classification of the simple finite-dimensional noncommutative Jordan superalgebras was posted in [6, Problem 3.100 a)] (see [26] as well). The present article continues our previous articles [23]–[24], where we classified the central simple finite-dimensional noncommutative Jordan superalgebras of characteristic 0. Now we consider the remaining modular case.

The class of noncommutative Jordan algebras is extremely extensive: it includes alternative, Jordan, quasi-associative, quadratic flexible, and anticommutative algebras.

The classification of finite dimensional simple noncommutative Jordan algebras of characteristic 0 was done by R. Schafer [27], who proved that these algebras are either commutative

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Jordan algebras or quasi-associative algebras or flexible algebras of degree 2. R. Oehmke [21] extended the Schafer classification to the case of flexible strictly power-associative algebras of characteristic $\neq 2, 3$; K. McCrimmon [18, 19] did it for noncommutative Jordan algebras of degree > 2 and of characteristic $\neq 2$, and K. Smith [31] described such algebras of degree two. The case of nodal simple algebras of positive characteristic was mainly considered by L. Kokoris [15, 16], and the case of infinite-dimensional noncommutative Jordan algebras was studied in the papers by I. P. Shestakov [28] and V. G. Skosyrskiy [30].

The finite-dimensional simple Jordan superalgebras over algebraically closed fields of characteristic zero were classified by V. Kac [10] and I. Kantor [11]. The study of Jordan superalgebras of positive characteristics was initiated by I. Kaplansky [13]. M. Racine and E. Zelmanov [25] classified the finite-dimensional simple Jordan superalgebras of characteristics $\neq 2$ with semisimple even part; C. Martinez and E. Zelmanov considered the case when the even part is not semisimple but unital [17], and E. Zelmanov did it for the remaining nonunital case [32].

In the first section of our paper we give some known results on the noncommutative Jordan superalgebras needed further. In particular, we present their defining identities, the Peirce decomposition, the coordinatization theorem, and some facts about such simple superalgebras.

In the second section we reduce the classification problem to the problem of description of Leibniz brackets on the simple Jordan superalgebras with $n < 3$ connected orthogonal idempotents and on $B(m, n)$. To this end we prove an analog of Oehmke's theorem.

In sections 3 – 5 we investigate the required cases of simple Jordan superalgebras. In particular, we describe Leibniz brackets on central simple finite-dimensional Jordan superalgebras (up to Leibniz brackets on $Kan(B(m, n))$ of mixed type).

In what follows, U denotes a noncommutative Jordan superalgebra (not necessarily finite-dimensional) over F, where F always stands for a ground field of characteristic $\neq 2$. The symbol $:=$ denotes an equality by definition, $(x, y, z) := (xy)z - x(yz)$, and $\langle Y\rangle := \langle Y\rangle_F$ is the linear span of a set Y over F.

2 Preliminaries

Let $U = U_{\bar 0} \oplus U_{\bar 1}$ be a superalgebra, $(-1)^{xy} := (-1)^{p(x)p(y)}$, where $p(x)$ is the parity of x, that is, $p(x) = i$ if $x \in U_{\bar i}$. In what follows, if the parity of an element appears in a formula then this element is assumed to be homogeneous; the idempotents are assumed to be even. Denote by L_x and R_x the left and right multiplication operators by $x \in U$: $yR_x := yx,\ yL_x := (-1)^{xy}xy$; $[x, y] := xy - (-1)^{xy}yx$, $x \circ y := \frac{1}{2}(xy + (-1)^{xy}yx)$.

A superalgebra U is called a *noncommutative Jordan superalgebra* (or *NJ-superalgebra* for short) provided that the operator identities

$$[R_{x\circ y}, L_z] + (-1)^{x(y+z)}[R_{y\circ z}, L_x] + (-1)^{z(x+y)}[R_{z\circ x}, L_y] = 0, \tag{1}$$
$$[R_x, L_y] = [L_x, R_y] \tag{2}$$

hold for all $x, y, z \in U$. The second operator identity defines the class of flexible superalgebras. Note that (2) follows from (1) if U possesses the unity. If we assume that all elements in U are even then we arrive at a noncommutative Jordan algebra definition.

The flexibility identity may be written in the following forms:

$$(-1)^{xy}L_{xy} - L_yL_x = R_{yx} - R_yR_x, \tag{3}$$
$$(x,y,z) = -(-1)^{xy+xz+yz}(z,y,x). \tag{4}$$

Let Γ be the Grassmann superalgebra in generators $1,\ \xi_i,\ i \in \mathcal{I}$, where we admit the case $\mathcal{I} = \emptyset$. If $\mathcal{I} = \{1,\dots,n\}$ then we denote this superalgebra by Γ_n.

If $U = (U,\cdot)$ is a superalgebra and $\lambda \in F$ then the λ-*mutation* of U is the superalgebra $U^{(\lambda)} = (U,\cdot_\lambda)$, where $x \cdot_\lambda y = \lambda x\cdot y + (-1)^{xy}(1-\lambda)y\cdot x$. Put $U^{(+)} := U^{(\frac{1}{2})}$.

Lemma 1. *([23].) U is a NJ-superalgebra iff U is a flexible superalgebra such that $U^{(+)}$ is a Jordan superalgebra.*

Recall some usual facts about Peirce decompositions [1, 18, 23]. Let e be an idempotent of U. Put $U_i = \{x : ex + xe = ix\}$ with $i = 0,1,2$. Using the standard argument we get

$$U \quad = \quad U_0 \oplus U_1 \oplus U_2. \tag{5}$$

Denote by P_i the associated projections on U_i. Note that U_0, U_1, and U_2 satisfy the following relations:

$$U_i^2 \subseteq U_i,\ U_iU_1 + U_1U_i \subseteq U_1,\ U_0U_2 = U_2U_0 = 0, \tag{6}$$
$$x \in U_i \Rightarrow xe = ex = \frac{1}{2}ix\ (i=0,2);\ x,y \in U_1 \Rightarrow x\circ y \in U_0 + U_2. \tag{7}$$

Given $\lambda \in F$, define $U_1^{[\lambda]}(e) = \{x \in U_1 \mid ex = \lambda x\}$. One can easily check that the subspace $S_1^{[\phi]}(e) = U_1^{[\lambda]}(e) + U_1^{[1-\lambda]}(e)$ is uniquely defined by $\phi = \lambda(1-\lambda)$, namely: $S_1^{[\phi]}(e) = \{u \in U_1 \mid eue = \phi u\}$. For a pair of orthogonal idempotentes e_i, e_j we put $S_{ij}^{[\phi]} := S_1^{[\phi]}(e_i) \cap S_1^{[\phi]}(e_j)$. We say that e_i and e_j are *evenly connected* if there are $\phi \in F$ and even $v_{ij}, u_{ij} \in S_{ij}^{[\phi]}$ such that $v_{ij}u_{ij} = u_{ij}v_{ij} = e_i + e_j,\ i<j$. We say that e_i and e_j are *oddly connected* if there are $\phi \in F$ and odd $v_{ij}, u_{ij} \in S_{ij}^{[\phi]}$, such that $v_{ij}u_{ij} = -u_{ij}v_{ij} = e_i - e_j,\ i<j$. The idempotents e_i and e_j are said to be *connected* if they are either evenly or oddly connected.

A *split quasiassociative superalgebra* is a mutation $\mathcal{D}^{(\lambda)}$ of an associative superalgebra $\mathcal{D}$. A superalgebra U is *quasiassociative* if there is an extension Ω of F such that U_Ω is a split quasiassociative superalgebra over Ω: $U_\Omega = \mathcal{D}^{(\lambda)}$ for $\lambda \in \Omega$.

The following theorem and lemmas from [23]–[24] we use further.

Theorem 1. *(Coordinatization theorem.) Let U be a NJ-superalgebra. Assume that the unity $1 = \sum_{i=1}^n e_i \in U$ is a sum of $n \geq 3$ connected orthogonal idempotents. Then either U is a commutative Jordan superalgebra or $U = \mathcal{D}_n^{(\lambda)}$ is a split quasi-associative superalgebra determined by the superalgebra $\mathcal{D}_n$ of $n\times n$ matrices with entries in $\mathcal{D}$, where $\mathcal{D}$ is associative.*

Lemma 2. *If $u_i \in U_i$ $(i = 0,2)$, $z_1, w_1 \in U_1$ then*

$$2e(z_1\circ u_0) = 2ez_1\circ u_0 = z_1u_0,\ 2(u_0\circ z_1)e = 2u_0\circ z_1e = u_0z_1; \tag{8}$$
$$2e(u_2\circ z_1) = 2u_2\circ ez_1 = u_2z_1,\ 2(z_1\circ u_2)e = 2z_1e\circ u_2 = z_1u_2; \tag{9}$$
$$2P_2(ez_1\circ w_1) = 2P_2(z_1\circ w_1e) = P_2(z_1w_1),\ 2P_0(w_1\circ ez_1) = 2P_0(w_1e\circ z_1) = P_0(w_1z_1); \tag{10}$$
$$P_1(z_1w_1)\circ u_i = P_1(z_1(w_1\circ u_i)) = (-1)^{w_1u_i}P_1((z_1\circ u_i)w_1). \tag{11}$$

Lemma 3. *Let $(A,\cdot)$ be a flexible superalgebra. If $A^{(+)}$ possesses the unity $1=\sum_{i=1}^n e_i$ for some orthogonal even idempotents e_i then A possesses the same property.*

Lemma 4. *The mapping $d=[\cdot,x]$ is a superderivation in $U^{(+)}$ for every $x\in U$.*

Lemma 5. *If U is simple then $U^{(+)}$ is differentiably simple.*

A super-anticommutative binary operation $\{\,,\}$ we will call a *Leibniz bracket* on a superalgebra $(A,\cdot)$ provided that

$$\{a\cdot b,c\}=(-1)^{bc}\{a,c\}\cdot b+a\cdot\{b,c\}\quad\text{(Leibniz identity)}\tag{12}$$

holds for all homogeneous $a,b,c\in A$.

In [23]–[24] we called these brackets *generalized Poisson brackets*, but this term was used also by V. Kac in [4] for another algebra.

The category of NJ-superalgebras is isomorphic to the category of Jordan superalgebras with Leibniz (super)brackets.

Lemma 6. *Let $(J;\circ)$ be a Jordan superalgebra equipped with a Leibniz bracket $\{\,,\}$. Then the operation $ab=a\circ b+\frac{1}{2}\{a,b\}$ equips J with the structure of an NJ-superalgebra U such that $U^{(+)}=J$. Conversely, if U is an NJ-superalgebra then the supercommutator $[\,,]$ is a Leibniz bracket on the Jordan superalgebra $U^{(+)}$. Furthermore, the multiplication in U can be recovered from the Jordan multiplication $\circ$ in $U^{(+)}$ and the Leibniz bracket $[\,,]$ via $ab=a\circ b+\frac{1}{2}[a,b]$.*

3 An analog of Oehmke's theorem

Let $B(m)$ be the algebra of truncated polynomials in m variables, i.e., $B(m)$ is a quotient algebra of the polynomial algebra $F[t_1,\dots,t_m]$ by the ideal generated by $t_i^p,\ i=1,\dots,m$. Let $\mathit{\Gamma}_n$ be the Grassmann algebra in n variables $\xi_1,\dots,\xi_n$.

Theorem. *(Cheng-Kac Theorem [5].) Let A be a finite-dimensional differentially simple superalgebra over an algebraically closed field F of characteristic 0. Then $A\cong S\otimes_F \mathit{\Gamma}_n$, where S is some simple superalgebra.*

Slightly changing the proof of this theorem we obtain its modular analog:

Theorem. *Let A be a finite-dimensional differentially simple superalgebra over an algebraically closed field F of characteristic $p\geq 0$. Then $A\cong S\otimes_F B(m,n)$, where S is a simple superalgebra and $B(m,n):=B(m)\otimes \mathit{\Gamma}_n$.*

Thus, the adjoined Jordan superalgebra $U^{(+)}$ of a simple NJ-superalgebra U over an algebraically closed field F of characteristic $p\neq 2$ is a tensor product of a simple Jordan superalgebra and a superalgebra $B(m,n)$.

Proposition 1. *Let U be a simple finite-dimensional NJ-superalgebra such that $U^{(+)}\cong J\otimes B(m,n)$, and let $J=J_0\oplus J_1\oplus J_2$ be the Peirce decomposition of the simple Jordan superalgebra J for some idempotent $e\in J$. If for every nonzero $a\in J_i,\ i=0,1,2$, there exists $u_a\in J_1$ such that $a\circ u_a\neq 0$ then $m=n=0$, and $U^{(+)}=J$ is simple.*

Proof. Denote $B := B(m, n)$ and let B_+ denote the augmentation ideal of the superalgebra B, that is, B_+ is the subsuperalgebra of B generated by all $1 \otimes \xi_i$ and $t_j \otimes 1$; then $B = F + B_+$. Assume that $B_+ \neq 0$, that is, either $m \neq 0$ or $n \neq 0$. Show that $I := J \otimes B_+ \lhd U$.

We have $U_k = J_k \otimes B,\ k = 0, 1, 2$. Let $I_k = J_k \otimes B_+$, then $I = I_2 + I_1 + I_0$. It is clear that $I \lhd U^{(+)}$. Thus it suffices to prove that $IU \subseteq I$.

From (8) and (9) we get

$$I_k U_1 \subseteq I \circ U \subseteq I \text{ for } k \in \{0, 2\}. \tag{13}$$

Furthermore, by (6) we have $I_k U_j = 0$ for $k \neq j \in \{0, 2\}$. Therefore, it sufficent to prove that $I_1 U \subseteq I$ and that $I_k U_k \subseteq I$ for $k = 0, 2$.

To prove (13) for $k = 1$, in view of (10) it suffices to show that $P_1(xy) \in I$ when $x \in I_1,\ y \in U_1$. Assume that $P_1(xy) = z_0 \otimes 1 + \sum z_i \otimes f_i$ for some $z_i \in J_1,\ z_0 \neq 0,\ f_i \in B_+$. Denote $t_1^{p-1} \ldots t_m^{p-1} \otimes \xi_1 \ldots \xi_n$ by $t\xi$. Take $h = e \otimes t\xi$. Then $h \circ P_1(xy) \neq 0$, which contradicts (11).

Let us next prove that the subspace $J_1 \otimes t\xi$ is invariant by multiplication on e. It is clear that $(J_1 \otimes t\xi) \circ e \subseteq J_1 \otimes t\xi$, hence it suffices to show that $[J_1 \otimes t\xi, e] \subseteq J_1 \otimes t\xi$.

Observe that

$$[e \otimes a, e] = 0 \text{ for any } a \in B. \tag{14}$$

In fact, it follows easily from the Leibniz identity that

$$[a^2, x] = 2[a, a \circ x] \text{ for any } a \in U_{\bar{0}}.$$

Therefore,

$$[e \otimes a, e] = [e \otimes a, e^2] = 2[(e \otimes a) \circ e, e] = 2[e \otimes a, e],$$

which proves (14). Let now $u \in J_1$, consider

$$\begin{aligned} [u \otimes t\xi, e] &= 2[(u \otimes 1) \circ (e \otimes t\xi), e] = 2(u \otimes 1) \circ [e \otimes t\xi, e] \pm 2(e \otimes t\xi) \circ [u \otimes 1, e] = \\ &= (\text{ by } (14)) = \pm 2(e \otimes t\xi) \circ [u \otimes 1, e] \in (e \otimes t\xi) \circ (J_1 \otimes B) \in J_1 \otimes t\xi. \end{aligned}$$

Let now $u \in I_0 \cup I_2$. By (8), (9) we have

$$u(J_1 \otimes t\xi) \subseteq u \circ ((J_1 \otimes t\xi)e) + u \circ (e(J_1 \otimes t\xi)) \subseteq u \circ (J_1 \otimes t\xi) = 0,$$

since evidently $I \circ (J \otimes t\xi) = 0$. Therefore,

$$(I_0 + I_2)(J_1 \otimes t\xi) = (J_1 \otimes t\xi)(I_0 + I_2) = 0. \tag{15}$$

The same arguments show that

$$(U_0 + U_2)(J_1 \otimes t\xi) + (J_1 \otimes t\xi)(U_0 + U_2) \subseteq J_1 \otimes t\xi. \tag{16}$$

Prove now that $I_1(J_1 \otimes t\xi) = 0$. For any $a \in J_1,\ f \in B_+$ and $h \in J_1 \otimes t\xi$ we have

$$\begin{aligned} (a \otimes f)h &= (a \otimes f) \circ h + \tfrac{1}{2}[a \otimes f, h] = \tfrac{1}{2}[a \otimes f, h] = [(a \otimes 1) \circ (e \otimes f), h] = \\ &= (a \otimes 1) \circ [e \otimes f, h] \pm (e \otimes f) \circ [a \otimes 1, h]. \end{aligned}$$

By (15), $[e\otimes f,h]\in[I_2,J_1\otimes t\xi]=0$. Furthermore,

$$(e\otimes f)\circ[a\otimes 1,h]=(e\otimes f)\circ P_2([a\otimes 1,h])+(e\otimes f)\circ P_1([a\otimes 1,h])).$$

By (10),

$$P_2([a\otimes 1,h]) \;=\; 2P_2((a\otimes 1)\circ(eh\pm he))\in P_2((a\otimes 1)\circ(J_1\otimes t\xi))\subseteq J_2\otimes t\xi,$$

hence $(e\otimes f)\circ P_2([a\otimes 1,h])\in I_2\circ(J_2\otimes t\xi)=0$. Similarly, by (11),

$$P_1([a\otimes 1,h])\circ(e\otimes f) \;=\; P_1((a\otimes 1)(h\circ(e\otimes f))\pm((e\otimes f)\circ h)(a\otimes 1))=0.$$

Therefore,

$$I_1(J_1\otimes t\xi)=0. \tag{17}$$

Take now $a\in I_k,\ b\in U_i,\ i\in\{0,2\}$. Assume that $ab=c\otimes 1+f$, where $0\neq c\in J_k,\ f\in I$. Take $h=u_c\otimes t\xi$, then $h\circ(ab)=(u_c\circ c)\otimes t\xi\neq 0$. On the other hand,

$$h\circ(ab)=h\circ(a\circ b)+\tfrac{1}{2}h\circ[a,b].$$

It is clear that $h\circ(a\circ b)\in h\circ I=0$. Furthermore, by (16)

$$h\circ[a,b] \;=\; [h\circ a,b]\pm a\circ[h,b]=\pm a\circ[h,b]\in a\circ(J_1\otimes t\xi)=0.$$

Hence $h\circ(ab)=0$ and $c=0$. Therefore, $I_k(U_0+U_2)\subseteq I$. ■

We say that a noncommutative Jordan superalgebra U is of *degree* k if k is a maximal possible number of pairwise orthogonal idempotents in $U\otimes_F\bar{F}$, where $\bar{F}$ is the algebraical closure of the ground field F.

The theorem below was proved in [23, Theorem 2.4] for superalgebras over a field of characteristic zero, but the same proof is valid for the case of characteristic $p\neq 2$; one has only to use the mentioned above modular modification of the Cheng-Kac Theorem instead of the original theorem.

Theorem 2. *Let U be a finite dimensional central simple NJ-superalgebra of characteristic $p\neq 2$. Then one of the following cases holds:*

(a) U is a superalgebra of degree ≤ 2;

(b) U is a quasi-associative superalgebra;

(c) U is a Jordan superalgebra.

The next theorem is an analog of the well-known Oehmke's theorem from [21]:

Theorem 3. *Let U be a finite-dimensional central simple NJ-superalgebra of degree >1 over an algebraically closed field F of characteristic $p\neq 2$. Then $U^{(+)}$ is simple.*

Proof. By Lemma 3, the superalgebra $U^{(+)}$ has the same degree as U. Hence it suffices to verify that the assumptions of Proposition 1 are fulfilled for all simple Jordan superalgebras of degree >1.

We recall the list of these superalgebras (see [17, 25, 29, 32] for the definitions):

- Jordan matrix superalgebras: $M_{n|m}^{(+)}$, $M_n[\sqrt{1}]^{(+)}$ (or $P(n)$), $Josp(n,2m) = H(M_{n,2m}, osp)$, $Jtrp(n) = H(M_{n|n}, trp)$ (or $Q(n)$);
- Superalgebra of bilinear form $J(V,f)$;
- Kaplansky superalgebra K_3 and 4-dimensional superalgebra D_t;
- Kac superalgebras K_{10} and K_9 (the last in the case $char\, F = 3$);
- Kantor doubles $Kan(B(m,n))$, $n > 0$, $m \geq 0$, ($m > 0$ only if $char\, F > 0$),
- Exceptional matrix superalgebras $H_3(B(1,2))$, $H_3(B(4,2))$, $char\, F = 3$;
- Cheng-Kac superalgebra $CK(Z,d)$, $char\, F > 0$;
- Semi-unital superalgebra $V_{1/2}(Z,d)$, $char\, F > 0$.

For every case the verification is simple, it suffices to consider the Peirce decomposition and make simple computations. ■

In the next sections we will determine the structure of a simple NJ-superalgebra U according to the type of the simple Jordan superalgebra $J = U^{(+)}$.

Some of the cases were already investigated in [23] and [24] when $char\, F = 0$. We will refer to these cases when they are the same for the modular case.

4 Superalgebras of degree ≥ 3

Consider first simple NJ-superalgebras U of degree ≥ 3 for which the superalgebra $U^{(+)}$ is of matrix type. As we have already observed, by Lemma 3 the degree of $U^{(+)}$ is also ≥ 3.

By Theorem 2, if $U \neq U^{(+)}$ then U is quasi-associative. In this case $U \subseteq A^{(\lambda)}$ for an associative superalgebra A, hence the superalgebra $U^{(+)} \subseteq A^{(+)}$ is special. Therefore, if $U^{(+)}$ is one of the exceptional matrix superalgebras $B(1,2)$, $B(4,2)$ then U is Jordan, $U = U^{(+)}$.

If $U^{(+)}$ is one of the superalgebras $M_{n|m}^{(+)}$ $(n+m > 2)$, $M_n[\sqrt{1}]^{(+)}$ $(n > 2)$ and $U \neq U^{(+)}$ then U is quasi-associative, $U \cong M_{n|m}^{(\lambda)}$ or $U \cong M_n[\sqrt{1}]^{(\lambda)}$.

Now, let $U^{(+)} \cong Josp\,(n,2m) = H(M_{n,2m}, osp)$ where $n + m > 2$. Assume that U is quasi-associative, then $U = M_{p|q}^{(\lambda)}$ or $U = M_k[\sqrt{1}]^{(\lambda)}$. In the second case $(U^{(+)})_{\bar{0}} = M_k^{(+)}$ is a simple algebra while $Josp\,(n,2m)_{\bar{0}} = H(M_n, trp) \oplus H(M_{2m}, sp)$ is a direct sum of two simple algebras, a contradiction. Furthermore, if $U = M_{p|q}^{(\lambda)}$ then $(U^{(+)})_{\bar{0}} = M_p^{(+)} \oplus M_q^{(+)} = H(M_n, trp) \oplus H(M_{2m}, sp)$. But it is easy to see that the alebra $H(M_n, trp)$ is not isomorphic to any algebra $M_p^{(+)}$. In fact, in case of isomorphism these algebras should have the same degree, that is, n should be equal to p, but then $n^2 = \dim M_n^{(+)} \neq \frac{n(n+1)}{2} = \dim H(M_n, trp)$, a contradiction.

Consider now the case $U^{(+)} = Jtrp(n) = H(M_{n|n}, trp)$. Then $(U^{(+)})_{\bar{0}} \cong M_n^{(+)}$, and the odd part $(U^{(+)})_{\bar{1}}$ considered as a Jordan bimodule over $(U^{(+)})_{\bar{0}}$ is a direct sum of two irreducible bimodules, corresponding to the subspaces of symmetric and skewsymmetric $n \times n$ matrices. If U were quasi-associative then evidently $U = M_n[\sqrt{1}]^{(\lambda)}$. In this case we would have $(U^{(+)})_{\bar{0}} \cong M_n^{(+)}$ again, but the odd part $(U^{(+)})_{\bar{1}}$ in this case would be an irreducible bimodule over $(U^{(+)})_{\bar{0}}$ isomorphic to the regular bimodule $Reg\, M_n^{(+)}$. Therefore, U is not quasi-associative and $U = U^{(+)}$.

It remains to consider a superalgebra U of degree 3 for which $U^{(+)} \cong K_{10}$. Since K_{10} is an exceptional supperalgebra, U can not be quasi-associative. Therefore, by Theorem 2, U is Jordan and $U = U^{(+)}$.

Remark 1. *The last result is equivalent to the fact that the Jordan superalgebra K_{10} does not admit a non-trivial Leibniz bracket. In the next section we will give a unified proof of this fact for the superalgebras K_{10} and K_9, where the last one is of degree 2 and can not be handled by Theorem 2.*

We summarize the obtained results in the following theorem:

Theorem 4. *Let U be a finite-dimensional central simple NJ-superalgebra of degree > 2 over an algebraically closed field F of characteristic $p \neq 2$. Then $U = U^{(+)}$ is Jordan, except the cases $U^{(+)} \in \{M_{n|m}^{(+)},\ M_n[\sqrt{1}]^{(+)}\}$ when it is possible that U were quasi-associative, $U \in \{M_{n|m}^{(\lambda)},\ M_n[\sqrt{1}]^{(\lambda)}\}$.*

5 Superalgebras of degree 2

5.1 The cases $U^{(+)} \cong D_t,\ K_3,\ M_{1|1}^{(+)},\ Josp(1,2),\ (M_2[\sqrt{1}])^{(+)},\ Jtrp(2)$.

Recall that $D_t = (Fe_1 + Fe_2) + (Fx + Fy)$ with $(D_t)_{\bar{0}} = Fe_1 + Fe_2$, $(D_t)_{\bar{1}} = Fx + Fy$, where $e_i^2 = e_i,\ e_1 \circ e_2 = 0,\ e_i \circ x = \frac{1}{2}x,\ e_i \circ y = \frac{1}{2}y,\ x \circ y = e_1 + te_2,\ t \in F,\ x \circ x = y \circ y = 0$.

In order to determine NJ-superalgebras U for which $U^{(+)} \cong D_t$, it suffices to classify possible Leibniz (super)brackets $\{,\}$ on D_t. It is easy to see that $\{e_i, e_j\} = 0$. Let $\{e_1, x\} = \alpha x + \beta y,\ \{e_1, y\} = \gamma x + \delta y;\ \alpha, \beta, \gamma, \delta \in F$. We have

$$0 = \{e_1, x \circ y\} = x \circ \{e_1, y\} + \{e_1, x\} \circ y = (\alpha + \delta)(x \circ y) = (\alpha + \delta)(e_1 + te_2),$$

from where $\delta = -\alpha$. Let, furthermore, $\{x, x\} = \lambda e_1 + \mu e_2$. We have

$$\tfrac{1}{2}\{x, x\} = \{x, e_1 \circ x\} = e_1 \circ \{x, x\} + \{x, e_1\} \circ x = \lambda e_1 + \beta(x \circ y) = (\lambda + \beta)e_1 + \beta te_2,$$

which implies $\lambda = -2\beta$, $\mu = 2\beta t$, that is, $\{x, x\} = -2\beta(e_1 - te_2)$. Similarly, we obtain $\{y, y\} = 2\gamma(e_1 - te_2),\ \{x, y\} = 2\alpha(e_1 - te_2)$. Therefore, the Leibniz bracket $\{,\}$ on D_t is completely determined by the action of e_1 on $(D_t)_1$ via $z \mapsto \{e_1, z\}$, that is, by the parameters α, β, γ. We denote the corresponding NJ-superalgebra as $D_t(\alpha, \beta, \gamma)$.

If F is algebraically closed then we can find a Jordan basis x', y' of $(D_t)_{\bar{1}}$, where the matrix of this action has one of the form:

$$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix},\ \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix}. \qquad (18)$$

Dividing x', y' on the determinant of the matrix of their coordinates in the base (x, y), we will have that $x' \circ y' = x \circ y$, hence without lost of generality we may assume that $x = x',\ y = y'$. Our previous consideration show that in the first matrix in (18) we should have $\alpha = 0$, and in the second $\gamma = -\alpha$. Resuming, we have

Theorem 5. *Let U be a simple central NJ-superalgebra over a field F of characteristic $\neq 2$ for which $U^{(+)} \cong D_t$. Then U is isomorphic to the superalgebra $D_t(\alpha, \beta, \gamma)$. Moreover, is F is algebraically closed then U is isomorphic to one of the NJ-superalgebras defined by the following nonzero Leibniz brackets on D_t:*

- $\{e_1, x\} = \alpha x,\ \{e_1, y\} = -\alpha y,\ \{x, y\} = -\{y, y\} = 2\alpha(e_1 - te_2),\ \ \alpha \in F;$
- $\{e_1, x\} = y, \quad \{x, x\} = -2(e_1 - te_2).$

We denote the corresponding NJ-superalgebras as $D_t(\alpha)$ and ND_t.

Recall that the Kaplansky superalgebra K_3 may be identified with the subspace $Fe_1 + Fx + Fy$ of the superalgebra D_0 which in this case is an algebra direct summond of D_0: $D_0 \cong K_3 \oplus Fe_2$. Using this fact, we obtain easily the following

Corollary 1. *Let U be a simple central NJ-superalgebra over an algebraically closed field F of characteristic $\neq 2$ for which $U^{(+)} \cong K_3$. Then U is isomorphic to one of the NJ-superalgebras $K_3(\alpha)$, NK_3, defined by the restriction of the Leibniz brackets from Theorem 5 on the subalgebra K_3 of D_0.*

Furthermore, one can easily check the isomorphisms:

$$M_{1|1}^{(+)} \cong D_{-1},\ \ Josp(1, 2) \cong D_{-\frac{1}{2}}.$$

Therefore, we have

Corollary 2. *Let U be a simple central NJ-superalgebra over an algebraically closed field F of characteristic $\neq 2$ for which $U^{(+)} \cong M_{1|1}^{(+)}$ or $U^{(+)} \cong Josp(1, 2)$. Then U is isomorphic to the NJ-superalgebras $D_{-1}(\alpha)$, ND_{-1} or to $D_{-\frac{1}{2}}(\alpha)$, $ND_{-\frac{1}{2}}$, respectively.*

Finally, we refer the following result:

Theorem 6. [24]. *Let U be a simple central NJ-superalgebra over an algebraically closed field F of characteristic $\neq 2$.*
a) If $U^{(+)} \cong (M_2[\sqrt{1}])^{(+)}$ then $U = (M_2[\sqrt{1}])^{(\lambda)}$ is quasi-associative.
b) If $U^{(+)} \cong Jtrp(2)$ then $U = U^{(+)}$ is Jordan.

5.2 The cases $U^{(+)} \cong K_9,\ K_{10}$.

Recall the definitions of the simple Jordan superalgebras K_{10} and K_9 over a field F (see [20, 29]).

The odd part of K_{10} is $M = \langle u, v, w, z\rangle$, the even part $A = A_1 \oplus A_2 = \langle e_1, uz, vz, uw, vw\rangle \oplus \langle e_2\rangle$ is a direct sum of ideals (of A). The unity in A_1 is e_1, and $e_i \cdot m = \frac{1}{2}m$ for every $m \in M$. Now,

$$u \cdot z = uz, u \cdot w = uw, v \cdot z = vz, v \cdot w = vw,$$

$$z \cdot w = e_1 - 3e_2, uz \cdot w = -u, vz \cdot w = -v, uz \cdot vw = 2e_1,$$

and the remaining nonzero products may be obtained either by applying the skew-symmetries $z \leftrightarrow w$, $u \leftrightarrow v$, or by the substitution $z \leftrightarrow u$, $w \leftrightarrow v$. For the convenience of reader, we write them:

$$vw \cdot z = v, vw \cdot u = -w, uz \cdot v = z, uw \cdot z = u, uw \cdot v = w, vz \cdot u = -z,$$

$$uw \cdot vz = -2e_1, u \cdot v = e_1 - 3e_2.$$

If the characteristic of F is not 3, the superalgebra K_{10} is simple; but in the case of characteristic 3 it contains a simple subsuperalgebra $K_9 = A_1 \oplus M$.

Note that the following theorem was formulated in [23], but it was given without a proof, and here we give a proof, which is independent of the characteristic, and it works even in the case of K_9.

Theorem 7. *Let U be a noncommutative Jordan superalgebra such that either $U^{(+)} \cong K_{10}$ or $U^{(+)} \cong K_9$. Then $U \cong K_{10}$ or K_9, respectively.*

Proof. For the proof we show that every Leibniz bracket on $J := K_{10}$ (K_9) is zero. We give the proof for the case of K_{10}, since in the case of the superalgebra K_9 the proof is the same (indeed, it is easier since we don't have to think about the idempotent e_2).

From the multiplication table in K_{10} we have:

$$Au \subseteq \langle u, w, z\rangle, \; Az \subseteq \langle z, u, v\rangle, \; Av \subseteq \langle v, z, w\rangle, \; Aw \subseteq \langle w, u, v\rangle.$$

By properties of Leibniz bracket we get:

$$\begin{aligned}\{u, u\} &= \{uw \cdot z, u\} = uw \cdot \{z, u\} - \{uw, u\}z \in uw \cdot A + uA \cdot z + wA \cdot z \\ &\subseteq \langle e_1, e_2, vz, uz, uw\rangle.\end{aligned}$$

Similarly, $\{u, u\} = -\{uz \cdot w, u\} \in \langle e_1, e_2, vw, uz, uw\rangle$, and $\{u, u\} = \alpha e_1 + \beta e_2 + \gamma uw + \delta uz$ for some $\alpha, \beta, \gamma, \delta \in F$.

Analogously, $\{v, v\} = \alpha' e_1 + \beta' e_2 + \gamma' vw + \delta' vz$ for some $\alpha', \beta', \gamma', \delta' \in F$.

Consider now

$$\begin{aligned}\{u, v\} &= \{uw \cdot z, v\} = uw \cdot \{z, v\} + z\{uw, v\} \in uw \cdot A + (uA + wA) \cdot z \\ &\subseteq \langle e_1, e_2, vz, uz, uw\rangle.\end{aligned}$$

Similarly, $\{u, v\} = -\{uz \cdot w, v\} \in \langle e_1, e_2, vw, uz, uw\rangle$, and $\{u, v\} \in \langle e_1, e_2, uz, uw\rangle$. Substituting $v = vw \cdot z = -vz \cdot w$, we analogously get $\{u, v\} \in \langle e_1, e_2, vw, vz\rangle$, and finally $\{u, v\} = \lambda_1 e_1 + \lambda_2 e_2$ for some $\lambda_1, \lambda_2 \in F$.

Furthermore,

$$\begin{aligned}0 &= \{uv, uv\} = u\{v, uv\} + \{u, uv\}v \\ &= -u(u\{v, v\}) + u(\{v, u\}v) - (u\{u, v\})v + (\{u, u\}v)v \\ &= -u(u(\gamma' vw + \delta' vz)) + \frac{\lambda_1 + \lambda_2}{2} uv - \frac{\lambda_1 + \lambda_2}{2} uv + ((\gamma uw + \delta uz)v)v \\ &= \gamma' uw + \delta' uz + \gamma wv + \delta zv.\end{aligned}$$

Therefore, $\gamma = \delta = \gamma' = \delta' = 0$, and we have

$$\{u, u\} = \alpha e_1 + \beta e_2, \; \{v, v\} = \alpha' e_1 + \beta' e_2.$$

Similarly, $\{z, z\}, \{w, w\}, \{z, w\} \in \langle e_1, e_2\rangle$.

Since e_1 and e_2 are idempotents, $\{e_i, a\} = 0$ for all $a \in A$, $i = 1, 2$. Therefore,

$$\begin{aligned}0 &= \{uv, uz\} = u\{v, uz\} + \{u, uz\}v \\ &= -\{v, u \cdot uz\} + \{v, u\}uz + \{u, uz \cdot v\} - uz\{u, v\} = \{u, z\}.\end{aligned}$$

Analogously, $\{v,z\}=\{u,w\}=\{v,w\}=0$.

Furthermore,

$$\begin{aligned}\{v,v\} &= \{vw\cdot z,v\}=(\{v,v\}w)z=\frac{\alpha'+\beta'}{2}(3e_2-e_1)=\alpha' e_1+\beta' e_2;\\ 0 &= \{v,z\}=\{v,uz\cdot v\}=\{v,uz\}v+uz\{v,v\}\\ &= (\{v,u\}z)v+uz\{v,v\}=\frac{\lambda_1+\lambda_2}{2}zv+\alpha' uz,\end{aligned}$$

whence $\alpha'=\beta'=0,\ \lambda_1+\lambda_2=0$. Then $\{u,v\}=\{uw\cdot z,v\}=(\{u,v\}w)z=0$. Now,

$$\begin{aligned}\{u,u\} &= \{uw\cdot z,u\}=(\{u,u\}w)z=\frac{\alpha+\beta}{2}(3e_2-e_1);\\ 0 &= \{u,z\}=\{u,uz\cdot v\}=\{u,uz\}v=\{u,u\}zv=-\frac{\alpha+\beta}{2}zv.\end{aligned}$$

Therefore, $\{u,u\}=0$.

Similarly, $\{z,z\}=\{z,w\}=\{w,w\}=0$. Since M generates J, $\{J,J\}=0$. ■

5.3 The case $U^{(+)}\cong CK(Z,d)$.

Recall the definition of the simple Jordan Cheng — Kac superalgebra $CK(Z,d)$, where Z is a unital associative commutative algebra, and $d:Z\mapsto Z$ is a derivation (see [17]).

The even part is $A=\langle 1,w_1,w_2,w_3\rangle$, and the odd part is $M=\langle x,x_1,x_2,x_3\rangle$, which are free Z-modules of rank 4. The multiplication $\circ$ in A is Z-linear, $w_i\circ w_j=0$ when $i\neq j$, $w_1^2=w_2^2=1, w_3^2=-1$. Denote $x_{i\times i}=0, x_{1\times 2}=-x_{2\times 1}=x_3, x_{1\times 3}=-x_{3\times 1}=x_2, x_{3\times 2}=-x_{2\times 3}=x_1$, and do analogously with $i\times s$ ($i,s\in S:=\{1,2,3\}$). The A-bimodule structure on M is given by

$\circ$	g	gw_j
fx	$(fg)x$	$(fg^d)x_j$
fx_i	$(fg)x_i$	$(fg)x_{i\times j}$

The product on M is defined by the rules

$\circ$	gx	gx_j
fx	f^dg-fg^d	$-(fg)w_j$
fx_i	$(fg)w_i$	0

The superalgebra $CK(Z,d)$ is simple iff Z is D-simple. Therefore, in the modular case by [3] we may assume that $Z\cong B(m)$.

Observe that $e_1=\frac{1-w_1}{2}$, $e_2=\frac{1+w_1}{2}$ are orthogonal idempotents in $CK(Z,d)$ and $e_1+e_2=1$, hence $CK(Z,d)$ is of degree 2.

Proposition 2. *Let Z be a d-simple associative commutative ring ($d\in Der(Z), d\neq 0$). Let $\{\,,\}$ be a Leibniz bracket on Z such that*

$$\{f^d,g\} = \{f,g\}^d=\{f,g^d\} \tag{19}$$

hold for all $f,g\in Z$. Then $\{Z,Z\}=0$.

Proof. It is easy to see that Z has no absolute divisors of zero (for example, see[22]). Assume that $\{Z,Z\}\neq 0$. Act by d on $\{fg,h\}=\{f,h\}g+f\{g,h\}$ and use (19). For all $f,g,h\in Z$, we obtain

$$\begin{aligned}&\{fg,h^d\}=\{f,h^d\}g+\{f,h\}g^d+f^d\{g,h\}+f\{g,h^d\},\\&\{f,h\}g^d+f^d\{g,h\}=0,\{f,h\}f^d=0.\end{aligned}\tag{20}$$

Now, by (20) $0=\{f,g^d\}f^d=\{f^d,g\}f^d$ for all $f,g\in Z$. Therefore, $\{Z^dZ^d,Z\}=0$.

Note that by the differential simplicity of Z we have $Z^dZ=Z$ and $\{Z,Z\}Z=Z$, whence $Z^dZ^d\neq 0$ and the equality $x\{Z,Z\}=0$ implies $x=0$.

If $a\neq 0$ and $\{a,Z\}=0$ then by (20) $a^d\{Z,Z\}=0$ and $a^d=0$. Consequently, $aZ=Z$.

By above, there exists $b=a^d\in Z^d$ such that $b^2Z=Z$, i.e., $bZ=Z$. By (20) with $f=a$ we have $\{a,Z\}=0$. Again by (20) with $g=a$ we get $\{f,h\}b=0$ and $\{f,h\}=0$ for all $f,h\in Z$, which is a contradiction. ■

Theorem 8. *Let U be a central simple noncommutative Jordan superalgebra such that $U^{(+)}\cong CK(Z,d)$. Then $U\cong CK(Z,d)$.*

Proof. We have to show that every Leibniz bracket $\{\,,\}$ on $J=CK(Z,d)$ is zero. Note that if $d=0$ and Z is the main field (by the centrality) then $CK(Z,d)\cong (M_2[\sqrt{1}])^{(+)}$. Thus, we assume that $d\neq 0$.

As 1 is the unity in $U^{(+)}$, $\{1,u\}=0$ for every $u\in U$. Since A is the even part, we have $\{a,a\}=0$ for all $a\in A$. Recall that supercommutator in U defines a Leibniz bracket in $U^{(+)}$ and the superderivation $\{\circ,u\}(x)=\{x,u\}$ in $U^{(+)}$. Note that the description of derivations of $CK(Z,d)$ was obtained in [**?**].

Firstly, we prove that every Leibniz bracket $\{\,,\}$ is zero on the Z-generators of U. In what follows, we put $w_0:=1, x_0:=x$, and we use i,j,k for some pairwise different indices in S.

Let $\{w_i,w_j\}=\sum_{r=0}^3\alpha_rw_r$ for some $\alpha_r\in Z$. The relations $0=\{w_i^2,w_j\}=2\{w_i,w_j\}\circ w_i$ give $(\sum_{r=0}^3\alpha_rw_r)\circ w_i=\alpha_0w_i\pm\alpha_i=0$, i.e., $\{w_i,w_j\}=\alpha_{ij}w_{i\times j}$, where $\alpha_{ij}=\alpha_{ji}\in Z$. Now, $0=\{w_1\circ w_2,w_3\}=\alpha_{13}w_{1\times 3}\circ w_2+\alpha_{23}w_1\circ w_{2\times 3}=\alpha_{13}-\alpha_{23}$; $0=\{w_1\circ w_3,w_2\}=-\alpha_{12}+\alpha_{32}$. Hence, $\alpha_{12}=\alpha_{13}=\alpha_{23}:=\kappa\in Z$. Thus,

$$\{w_i,w_j\}=\kappa w_{i\times j}\text{ for all } i,j\in S.$$

Let $\{x,w_i\}=\sum_{r=0}^3\theta_r^{(i)}x_r$ for some $\theta_r^{(i)}\in Z$. Then

$$\begin{aligned}0&=\{x\circ w_i,w_i\}=(\sum_{r=0}^3\theta_r^{(i)}x_r)\circ w_i=\sum_{r=1}^3\theta_r^{(i)}x_{r\times i}\ \Rightarrow\ \{x,w_i\}\in\langle x,x_i\rangle_Z\,;\\0&=\{x\circ w_j,w_i\}=(\theta_0^{(i)}x+\theta_i^{(i)}x_i)\circ w_j+x\circ\kappa w_{j\times i}=\theta_i^{(i)}x_{i\times j}-\kappa^dx_{i\times j}.\end{aligned}$$

Hence, $\theta_i^{(i)}=\kappa^d$ for all $i\in S$.

Let $\{x_i,w_j\}=\sum_{r=0}^3\gamma_r^{ij}x_r$ for some $\gamma_r^{ij}\in Z$. Then

$$0=\{x_i,w_j^2\}\Rightarrow 0=\{x_i,w_j\}\circ w_j=\sum_{r=1}^3\gamma_r^{ij}x_{r\times j}\Rightarrow\{x_i,w_j\}\in\langle x,x_j\rangle_Z\,.$$

Therefore,

$$\begin{aligned}\{w_i, x_j\} &= \pm\{w_i, x_k \circ w_i\} = \pm\{w_i, x_k\} \circ w_i = 0,\\ \{w_i, x_i\} &= \pm\{w_i, x_j \circ w_k\} = \pm x_j \circ \{w_i, w_k\} = 0.\end{aligned}$$

Since

$$\kappa w_{i\times j} = \{w_i, w_j\} = \{x_i \circ x, w_j\} = x_i \circ \{x, w_j\} = x_i \circ (\theta_0^{(j)} x) = \theta_0^{(j)} w_i;$$

therefore, $\kappa = 0, \theta_0^{(j)} = 0$ for all $j \in S$. Finally,

$$\{x, w_i\} = \{w_i, w_s\} = \{x_i, w_s\} = 0$$

for all $i, s \in S$.

Let $\{x, x\} = \sum_{i=0}^{3} \delta_r w_r$. Then $0 = \{x \circ w_i, x\} = \delta_0 w_i \pm \delta_i$ with $i \in S$, whence $\{x, x\} = 0$. Analogously, $0 = \{x \circ w_i, x_s\} = \{x, x_s\} \circ w_i$, and $\{x, x_s\} = 0$ for all $s \in S$.

Now, for every $f \in Z$ we have $\{f, x\} = \sum_{r=0}^{3} \psi_f^{(r)} x_r$ for some $\psi_f^{(r)} \in Z$, whence $0 = \{f w_i \circ w_j, x\} = (\{f, x\} \circ w_i) \circ w_j$ and $\{f, x\} = \psi_f x$ for $\psi_f := \psi_f^{(0)} \in Z$.

Let $\{f, x_s\} = \sum_{r=0}^{3} \phi_f^{s,r} x_r$ for some $\phi_f^{s,r} \in Z$. Then the equalities $0 = \{f, x_i \circ x_j\} = \{f, x_i\} \circ x_j + x_i \circ \{f, x_j\}$ imply $\phi_f^{s,0} = 0$ for all $f \in Z, s \in S$. Now, as above

$$0 = \{f w_i \circ w_j, x_s\} = (\{f, x_s\} \circ w_i) \circ w_j \text{ and } \{f, x_s\} = 0.$$

Note that $\{f, w_i\} = \{f, x_i \circ x\} = x_i \circ \psi_f x = \psi_f w_i$, and $\{f, w_i^2\} = 2\{f, w_i\} \circ w_i = \pm 2\psi_f$, whence $\psi_f = 0$. Finally,

$$\{f, x\} = \{f, x_i\} = \{f, w_i\} = 0 \text{ for all } i \in S.$$

Show that $\{x_i, x_j\} = 0$ for all $i, j \in S$.

Let $\{x_1, x_2\} = \sum_{i=0}^{3} \beta_i w_i$. Then

$$\begin{aligned}\{x_1, x_2\} &= -\{x_1, x_3 \circ w_1\} = -\{x_1, x_3\} \circ w_1,\\ \{x_1, x_3\} &= -\{x_1, x_2 \circ w_1\} = -\{x_1, x_2\} \circ w_1.\end{aligned}$$

Therefore, $\{x_1, x_2\} = (\{x_1, x_2\} \circ w_1) \circ w_1 = (\beta_0 w_1 + \beta_1) \circ w_1 = \beta_0 + \beta_1 \circ w_1$. Analogously we get

$$\begin{aligned}\{x_2, x_1\} &= \{x_2, x_3 \circ w_2\} = \{x_2, x_3\} \circ w_2,\\ \{x_2, x_3\} &= \{x_2, x_1 \circ w_2\} = \{x_2, x_1\} \circ w_2,\end{aligned}$$

whence

$$\begin{aligned}\{x_1, x_2\} &= \beta_0, \{x_2, x_3\} = \beta_0 w_2, \{x_1, x_3\} = -\beta_0 w_1,\\ \{x_2, x_3\} &= \{x_1 \circ w_3, x_3\} = \{x_1, x_3\} \circ w_3 = 0, \text{ and } \beta_0 = 0.\end{aligned}$$

Finally, $\{x_i, x_i\} = \pm\{x_i, x_j \circ w_k\} = 0$.

Note the following equalities

$$\begin{aligned}\{x \circ g w_i, h\} &= \{g^d x_i, h\} = x_i \circ \{g^d, h\},\\ \{x \circ g w_i, h\} &= x \circ \{g w_i, h\} = x \circ (w_i \circ \{g, h\}).\end{aligned}$$

Let $\{g,h\}=\sum_{i=0}^{3}\{g,h\}_i w_i$, where $\{g,h\}_i\in Z$. Then

$$x\circ(w_i\circ\{g,h\})=x\circ(\{g,h\}_0 w_i\pm\{g,h\}_i)=\{g,h\}_0^d x_i\pm\{g,h\}_i x.$$

Since $x_i\circ\{g^d,h\}\in\langle x_1,x_2,x_3\rangle_Z$, we have $\{g,h\}_i=0$ for all $i\in S$ and $g,h\in Z$, i.e., $\{g,h\}\in Z$ for all $g,h\in Z$. We also infer that (19) hold in Z. Hence, by Proposition 2 $\{Z,Z\}=0$. As we see, all defined above Leibniz brackets are zero. Since a Leibniz bracket is completely defined by its values on the generators (and Z, in our case), $\{x,y\}=0$ for all $x,y\in J$. Thus, every Leibniz bracket on $CK(Z,d)$ is zero. ■

6 The cases $U^{(+)}\cong J(V,f),\ Kan(A),\ V_{1/2}(Z,d),\ B(m,n)$

6.1 The case $J(V,f),\ V=V_{\bar 1}$.

The case when U is a simple noncommutative Jordan superalgebra such that $U^{(+)}\cong J(V,f)$ is a Jordan superalgebra determined by a supersymmetric nondegenerate bilinear form f on the superspace $V=V_{\bar 0}+V_{\bar 1}$ was considered in [23, Lemma 4.4]. It was proved there that this condition is always true when $J(V,f)$ is of degree 2. Here we consider the remaining subcase when $J(V,f)$ is of degree 1. In this case $J(V,f)=F\oplus V$, where $V=V_{\bar 1}$ is the odd part of J ($dim\, V\geq 2$) and $f(v,v)=0$ for every $v\in V$.

Proposition 3. *Let U be a simple noncommutative Jordan superalgebra such that $U^{(+)}\cong J(V,f)\otimes B(m,n)$, where $V=V_{\bar 1}$. Then $m=n=0$ and $U^{(+)}\cong J(V,f)$.*

Proof. As above, denote by B_+ the augmentation ideal of the superalgebra $B=B(m,n)$, that is, the subsuperalgebra in B generated by all $1\otimes\xi_i$ and $t_j\otimes 1$. As in Section 3, show that $I:=J(V,f)\otimes B_+\lhd U$. Put $I_1:=\{g\in I : g=\sum g_i\otimes b_i,\ b_i\in B_{\bar 1}\}$. Write $a\equiv_I b$ if $a-b\in I$ (analogously for I_1). Let $\{\,,\}$ be a Leibniz bracket on $J:=U^{(+)}$. Show that $\{J,J\}\subseteq I$.

Let $\{u,u\}\equiv_{I_1}1\otimes a,\ \{u,v\}\equiv_{I_1}1\otimes b$, where u and v are arbitrary linearly independent elements in V, $a,b\in B_{\bar 0}$. Then

$$0=\{u\circ v,u\}=-\{u,u\}\circ v+u\circ\{v,u\}\equiv_{I_1}-v\otimes a+u\otimes b,$$

whence $a=b=0$. Note that if $m=n=0$ then we infer that $\{\,,\}$ is zero on J. Assume further that either $m\neq 0$ or $n\neq 0$. Let $v\in V$ and $\{v,t_i\}\equiv_I v_i\otimes 1$ for some $v_i\in V$. Then

$$0=-\{v\circ vt_i,vt_i\}=\{v,vt_i\}\circ vt_i-v\circ\{vt_i,vt_i\}\equiv_{I^2}$$
$$-(v\circ\{v,t_i\})\circ vt_i-v\circ(\{v,vt_i\}\circ t_i+v\circ\{t_i,vt_i\})\equiv_{I^2}$$
$$-(v\circ v_i)\circ vt_i-v\circ((-v\circ\{v,t_i\})\circ t_i+v\circ(\{t_i,v\}\circ t_i))\equiv_{I^2}$$
$$-(v\circ v_i)\circ vt_i+v\circ((v\circ v_i)\circ t_i+v\circ(v_i\circ t_i))=$$
$$-(v\circ v_i)\circ vt_i+2v\circ(v\circ v_i)t_i=(v\circ v_i)vt_i.$$

(Here and further, vb stands for $v\otimes b$ for $v\in V, b\in B$). Therefore, $v\circ v_i=0$. Now, take some $v,u\in V$ such that $u\circ v=1$. We have $\{v,t_i\}=\{vt_i\circ u,v\}=-\{vt_i,v\}\circ u+vt_i\circ\{u,v\}\equiv_I-(\{v,v\}\circ t_i)\circ u-(v\circ\{t_i,v\})\circ u\equiv_I(v\circ\{v,t_i\})\circ u$, whence $v_i=(v\circ v_i)\circ u=0$. Thus, $\{v,t_i\}\equiv_I 0$.

Let $\{v, \xi_i\} \equiv_{I_1} 1 \otimes g_i$, $\{u, \xi_i\} \equiv_{I_1} 1 \otimes h_i$, where u and v are arbitrary linearly independent elements in V, $g_i, h_i \in B_{\bar{0}}$. Then

$$0 = \{v \circ u, \xi_i\} = -\{v, \xi_i\} \circ u + v \circ \{u, \xi_i\}; -u \otimes g_i + v \otimes h_i = 0,$$

whence $g_i = h_i = 0$.

Now, fix i, j and let $\{t_i, \xi_j\} = 1 \otimes a + u \otimes 1 + \sum v_i \otimes h_i$, where $a \in B_{\bar{1}}$, $v_i, u \in V$, $h_i \in B_{\bar{0}}$, h_i is not invertible for every i. Take $v \in V$ such that $v \circ u = 1$. Then

$$\{t_i, \xi_j\} = \{v \circ ut_i, \xi_j\} = -\{v, \xi_j\} \circ ut_i + v \circ \{ut_i, \xi_j\} \equiv_I v \circ (\{u, \xi_j\} \circ t_i + u \circ \{t_i, \xi_j\}) \equiv_I$$

$$v \circ (u \circ \{t_i, \xi_j\}) = v \circ (u \otimes a + \sum (u \circ v_i) \otimes h_i) \equiv_I 1 \otimes a,$$

whence $u = 0$. Moreover,

$$\{v \circ ut_i, ut_j\} = \{t_i, u \circ t_j\} \equiv_I u \circ \{t_i, t_j\};$$

$$\{v \circ ut_i, ut_j\} = v \circ \{ut_i, ut_j\} - \{v, ut_j\} \circ ut_i \equiv_I v \circ \{ut_i, ut_j\};$$

$$\{ut_i, ut_j\} \equiv_I u \circ \{t_i, ut_j\} \equiv_I u \circ (u \circ \{t_i, t_j\}) \equiv_I 0,$$

i. e., $u \circ \{t_i, t_j\} \equiv_I 0$, and $\{t_i, t_j\} \equiv_I 0$. We proceed analogously with ξ_i, ξ_j. Thus, $\{J, J\} \subseteq I$, $U^2 \subseteq I$ and $I = J(V, f) \otimes B_+ \lhd U$. Hence U is simple, we have $I = 0$ and $U^{(+)} = J(V, f)$. ∎

The Proposition 3 and [23, Lemma 4.4] imply

Theorem 9. *Let U be a noncommutative Jordan superalgebra such that $U^{(+)}$ is the Jordan superalgebra $J(V, f) = F + V$ of a nondegenerate supersymmetric bilinear superform f on a vector superspace V. Realizing a quadratic extension of the field F if necessary, the multiplication in U is given by*

$$uv = f(u, v) + u \times v,$$

where $u \times v$ is a superanticommutative multiplication on V such that $f(u \times v, w) = f(u, v \times w)$ for any $u, v, w \in V$.

We will denote the superalgebra constructed in this theorem as $J(V, f, \times)$. An example of a simple superalgebra of type $J(V, f, \times)$ one can obtain by taking for $(V, \times)$ a classical semisimple Lie superalgebra and for f the Killing form on V.

6.2 $U^{(+)} \cong Kan(A)$.

Recall the definition of Kantor double $Kan(A)$.

Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be an associative super-commutative superalgebra. Assume that on A another operation $\{\,,\}$ (*a bracket*) is defined which is super-anticommutative. Consider the direct sum of vector spaces

$$Kan(A) = A \oplus \bar{A},$$

where $\bar{A}$ is an isomorphic copy of A. Define a product on $Kan(A)$. Given $a, b \in A$, their product in $Kan(A)$ coincides with the product ab in A, and

$$a\bar{b} = \overline{ab},\ \bar{b}a = (-1)^a \overline{ba},\ \bar{a}\bar{b} = (-1)^b \{a, b\}.$$

The superalgebra structure on $J = Kan(A)$ is given by: $J_{\bar{0}} = A_{\bar{0}} \oplus \bar{A}_{\bar{1}}$, $J_{\bar{1}} = A_{\bar{1}} \oplus \bar{A}_{\bar{0}}$. The obtained superalgebra is called the *Kantor double* of A. The bracket $\{,\}$ on A is called *Jordan* if $Kan(A)$ is a Jordan superalgebra. Some important properties of Jordan brackets may be found in [14]. In particular, the superalgebra $Kan(A)$ is simple if and only if A is *bracket simple*, that is, A does not contain nontrivial $\{,\}$-invariant ideals. Moreover, if $\{,\}$ is a Jordan bracket in A then the mapping $D\colon a \to \{a, 1\}$ is a derivation of A that satisfies the identity

$$\{ab, c\} - a\{b, c\} - (-1)^{bc}\{a, c\}b - abD(c) = 0. \tag{21}$$

Two important particular examples of Jordan brackets are *Poisson bracket* when $D = 0$ (see [12]), and *vector type bracket* which has a form $\{a, b\} = D(a)b - aD(b)$. A superalgebra with a Poisson bracket is called a *Poisson superalgebra*.

The smallest Poisson superalgebra with a non-trivial odd part is the Grassmann superalgebra $\Gamma_1 = F \oplus F\xi$, where ξ is an odd element, $\xi^2 = 0$, $\{\xi, \xi\} = 1$. One can check that $Kan(\Gamma_1) \cong D_{-1} \cong M_{1|1}^{(+)}$.

Let $\{,\}$ be a bracket on a superalgebra A. Denote by $Z(A, \{,\}) = \{z \in A \mid \{z, A\} = 0\}$ the *bracket–center* of the superalgebra A. If A is a Poisson superalgebra then its bracket-center is a subsuperalgebra, but in general it is not true. Nevertheless, we have

Lemma 7. *Let $\{,\}$ be a Jordan bracket on the superalgebra $B = B(m, n)$. If $Kan(B)$ is simple then the bracket–center $Z(B, \{,\})$ coincides with F.*

Proof. We have already notice above that if $Kan(B)$ is simple then B is bracket–simple, that is, B does not contains ideals invariant with respect to the bracket $\{,\}$. Let $z \in Z(B, \{,\})$, then for any $a, b \in B$ we have by (21)

$$\{za, b\} = z\{a, b\} + zaD(b) \in zB,$$

which shows that the ideal zB of B is $\{,\}$-invariant. Therefore, $zB = B$ and z is invertible, $z = \alpha + n$, where $0 \neq \alpha \in F$ and n is nilpotent. Finally, $n = z - \alpha \in Z(B, \{,\})$ hence $n = 0$ and $z \in F$. ∎

Let $U^{(+)} \cong Kan(A)$. Fix an element $\alpha \in Kan(A)_{\bar{0}}$ and define a new product $[,] = [,]_\alpha$ on $Kan(A)$ by the rules:

$$[a, b] = [\bar{a}, b] = [a, \bar{b}] = 0, \;\; [\bar{a}, \bar{b}] = (-1)^b ab\,\alpha.$$

It is easy to check (see also [24, Theorem 3]) that the product $[,]_\alpha$ is a Leibniz bracket on $Kan(A)$. We will denote the corresponding NJ-superalgebra U as $Kan(A, \alpha)$.

Let $[x, y]$ be a Leibniz bracket on $J = Kan(A)$. Observe that in general this bracket is not homogeneous with respect to the Z_2-grading of J given by bar: $J = A \oplus \bar{A}$; that is, for instance, $[A, A]$ may not lie in A. We know only that $[J_{\bar{i}}, J_{\bar{j}}] \subseteq J_{\bar{i}+\bar{j}}$. Therefore, we have $[,] = [,]_0 + [,]_1$, where $[,]_i$ is a bracket homogeneous of degree i with respect to the Z_2-grading $J = A \oplus \bar{A}$. In other words,

$$[A, A]_0 + [\bar{A}, \bar{A}]_0 \subseteq A, \;\; [A, \bar{A}]_0 \subseteq \bar{A},$$
$$[A, A]_1 + [\bar{A}, \bar{A}]_1 \subseteq \bar{A}, \;\; [A, \bar{A}]_1 \subseteq A.$$

It is easy to check that the two brackets $[,]_0$ and $[,]_1$ satisfy the Leibniz identity (12), that is, the both are Leibniz brackets on J.

Consider first the case of an even Leibniz bracket $[,]$ on $Kan(A)$. Observe that for any $a \in A$ we have

$$\bar{a}\bar{1} = \{a, 1\} = D(a). \tag{22}$$

Denote $[a, \bar{1}] = E(a)\bar{1}$, then $E : A \to A$ is an even derivation of the superalgebra A. Set also $\alpha = [\bar{1}, \bar{1}] \in A_{\bar{0}}$.

Proposition 4. *Let $[,]$ be an even Leibniz bracket on $Kan(A)$. Then for any $a, b, c \in A$ the following identities hold*

$$\begin{aligned}
[\bar{a}, \bar{b}] &= (-1)^b(ab\alpha - aD(E(b)) - D(E(a)b) - D([a, b])), && (23)\\
[a, \bar{b}] &= (E(a)b + [a, b])\bar{1}, && (24)\\
[\{a, b\}, c] &= -E(\{a, b\})c - aE(b)D(c) - aD(E(b))c - aD([b, c]) \\
&+ E(a)bD(c) + D(E(a))bc + (-1)^{bc}D([a, c])b, && (25)\\
\{[a, b], c\} &= D([a, b])c + [a, b]D(c) + E(a)D(b)c + D(E(a))bc \\
&+ D(a)bE(c) - D(a)[b, c] - E(a)\{b, c\} - (-)^{bc}\{E(a), c\}b, && (26)\\
[\{a, b\}, c] &= \{a, [b, c] - bE(c)\} - (-1)^{ab}(\{b, [a, c] - aE(c)\}). && (27)
\end{aligned}$$

Conversely, let $[,]$ be a Leibniz bracket on A and E be an even derivation of A which satisfy idenities (25), (26), (27). Then for any $\alpha \in A_{\bar{0}}$ the super-anticommutative extension of $[,]$ on $Kan(A)$ given by (23) and (24) defines a Leibniz bracket on $Kan(A)$.

Proof. Let $a, b \in A$, consider

$$\begin{aligned}
[\bar{a}, \bar{b}] &= [a\bar{1}, b\bar{1}] = a[\bar{1}, b\bar{1}] + (-1)^{\bar{b}}[a, b\bar{1}]\bar{1} = \\
&= (-1)^b ab\alpha + a([\bar{1}, b]\bar{1}) - (-1)^{b+ab}(b[a, \bar{1}])\bar{1} - (-1)^b([a, b]\bar{1})\bar{1} \\
&= (-1)^b ab\alpha - (-1)^b aD(E(b)) - (-1)^b D(E(a)b) - (-1)^b D([a, b]),
\end{aligned}$$

which proves (23). Furthermore,

$$[a, \bar{b}] = [a, b\bar{1}] = (-1)^{ab}b[a, \bar{1}] + [a, b]\bar{1} = (E(a)b + [a, b])\bar{1},$$

proving (24).

Consider now

$$\begin{aligned}
[\bar{a}\bar{b}, \bar{c}] &= \bar{a}[\bar{b}, \bar{c}] + (-1)^{\bar{b}\bar{c}}[\bar{a}, \bar{c}]\bar{b} = (-1)^c\bar{a}(bc\alpha - bD(E(c)) - D(E(b)c) - D([b, c])) \\
&+ (-1)^{\bar{b}\bar{c}+c}(ac\alpha - aD(E(c)) - D(E(a)c) - D([a, c]))\bar{b} \\
&= (-1)^b(abc\alpha - abD(E(c)) - aE(b)D(c) - aD(E(b))c - aD([b, c]))\bar{1} \\
&+ ((-1)^{\bar{b}}(abc\alpha - abD(E(c)) - E(a)bD(c) - D(E(a))bc) + (-1)^{bc+b}D([a, c])b)\bar{1} \\
&= (-1)^b(-aE(b)D(c) - aD(E(b))c - aD([b, c]) + E(a)bD(c) + D(E(a))bc \\
&+ (-1)^{bc}D([a, c])b)\bar{1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[\bar{a}\bar{b}, \bar{c}] &= (-1)^b[\{a, b\}, c\bar{1}] = (-1)^b([\{a, b\}, c]\bar{1} + (-1)^{c(a+b)}c[\{a, b\}, \bar{1}]) \\
&= (-1)^b([\{a, b\}, c] + E(\{a, b\})c)\bar{1}.
\end{aligned}$$

Thus we have an equality

$$\begin{aligned}[\{a,b\},c]+E(\{a,b\})c &= -aE(b)D(c)-aD(E(b))c-aD([b,c]) \\ &+ E(a)bD(c)+D(E(a))bc+(-1)^{bc}D([a,c])b,\end{aligned}$$

which implies (25).

Consider next

$$\begin{aligned}[a\bar{b},\bar{c}] &= a[\bar{b},\bar{c}]+(-1)^{\bar{b}\bar{c}}[a,\bar{c}]\bar{b}=(-1)^{c}a(bc\alpha-bD(E(c))-D(E(b)c)-D([b,c])) \\ &+ (-1)^{\bar{b}\bar{c}+b}\{([a,c]+E(a)c),b\}.\end{aligned}$$

On the other hand,

$$[a\bar{b},\bar{c}] = [\overline{ab},\bar{c}]=(-1)^{c}(abc\alpha-abD(E(c))-D(E(ab)c)-D([ab,c])),$$

which implies

$$\begin{aligned}0 &= (-1)^{c}(-aD(E(b))c-aE(b)D(c)-aD([b,c])) \\ &- (-1)^{bc+c}(\{([a,c],b\}+E(a)\{c,b\}+(-1)^{bc}\{E(a),b\}c+E(a)cD(b)) \\ &+ (-1)^{c}((E(a)b+aE(b))D(c)+D(E(a)b+aE(b))c+D(a[b,c]+(-1)^{bc}[a,c]b)) \\ &= -(-1)^{bc+c}(\{([a,c],b\}+E(a)\{c,b\}+(-1)^{bc}\{E(a),b\}c \\ &+ (-1)^{c}(E(a)bD(c)+D(E(a))bc+D(a)E(b)c \\ &+ D(a)[b,c]+(-1)^{bc}D([a,c])b+[a,c]D(b))\end{aligned}$$

or

$$\begin{aligned}&E(a)bD(c)+D(E(a))bc+D(a)E(b)c+D(a)[b,c]+E(a)\{b,c\}-\{E(a),b\}c \\ &=(-1)^{bc}(\{[a,c],b\}-D([a,c])b-[a,c]D(b)).\end{aligned}$$

proving (26). Finally, consider

$$\begin{aligned}[\bar{a}\bar{b},c] &= \bar{a}[\bar{b},c]+(-1)^{\bar{b}c}[\bar{a},c]\bar{b} \\ &= (-1)^{c}\bar{a}(([b,c]-bE(c))\bar{1})+(-1)^{bc}(([a,c]-aE(c))\bar{1})\bar{b} \\ &= (-1)^{b}(\{a,[b,c]-bE(c)\}-(-1)^{ab}(\{b,[a,c]-aE(c)\})).\end{aligned}$$

On the other hand,

$$[\bar{a}\bar{b},c]=(-1)^{b}[\{a,b\},c],$$

which implies (27).

The last statement of the proposition follows easily from the proof of identities (25), (26), (27).

■

Consider now the case when the Leibniz bracket $[,]$ is odd. In this case we have $[A,A]+[\bar{A},\bar{A}]\subseteq\bar{A}$ and $[\bar{A},A]\subseteq A$. Denote

$$[\bar{1},\bar{1}]=\alpha\bar{1},\ [a,\bar{1}]=a^{E},\ [a,b]=\langle a,b\rangle\bar{1}=\overline{\langle a,b\rangle},$$

then $\alpha\in A_{\bar{1}}$, $E\colon A\to A$ is an odd derivation of A, and $\langle a,b\rangle$ is an odd Leibniz bracket on A, that is,

$$\begin{aligned}E(A_i)\subseteq A_{1-i}, &\qquad (ab)^{E}=ab^{E}+(-1)^{b}a^{E}b, \\ \langle A_i,A_j\rangle\subseteq A_{i+j-1}, &\qquad \langle ab,c\rangle=a\langle b,c\rangle+(-1)^{bc+b}\langle a,c\rangle b.\end{aligned}$$

Proposition 5. *Let $[,]$ be an odd Leibniz bracket on $Kan(A)$. Then for any $a, b, c \in A$ the identities hold*

$$\begin{aligned} [\bar{a}, \bar{b}] &= (-1)^b (ab\alpha - (ab)^E)\bar{1}, & (28)\\ D(\langle a, b\rangle) &= 0, & (29)\\ [a, \bar{b}] &= (-1)^b a^E b, & (30)\\ \langle \{a, b\}, c\rangle &= 0, & (31)\\ \{a, b\}^E c &= (-1)^{c+1}(\{a, bc\alpha - (bc)^E\} - (-1)^{ab}\{b, ac\alpha - (ac)^E\}). & (32) \end{aligned}$$

Conversely, let $\langle,\rangle$ be an odd Leibniz bracket on A and E be an odd derivation of A which satisfy idenities (31), (32). Then for any $\alpha \in A_{\bar{1}}$ the super-anticommutative bracket $[,]$ on $Kan(A)$ given by relations $[a, b] = \langle a, b\rangle\bar{1}$, (28), (30), defines an odd Leibniz bracket on $Kan(A)$.

Proof. Consider

$$\begin{aligned} [\bar{a}, \bar{b}] &= [a\bar{1}, b\bar{1}] = a[\bar{1}, b\bar{1}] + (-1)^{\bar{b}}[a, b\bar{1}]\bar{1}\\ &= a((-1)^b b\alpha\bar{1} + [\bar{1}, b]\bar{1}) + (-1)^{\bar{b}+ab} ba^E\bar{1} + (-1)^{\bar{b}}(\overline{\langle a, b\rangle}\bar{1})\bar{1}\\ &= (-1)^b(ab\alpha - ab^E - (-1)^b a^E b - D(\langle a, b\rangle))\bar{1}\\ &= (-1)^b(ab\alpha - (ab)^E - D(\langle a, b\rangle))\bar{1}. \end{aligned}$$

On the other hand,

$$\begin{aligned} [\bar{a}, \bar{b}] &= -(-1)^{\bar{a}\bar{b}}[\bar{b}, \bar{a}] = (-1)^{ab+b+a+a}(ba\alpha - (ba)^E - D(\langle b, a\rangle))\bar{1}\\ &= (-1)^b(ab\alpha - (ab)^E + D(\langle a, b\rangle))\bar{1}, \end{aligned}$$

which implies (28) and (29).

Furthermore,

$$[a, \bar{b}] = [a, b\bar{1}] = (-1)^{ab} ba^E + [a, b]\bar{1} = (-1)^b a^E b + D(\langle a, b\rangle) = (-1)^b a^E b,$$

proving (30). Now

$$\begin{aligned} [\bar{a}\bar{b}, c] &= \bar{a}[\bar{b}, c] + (-1)^{\bar{b}c}[\bar{a}, c]\bar{b}\\ &= (-1)^{c+1}\bar{a}(bc^E) + (-1)^{\bar{b}c+c+1}(ac^E)\bar{b}\\ &= (-1)^b(abc^E)\bar{1} + (-1)^{b+1}(abc^E)\bar{1} = 0. \end{aligned}$$

On the other hand,

$$[\bar{a}\bar{b}, c] = (-1)^b[\{a, b\}, c],$$

hence $[\{a, b\}, c] = 0$, proving (31).

Finally, consider

$$\begin{aligned} [\bar{a}\bar{b}, \bar{c}] &= \bar{a}[\bar{b}, \bar{c}] + (-1)^{\bar{b}\bar{c}}[\bar{a}, \bar{c}]\bar{b} = (-1)^c\bar{a}((bc\alpha - (bc)^E)\bar{1}) + (-1)^{\bar{b}\bar{c}+c}((ac\alpha - (ac)^E)\bar{1})\bar{b}\\ &= (-1)^{b+1}\{a, bc\alpha - (bc)^E\} + (-1)^{\bar{c}\bar{b}+c+b}\{ac\alpha - (ac)^E, b\}\\ &= (-1)^{b+1}(\{a, bc\alpha - (bc)^E\} - (-1)^{ab}\{b, ac\alpha - (ac)^E\}). \end{aligned}$$

On the other hand,

$$\begin{aligned}
[\bar{a}\bar{b}, \bar{c}] &= (-1)^b[\{a,b\}, c\bar{1}] = (-1)^b([\{a,b\},c]\bar{1} + (-1)^{c(a+b)}c[\{a,b\},\bar{1}]) \\
&= (-1)^b D(\langle\{a,b\},c\rangle) + (-1)^{c+b}\{a,b\}^E c = (-1)^{c+b}\{a,b\}^E c,
\end{aligned}$$

which proves (32).

To prove the last statement, one has to check only that $[a\bar{b},\bar{c}] = a[\bar{b},\bar{c}] + (-1)^{\bar{b}\bar{c}}[a,\bar{c}]\bar{b}$. In fact,

$$\begin{aligned}
& a[\bar{b},\bar{c}] + (-1)^{\bar{b}\bar{c}}[a,\bar{c}]\bar{b} = (-1)^c a((bc\alpha - (bc)^E)\bar{1}) + (-1)^{\bar{b}\bar{c}+c}a^E c\bar{b} \\
= \; & (-1)^c(abc\alpha - a(bc)^E)\bar{1} + (-1)^{b+1}a^E bc\bar{1} = (-1)^c(abc\alpha - (abc)^E)\bar{1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[a\bar{b},\bar{c}] &= [\overline{ab},\bar{c}] = (-1)^c(abc\alpha - (abc)^E)\bar{1} \\
&= (-1)^c(abc\alpha - a(bc)^E)\bar{1} + (-1)^{b+1}a^E bc\bar{1} = (-1)^c(abc\alpha - (abc)^E)\bar{1},
\end{aligned}$$

proving the proposition.

■

6.2.1 $U^{(+)} \cong Kan(A)$, the case of Poisson brackets.

Let us call a unital Poisson superalgebra A over a field F *central* if its bracket-center $Z(A,\{,\})$ coincides with F.

Theorem 10. *Let $A = (A,\cdot,\{,\})$ be a central Poisson superalgebra over a field F such that $\dim A_{\bar{1}} > 1$. Then every Leibniz bracket $[,]$ on $Kan(A)$ has a form $[,]_\alpha$ where $\alpha = [\bar{1},\bar{1}] \in A_{\bar{0}}$.*

Proof. We will prove that every even Leibniz bracket $[,]$ on $Kan(A)$ has a form $[,]_\alpha$ and every odd Leibniz bracket is trivial.

Consider first the case of even bracket. We have $D = 0$, hence (23) implies

$$[\bar{a},\bar{b}] = (-1)^b ab\,\alpha,$$

where $\alpha = [\bar{1},\bar{1}] \in A_{\bar{0}}$. Furthermore, putting $b = 1$ in (27) we get $\{E(a),c\} = 0$, which means that $E(A) \subseteq Z(A,\{,\}) = F$. In particular, $E(A_{\bar{1}}) = 0$. Assume that $E(A_{\bar{0}}) \neq 0$ and let $a \in A_{\bar{0}}$, $E(a) \neq 0$. Since E is a derivation, we have $E(a^2) = 2aE(a)$ which implies that $a \in F$. But $E(1) = 0$ since E is a derivation, hence $E(F) = 0$ and $E(a) = 0$, a contradiction. Therefore, $E = 0$. Now from (26) we have $\{[a,b],c\} = 0$, that is, $[A,A] \subseteq F$. As a corollary, $[A_{\bar{1}},A_{\bar{0}}] = 0$. Assume that $[A,A] \neq 0$ and let $a,b \in A_{\bar{0}}$ such that $[a,b] \neq 0$. Then $F \ni [a,b^2] = 2[a,b]b$ and $b \in F$, a contradiction. Hence $[A_{\bar{0}},A_{\bar{0}}] = 0$. Furthermore, assume that $[x,x] \neq 0$ for some $x \in A_{\bar{1}}$. Then for any $y \in A_{\bar{1}}$ we have $[x,x]y = [x,xy] - (-1)^{xy}[x,y]x = -(-1)^{xy}[x,y]x$, which implies that $A_{\bar{1}} = Fx$, a contradiction. Hence $[A,A] = 0$.

Finally, by (24) $[a,\bar{b}] = \overline{[a,b]} = 0$, hence $[A,\bar{A}] = 0$ and $[,] = [,]_\alpha$.

Consider now the case when the Leibniz bracket $[,]$ is odd. In this case we have $[A,A] + [\bar{A},\bar{A}] \subseteq \bar{A}$ and $[\bar{A},A] \subseteq A$. Denote, as above,

$$[\bar{1},\bar{1}] = \alpha\bar{1},\; [a,\bar{1}] = a^E,\; [a,b] = \langle a,b\rangle\bar{1} = \overline{\langle a,b\rangle},$$

then $\alpha \in A_{\bar{1}}$, $E\colon A \to A$ is an odd derivation of A, and $\langle a, b\rangle$ is an odd Leibniz bracket on A.

Substituting $b = c = 1$ in (32), we get $\{a, \alpha\} = 0$, that is, $\alpha \in Z(A, \{,\}) = F$, which gives $\alpha = 0$. Now for $b = 1$ the same identity gives $\{a, c^E\} = 0$, that is, $A^E \subseteq F$, In particular, we have $(A_{\bar{0}})^E = 0$.

Assume that there exists $x \in A_{\bar{1}}$ with $x^E \neq 0$. Then for any $y \in A_{\bar{1}}$ we have $0 = (xy)^E = xy^E - x^E y$, which implies $y \in Fx$ and $\dim A_{\bar{1}} = 1$, a contradiction. Therefore, $E = 0$, and from (28), (30) we have

$$[A, \bar{A}] = [\bar{A}, \bar{A}] = 0.$$

Finally, consider

$$(-1)^c\{\langle a, b\rangle, c\} = \overline{\langle a, b\rangle}\bar{c} = [a, b]\bar{c} = [a, \overline{bc}] - (-1)^{ab} b[a, \bar{c}] = 0,$$

which implies $\langle A, A\rangle \subseteq F$. In particular, we have

$$[A_{\bar{0}}, A_{\bar{0}}] = [A_{\bar{1}}, A_{\bar{1}}] = 0.$$

Assume that there exist $a \in A_{\bar{0}}$, $x \in A_{\bar{1}}$ such that $F \ni \langle a, x\rangle \neq 0$. Then for any $y \in A_{\bar{1}}$ we have

$$\langle a, x\rangle y = \langle a, xy\rangle - x\langle a, y\rangle = -x\langle a, y\rangle,$$

which implies that $\dim A_{\bar{1}} = 1$, a contradiction. Therefore, $[A, A] = 0$, proving the theorem. ■

Observe that the restriction $\dim A_{\bar{1}} > 1$ is essential for the theorem. In fact, for the Grassmann superalgebra $\mathit{\Gamma}_1$ we have $\dim \mathit{\Gamma}_1 = 1$ and the superalgebra $Kan(\mathit{\Gamma}_1) \cong M_{1|1}^{(+)}$ has nontrivial odd Leibniz brackets different from $[,]_\alpha$ (see Corollary 2 and Theorem 5).

Corollary 3. *Let $\{,\}$ be a Poisson bracket on $B = B(m, n)$ such that $Kan(B)$ is a simple superalgebra. Then every Leibniz bracket on $Kan(B)$ has form $[,]_\alpha$ for some $\alpha \in B_{\bar{0}}$ except the case $m = 0$, $n = 1$, when $Kan\, B \cong M_{1\,|\,1}^{(+)}$ and Leibniz brackets on it are determined by Corollary 2 and Theorem 5.*

Proof. Since $Kan(B)$ is simple, the superalgebra B is bracket–simple, that is, it is a simple Poisson superalgebra. In this case the bracket-center $Z(B, \{,\}) \subseteq B_{\bar{0}}$ is a field, which coincides with F (since $B = F + B_+$ where B_+ is a nilpotent ideal). Thus B is central and by the theorem we may assume that $\dim B_{\bar{1}} = 1$, Clearly, this means that $n = 1, m = 0$, and $Kan\, B \cong M_{1\,|\,1}^{(+)}$. ■

The following result was proved in [24], but the case of odd Leibniz bracket was not considered there.

Corollary 4. *Let $U^{(+)} \cong Kan(\mathit{\Gamma}_n)$ for $n > 1$, then $U \cong Kan(\mathit{\Gamma}_n, \alpha)$ for some $\alpha \in (\mathit{\Gamma}_n)_{\bar{0}}$.*

For the proof it suffices to note that the only possible Jordan bracket on $\mathit{\Gamma}_n$ is a Poisson bracket (see [10, 11]).

6.2.2 $U^{(+)} \cong Kan(A)$, the case of vector brackets.

Recall that a bracket $\{,\}$ on a commutative superalgebra A is called of *vector type* if there exists a derivation $D \in Der\, A$ such that $\{a, b\} = D(a)b - aD(b)$. It is known that in this case $Kan(A)$ is simple if and only if A is D-simple, that is A does not contain non-trivial D-invariant ideals. Moreover, if A is D-simple then $A = A_{\bar{0}}$ [14] and A is unital [22].

Theorem 11. *Let $U^{(+)} \cong Kan(A)$ where (A, D) is a D-simple commutative algebra with the bracket of vector type. If A is bracket-central, that is, $Z(A, \{,\}) = F$, then $U \cong Kan(A, \alpha)$ for some $\alpha \in A = A_{\bar{0}}$.*

Proof. Let $[,]$ be a Leibniz bracket on $Kan(A)$. Denote again $[1, \bar{a}] = E(a)\bar{1}$, $[\bar{1}, \bar{1}] = \alpha$, then $E \in Der\, A$ and $\alpha \in A$. We will show that $E = 0$ and $[A, A] = [A, \bar{A}] = 0$.

From (25) for $c = 1$ we get

$$E(\{a, b\}) = -aD(E(b)) + D(E(a))b,$$

Returning to (25), we have

$$[\{a, b\}, c] = -aE(b)D(c) + bE(a)D(c) - aD([b, c]) + bD([a, c]),$$

which for $b = 1$ gives

$$[D(a), c] = E(a)D(c) + D([a, c]) \tag{33}$$

Substituting in the previos identity $\{a, b\} = D(a)b - aD(b)$, we get

$$\begin{aligned} &D(a)[b, c] + [D(a), c]b - a[D(b), c] - [a, c]D(b) = \\ &= -aE(b)D(c) + bE(a)D(c) - aD([b, c]) + bD([a, c]), \end{aligned}$$

which in view of (33) gives

$$D(a)[b, c] - D(b)[a, c] = 0.$$

Now we have

$$D(a)[b, c] = D(b)[a, c] = D(c)[a, b] = D(a)[c, b] = 0,$$

that is, $D(A)[A, A] = 0$. Since A is D-simple, this implies $[A, A] = 0$. Then (33) gives $E(A)D(A) = 0$, implying $E(A) = 0$. Finally, (23) gives $[A, \bar{A}] = 0$ and (23) finishes the proof.

■

One can easily check that every Jordan bracket on the polynomial algebra $F[x]$ (or on the truncated algebra $F[x]/(x^p)$ in case of characteristic $p > 0$) is of vector type for the derivation $D = \{x, 1\}\frac{d}{dx}$. Therefore, we have

Corollary 5. *If $U^{(+)} \cong Kan(B(1, 0))$ where $Kan(B(1, 0))$ is a simple Jordan superalgebra then $U \cong Kan(B(1, 0), \alpha)$ for some $\alpha \in B(1, 0)$.*

It is easy to see that in general the NJ-superalgebras $Kan(B(m,n),\alpha)$ are non-isomorphic for different α. For example, we may take $\alpha = 1, n \geq 1$, and in this case the Leibniz bracket is zero on some subspace in $Kan(B(m,n),\alpha)$ of codimension > 1; but for the case $\alpha = t_1^{p-1} \dots t_m^{p-1} \otimes \xi_1 \dots \xi_n$ $(n = 2k \geq 2)$ this codimension is 1.

At the moment, we have no examples of Leibniz brackets on $Kan(B(m,n))$ different from $[,]_\alpha$ except of the case $B(0,1)$ mentioned in Corollary 3.

Problem 1. *Describe Leibniz brackets on $Kan(B(m,n))$ in general case.*

Propositions 4 and 5 may be useful in such a description.

6.3 $U^{(+)} \cong V_{\frac{1}{2}}(A,D)$.

Recall the definition of the semi-unital Jordan superalgebra $V_{\frac{1}{2}}(A,D)$.

Let (A,D) be again an associative commutative algebra with a derivation D. Consider the direct sum of vector spaces

$$V_{\frac{1}{2}}(A,D) = A \oplus \bar{A},$$

where $\bar{A}$ is an isomorphic copy of A, and extend on it the product in A via

$$a\bar{b} = \bar{b}a = \tfrac{1}{2}\overline{ab},\ \bar{a}\bar{b} = -\bar{b}\bar{a} = \{a,b\},$$

with the vector bracket $\{a,b\} = D(a)b - aD(b)$. If we put $V_{\frac{1}{2}}(A,D)_{\bar{0}} = A$, $V_{\frac{1}{2}}(A,D)_{\bar{1}} = \bar{A}$ then $V_{\frac{1}{2}}(A,D)$ became a commutative superalgebra which turns out to be a Jordan superalgebra. Moreover, if A is D-simple then the superalgebra $V_{\frac{1}{2}}(A,D)$ is simple. Observe that the unit 1 of A acts on $\bar{A}$ as $\frac{1}{2}$, that is why we call this superalgebra *semi-unital.*

Theorem 12. *Let $U^{(+)} \cong V_{\frac{1}{2}}(A,D)$ where (A,D) is a D-simple commutative algebra with the bracket of vector type. If A is bracket-central, that is, $Z(A,\{,\}) = F$, then $U \cong V_{\frac{1}{2}}(A,D)$.*

Proof. Let $[,]$ be a Leibniz bracket on $V_{\frac{1}{2}}(A,D)$, we will prove that it is trivial. The proof follows that of Theorem 11. Set again

$$[a,\bar{1}] = E(a)\bar{1} = \tfrac{1}{2}\overline{E(a)},\ [\bar{1},\bar{1}] = \alpha.$$

Observe that E is a derivation of A. We have

$$\begin{aligned}
[\bar{a},\bar{b}] &= 4[a\bar{1},b\bar{1}] = 4a[\bar{1},b\bar{1}] - 4[a,b\bar{1}]\bar{1} = \\
&= 4ab[\bar{1},\bar{1}] + 4a([\bar{1},b]\bar{1}) - 4(b[a,\bar{1}])\bar{1} - 4([a,b]\bar{1})\bar{1} \\
&= 4ab\alpha - 2aD(E(b)) - D(E(a)b) - 2D([a,b]), \\
[a,\bar{b}] &= 2[a,b\bar{1}] = 2(E(a)b + [a,b])\bar{1}.
\end{aligned}$$

Consider

$$\begin{aligned}
[\bar{a}\bar{b},\bar{c}] &= \bar{a}(4bc\alpha - 2bD(E(c)) - D(E(b)c) - 2D([b,c])) \\
&+ (-4ac\alpha + 2aD(E(c)) + D(E(a)c) + 2D([a,c]))\bar{b} \\
&= (-aD(E(b)c) + bD(E(a)c) - 2aD([b,c]) + 2bD([a,c]))\bar{1}.
\end{aligned}$$

On the other hand,

$$[\bar{a}\bar{b},\bar{c}] = [\{a,b\},\bar{c}] = 2(E(\{a,b\})c + [\{a,b\},c])\bar{1},$$

which gives

$$2(E(\{a,b\})c + [\{a,b\},c]) = -aD(E(b)c) + bD(E(a)c) - 2aD([b,c]) + 2bD([a,c]). \quad (34)$$

Putting here $c = 1$ we get

$$2E(\{a,b\}) = -aD(E(b)) + bD(E(a)).$$

Returning to (34) we have

$$[\{a,b\},c] = -aE(b)D(c) + bE(a)D(c) - 2aD([b,c]) + 2bD([a,c]),$$

which for $b = 1$ gives

$$[D(a),c] = E(a)D(c) + 2D([a,c]). \quad (35)$$

Substituting in the previos identity $\{a,b\} = D(a)b - aD(b)$, we get

$$\begin{gathered} D(a)[b,c] + [D(a),c]b - a[D(b),c] - [a,c]D(b) = \\ = -aE(b)D(c) + bE(a)D(c) - 2aD([b,c]) + 2bD([a,c]), \end{gathered}$$

which in view of (35) gives

$$D(a)[b,c] = D(b)[a,c].$$

As in the proof of Theorem 11, this implies $[A,A] = 0$. Now (35) gives $E(A)D(A) = 0$ and $E = 0$.

Finally, consider

$$\alpha = 2[1\cdot\bar{1},\bar{1}] = 2[\bar{1},\bar{1}] - 2[1,\bar{1}]\bar{1} = 2\alpha - 2(E(1)\bar{1})\bar{1} = 2\alpha,$$

hence $\alpha = 0$ and the braket $[,]$ is trivial.

■

6.4 $U^{(+)} \cong B(m,n)$.

It remains to consider the case when U is a simple noncommutative Jordan superalgebra such that $U^{(+)} \cong B(m,n)$.

Fix three sets of elements

$$\begin{aligned} \mathcal{A} &= \{a_{ij} = -a_{ji} \in B(m,n)_{\bar{0}},\ 1 \le i < j \le m\}, \\ \mathcal{B} &= \{b_{ij} = b_{ji} \in B(m,n)_{\bar{0}},\ i,j = 1,\ldots,n\}, \\ \mathcal{C} &= \{c_{ij} \in B(m,n)_{\bar{1}},\ i = 1,\ldots,m,\ j = 1,\ldots,n\}, \end{aligned}$$

and define a Leibniz braket on $B(m,n)$ by setting

$$[t_i,t_j] = a_{ij},\ [t_i,\xi_k] = c_{ik},\ [\xi_k,\xi_l] = b_{kl}; \quad (36)$$

$$i,j = 1,\ldots,m;\ k,l = 1,\ldots,n. \quad (37)$$

then the space $B(m,n)$ with the multiplication $a * b = a \cdot b + \frac{1}{2}[a,b]$ forms a *nodal NJ*-superalgebra $B(m,n)(\mathcal{A},\mathcal{B},\mathcal{C})$ (compare [15, 16]).

For every $i = 1,\dots,m;, j = 1\dots,n$ define even and odd derivations on $B(m,n)(\mathcal{A},\mathcal{B},\mathcal{C})$:

$$D_i : x \to [x, t_i], \tag{38}$$

$$E_k : x \to [x, \xi_k]. \tag{39}$$

Then the superalgebra $B(m,n)(\mathcal{A},\mathcal{B},\mathcal{C})$ is simple iff the superalgebra $B(m,n)$ is $\mathcal{D}$-simple relative to the set of derivations $\mathcal{D} = \{D_i,\, E_k, \,|\, 1 \leq i \leq m,\, 1 \leq k \leq n\}$.

Theorem 13. *Let A be a NJ-superalgebra such that $A^{(+)} \cong B(m,n)$. Then $A \cong B(m,n)(\mathcal{A},\mathcal{B},\mathcal{C})$ for certain $\mathcal{A} \in M_m(B(m,n)_{\bar 0})$, $\mathcal{B} \in M_n(B(m,n)_{\bar 0})$, $\mathcal{C} \in M_{m,n}(B(n,m)_{\bar 1})$.*

Note that by analogy with [24] we may effectively answer the question about the D-simplicity of $B(m,n)$ for every fixed derivation D of $B(m,n)$. Also some examples of non-isomorphic superalgebras $B(m,n)(\mathcal{D})$ for different $\mathcal{D}$ may be found in [24].

Finally, we may state the main theorem of the article.

Theorem 14. *Let U be a finite-dimensional central simple NJ-superalgebra over an algebraically closed field field F of characteristic $\neq 2$. Assume that U is neither quasi-associative nor supercommutative. Then either $U \cong K_3(\alpha), NK_3,\ D_t(\alpha), ND_t,\ J(V,f,\times)$, $Kan(B(m,n),\alpha)$, $B(m,n)(\mathcal{A},\mathcal{B},\mathcal{C})$ or $U \cong (Kan(B(m,n)),[,])$ where the Leibniz bracket $[,]$ on $Kan(B(m,n))$ is not of type $[,]_\alpha$.*

We do not have examples of Leibniz bracket of the last type except the case $Kan(B(0,1))$ considered in Corollary 2. As we have seen, it may appear only when the Jordan bracket on $B(m,n)$ is of mixed (nor Poisson neither vector) type. In particular, for $n > 1$ one should necessary have $m > 0$.

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