THE BLOCK STRUCTURE SPACES OF REAL PROJECTIVE SPACES AND ORTHOGONAL CALCULUS OF FUNCTORS

TIBOR MACKO

Abstract. Given a compact manifold X, the set of manifold structures on $X \times \Delta^k$ relative to the boundary can be viewed as the k-th homotopy group of a space $\widetilde{\mathcal{S}}^s(X)$. This space is called the block structure space of X.

We study the block structure spaces of real projective spaces. Generalizing Wall's join construction we show that there is a functor from the category of finite dimensional real vector spaces with inner product to the category of pointed spaces which sends the vector space V to the block structure space of the projective space of V. We study this functor from the point of view of orthogonal calculus; we show that it is polynomial of degree ≤ 1 in the sense of orthogonal calculus.

This result suggests an attractive description of the block structure space of the infinite-dimensional real projective space via the Taylor tower of orthogonal calculus. This space is defined as a colimit of block structure spaces of projective spaces of finite-dimensional real vector spaces and is closely related to some automorphisms spaces of real projective spaces.

Introduction

The central objects of study in this paper are the block structure spaces of certain topological manifolds. These spaces arise as follows. For a closed topological manifold X, a *simple manifold structure* on X is a simple homotopy equivalence f from some closed manifold M to X. Two simple manifold structures $f_0 \colon M_0 \to X$ and $f_1 \colon M_1 \to X$ are *equivalent* if there exists a homeomorphism $g \colon M_0 \to M_1$ and a homotopy from $f_1 \circ g$ to f_0. The set of equivalence classes is called the *simple structure set* of X and is denoted by $\mathbb{S}^s(X)$. More generally, if X is a compact manifold with boundary, there is a simple structure set $\mathbb{S}^s_\partial(X)$ whose elements are represented by simple homotopy equivalences of pairs $f \colon (M, \partial M) \to (X, \partial X)$ where the restricted map $\partial M \to \partial X$ is a homeomorphism.

For a given compact manifold X, Quinn constructed in [Qu] a space $\widetilde{\mathcal{S}}^s(X)$, the *block structure space* of X, such that

$$\pi_k(\widetilde{\mathcal{S}}^s(X)) \cong \mathbb{S}^s_\partial(X \times \Delta^k). \tag{1}$$

This space is related to the automorphism spaces of X via the homotopy fibration sequence

$$\widetilde{\mathrm{TOP}}(X) \to \mathrm{G}^s(X) \to \widetilde{\mathcal{S}}^s(X). \tag{2}$$

Here the space $\widetilde{\mathrm{TOP}}(X)$ is the block version of the space $\mathrm{TOP}(X)$ of self homeomorphisms of X and $\mathrm{G}^s(X)$ is the space of simple self homotopy equivalences of X. See [WW], [BLR].

The structure sets $\mathbb{S}^s(X)$ have been studied in many special cases. In particular they were calculated for real projective spaces, lens spaces and complex projective spaces by work of many authors, especially Wall [Wa, chapter 14], Browder [Br], Browder-Livesay [BL] and López de Medrano [LdM]. A useful tool used by these authors is the *join construction* (see [Wa, chapter 14]). It provides us with maps

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between structure sets in different dimensions. For example in the case of real projective spaces we get maps $\mathbb{S}^s(\mathbb{R}P^n) \to \mathbb{S}^s(\mathbb{R}P^{n+m})$. We generalize the join construction to define maps between the block structure spaces.

In principle the homotopy type of the block structure space of any manifold is well understood as the homotopy fiber in the Sullivan-Wall-Quinn-Ranicki homotopy fibration sequence of surgery (see [WW, Theorem 2.3.1.]). Unfortunately, if $m > 2$ this description of the block structure space does not easily extend to an illuminating description of the maps $\widetilde{\mathcal{S}}^s(\mathbb{R}P^n) \to \widetilde{\mathcal{S}}^s(\mathbb{R}P^{n+m})$ induced by the join construction (Ranicki showed me how to handle the cases $m = 1, 2$). We propose to remedy this defect by using ideas from functor calculus.

For this purpose we formulate the join construction as a certain continuous functor. In the case of real projective spaces it is a functor, which we denote F, from the category J of real finite dimensional vector spaces with inner product to the category Spaces_* of pointed spaces which sends the vector space V to the block structure space of $\mathbb{R}P(V)$, the real projective space of V. There are variants of this for the lens spaces and complex projective spaces. The general methods we use apply to all cases. However, in the paper we specialize to the case of real projective spaces. We make remarks on the other cases at the end of the introduction.

The advantage of the formulation of the join construction as a continuous functor is that it enables us to use the orthogonal calculus of functors to study the block structure spaces of real projective spaces. The orthogonal calculus provides us with a general theory for studying continuous functors from J (as above) to Spaces_*. It was developed by Weiss in [We]. He associates to a given continuous functor E the tower of functors

$$\cdots \to T_k E \to T_{k-1} E \to \cdots \to T_0 E,$$

called the *Taylor tower*. For each k the functor $T_k E$ is a *polynomial functor of degree* $\leq k$ in a certain sense and it is also the best approximation of E by such a functor. Therefore it should be thought as the k-th Taylor polynomial of E. For example, the functor $T_0 E$ is defined by $T_0 E(V) = \text{hocolim}_n E(V \oplus \mathbb{R}^n)$. It is the best approximation of E by an essentially constant functor (all induced maps $T_0 E(V) \to T_0 E(W)$ are homotopy equivalences). Thus the expansion is at *infinity*. The functor E is related to its Taylor tower via natural transformations $E \to T_k E$ which fit together to form a commutative diagram. In favorable circumstances the maps $E(V) \to T_k E(V)$ are highly connected for k large.

The idea of the calculus of functors is to obtain some information about the functor E by investigating its Taylor tower. For this purpose we need to estimate the connectivity of the maps $E(V) \to T_k E(V)$. The best situation happens if the functor E is polynomial of some degree $\leq k$. Then the maps $E(V) \to T_k E(V)$ are homotopy equivalences for all $V \in J$.

A continuous functor E is called polynomial of degree $\leq k$ if the canonical map

$$E(V) \to \underset{0 \neq U \subseteq \mathbb{R}^{k+1}}{\text{holim}} E(V \oplus U)$$

is a homotopy equivalence for all $V \in J$. The homotopy limit is over the topological poset of all non-zero vector subspaces of $\mathbb{R}^{k+1}$. There is a way how to verify that a given functor is polynomial of degree $\leq k$; in the case $k = 1$ we discuss it in detail in the paper.

Once the convergence question is established (that means the connectivity of maps $E(V) \to T_k E(V)$ is known), the goal is to determine the Taylor tower itself. This can be done via an important theorem in [We] which says that the difference, $\text{hofiber}_*(T_k E(V) \to T_{k-1} E(V))$, between the stages of the Taylor tower of the functor E is determined by a certain spectrum $\mathbf{\Theta} E^{(k)}$ with an action of the orthogonal group $O(k)$. This spectrum should therefore be thought of as the k-th derivative of E at *infinity*.

The main theorem of this paper says that the functor F defined above is essentially polynomial of degree ≤ 1. It can be formulated as follows.

Theorem 6. *Let $F\colon J \to \mathrm{Spaces}_*$ be the functor defined by $V \mapsto \widetilde{\mathcal{S}}^s(\mathbb{R}P(V))$. Then the canonical maps*

$$F(V) \to \operatorname*{holim}_{0\neq U\subseteq\mathbb{R}^2} F(V\oplus U)$$

are homotopy equivalences if $\dim(V) \geq 6$.

This implies that the maps $F(V) \to T_1F(V)$ are homotopy equivalences if $\dim(V) \geq 6$ (see Corollary 1) and therefore in high enough dimensions the Taylor tower of F has only two interesting stages, namely T_1F and T_0F. As mentioned above, the difference between them is measured by the first derivative of F at infinity, which is a certain spectrum $\mathbf{\Theta}F^{(1)}$ with an action of the orthogonal group $O(1)$. The homotopy groups of the spectrum $\mathbf{\Theta}F^{(1)}$ are also determined.

Theorem 7. *We have*

$$\pi_k(\mathbf{\Theta}F^{(1)}) \cong L_k(1) \text{ for } k \in \mathbb{Z},$$

where the groups $L_k(1)$ are the 4-periodic L-groups from surgery theory associated to the trivial group.

There is another spectrum $\mathbf{L}_\bullet$ whose homotopy groups are $\pi_k(\mathbf{L}_\bullet) \cong L_k(1)$ for $k \in \mathbb{Z}$ (see [Ra2, chapter 13]). The relationship between $\mathbf{\Theta}F^{(1)}$ and $\mathbf{L}_\bullet$ is not yet clear to us, because we do not know any map between the two spectra.

The results of Theorems 6., 7. can be used to provide a functorial description of spaces $F(V) \simeq T_1F(V)$ via the Taylor tower of F, if $\dim(V) \geq 6$. For example by comparing the Taylor tower of F evaluated at 0 and at V we obtain the homotopy fibration sequence:

$$\Omega^\infty[(S(V)_+ \wedge \mathbf{\Theta}F^{(1)})_{hO(1)}] \to T_1F(0) \to T_1F(V), \tag{3}$$

where $S(V)$ is the unit sphere in V with the antipodal involution and the subscript $hO(1)$ denotes the homotopy orbit spectrum. We expect to determine the homotopy type of $T_1F(0)$ and the first map of the homotopy fibration sequence (3) jointly with Weiss in [MW]. In any case, this yields a surprising similarity of the homotopy fibration sequence (3) with the Sullivan-Wall-Quinn-Ranicki homotopy fibration sequence of surgery theory for $\mathbb{R}P(V)$ (see [WW, Theorem 2.3.1]). However, an important difference between the two homotopy fibration sequences is that (3) is functorial in V whereas the surgery theoretic description is NOT.

The Taylor tower of F can also be used to give a satisfying description of the colimit

$$F(\mathbb{R}^\infty) \simeq T_0F(0) = \operatorname*{colim}_{n\in\mathbb{N}} F(\mathbb{R}^n)$$

which can be thought as the block structure space of the infinite-dimensional real projective space. Namely, the Taylor tower of F evaluated at 0 becomes the homotopy fibration sequence:

$$\Omega^\infty[(\mathbf{\Theta}F^{(1)})_{hO(1)}] \to T_1F(0) \to T_0F(0) = F(\mathbb{R}^\infty). \tag{4}$$

If the promised results of [MW] are obtained, this yields the desired description of $F(\mathbb{R}^\infty)$.

The ultimate goal in the study of the block structure space of a manifold X is to provide some information about the automorphism spaces of X. As mentioned before these spaces are related via the homotopy fibration sequence (2). For the real projective spaces we obtain via the join construction also the functors $V \mapsto \widetilde{\mathrm{TOP}}(\mathbb{R}P(V))$ and $V \mapsto \mathrm{G}^s(\mathbb{R}P(V))$ and we can define the stable versions of the automorphism spaces

$$\widetilde{\mathrm{TOP}}(RP^\infty) = \operatorname*{colim}_{n\in\mathbb{N}} \widetilde{\mathrm{TOP}}(\mathbb{R}P(\mathbb{R}^n)), \quad \mathrm{G}^s(\mathbb{R}P^\infty) = \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{G}^s(\mathbb{R}P(\mathbb{R}^n)).$$

The join maps commute with the maps in (2) and so we obtain the homotopy fibration sequence

$$\widetilde{\mathrm{TOP}}(\mathbb{R}P^\infty) \to \mathrm{G}^s(\mathbb{R}P^\infty) \to \widetilde{\mathcal{S}}^s(\mathbb{R}P^\infty).$$

The spaces $\mathrm{G}^s(\mathbb{R}P(V))$ have been studied in [BS], in particular the homotopy type of the stable version $\mathrm{G}^s(RP^\infty)$ has been completely determined. Thus any information about the homotopy type of $\widetilde{\mathcal{S}}^s(\mathbb{R}P^\infty)$ gives us also information about $\widetilde{\mathrm{TOP}}(\mathbb{R}P^\infty)$ which is closely related to the stable version $\mathrm{TOP}(RP^\infty)$ of the spaces of honest self homeomorphisms of the real projective spaces (see [WW] for the relationship between the spaces of block self homeomorphisms and honest self homeomorphisms).

Now we would like to discuss briefly the study of the block structure spaces of lens spaces and complex projective spaces. Using the join construction we obtain in these cases a functor from the category of complex vector spaces with inner product to Spaces_* which sends the vector space V to $\widetilde{\mathcal{S}}^s(L(V))$ or $\widetilde{\mathcal{S}}^s(\mathbb{C}P(V))$. Here we can employ the unitary calculus which is the complex analogue of orthogonal calculus and we can try to run the study parallel to the case of real projective spaces. We note that the main ingredient in the proof of Theorems 6., 7. is codimension 1 surgery theory. In the case of lens spaces and complex projective spaces we have to use codimension 2 surgery theory which is considerably more complicated (see [Ra1, chapter 7.8]). At the moment we are unable to carry over the whole proofs of Theorems 6., 7. We hope to do this in the future. We would also like to note that the block structure spaces of lens spaces were already studied by Madsen in [Mad]. There he showed that if the fundamental group $\pi_1(L(V))$ is fixed of odd order and when localized at 2 and away from 2 all spaces $\widetilde{\mathcal{S}}^s(L(V))$ are homotopy equivalent. An intriguing question is whether the join construction induces a homotopy equivalence.

The paper is organized as follows. In section 1 we introduce the block structure spaces. In section 2 we construct the functor F. In section 3 we give an overview of the basic constructions from orthogonal calculus that we use in the paper. In particular for a continuous functor $E\colon J \to \mathrm{Spaces}_*$ we recall a necessary and sufficient condition for E to be polynomial of degree ≤ 1 (Proposition 3). This involves studying a certain functor $E^{(1)}\colon J \to \mathrm{Spaces}_*$ with some additional structure, which is called the first derivative functor of E. Section 4 contains background material about codimension 1 surgery theory, which is used to give a description of the first derivative functor $F^{(1)}$. Up to now we claim no originality. The description of $F^{(1)}$ will then be the main tool in the proofs of Theorems 6., 7. which are given in section 5. The proofs are our contribution and they are the core of the paper. We note, that the section 3 is a short survey of orthogonal calculus, however, for complete information we refer the reader to the original source [We]. Throughout the paper we also assume basic knowledge of surgery theory (that means the book [Wa] up to chapter 10), but we provide some background about the less standard parts, such as codimension 1 surgery theory.

Notation 1. For $V \in \mathrm{Ob}(J)$ by $\mathbb{R}P(V)$ is denoted the real projective space of V and by $S(V)$ is denoted the unit sphere in V.

1. The Block Structure Space of a Manifold

In this section we introduce the block structure space of a closed manifold X in general and we also introduce another model for the block structure spaces of the real projective spaces which will be functorial.

In the introduction we have defined for a given closed manifold X its simple structure set $\mathbb{S}^s(X)$ and more generally for a compact manifold with boundary $(X, \partial X)$ the simple structure set $\mathbb{S}^s_\partial(X)$. These are the basic objects of study of surgery theory which have been calculated in many special cases (see for example [Wa], [LdM]).

For a compact manifold X the structure sets $\mathbb{S}^s_\partial(X \times \Delta^k)$ become groups when $k \geq 1$. The block structure space of X is designed to be a space such that

$$\pi_k(\widetilde{\mathcal{S}}^s(X)) \cong \mathbb{S}^s_\partial(X \times \Delta^k). \tag{5}$$

It is constructed as a Δ-set (see [RS]). If X is a closed manifold then a k-simplex of $\widetilde{\mathcal{S}}^s(X)$ is a simple homotopy equivalence $f\colon M \to X \times \Delta^k$ between $(n+k)$-dimensional manifold $(k+2)$-ads, where $n = \dim(X)$ (the notion of a manifold $(k+2)$-ad is discussed below). The face operators are given by restrictions and the base point in dimension k is $\mathrm{id}\colon X \times \Delta^k \to X \times \Delta^k$.

The construction of the space $\widetilde{\mathcal{S}}^s(X)$ was given by Quinn in [Qu]. As indicated in the introduction the main motivation for studying this space comes from the fact that it is closely related to the automorphism spaces of X. We refer the reader to the survey article [WW] for more information about this application. In [WW, Theorem 2.3.1.] it is also stated that $\widetilde{\mathcal{S}}^s(X)$ fits into the homotopy fibration sequence of surgery theory of Sullivan-Wall-Quinn-Ranicki, which in principle determines its homotopy type.

A disadvantage of this model for $\widetilde{\mathcal{S}}^s(X)$ is that the assignment $V \mapsto \widetilde{\mathcal{S}}^s(\mathbb{R}P(V))$ is not functorial. Therefore we introduce another model for $\widetilde{\mathcal{S}}^s(\mathbb{R}P(V))$ which is functorial in V in Construction 4. In order to show that this model has the correct homotopy type we analyze Quinn's definition in a more detail.

Firstly we elaborate on the definition of a manifold $(k+2)$-ad. It is a generalization of a cobordism between manifolds, namely a cobordism is a special manifold 3-ad. According to [Mi] (or [Lu]) a cobordism between two n-dimensional manifolds M_0, M_1 is a 5-tuple (M, M_0, M_1, h_0, h_1), where M is an $(n+1)$-dimensional manifold with two disjoint pieces of boundary $\partial_0 M$, $\partial_1 M$ and $h_i\colon M_i \to \partial_i M$ for $i = 0, 1$ are homeomorphisms of n-dimensional manifolds. A manifold $(k+2)$-ad is a generalization where the manifold M has $(k+1)$ pieces of boundary which fit together in a nice way. We formalize this notion below.

More precisely we give an abstract construction of a Δ-set which consists of objects formally resembling manifold $(k+2)$-ads as an associated Δ-set to a certain Δ-groupoid. This is done in Construction 2 and then applied to the special case of the block structure spaces in Construction 3.

We recall basic definitions for Δ-sets of [RS] (see also [Ra2, chapter 11]). A Δ-set K is a functor $K\colon \Delta^{op} \to \mathrm{Sets}$, where Δ is the category with objects finite totally ordered sets $\mathbf{k} = \{0, \dots, k\}$ and morphisms injective monotone functions $\alpha\colon \mathbf{m} \to \mathbf{k}$ for $m \leq k$. Thus a Δ-set K can be described as a collection of sets $\{K_k\}_{k \in \mathbb{N}}$ together with face maps $\partial_\alpha\colon K_k \to K_m$. The standard k-simplex is a Δ-set in the following way: the set of m-simplices Δ^k_m is the set of monotone injective functions $\sigma\colon \mathbf{m} \to \mathbf{k}$ and for $\beta\colon \mathbf{l} \to \mathbf{m}$ the face map $\partial_\beta\colon \Delta^k_m \to \Delta^k_l$ is defined by $\partial_\beta(\sigma) = \sigma \circ \beta$. Also $\alpha\colon \mathbf{m} \to \mathbf{k}$ defines a map of Δ-sets $\Delta^m \to \Delta^k$. More generally, a Δ-groupoid K is a functor $K\colon \Delta^{op} \to \mathrm{Groupoids}$, that means for each k the set of k-simplices K_k is a groupoid (a category where all morphisms are isomorphisms) and the face maps ∂_α are functors. Note that the standard simplex Δ^k is also a

Δ-groupoid in a trivial way. A Δ-set K is *pointed* if for every $k \in \mathbb{N}$ there is a base simplex $* \in K_k$ such that $\partial_\alpha * = *$ for all α.

Now we recall the following well known construction for the special case of a Δ-groupoid.

Construction 1. [Dw] Let K be a Δ-groupoid. The *Grothendieck construction* $Gr(K)$ of K is the category with objects pairs $(\mathbf{k}, x)$, where $\mathbf{k} \in \Delta$, $x \in K_k$, and morphisms $(\mathbf{k}, x) \to (\mathbf{m}, y)$ are pairs (α, h), where $\alpha \colon \mathbf{m} \to \mathbf{k}$ is a morphism in Δ and $h \colon \partial_\alpha x \to y$ is a morphism in K_m. We also define the degree of the object $(\mathbf{k}, x)$ to be $k \in \mathbb{N}$. Note that the construction is functorial. In particular for $\alpha \colon \mathbf{m} \to \mathbf{k}$ we have a functor $Gr(\alpha) \colon Gr(\Delta^m) \to Gr(\Delta^k)$.

Construction 2. Let K be a Δ-groupoid. We define the *associated Δ-set* of K to be the Δ-set $K^\sharp$ defined as follows. The set of k-simplices of the Δ-set $K^\sharp$ is the set of degree preserving functors $\sigma \colon Gr(\Delta^k) \to Gr(K)$ which satisfy the extra condition that for $\beta \colon \mathbf{l} \to \mathbf{m}$ the morphism $\sigma(\beta, id)$ equals (β, h_β) for some $h_\beta \in K_l$. For $\alpha \colon \mathbf{m} \to \mathbf{k}$ the face map $\partial^\sharp_\alpha \colon K^\sharp_k \to K^\sharp_m$ is defined by $\partial^\sharp_\alpha \sigma = \sigma \circ Gr(\alpha)$.

A k-simplex $\sigma \in K^\sharp$ can be described by a list $\sigma = (x_\alpha, h_\alpha)$, where $x_\alpha \in K_m$ and $h_\alpha \colon \partial_\alpha x \to x_\alpha$ is an isomorphism in K_m for $\alpha \colon \mathbf{m} \to \mathbf{k}$ a morphism in Δ, satisfying various compatibility conditions. Here $x = x_{\mathrm{id}}$. Note that if K is a trivial Δ-groupoid (all morphisms are the identity), then a k-simplex of $K^\sharp$ is just a collection $\sigma = (x_\alpha, \mathrm{id})$, where $x_\alpha = \partial_\alpha x$. Thus we recover the underlying Δ-set of K. If K is a non-trivial Δ-groupoid a k-simplex of $K^\sharp$ is a collection $\sigma = (x_\alpha, h_\alpha)$, where objects x_α are just isomorphic to $\partial_\alpha x$.

Remark 1. The assignment $K \mapsto K^\sharp$ is a functor from Δ-groupoids to Δ-sets. Suppose that K is a topological Δ-groupoid, that means the category K_k is a topological category for each k and the face operators are continuous functors. Then $K^\sharp_k$ can also be endowed with a topology (the subspace topology of the product topology) and $K^\sharp$ becomes a Δ-space, also in a functorial way. The geometric realization then converts the Δ-set (or the Δ-space) $K^\sharp$ into a genuine space again in a functorial way. In the notation we do not distinguish between the underlying Δ-set (or Δ-space) and its geometric realization. We note that all these constructions have pointed versions.

Definition 1. A *k-block* M is a manifold with boundary such that the boundary ∂M is decomposed into codimension 0 submanifolds $\partial_i M$, for $i \in \mathbf{k}$, whose intersections $\partial_i M \cap \partial_j M$ are codimension 0 submanifolds of the boundaries of $\partial_i M$ and $\partial_j M$, and so on. For $\alpha \colon \mathbf{m} \to \mathbf{k}$ a morphism in Δ we denote $\partial_\alpha M = \cap_{i \in \mathrm{im}(\alpha)} \partial_i M$. A k-block is *special* if $\cap_{i \in \mathbf{k}} \partial_i M = \emptyset$.

An example of a special k-block is $X \times \Delta^k$ for X a closed manifold. Now we are ready for the definition of the block structure space $\widetilde{\mathcal{S}}^s(X)$.

Construction 3. Let X be a closed n-dimensional manifold. The block structure space $\widetilde{\mathcal{S}}^s(X)$ of X is the associated pointed Δ-set of the pointed Δ-groupoid $\widetilde{\mathcal{S}}^s_{gr}(X)$ defined as follows. The objects of $\widetilde{\mathcal{S}}^s_{gr}(X)$ in degree k are pairs (M, f), where M is a special k-block and $f \colon M \to X \times \Delta^k$ is a simple homotopy equivalence between $(n+k)$-dimensional k-blocks, that is, f respects faces. Face operators ∂_α are obtained by taking the face $\partial_\alpha M$ and restricting f to $f|_{\partial_\alpha M} \colon \partial_\alpha M \to X \times \partial_\alpha \Delta^k$. Morphisms are homeomorphisms $M \to M'$ of k-blocks over $X \times \Delta^k$. The base point in dimension k is $\mathrm{id} \colon X \times \Delta^k \to X \times \Delta^k$.

A k-simplex of $\widetilde{\mathcal{S}}^s(X)$ can be described by a list $((M_\alpha, f_\alpha), h_\alpha)$, where for $\alpha \colon \mathbf{m} \to \mathbf{k}$ the M_α is a special m-block, the map $f_\alpha \colon M_\alpha \to X \times \partial_\alpha \Delta^k$ is a simple homotopy equivalence of m-blocks, $h_\alpha\colon M_\alpha \to \partial_\alpha M$ is a homeomorphism of m-blocks and the maps f_α, h_α satisfy various compatibility conditions. The list (M_α, h_α) is the correct formalization of a manifold $(k+2)$-ad. In the case $k=1$, the manifold 3-ad is a cobordism in the sense of [Mi], [Lu] as described earlier.

The block structure space $\widetilde{\mathcal{S}}^s(X)$ as defined above is a Kan Δ-set for any manifold X. For Kan Δ-sets there is a convenient combinatorial formula for homotopy groups (see [Ra2, chapter 11]) and using it we immediately obtain the isomorphisms (5).

Now we would like to specialize to the real projective spaces. Note firstly that any homotopy equivalence $f\colon M \to \mathbb{R}P(V) \times \Delta^k$ is simple because $\mathrm{Wh}(\mathbb{Z}_2) = \{0\}$ (see [Lu]). Therefore we drop the word simple when dealing with the real projective spaces. As mentioned above the assignment $V \mapsto \widetilde{\mathcal{S}}^s(\mathbb{R}P(V))$ is not functorial for $\widetilde{\mathcal{S}}^s(-)$ as defined in Construction 3. Therefore we introduce another model.

Construction 4. Let $\widetilde{\mathcal{S}}^{s,1}(\mathbb{R}P(V))$ be the associated Δ-set of the Δ-groupoid $\widetilde{\mathcal{S}}^{s,1}_{gr}(\mathbb{R}P(V))$ defined as follows. A k-simplex of $\widetilde{\mathcal{S}}^{s,1}_{gr}(\mathbb{R}P(V))$ is a pair (T,f) where:

- T is a free involution on $S(V)\times\Delta^k$, which respects faces,
- $f\colon S(V)\times\Delta^k \to S(V)\times\Delta^k$ is an equivariant homotopy equivalence w.r.t. T on the source and to the antipodal involution on the target such that the quotient is also a homotopy equivalence and it again respects faces.

Face maps are defined by restrictions and isomorphisms are equivariant homeomorphisms over $S(V)\times\Delta^k$ with the antipodal involution. The base point is the pair (T_a, id) where T_a is given by the antipodal involution on $S(V)$ and by the trivial involution on Δ^k.

There is a map $\widetilde{\mathcal{S}}^{s,1}_{gr}(\mathbb{R}P(V)) \to \widetilde{\mathcal{S}}^{s}_{gr}(\mathbb{R}P(V))$ defined by taking quotients. This induces a map of the associated Δ-sets $\widetilde{\mathcal{S}}^{s,1}(\mathbb{R}P(V)) \to \widetilde{\mathcal{S}}^{s}(\mathbb{R}P(V))$.

Proposition 1. *The map $\widetilde{\mathcal{S}}^{s,1}(\mathbb{R}P(V)) \to \widetilde{\mathcal{S}}^{s}(\mathbb{R}P(V))$ is a homotopy equivalence if $\dim(V) \geq 5$.*

We use the following lemma.

Lemma 1. *Let $L \to K$ be a map of pointed Δ-groupoids such that $L^\sharp$ and $K^\sharp$ are Kan Δ-sets. If $L_k \to K_k$ is an equivalence of categories for every k, then the induced map $L^\sharp \to K^\sharp$ is a homotopy equivalence.*

Proof. Using the combinatorial expression of π_k it is easy to see that the map $L^\sharp \to K^\sharp$ induces isomorphisms on homotopy groups. □

Proof of Proposition 1. Using Lemma 1 it is enough to show that for each $k \in \mathbb{N}$ the map $\widetilde{\mathcal{S}}^{s,1}_{gr}(\mathbb{R}P(V))_k \to \widetilde{\mathcal{S}}^{s}_{gr}(\mathbb{R}P(V))_k$ is an equivalence of categories. This follows from the s-cobordism theorem, the generalized Poincaré conjecture and from the fact that the orbit space of any free action of $\mathbb{Z}_2$ on $S(V)$ is homotopy equivalent to $\mathbb{R}P(V)$ (see [Wa, chapter 13A]). □

Remark 2. Now we have two models for the block structure space of a real projective space. We will see that the model from Construction 4 has better functorial properties. However, the model from Construction 3 fits better with various surgery theoretic constructions. We will switch between these two models throughout the paper.

2. The Join Construction

In this section we show that the assignment $V \mapsto \widetilde{\mathcal{S}}^{s,1}(\mathbb{R}P(V))$ can be made into a continuous functor from J to Spaces_*. Firstly we show that this assignment can be made into a functor from J to $\Delta-\mathrm{sets}_*$, then we show that after the geometric realization we obtain a continuous functor $J \to \text{Spaces}_*$ (see Definition 2 for the definition of a continuous functor).

The following theorem is a generalization of Wall's *join construction*, discussed in [Wa, chapter 14A], to the level of block structure spaces.

Theorem 1. *The rule* $F\colon V \mapsto \widetilde{\mathcal{S}}^{s,1}(\mathbb{R}P(V))$ *is a functor from* J *to* Spaces_*.

Proof. In view of Remark 1 it is enough to show that there exists a functor (which we also denote by F, abusing notation) which to a vector space V associates the Δ-groupoid $\widetilde{\mathcal{S}}^{s,1}_{gr}(\mathbb{R}P(V))$.

To define F on morphisms, take $\xi\colon V \to W$, a linear monomorphism respecting inner products on V, W. The map ξ induces a splitting $W = V \oplus V^\perp$, here we identify V with $\xi(V)$. We need to define maps $F(\xi)_k\colon F(V)_k \to F(W)_k$ for each $k \in \mathbb{N}$ in a compatible way. We start with a k-simplex (T, f) in $\widetilde{\mathcal{S}}^{s,1}_{gr}(\mathbb{R}P(V))$. Regarding $S(V) \times \Delta^k$ as a subspace of $\widetilde{\mathcal{S}}^s(W) \times \Delta^k$ we want to extend both T and f to T', f' defined on $S(V) \times \Delta^k$. We distinguish cases $k = 0$ and $k > 0$.

Case $k = 0$**.** This is the *join construction* of Wall [Wa, chapter 14A]. Note that with an appropriate definition of a join of spheres embedded in an Euclidean space we can make an identification $S(W) = S(V) * S(V^\perp)$ (as sets). On $S(V)$ there is the pair (T, f), on $S(V^\perp)$ we have the pair (T_a, id), where T_a is the antipodal involution with id as an equivariant map. Both of these extend to the join, the resulting involution being free and the join of maps $f * \mathrm{id}$ being an equivariant map.

Case $k > 0$**.** By the previous discussion we have $S(W) \times \Delta^k = \big(S(V) * S(V^\perp)\big) \times \Delta^k$. However we cannot extend (T, f) by the same argument as before since both T and f might not be maps over Δ^k.

Observe instead the following expression:

$$S(W) = S(V) * S(V^\perp) = \bigcup_{y \in S(V^\perp)} C_y\big(S(V)\big)$$

where $C_y\big(S(V)\big)$ is the cone on $S(V)$ with a cone point $y \in S(V^\perp)$. Note that in the previous case when $k = 0$ we could express the extension of T and f in two steps: first extend canonically to each cone (this produces an involution which is not free), then compose with the antipodal involution on $S(V^\perp)$. This produces the same free involution as described before.

For $k > 0$ we have

$$S(W) \times \Delta^k = \bigcup_{y \in S(V^\perp)} C_y\big(S(V)\big) \times \Delta^k,$$

and we use the same approach.

First we extend (T, f) to $C_y(S(V)) \times \Delta^k$ for each $y \in S(V^\perp)$; note that the extended involution will not be free. We proceed inductively. We extend (T, f) to $C_y(S(V)) \times (\Delta^k)_0$ using the canonical extension to a cone. Now assume the existence of the extension to $C_y(S(V)) \times \partial\Delta^k$. Observe the following simple fact. The product $C_y\big(S(V)\big) \times \Delta^k$ can be expressed as the cone on its boundary

$$\partial\big(C_y\big(S(V)\big) \times \Delta^k\big) = \Big(\big(S(V) \times \Delta^k\big) \cup \big(C_y(S(V)) \times \partial\Delta^k\big)\Big).$$

By the induction hypothesis (T, f) are extended to the boundary and we use the canonical extension to a cone again to extend (T, f) to the whole product. The fixed point set of the extended involution is the set of cone points $\{y\} \times \Delta^k$.

Secondly, compose with the antipodal involution on $S(V^\perp)$ which produces a free involution T' on the product $S(V) \times \Delta^k$.

It is clear that the maps $F(\xi)_k$ are compatible w.r.t. the face operators and that the map $F(\xi)$ is a map of pointed Δ-sets. Functoriality of the construction follows from the associativity of the join construction and from the fact that the join of two antipodal involutions is again the antipodal involution. $\square$

Before we discuss continuity of the functor F observe that the set of k-simplices of the Δ-groupoid $\widetilde{\mathcal{S}}^{s,1}_{gr}(\mathbb{R}P(V))$ can be endowed with a topology, namely the subspace topology of the product topology of the compact-open topology on the space of self-maps on $S(V)\times\Delta^k$. Then we obtain a topological Δ-groupoid and the associated Δ-set $\widetilde{\mathcal{S}}^{s,1}(\mathbb{R}P(V))$ becomes a Δ-space.

Lemma 2. *The geometric realization of $\widetilde{\mathcal{S}}^{s,1}(\mathbb{R}P(V))$ as a Δ-space is homotopy equivalent to the geometric realization of the same object as a Δ-set.*

Proof. To see this we resolve the Δ-space version of $\widetilde{\mathcal{S}}^{s,1}(\mathbb{R}P(V))$ by taking the singular simplicial set of the space of k-simplices for each $k\in\mathbb{N}$. We obtain a combined simplicial-Δ-set, which has its geometric realization homotopy equivalent to the geometric realization of the Δ-space. We can 'change the order' of realization and realize first spaces $\mathrm{maps}(\Delta^k,\widetilde{\mathcal{S}}^{s,1}(\mathbb{R}P(V)))$ for each $k\in\mathbb{N}$ as Δ-sets and then realize the resulting simplicial set. The lemma will be proved if we show that all degeneracy and face maps between $\mathrm{maps}(\Delta^k,\widetilde{\mathcal{S}}^{s,1}(\mathbb{R}P(V)))$ and $\mathrm{maps}(\Delta^{k+1},\widetilde{\mathcal{S}}^{s,1}(\mathbb{R}P(V)))$ induce homotopy equivalences. This can be verified using the Kan property of $\widetilde{\mathcal{S}}^{s,1}(\mathbb{R}P(V))$. $\square$

Now we show that the functor F is a continuous functor in the sense of Definition 2. Firstly, note that one can define a continuous functor from J to topological Δ-groupoids or to Δ-spaces to be a functor for which for all k the evaluation maps

$$\mathrm{ev}_k\colon \mathrm{mor}(V,W)\times E(V)_k\to E(W)_k$$

are continuous. Given such a continuous functor from $E\colon J\to \mathrm{top}-\Delta-\mathrm{groupoids}_*$, composing with the functor $E\mapsto E^\sharp$ and taking the geometric realization of $E^\sharp$ we obtain a continuous functor to Spaces_*.

Theorem 2. *The functor $F\colon J\to \mathrm{Spaces}_*$ is a continuous functor.*

Proof. In view of the previous remarks it is enough to show that the functor $F\colon V\mapsto \widetilde{\mathcal{S}}^{s,1}_{gr}(\mathbb{R}P(V))$ is continuous. That means we need to show that the evaluation map

$$\mathrm{ev}_k\colon \mathrm{mor}(V,W)\times F(V)_k\to F(W)_k$$

is continuous for each k.

If $\dim(V)=\dim(W)$ then ev_k is given by $(\xi,(T,f))\mapsto(T\circ\xi^{-1},\xi\circ f\circ\xi^{-1})$ which clearly gives continuity.

For the case when $\dim(V)<\dim(W)$ we work locally and reduce to the above case as follows. Choose $\xi\colon V\to W$. We prove that the evaluation map is continuous on some neighborhood V_ξ of ξ. There is a map $\mathrm{mor}(W,W)\to\mathrm{mor}(V,W)$ given by $\zeta\mapsto\zeta\circ\xi$, which is a bundle map. Therefore there is a neighborhood $V_\xi\subset\mathrm{mor}(V,W)$ of ξ over which we have a section s of this bundle. Then the evaluation map ev_k can be factored as

$$V_\xi\times F(V)_k\xrightarrow{s\times F(\xi)_k}\mathrm{mor}(W,W)\times F(W)_k\xrightarrow{\mathrm{ev}_k}F(W)_k.$$

The first map is continuous because by the assumption both of its factors are and the second one is the previous case. $\square$

3. Orthogonal Calculus

Our method of study of the functor F defined in the previous section is the orthogonal calculus of functors. In this section we review some of the terms and tools of orthogonal calculus which we use in this paper. It is intended as a short survey which describes the context in which we work. For the full account we refer the reader to the paper of Weiss [We] in which orthogonal calculus was developed.

We recall some notation. By J is denoted the category of finite-dimensional real vector spaces with (positive definite) inner product. Morphisms are linear maps preserving inner product, in particular they are linear inclusions. The objects of J are usually denoted by letters U, V, W. Morphism sets $\mathrm{mor}(V, W)$ are Stiefel manifolds and so have a topology. For $U \in \mathrm{Ob}(J)$, U^c denotes the one-point compactification of U. Furthermore $\mathbb{R}_+ = \{u \in \mathbb{R} \mid u \geq 0\}$ and $\mathbb{R}_+^c = \mathbb{R}_+ \cup \{\infty\}$. We also write $k \cdot U$ for $\mathbb{R}^k \otimes U$.

The orthogonal calculus works for functors $E \colon J \to \mathrm{Spaces}_*$ satisfying the following axiom.

Definition 2. [We] A functor $E \colon J \to \mathrm{Spaces}_*$ is called *continuous* if for each $V, W \in \mathrm{Ob}(J)$ the evaluation map

$$\mathrm{ev} \colon \mathrm{mor}(V, W) \times E(V) \to E(W)$$

is continuous.

We have shown in Theorem 2 that the functor F satisfies this axiom. There are many other functors of this type. For example, the functor $V \mapsto \mathrm{BO}(V)$ where $\mathrm{BO}(V)$ is the classifying space of the orthogonal group $\mathrm{O}(V)$ has been studied in [Ar]. Other examples come up in the study of automorphisms of manifolds, see papers [We], [WW].

We have already presented the basic philosophy of orthogonal calculus in the introduction. Recall that a continuous functor $E \colon J \to \mathrm{Spaces}_*$ is studied by means of its Taylor tower which is a tower of functors

$$\cdots \to T_k E \to T_{k-1} E \to \cdots \to T_0 E, \tag{6}$$

where the functor $T_k E$ is the k-th polynomial approximation of E. The tower should be thought as the expansion at infinity. There are two aspects of the Taylor tower which one needs to understand. Firstly, one needs to find out whether the Taylor tower converges to E, that means to estimate the connectivity of the canonical maps $E(V) \to T_k E(V)$. Secondly, one needs to determine the differences, $\mathrm{hofiber}_*(T_k E(V) \to T_{k-1} E(V))$, between the stages of the tower.

Both issues can be investigated using the derivatives in orthogonal calculus. These arise as follows. Given a continuous functor $E \colon J \to \mathrm{Spaces}_*$ there is defined for each $k \in \mathbb{N}$ its k-th *derivative functor* $E^{(k)}$ which is again a continuous functor from J to Spaces_* with the additional structure of a coordinate free spectrum of multiplicity k. That means there are maps

$$\sigma_k \colon (k \cdot U)^c \wedge E^{(k)}(V) \to E^{(k)}(V \oplus U).$$

On objects $E^{(k+1)}$ is defined inductively as

$$E^{(k+1)}(V) = \mathrm{hofiber}_*(\tilde{\sigma}_k \colon E^{(k)}(V) \to \Omega^k E^{(k)}(V \oplus \mathbb{R})),$$

where $\tilde{\sigma}_k$ is the adjoint of σ_k. For the definition on morphisms, take $\xi \in \mathrm{mor}(V, W)$ and define $\xi' \in \mathrm{mor}(V \oplus \mathbb{R}, W \oplus \mathbb{R})$ by $\xi'(v, x) = (\xi(v), x)$. The pair (ξ, ξ') then defines the map $E^{(k+1)}(\xi)$ between the homotopy fibers $E^{(k+1)}(V) \to E^{(k+1)}(W)$. In Construction 5 we confine ourselves to a description of the maps σ_k for $k = 1$ as we will not need the maps σ_k for $k > 1$.

Construction 5. The values $E^{(1)}(V)$ are defined as homotopy fibers of the stabilization maps $E(\iota_V)\colon E(V) \to E(V \oplus \mathbb{R})$ where $\iota_V \colon V \to V \oplus \mathbb{R}$ is the linear inclusion. A point in $E^{(1)}(V)$ is described as a pair (x, λ), where $x \in E(V)$ and $\lambda\colon \mathbb{R}_+^c \to E(V{\oplus}\mathbb{R})$ satisfies $\lambda(0) = E(\iota_V)(x)$ and $\lambda(\infty) = *$.

We give a description of the adjoints of the maps σ_1, that means the maps

$$\tilde{\sigma}_1 \colon E^{(1)}(V) \to \Omega^U E^{(1)}(V \oplus U).$$

Given $(x, \lambda) \in E^{(1)}(V)$ we describe the image $(x', \lambda') := \tilde{\sigma}(x, \lambda) \in \Omega^U E^{(1)}(V \oplus U)$ where

$$x' \colon U^c \to E(V \oplus U),$$
$$\lambda' \colon (U \oplus \mathbb{R}_+)^c \to E(V \oplus U \oplus \mathbb{R}),$$

such that $x'(\infty) = *$, $\lambda'|_{(U\oplus\{0\})^c} = E(\iota_{V\oplus U})(x')$ and $\lambda'(\infty) = *$.

Each vector $u \in U$, such that $\|u\| = 1$, defines a morphism $\xi_u \colon V{\oplus}\mathbb{R} \to V \oplus U$ in J by $\xi_u \colon (v, 1) \mapsto (v, u)$.

Similarly, each vector $w \in U \oplus \mathbb{R}_+$, such that $\|w\| = 1$, defines a morphism $\zeta_w \colon V{\oplus}\mathbb{R} \to V \oplus U \oplus \mathbb{R}$ in J by $\zeta_w \colon (v, 1) \mapsto (v, w)$.

All such unit vectors $u \in U$, $w \in U \oplus \mathbb{R}_+$ define rays $\langle u \rangle$ and $\langle w \rangle$ from 0 trough u and w, which cover U and $U \oplus \mathbb{R}_+$ respectively. We define x', λ' on the closures $\langle u \rangle^c \subseteq U^c$, $\langle w \rangle^c \subseteq (U \oplus \mathbb{R}_+)^c$ as the compositions

$$x' \colon \langle u \rangle^c \xrightarrow{\cong} \mathbb{R}_+^c \xrightarrow{\lambda} E(V{\oplus}\mathbb{R}) \xrightarrow{E(\xi_u)} E(V \oplus U),$$

$$\lambda' \colon \langle w \rangle^c \xrightarrow{\cong} \mathbb{R}_+^c \xrightarrow{\lambda} E(V{\oplus}\mathbb{R}) \xrightarrow{E(\zeta_w)} E(V \oplus U \oplus \mathbb{R}).$$

Here the fact that the functor E is continuous is used. It guarantees that the maps x', λ' are continuous.

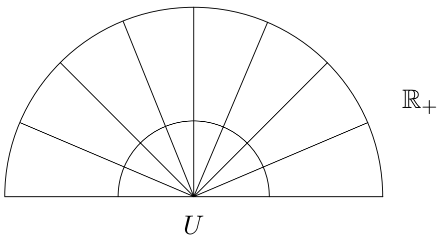

FIGURE 1. Schematic picture of the map $\tilde{\sigma}_1$

Figure 1 describes the map $\tilde{\sigma}_1$. Lines inside the half-disk represent the rays $\langle w \rangle$ and the half-circle represents the unit hemisphere inside $U \oplus \mathbb{R}_+$. The morphism $\zeta_w \in \mathrm{mor}(V{\oplus}U, V{\oplus}U \oplus \mathbb{R})$ which is used to define λ' on each ray $\langle w \rangle$ is determined by the point on the unit hemisphere which the ray $\langle w \rangle$ intersects.

Now we turn our attention to the convergence question. We will not address this issue in full generality, that means we will not discuss a notion of an analytic functor analogous to such a notion in [Go]. We will only consider the case when a functor E is polynomial of some degree $\leq k$. In this case the Taylor tower 'stops' at the k-stage, that means the canonical maps $E(V) \to T_k E(V)$ are homotopy equivalences and thus the k-th stage $T_k E$ gives us complete homotopy theoretical information about the functor E. Polynomial functors are defined as follows.

Definition 3. A continuous functor $E\colon J \to \mathrm{Spaces}_*$ is called *polynomial of degree* $\leq k$ if for every $V \in J$ the following canonical map is a homotopy equivalence

$$E(V) \longrightarrow \operatorname*{holim}_{0 \neq U \subseteq \mathbb{R}^{k+1}} E(V{\oplus}U).$$

The homotopy limit is taken over the topological poset of all non-zero vector subspaces of $\mathbb{R}^{k+1}$ (see [We, section 5]).

This property is really an extrapolation property. Notice that the value of the functor E at V is determined by the values of E at vector spaces between V and $V\oplus\mathbb{R}^{k+1}$ which are strictly greater than V. For more details see [We, section 5].

The derivatives are related to the polynomial functors via the following proposition [We, Proposition 5.3.].

Proposition 2. *For any $V \in J$ the value $E^{(k+1)}(V)$ is homotopy equivalent to the homotopy fiber of the canonical map*

$$E(V) \to \operatorname*{holim}_{0\neq U\subseteq\mathbb{R}^{k+1}} E(V\oplus U)$$

over the base point of the target.

Thus if the functor E is polynomial of degree $\leq k$ then the $(k+1)$-th derivative functor $E^{(k+1)}$ vanishes. If the target is path-connected for all V, then the converse is also true. If the target is not path-connected, then we have to investigate also the homotopy fibers over other points than the base point. This leads to a necessary and sufficient condition for E to be polynomial of degree $\leq k$ which we state in Theorem 3 in the case $k = 1$. It is due to Weiss (but did not appear in [We]) and we are grateful to him for showing it to us. Before we state it we need to define a modification of the map $\tilde{\sigma}_1$ presented in Construction 6.

We also need some new notation because we will use compactifications of Euclidean spaces other than the one-point compactification. By $\overline{\mathbb{R}}$ is denoted the 0-sphere (two-point) compactification of $\mathbb{R}$, the points at infinity are denoted $\pm\overline{1}$. By $\overline{\mathbb{R}\oplus\mathbb{R}_+}$ is denoted the 1-disk compactification of $\mathbb{R}\oplus\mathbb{R}_+$, points at infinity are denoted $\overline{w}$ where $w \in \mathbb{R}\oplus\mathbb{R}_+$, $\|w\| = 1$.

Construction 6. Denote $E^{(1)}(V, y) = \mathrm{hofiber}_y(E(V) \to E(V\oplus\mathbb{R}))$, where $\mathrm{hofiber}_y$ means the homotopy fiber taken over some point $y \in E(V\oplus\mathbb{R})$.

Let X_y denote the space of pairs of maps (x', λ') where

$$x' \colon \overline{\mathbb{R}} \to E(V\oplus\mathbb{R}),$$
$$\lambda' \colon \overline{\mathbb{R}\oplus\mathbb{R}_+} \to E(V\oplus\mathbb{R}^2),$$

such that $x'(\pm\overline{1}) = \pm y$, where $-y$ denotes the image of the point y under the self-map of $E(V\oplus\mathbb{R})$ induced by the reflection of $\mathbb{R}$. Furthermore we require that $\lambda'|_{\overline{\mathbb{R}}\oplus\{0\}} = E(\iota_{V\oplus\mathbb{R}})(x')$ and $\lambda'(\overline{w}) = E(\zeta_w)(y)$ where ζ_w is as in Construction 5.

Notice that in the case $y = *$ the space X_y is homotopy equivalent to the loop space $\Omega E^{(1)}(V\oplus\mathbb{R})$. If $y \neq *$ the space X_y can be thought as a certain space of paths in $E^{(1)}(V\oplus\mathbb{R}, y')$ where $y' = E(\zeta_{w'})(y)$ where $w' = (0, 0, 1)$. This is depicted in Figure 2.

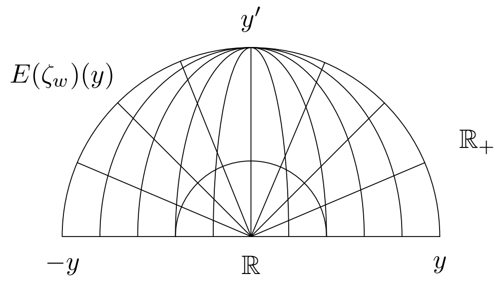

FIGURE 2. Schematic picture of λ' and of the map $\tilde{\sigma}_1$.

The map $\tilde{\sigma}_1\colon E^{(1)}(V,y) \to X_y$ is defined similarly as the map $\tilde{\sigma}_1$ in Construction 5. For $(x,\lambda) \in E^{(1)}(V,y)$ we define $(x',\lambda') = \tilde{\sigma}_1(x,\lambda) \in X_y$ as follows. Given unit vectors $u \in \mathbb{R}$, $w \in \mathbb{R}\oplus\mathbb{R}_+$ we now have rays $\langle 0,\overline{u}\rangle$, $\langle 0,\overline{w}\rangle$ from 0 to $\overline{u}$ or $\overline{w}$. These rays cover $\overline{\mathbb{R}}$, $\overline{\mathbb{R}\oplus\mathbb{R}_+}$ respectively. Define

$$x'\colon \langle 0,\overline{u}\rangle \xrightarrow{\cong} \mathbb{R}^c_+ \xrightarrow{\lambda} E(V\oplus\mathbb{R}) \xrightarrow{E(\xi_u)} E(V\oplus\mathbb{R}),$$

$$\lambda'\colon \langle 0,\overline{w}\rangle \xrightarrow{\cong} \mathbb{R}^c_+ \xrightarrow{\lambda} E(V\oplus\mathbb{R}) \xrightarrow{E(\zeta_w)} E(V\oplus\mathbb{R}^2).$$

If $y = *$ then the map $\tilde{\sigma}_1$ obviously coincides with the map $\tilde{\sigma}_1$ from Construction 5, see Figure 2 again.

Now we are ready to state the promised necessary and sufficient condition.

Theorem 3. *A continuous functor $E\colon J \to \mathrm{Spaces}_*$ is polynomial of degree ≤ 1 if and only if for all $V \in J$ and for all $y \in E(V\oplus\mathbb{R})$ the map $\tilde{\sigma}_1\colon E^{(1)}(V,y) \to X_y$ defined in Construction 6 is a homotopy equivalence.*

Proof. We sketch the proof. We need to show that the map

$$E(V) \to \operatorname*{holim}_{0\neq U\subset\mathbb{R}^2} E(V\oplus U) \tag{7}$$

is a homotopy equivalence if and only if for all choices of V and y the map $\tilde{\sigma}_1$ is.

Consider the map $E(V) \to E(V\oplus\mathbb{R})$ induced by $\iota_V\colon V \to V\oplus\mathbb{R}$ and the map $\operatorname{holim}_{0\neq U\subset\mathbb{R}^2} E(V\oplus U) \to E(V\oplus\mathbb{R})$ obtained by the restriction. For $y \in E(V\oplus\mathbb{R})$ consider the induced map, say $\tilde{\sigma}_y$, between the homotopy fibers over y of both maps. The second map is a fibration so the target homotopy fiber can be replaced by the fiber over y. It is not difficult to see that the space X_y is homotopy equivalent to this fiber and the map $\tilde{\sigma}_y$ can be identified with the map $\tilde{\sigma}_1$. But clearly the map (7) is a homotopy equivalence if and only if for all choices of y the map $\tilde{\sigma}_y$ is. □

Remark 3. Note that if we want to use Theorem 3 to prove that a certain functor E is polynomial of degree ≤ 1 there are two possibilities for each $y \in E(V\oplus\mathbb{R})$; the spaces $E^{(1)}(V,y)$, X_y might be either empty or not. So the proof splits into two parts. Namely,

(1) show that $E^{(1)}(V,y)$ is empty if and only if X_y is empty, and
(2) if $E^{(1)}(V,y)$ is not empty, show that $\tilde{\sigma}_1\colon E^{(1)}(V,y) \to X_y$ is a homotopy equivalence.

This finishes the discussion of polynomial functors. Now we turn to the Taylor tower. We have already mentioned that for a continuous functor E the functors T_kE are its polynomial approximations. This is the content of Theorem 6.3 of [We] which in a slightly weaker form reads as:

Theorem 4. *Let $E\colon J \to \mathrm{Spaces}_*$ be a continuous functor. Then for $k \geq 0$ there exists a functor $T_kE\colon J \to \mathrm{Spaces}_*$ and a natural transformation $\eta_k\colon E \to T_kE$ such that:*

(1) *T_kE is polynomial of degree $\leq k$,*
(2) *if E already is polynomial of degree $\leq k$ then the map*

$$\eta_k\colon E(V) \to T_kE(V)$$

is a homotopy equivalence for all V.

The formula for T_kE is given in the proof of Theorem 6.3 in [We]. For example, if $k = 0$ we have for all $V \in \mathrm{Ob}(J)$

$$T_0E(V) = E(V\oplus\mathbb{R}^\infty) = \operatorname*{hocolim}_{m\in\mathbb{N}} E(V\oplus\mathbb{R}^m).$$

Now we address the issue how to determine the layers of the Taylor tower. They are determined by the k-th *derivative spectrum* $\mathbf{\Theta}E^{(k)}$ which is the 'ordinary' spectrum associated to the coordinate free spectrum given by the k-th derivative functor $E^{(k)}$. Namely $(\mathbf{\Theta}E^{k})_{km} = E^{(k)}(\mathbb{R}^m)$ and the structure maps are specializations:

$$\sigma_k \colon (\mathbb{R}^k)^c \wedge (\mathbf{\Theta}E^{(k)})_{km} \to (\mathbf{\Theta}E^{(k)})_{k(m+1)}.$$

As we have mentioned before the tower is the expansion at infinity and therefore the spectrum $\mathbf{\Theta}E^{(k)}$ should be thought as the k-th derivative of E at *infinity*.

We have also already indicated that there is an action of the orthogonal group $O(k)$ on the k-th derivative spectrum associated to $E^{(k)}$. The relevant statement in [We] is Proposition 3.1 and simplified it reads as:

Proposition 3. *There exists a unique family $\{\alpha_V\}$ of left actions*

$$\alpha_V \colon O(k) \times E^{(k)}(V) \to E^{(k)}(V)$$

which makes the maps

$$\sigma_k \colon (k \cdot U)^c \wedge E^{(k)}(V) \to E^{(k)}(V \oplus U)$$

into $O(k)$-maps. Here $O(k)$ acts diagonally on the domain of the map σ_k, it acts on $(k \cdot U) = \mathbb{R}^k \otimes U$ because it acts on $\mathbb{R}^k$.

As we have said before, the polynomial approximations T_kE fit together to form the *Taylor tower* of E which is the diagram (6). The natural transformation $T_kE \to T_{k-1}E$ is essentially given as the $(k-1)$-st polynomial approximation of T_kE. The usefulness of the Taylor tower follows from the fact that the differences between its stages, $\mathrm{hofiber}_*(T_kE(V) \to T_{k-1}E(V))$, which are called the layers of the tower, can be described using the derivatives. The formula is given in Theorem 9.1 of [We]. It states:

Theorem 5. *For a continuous functor $E \colon J \to \mathrm{Spaces}_*$, any $k > 0$ and any $V \in J$ there is the following homotopy fibration sequence:*

$$\Omega^\infty[((k \cdot V)^c \wedge \mathbf{\Theta}E^{(k)})_{hO(k)}] \to T_kE(V) \xrightarrow{r_k} T_{k-1}E(V)$$

where the subscript $hO(k)$ denotes the homotopy orbit spectrum. The group $O(k)$ acts diagonally on the smash product, it acts on $\mathbf{\Theta}E^{(k)}$ by Proposition 3 and it acts on $k \cdot V = \mathbb{R}^k \otimes V$ because it acts on $\mathbb{R}^k$.

Once we have established the convergence question and we know the layers of the Taylor tower we can use the Taylor tower to obtain some information about the functor E. One way how to do this is suggested in the introduction for our functor F. The input for the Taylor tower of F is given in Theorems 6. and 7. For examples how the Taylor tower can be used to give information about other functors see [We, section 10].

Remark 4. The functor F defined in section 2 is a continuous functor $J \to \mathrm{Spaces}_*$ which is obtained from the functor to the category $\Delta-\mathrm{sets}_*$ (or $\Delta-\mathrm{spaces}_*$) by the geometric realization. Although we do not generalize the machinery of orthogonal calculus to this setting we often use constructions in the category $\Delta-\mathrm{sets}_*$ (or $\Delta-\mathrm{spaces}_*$), for example the construction of the homotopy fiber of a map or other constructions in section 5 in Construction 9. After geometric realization these constructions always commute (at least up to homotopy) with the corresponding constructions in the category Spaces_* which justifies their use.

4. The First Derivative Functor $F^{(1)}$

Now we start to apply the machinery of orthogonal calculus to study the functor F defined in section 2. The aim of this section is to identify the homotopy type of the spaces $F^{(1)}(V, y)$. This is done in Proposition 4 where we give a criterion for $F^{(1)}(V, y)$ to be empty and we show that when it is not empty it is homotopy equivalent to a certain space $\mathbf{LN}_n(\phi_V)$. The homotopy groups of the space $\mathbf{LN}_n(\phi_V)$ are the groups $LN_{n+*}(\phi_V)$ which are special cases of the LS-groups which are obstruction groups in codimension 1 surgery theory. The groups $LN_{n+*}(\phi_V)$ are well known to be isomorphic to the L-groups of the trivial group from 'ordinary' surgery theory. Thus we also obtain a calculation of the homotopy groups of the spaces $F^{(1)}(V, y)$.

Relating the comparison between the block structure spaces of real projective spaces of different dimensions with codimension 1 surgery theory is not our original idea. At the level of structure sets it was used by Browder-Livesay, López de Medrano and Wall in order to calculate the structure sets of the real projective spaces (see [BL], [LdM], [Wa, chapter 14C]). The statements at the level of block structure spaces which we review here follow from general theory of [Wa, chapter 17A] and from the treatment of codimension q surgery theory in [Ra1, chapter 7].

In this section we use the definition of the block structure space $\widetilde{\mathcal{S}}^s(X)$ from Construction 1 and we work on the level of Δ-sets. This is legal by Remark 4. We also adopt the convention that an embedding of manifolds will always mean locally flat embedding. This condition is also fulfilled in all examples that we mention.

4.1. Codimension q surgery theory.

We include a brief review of the codimension q surgery theory in which we fix the terminology, the notation and we also introduce some constructions that will be used in section 5. The reference for the general theory is [Wa, chapter 11]. The treatment there is slightly more general, see remarks by the editor at the beginning of the chapter. Another useful reference is [Ra1, chapter 7]. We do assume however a basic knowledge of 'ordinary' surgery theory, in particular, the algebraic definition of the L-groups and the construction of the surgery obstruction for a degree one normal map (see [Wa, chapters 5,6]).

First we need the following construction:

Construction 7. Let $M \hookrightarrow X$ be an embedding of manifolds. It induces the following decomposition of the ambient manifold X. The submanifold M has a normal block bundle ν, with total space $E(\nu)$ and with the total space of the associated sphere bundle denoted by $S(\nu)$. The closure of the complement of $E(\nu)$ in X is denoted by C. Thus we have

$$X = E(\nu) \cup_{S(\nu)} C.$$

Definition 4. Let $M \hookrightarrow X$ be an embedding of manifolds with the normal block bundle ν. A simple homotopy equivalence $f \colon Y \to X$ from a manifold Y of the same dimension as X is said to be *split along* M w.r.t ν, if f is transverse to M w.r.t. ν and the following restrictions are simple homotopy equivalences:

- $f|_{M'} \colon M' = f^{-1}(M) \to M$,
- $f|_{C'} \colon C' = f^{-1}(C) \to C$.

Definition 5. Let $M \hookrightarrow X$ be an embedding of manifolds as above. We will refer to a simple homotopy equivalence $f \colon Y \to X$ as a *splitting problem* (*along* M).

Given a splitting problem f along M the task is to change f by a homotopy to a simple homotopy equivalence which is split along M. If $q = (\dim(X) - \dim(M)) \geq 3$ the obstruction theory reduces to the ordinary surgery theory with the L-groups as obstruction groups ([Wa, Theorem 11.3]). In cases $q = 1, 2$ the obstruction theory is different. Firstly, we can assume that ν is a vector bundle. Then given a splitting problem f along M there is an obstruction $\theta(f)$ in a group $LS_n(\Phi)$. The obstruction vanishes if and only if the problem can be solved. Here $n = \dim(M)$ and Φ is the following diagram of fundamental groupoids:

$$\begin{tikzcd} \pi(S(\nu)) \arrow[r] \arrow[d] & \pi(C) \arrow[d] \\ \pi(M) \arrow[r] & \pi(X). \end{tikzcd}$$

The groups $LS_n(\Phi)$ depend only on Φ, dimension n, codimension q and the orientation characters of M and X. See [Wa, chapter 11] for the details of the construction of the LS-groups.

In the special case when $\pi(M) \to \pi(X)$ is an isomorphism also the other horizontal morphism in Φ is an isomorphism and thus the diagram depends only on the vertical morphism $\phi\colon \pi(C) \to \pi(X)$. In this case the group $LS_n(\Phi)$ is denoted $LN_n(\phi)$. This will be the most important case for us.

Suppose now we are given an embedding of manifolds $M^n \hookrightarrow X^{n+q}$ which induces the diagram of fundamental groupoids Φ. Following general philosophy of [Wa, chapter 17A] to 'spacify' all obstruction groups in surgery theory we can define a space $\mathbf{LS}_n(\Phi)$ such that $\pi_k(\mathbf{LS}_n(\Phi)) \cong LS_{n+k}(\Phi)$ (see also [Ha]).

We will not define this space in general, instead we look at the special case of an embedding $M^n \hookrightarrow E(\xi)$ of a manifold as the zero section of the total space of some disk bundle ξ. Let $\phi\colon \pi(S(\xi)) \to \pi(E(\xi))$ be the induced homomorphism of fundamental groupoids. The embedding corresponds to the LN-situation, so we will talk about the space $\mathbf{LN}_{n+1}(\phi)$.

Notice that we are provided with a map $\xi^!\colon \widetilde{\mathcal{S}}^s(M) \to \widetilde{\mathcal{S}}^s(E(\xi), S(\xi))$. On a k-simplex $f\colon M' \to M \times \Delta^k$ it is given by pulling back the disk bundle $E(\xi) \to M$ along f. Let y be a 0-simplex in $\widetilde{\mathcal{S}}^s(E(\xi), S(\xi))$, that means y is a simple homotopy equivalence $f_y\colon (E,S) \to (E(\xi), S(\xi))$ which can also be seen as a splitting problem along M. The splitting obstruction $\theta(f_y)$ lives in the group $LN_n(\phi)$. Denote

$$\widetilde{\mathcal{S}}^s(\xi, y) = \mathrm{hofiber}_y\big(\xi^!\colon \widetilde{\mathcal{S}}^s(M) \to \widetilde{\mathcal{S}}^s(E(\xi), S(\xi))\big).$$

By definition (see [Ra2, chapter 11]) a k-simplex of $\widetilde{\mathcal{S}}^s(\xi, y)$ is a simple homotopy equivalence $f\colon (E,S) \times \Delta^{k+1} \to (E(\xi), S(\xi)) \times \Delta^{k+1}$ which respects faces, such that $f_y = \partial_\alpha f\colon (E,S) \times \partial_\alpha \Delta^{k+1} \to (E(\xi), S(\xi)) \times \partial_\alpha \Delta^{k+1}$ for $\alpha\colon \mathbf{0} \to \mathbf{k{+}1}$, such that $\alpha(0) = k+1$, and $\partial_{k+1} f\colon (E,S) \times \partial_{k+1}\Delta^{k+1} \to (E(\xi), S(\xi)) \times \partial_{k+1}\Delta^{k+1}$ is split along M where $k+1\colon \mathbf{k} \to \mathbf{k{+}1}$ is obtained by omitting $k+1$ in the target.

It is not hard to see that,

- if $\theta(f_y) \neq 0$ then $\widetilde{\mathcal{S}}^s(\xi, y)$ is empty,
- if $\theta(f_y) = 0$ then $\widetilde{\mathcal{S}}^s(\xi, y) \simeq \mathbf{LN}_{n+1}(\phi)$.

See [Ra1, chapter 7.2] for the reference.

4.2. **The homotopy type of $F^{(1)}(V, y)$.** Consider now the embedding $\mathbb{R}P(V) \hookrightarrow \mathbb{R}P(V \oplus \mathbb{R})$. We have the decomposition

$$\mathbb{R}P(V \oplus \mathbb{R}) = E(\nu) \cup_{S(\nu)} C$$

where ν is the normal disk bundle of $\mathbb{R}P(V)$ and $C \cong D^n$, $n = \dim(V)$.

Consider now the embedding $\mathbb{R}P(V) \hookrightarrow E(\nu)$, the induced homomorphism $\phi_V\colon \pi_1(S(\nu)) \to \pi_1(E(\nu))$ and a homotopy equivalence $y = f_y\colon (E,S) \to (E(\nu), S(\nu))$ with the splitting obstruction $\theta(f_y) \in LN_{n-1}(\phi_V)$. Consider

$$\widetilde{\mathcal{S}}^s(\nu, y) = \mathrm{hofiber}_y\big(\nu^!\colon \widetilde{\mathcal{S}}^s(\mathbb{R}P(V)) \to \widetilde{\mathcal{S}}^s(E(\nu), S(\nu))\big).$$

By the above discussion we have that,

- if $\theta(f_y) \neq 0$ then $\widetilde{\mathcal{S}}^s(\nu, y)$ is empty,
- if $\theta(f_y) = 0$ then $\widetilde{\mathcal{S}}^s(\nu, y) \simeq \mathbf{LN}_n(\phi_V)$.

Now let $y = f_y \colon M \to \mathbb{R}P(V \oplus \mathbb{R})$ be a 0-simplex in $\widetilde{\mathcal{S}}^s(\mathbb{R}P(V \oplus \mathbb{R}))$. Consider the space $F^{(1)}(V, y) = \mathrm{hofiber}_y(\widetilde{\mathcal{S}}^s(\mathbb{R}P(V)) \to \widetilde{\mathcal{S}}^s(\mathbb{R}P(V \oplus \mathbb{R})))$. A k-simplex of this space can be viewed as a splitting problem over $\mathbb{R}P(V \oplus \mathbb{R}) \times \Delta^{k+1}$ along $\mathbb{R}P(V) \times \Delta^{k+1}$ satisfying certain conditions on the boundary. The homotopy type of $F^{(1)}(V, y)$ is identified in the following Proposition.

Proposition 4. *Let $\dim(V) = n \geq 6$ and consider the homotopy equivalence $y = f_y \colon M \to \mathbb{R}P(V \oplus \mathbb{R})$ as a splitting problem along $\mathbb{R}P(V)$ with the splitting obstruction $\theta(f_y) \in LN_{n-1}(\phi_V)$. Then the following holds*

- *if $\theta(f_y) \neq 0$ then $F^{(1)}(V, y)$ is empty,*
- *if $\theta(f_y) = 0$ then $F^{(1)}(V, y) \simeq \mathbf{LN}_n(\phi_V)$.*

Proof. Firstly notice that there is a map $\widetilde{\mathcal{S}}^s(E(\nu), S(\nu)) \to \widetilde{\mathcal{S}}^s(\mathbb{R}P(V \oplus \mathbb{R}))$ given by 'coning off'. This map is a homotopy equivalence by the following argument. Consider the embedding $S(\nu) \times \Delta^k \hookrightarrow \mathbb{R}P(V \oplus \mathbb{R}) \times \Delta^k$. This is an embedding of a codimension 1 separating submanifold. By [Wa, Theorem 12.1], if $\dim(V) \geq 6$, any homotopy equivalence mapping into $\mathbb{R}P(V \oplus \mathbb{R}) \times \Delta^k$ can be split along $S(\nu) \times \Delta^k$. This gives rise to the fibration sequence

$$\widetilde{\mathcal{S}}^s_\partial(C) \to \widetilde{\mathcal{S}}^s(\mathbb{R}P(V \oplus \mathbb{R})) \to \widetilde{\mathcal{S}}^s(E(\nu), S(\nu)),$$

where the map from the total space to the base is a Kan fibration, it is onto on π_0 and the fiber over each 0-simplex of $\widetilde{\mathcal{S}}^s(E(\nu), S(\nu))$ is contractible. So it is a homotopy equivalence. The map given by coning off is the homotopy inverse of this map and hence is also a homotopy equivalence.

Abusing notation we denote by y also the image of $y \in \widetilde{\mathcal{S}}^s(\mathbb{R}P(V \oplus \mathbb{R}))$ under some homotopy inverse of the coning off map. We have the following diagram which commutes up to homotopy

$$\begin{tikzcd}
\widetilde{\mathcal{S}}^s(\nu, y) \arrow[r] \arrow[d, dashed] & \widetilde{\mathcal{S}}^s(\mathbb{R}P(V)) \arrow[r, "\nu^!"] \arrow[d, equal] & \widetilde{\mathcal{S}}^s(E(\nu), S(\nu)) \arrow[d] \\
F^{(1)}(V, y) \arrow[r] & \widetilde{\mathcal{S}}^s(\mathbb{R}P(V)) \arrow[r, "F(\iota_V)"] & \widetilde{\mathcal{S}}^s(\mathbb{R}P(V \oplus \mathbb{R}))
\end{tikzcd}$$

The left hand vertical map is induced by the right hand vertical map which is a homotopy equivalence and so the left hand map is also a homotopy equivalence.

By the criterion preceding the theorem we get the desired statement. $\square$

Proposition 5. *If $\dim(V) = n \geq 6$ we have*

$$\pi_k(\mathbf{LN}_n(\phi_V)) \cong LN_{n+k}(\phi_V) \cong L_{k-n}(1)$$

Proof. We have the homomorphism $\phi_V \colon 1 \to \mathbb{Z}_2$, where the subscript V stands for the orientation character. This equals $(-1)^n$. By [Wa, Corollary 12.9.1.] we have that $LN_*(1 \to \mathbb{Z}_2) \cong L_{*+\varepsilon}(1)$ where $\varepsilon = 0$ if the orientation character is -1 and $\varepsilon = 2$ if the orientation character is $+1$. This together with the 4-periodicity of the L-groups yields the desired statement. $\square$

4.3. **An isomorphism from $\pi_k(\mathbf{LN}_n(\phi_V))$ to $L_{k-n}(1)$.** Following [Wa, chapter 12C] we now give a concrete and explicit description of the isomorphism

$$\theta \colon \pi_k(\mathbf{LN}_n(\phi_V)) \to L_{k-n}(1).$$

By definition a k-simplex in $\mathbf{LN}_n(\phi_V)$ is a splitting problem $f\colon (E,S)\times\Delta^{k+1}\to (E(\nu),S(\nu))\times\Delta^{k+1}$ along $\mathbb{R}P(V)\times\Delta^{k+1}$ satisfying certain conditions on the boundary. The isomorphism θ sends the homotopy class of f to the splitting obstruction rel $\partial\Delta^{k+1}$ of f which lives in the group $L_{k-n}(1)$. Simultaneously we give a description of the map

$$\theta\colon \pi_0(\widetilde{\mathcal{S}}^s(E(\nu),S(\nu)))\to L_{-1-n}(1).$$

A 0-simplex of $\widetilde{\mathcal{S}}^s(E(\nu),S(\nu))$ is a splitting problem $f\colon (E,S)\to (E(\nu),S(\nu))$ along $\mathbb{R}P(V)$ and the map θ sends f to its splitting obstruction.

So we consider a splitting problem $f\colon (E,S)\times\Delta^{k+1}\to (E(\nu),S(\nu))\times\Delta^{k+1}$, where $k\geq -1$ and we find a splitting obstruction $\theta(f)$ in $L_{k-n}(1)$. If $(k-n)$ is odd, then $L_{k-n}(1)$ is trivial so it is enough to consider the case when $(k-n)$ is even and hence also $(n+k)$ is even.

The even dimensional L-groups are defined algebraically in [Wa, chapter 5]. For a group π with a homomorphism $w\colon \pi\to\mathbb{Z}_2$ and for $2m=(k-n)$ an element of the group $L_{k-n}(\pi)$ is represented by a *simple* $(-1)^m$*-hermitian form* (G,λ,μ) where G is a stably free $\mathbb{Z}[\pi]$-module and the maps

$$\lambda\colon G\times G\to\mathbb{Z}[\pi],\quad \mu\colon G\to\mathbb{Z}[\pi]/\{x-(-1)^l\overline{x}\},$$

where $x\mapsto\overline{x}$ is an involution on $\mathbb{Z}[\pi]$ induced by the homomorphism w, satisfy certain conditions, see [Wa, Theorem 5.2]. A submodule H of G is called a *lagrangian* of the form (G,λ,μ) if the map λ induces an isomorphism $G/H\to\mathrm{Hom}_{\mathbb{Z}[\pi]}(H,\mathbb{Z}[\pi])$, $\lambda(H\times H)=0$ and $\mu(H)=0$. The form (G,λ,μ) admits a lagrangian if and only if it represents a zero element in the L-group in which case it is called a *hyperbolic* form.

In [Wa, chapter 5] the surgery obstruction for converting an even dimensional degree one normal map into a simple homotopy equivalence is described as an element of such an L-group. Following [Wa, chapter 12C] we now describe how to associate a simple $(-1)^m$-hermitian form which represents an element in $L_{k-n}(1)$ to the splitting problem f as above.

Construction 8. So we start with $f\colon (E,S)\times\Delta^{k+1}\to (E(\nu),S(\nu))\times\Delta^{k+1}$ a homotopy equivalence. Denote by $N=f^{-1}(\mathbb{R}P(V)\times\Delta^{k+1})$ and by i the embedding $N\hookrightarrow (E,S)\times\Delta^{k+1}$ and note that i is a degree one normal map. The obstruction $\theta(f)$ coincides with the obstruction to make the embedding i into a homotopy equivalence by an ambient surgery on N inside $(E,S)\times\Delta^{k+1}$ (see [Wa, chapter 12C] or [Ra1, chapter 7.6.] for the definition of the ambient surgery on a submanifold).

This is defined as follows. Let $2l=(n+k)$. First make i into an l-connected map by an ambient surgery. The ordinary surgery obstruction associated to i is a simple $(-1)^l$-hermitian form (H,λ,μ) over $\mathbb{Z}[\mathbb{Z}_2]$, where $H=\pi_{l+1}(i)=\pi_{l+1}(E\times\Delta^{k+1},N)$. As usual the elements of H can be represented by l-dimensional spheres immersed in N, the form λ is given by the $\mathbb{Z}[\mathbb{Z}_2]$-intersection numbers of these spheres and μ is given by the $\mathbb{Z}[\mathbb{Z}_2]$-selfintersection numbers. Let us now instead look at the double covers and try to change $\widetilde{i}\colon\widetilde{N}\hookrightarrow\widetilde{E}\times\Delta^{k+1}$ by an equivariant ambient surgery. Here $\widetilde{E}$ denotes the non-trivial double cover of E. The ordinary non-equivariant surgery obstruction associated to $\widetilde{i}$ is a simple $(-1)^l$-hermitian form (H,λ',μ') over $\mathbb{Z}$, with the same H, only considered now as a $\mathbb{Z}$-module. The maps λ', μ' are now given by $\mathbb{Z}$-intersection, resp. $\mathbb{Z}$-selfintersection numbers of the representing immersed spheres. The form (H,λ',μ') is hyperbolic. More to the point, $\widetilde{N}$ is a separating submanifold and we have a decomposition $\widetilde{E}\times\Delta^{k+1}=A^+\cup_{\widetilde{N}}A^-$. The $\mathbb{Z}$-module $H=\pi_{l+1}(\widetilde{i})=\pi_{l+1}(\widetilde{E}\times\Delta^{k+1},\widetilde{N})$ splits as

$$H^+\oplus H^-=\pi_{l+1}(A^+,\widetilde{N})\oplus\pi_{l+1}(A^-,\widetilde{N}).$$

The involution T switches H^+ and H^- which are both lagrangians of (H,λ',μ').

The equivariant ambient surgery obstruction for $\widetilde{i}$ is now defined as follows. Denote by $\lambda_0(x,y) = \lambda'(x,ty)$, $\mu_0(x) = t\mu(x)$, where t is the nontrivial element of $\mathbb{Z}_2$. Then (H^+, λ_0, μ_0) is a simple $(-1)^m$-hermitian form over $\mathbb{Z}$ where $2m = (k-n)$. It represents the desired obstruction $\theta(f)$ in $L_{k-n}(1)$.

Furthermore we may assume that $H^+ \cong \pi_{l+1}(A^+, \widetilde{N})$ is a free $\mathbb{Z}$-module on generators $\{e_i\}_{i\in\mathbf{r}}$ whose boundaries are represented by the spheres S_i^l disjointly embedded in $\widetilde{N}$ (because H^+ is a lagrangian over $\mathbb{Z}$). Let $\widetilde{N}\times I$ be the normal disk bundle of $\widetilde{N}$ in $\widetilde{E}\times\Delta^{k+1}$. Then the generators e_i can be represented by the disks D_i^{l+1} disjointly embedded in $\widetilde{E}\times\Delta^{k+1}$ so that the disk D_i^{l+1} is attached to the normal bundle $\widetilde{N}\times I$ at the sphere $S_i^l\times\{1\}$ (see [BL]). Similarly H^- is generated by the elements $\{te_i\}_{i\in\mathbf{r}}$ whose boundaries are represented by the spheres TS_i^l and the elements te_i are themselves represented by the disks TD_i^{l+1} attached to $\widetilde{N}\times I$ at $TS_i^l\times\{-1\}$.

Remark 5. We did not prove that θ is an isomorphism if $k\geq 0$, we have only given a description of the map θ. However, from the fact that θ is an isomorphism if $k\geq 0$ follows an important realization theorem which says that if $k\geq 0$ then any element of $L_{k-n}(1)$ can be represented by a splitting problem over $(E(\nu), S(\nu))\times\Delta^{k+1}$ along $\mathbb{R}P(V)\times\Delta^{k+1}$. We refer to [Wa, chapter 12C] for the details.

Remark 6. In this section we have used a model of $F^{(1)}(V,y)$ such that a k-simplex was a splitting problem over $\mathbb{R}P(V\oplus\mathbb{R})\times\Delta^{k+1}$ along $\mathbb{R}P(V)\times\Delta^{k+1}$ satisfying certain conditions on the boundary. In the next section it will be more convenient to use another model where a k-simplex is an equivariant splitting problem over $S(V\oplus\mathbb{R})\times\Delta^{k+1}$ along $S(V)\times\Delta^{k+1}$ (it is clear what involutions are meant) satisfying corresponding conditions on the boundary.

5. The Functor F is polynomial of degree ≤ 1

In this section we prove the two main theorems of this paper, Theorem 6., 7. We start with:

Theorem 6. *Let $F\colon J\to \mathrm{Spaces}_*$ be the functor defined by $V\mapsto\widetilde{\mathcal{S}}^s(\mathbb{R}P(V))$. If $\dim(V)\geq 6$ then the canonical maps*

$$F(V)\to \operatorname*{holim}_{0\neq U\subset\mathbb{R}^2} F(V\oplus U) \tag{8}$$

are homotopy equivalences.

An immediate corollary of Theorem 6 is the following.

Corollary 1. *If $\dim(V)\geq 6$ then the canonical maps*

$$F(V)\to T_1F(V)$$

are homotopy equivalences.

Proof of Corollary 1. The definition of the functor T_1E for a general continuous functor $E\colon J\to\mathrm{Spaces}_*$ is given in [We, Theorem 6.3]. It says that T_kE is the homotopy colimit of the direct system

$$E\xrightarrow{\rho}\tau_1E\xrightarrow{\tau_1(\rho)}(\tau_1)^2E\xrightarrow{(\tau_1)^2(\rho)}\cdots,$$

where

$$\tau_1E(V) = \operatorname*{holim}_{0\neq U\subset\mathbb{R}^2} E(V\oplus U),$$

and the natural transformation $\rho\colon E\to\tau_1E$ is given by the canonical maps. Theorem 6 says that in the case of the functor F the maps $\rho\colon F(V)\to\tau_1F(V)$ are homotopy equivalences if $\dim(V)\geq 6$. The claim of Corollary 1 follows. $\square$

Overview of the proof of Theorem 6. The details of the proof of Theorem 6 are rather complicated. Therefore we first give an overview of the proof.

The pattern of the proof is to verify the condition of Theorem 3 for the map (8) to be a homotopy equivalence, if $\dim(V) \geq 6$. This condition says that we have to verify that a certain map $\tilde{\sigma}_1 \colon F^{(1)}(V,y) \to X_y$ is a homotopy equivalence for all choices of $y \in F(V \oplus \mathbb{R})$.

The space X_y and the map $\tilde{\sigma}_1$ were described for general continuous functor $E \colon J \to \mathrm{Spaces}_*$ in Construction 6. We work with the Δ-set models, so in Construction 9 we provide a Δ-set model of the space X_y and we also provide a Δ-set description of the map $\tilde{\sigma}_1$ which up to homotopy commutes with the geometric realization.

The homotopy type of the space $F^{(1)}(V,y)$ has been described in Proposition 4. It says that $F^{(1)}(V,y)$ is empty if and only if the splitting obstruction $\theta(f_y)$ of a certain equivariant splitting problem f_y over $S(V \oplus \mathbb{R})$ along $S(V)$ is non-zero (see also Remark 6). If $\theta(f_y) = 0$ then the space $F^{(1)}(V,y)$ is homotopy equivalent to a certain space $\mathbf{LN}_n(\phi_V)$ whose homotopy groups are calculated in Proposition 5.

We give a corresponding description of the homotopy type of the space X_y in Proposition 6. It says that X_y is empty if and only if the splitting obstruction $\theta(f'_y)$ of a certain equivariant splitting problem f'_y over $S(V \oplus \mathbb{R}^2) \times I$ along $S(V \oplus \mathbb{R}) \times I$ is non-zero. If $\theta(f'_y) = 0$ then the space X_y is homotopy equivalent to the space $\Omega\mathbf{LN}_{n+1}(\phi_{V\oplus\mathbb{R}})$ whose homotopy groups are also calculated by Proposition 5.

As we have indicated in Remark 3 the proof then splits into two parts. To prove the condition (1) we have to show that the space $F^{(1)}(V,y)$ is empty if and only if the space X_y is empty. This is the content of Proposition 7. It is proved using the criteria mentioned above. Notice that the equivariant splitting obstructions $\theta(f_y)$ and $\theta(f'_y)$ live in the same group $L_{-1-n}(1)$. Therefore it is enough to check that $\theta(f_y) = \theta(f'_y)$ in $L_{-1-n}(1)$. The proof of this statement is the crucial part of the whole proof. Unfortunately it is rather technical. It is given in a sequence of lemmas: Lemma 3, 4, 5.

To prove the condition (2) we have to show that if $F^{(1)}(V,y)$ is not empty then the map $\tilde{\sigma}_1 \colon F^{(1)}(V,y) \to X_y$ is a homotopy equivalence. This is done by showing that it induces isomorphisms on homotopy groups. Using Propositions 4, 6 and 5 we see that the spaces $F^{(1)}(V,y)$ and X_y have isomorphic homotopy groups. Thus we have to show that the map $\tilde{\sigma}_1$ induces isomorphisms between abstractly isomorphic groups. The idea here is the following. An element of $\pi_k(F^{(1)}(V,y)) \cong L_{k-n}(1)$ can be represented by a splitting obstruction $\theta(f)$ of some equivariant splitting problem f over $S(V\oplus\mathbb{R}) \times \Delta^{k+1}$ along $S(V) \times \Delta^{k+1}$. The image $\tilde{\sigma}_1(f)$ is a splitting problem over $S(V\oplus\mathbb{R}^2) \times \Delta^{k+1} \times I$ along $S(V\oplus\mathbb{R}) \times \Delta^{k+1} \times I$ with the splitting obstruction $\theta(\tilde{\sigma}_1(f))$ which represents the homotopy class of $\tilde{\sigma}_1(f)$ in $\pi_k(X_y) \cong L_{k-n}(1)$. The condition (2) is proved by showing that $\theta(f) = \theta(\tilde{\sigma}_1(f))$ in $L_{k-n}(1)$. This statement is the content of Proposition 8. Its proof turns out to be just a parameterized version of the proof of the statement of condition (1).

Construction 9. We need a $\Delta-$set model of the space X_y and a description of the map $\tilde{\sigma}_1$ on the level of Δ − sets. The descriptions given here and the descriptions from Construction 6 commute with the geometric realization up to homotopy.

Recall the Δ − set model for $F^{(1)}(V,y)$. Here $y = f_y \colon S(V \oplus \mathbb{R}) \to S(V \oplus \mathbb{R})$ is an equivariant splitting problem along $S(V)$. A k-simplex of $F^{(1)}(V,y)$ is an equivariant homotopy equivalence $f \colon S(V \oplus \mathbb{R}) \times \Delta^{k+1} \to S(V \oplus \mathbb{R}) \times \Delta^{k+1}$ w.r.t. the antipodal involution on the target and w.r.t. some free involution on the source, it respects faces, it is transverse to $S(V) \times \Delta^{k+1}$, for $\alpha \colon \mathbf{0} \to \mathbf{k{+}1}$, $\alpha(0) = k+1$, we have $f_y = \partial_\alpha f \colon S(V\oplus\mathbb{R}) \times \partial_\alpha \Delta^{k+1} \to S(V\oplus\mathbb{R}) \times \partial_\alpha \Delta^{k+1}$ and $\partial_{k+1} f$ is equivariantly split along $S(V) \times \Delta^k$.

Now we give a $\Delta -$ set model for X_y. A k-simplex of X_y is an equivariant homotopy equivalence $f'\colon S(V\oplus\mathbb{R}^2)\times\Delta^{k+1}\times I\to S(V\oplus\mathbb{R}^2)\times\Delta^{k+1}\times I$ w.r.t. the antipodal involution on the target and w.r.t. to some free involution on the source satisfying the following conditions on the boundary.

Firstly define

$$\partial_0(\Delta^{k+1}\times I)=(\partial_{k+1}\Delta^{k+1}\times I)\cup(\Delta^{k+1}\times\partial I),$$
$$\partial_1(\Delta^{k+1}\times I)=\partial_\alpha\Delta^{k+1}\times I,$$

where $\alpha\colon \mathbf{0}\to\mathbf{k+1}$ is again given by $\alpha(0)=k+1$.

The conditions on f' are the following. Firstly f' restricted to $S(V\oplus\mathbb{R}^2)\times\partial_0(\Delta^{k+1}\times I)$ is split along $S(V\oplus\mathbb{R})\times\partial_0(\Delta^{k+1}\times I)$. The restriction of f' to $S(V\oplus\mathbb{R}^2)\times\partial_1(\Delta^{k+1}\times I)$ is fixed and it is given as the composition

$$f'_y=(S(\zeta))^{-1}\circ(\Sigma(f_y)\times\mathrm{id})\circ S(\zeta). \tag{9}$$

Here Σ means the unreduced suspension map and the map $S(\zeta)\colon S(V\oplus\mathbb{R}^2)\times I\to S(V\oplus\mathbb{R}^2)\times I$ is a homeomorphism given by $(x,t)\mapsto(\zeta_t(x),t)$ where ζ_t is the unique rotation of $V\oplus\mathbb{R}^2$ which fixes V and sends $(0,1,0)$ to $(0,t,\sqrt{1-t^2})$.

The map $\tilde\sigma_1\colon F^{(1)}(V,y)\to X_y$ is given by the rule

$$\tilde\sigma_1\colon f\mapsto(S(\zeta))^{-1}\circ(\Sigma(f)\times\mathrm{id})\circ S(\zeta). \tag{10}$$

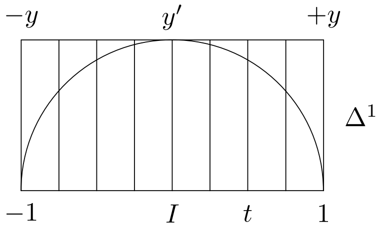

FIGURE 3. Schematic picture of a 0-simplex in X_y and of the map $\tilde\sigma_1$.

A 0-simplex of X_y is depicted in Figure 3. The top edge corresponds to the part $\partial_1(\Delta^1\times I)$ of the boundary. The closure of the complement of $\partial_1(\Delta^1\times I)$ in the boundary is $\partial_0(\Delta^1\times I)$. On each vertical line over $t\in I$ the restriction of the map $\tilde\sigma_1(f)$ is given by $(\zeta_t)^{-1}\circ\Sigma(f)\circ\zeta_t$. Compare this with Figure 2. The vertical lines here correspond after reparametrization to the rays from the origin in Figure 2.

Note that the map f'_y can be seen as an equivariant splitting problem along $S(V\oplus\mathbb{R})\times I$ with the splitting obstruction $\theta(f'_y)\in LN_{n+1}(\phi_{V\oplus\mathbb{R}})\cong L_{-1-n}(1)$. We need the following criterion.

Proposition 6. *We have that*

- *if* $\theta(f'_y)\neq 0$ *then the space* X_y *is empty,*
- *if* $\theta(f'_y)=0$ *then* $X_y\simeq\Omega\mathbf{LN}_{n+1}(\phi_{V\oplus\mathbb{R}})$.

Proof. If X_y is not empty then there exists a 0-simplex, say f, of X_y. Since f restricted to $(S(V\oplus\mathbb{R}^2)\times\partial_0(\Delta^{k+1}\times I))$ is by definition split along $S(V\oplus\mathbb{R})\times\partial_0(\Delta^{k+1}\times I)$ any such a 0-simplex f can be seen as a solution of the splitting problem f'_y and hence $\theta(f'_y)=0$. As we have remarked in Construction 6 if the space X_y is not empty it is homotopy equivalent to a certain space of paths in the space $F^{(1)}(V\oplus\mathbb{R},y')\simeq\mathbf{LN}_{n+1}(\phi_{V\oplus\mathbb{R}})$. It is not difficult to see that this path space is homotopy equivalent to $\Omega\mathbf{LN}_{n+1}(\phi_{V\oplus\mathbb{R}})$. □

Now we are ready to proceed with a verification of the condition of Theorem 3 for the functor F to be polynomial of degree ≤ 1. We start with the condition (1).

Proposition 7. *$F^{(1)}(V,y)$ is empty if and only if X_y is empty.*

Propositions 4, 6 tell us when the spaces $F^{(1)}(V,y)$ and X_y are empty. We see that both obstructions $\theta(f_y)$ and $\theta(f'_y)$ live in the group $L_{-1-n}(1)$. In order to prove the condition (1) it is enough to show that $\theta(f_y)=\theta(f'_y)$ in $L_{-1-n}(1)$. This is done in the sequence of the following three lemmas.

Remark 7. Note that a precomposition with a homeomorphism does not change the splitting obstruction of a splitting problem. Therefore when investigating $\theta(f'_y)$ it is enough to look at the splitting obstruction of the map $(S(\zeta))^{-1}\circ(\Sigma(f_y)\times \mathrm{id})$ and so from now on we denote

$$f'_y=(S(\zeta))^{-1}\circ(\Sigma(f_y)\times \mathrm{id}). \tag{11}$$

Lemma 3. *The map $f'_y\colon S(V\oplus\mathbb{R}^2)\times I\to S(V\oplus\mathbb{R}^2)\times I$ is transverse to the submanifold $S(V\oplus\mathbb{R})\times I$.*

Proof. One would expect that in order to find the splitting obstruction $\theta(f'_y)$ it is necessary to first adjust the map f'_y to make it transverse to $S(V\oplus\mathbb{R})\times I$. Amazingly, the map f'_y as constructed by the formula (11) already is transverse to $S(V\oplus\mathbb{R})\times I$! This is what we prove in this lemma, more precisely we show an equivalent statement that the map $(\Sigma(f_y)\times\mathrm{id})$ is transverse to $S(\zeta)\circ j$ where j denotes the embedding $S(V\oplus\mathbb{R})\times I\hookrightarrow S(V\oplus\mathbb{R}^2)\times I$.

We use an easier notion of transversality which is good enough for our purposes. It goes as follows. Let $M\hookrightarrow X$ be a codimension 1 embedding. Suppose that there exist an open neighborhood E of M in X and a submersion $g\colon E\to\mathbb{R}$ such that $g^{-1}(0)=M$. (This implies that M has a trivial normal bundle.) We say that a map $f\colon Y\to X$ is transverse to M if $g\circ f\colon f^{-1}(E)\to\mathbb{R}$ is a submersion in a neighborhood of $f^{-1}(M)$. Note that this condition can be verified locally.

In our case, $X=S(V\oplus\mathbb{R}^2)\times I$ and M is the image of $S(V\oplus\mathbb{R})\times I$ under the embedding $S(\zeta)\circ j$. Let $g=h\circ(S(\zeta))^{-1}$ where $h\colon S(V\oplus\mathbb{R}^2)\times I\to\mathbb{R}$ is the 'height function' defined by the projection on the second coordinate of $\mathbb{R}^2$. (This should be suitably restricted to a neighborhood of M in X.)

We use the following decompositions (12), (13) into (overlapping) codimension 0 (non-compact) submanifolds

$$S(V\oplus\mathbb{R})=E(\widetilde{\nu})\cup C_\pm \tag{12}$$

where $E(\widetilde{\nu})$ is the total space of the open normal vector bundle $\widetilde{\nu}$ of $S(V)$ in $S(V\oplus\mathbb{R})$, and $C_\pm=D_\pm(V\oplus\mathbb{R})\smallsetminus S(V)$, where $D_\pm(V\oplus\mathbb{R})=S(V\oplus\mathbb{R})\cap(V\oplus\mathbb{R}_\pm)$ with $\mathbb{R}_\pm=\{u\in\mathbb{R}\mid \pm u\geq 0\}$, and

$$S(V\oplus\mathbb{R}^2)=E(\widetilde{\mu})\cup D_\pm \tag{13}$$

where $E(\widetilde{\mu})$ is the total space of the open normal vector bundle $\widetilde{\mu}$ of $S(V)$ in $S(V\oplus\mathbb{R}^2)$ and $D_\pm=D_\pm(V\oplus\mathbb{R}^2)\smallsetminus S(V)$, where $D_\pm(V\oplus\mathbb{R}^2)=S(V\oplus\mathbb{R}^2)\cap(V\oplus\mathbb{R}\oplus\mathbb{R}_\pm)$.

Notice that the embedding $S(\zeta)\circ j$ restricts to the embeddings

$$E(\widetilde{\nu})\times I\hookrightarrow E(\widetilde{\mu})\times I,\quad C_\pm\times I\hookrightarrow D_\pm\times I.$$

We give two different arguments for the transversality on these two kinds of codimension 0 submanifolds of $S(V\oplus\mathbb{R}^2)\times I$

We start with the embedding $E(\widetilde{\nu})\times I\hookrightarrow E(\widetilde{\mu})\times I$. This is an embedding of a codimension 1 subbundle where both bundles are over $S(V)\times I$. The map g restricted to $E(\widetilde{\mu})\times I$ can be described as $g\colon(x,t)\mapsto(x-\mathrm{pr}_l(x))$ where pr_l is the orthogonal projection in the fiber of $E(\widetilde{\mu})\times\{t\}$ onto the image of $E(\widetilde{\nu})\times\{t\}$.

Now the map f_y was transverse to $S(V)$, therefore the map $\Sigma(f_y)$ is also transverse to $S(V)$. Let $\widetilde{N}=(f_y)^{-1}(S(V))$ and let $\widetilde{\nu}_{\widetilde{N}}$ be the open normal bundle of

$\widetilde{N}$ in $S(V\oplus\mathbb{R})$ and denote by $\widetilde{\mu}_{\widetilde{N}}$ the open normal bundle of $\widetilde{N}$ in $S(V\oplus\mathbb{R}^2)$. The restriction $\Sigma(f_y)|\colon E(\widetilde{\mu}_{\widetilde{N}}) \to E(\widetilde{\mu})$ is a bundle map. Now it is not difficult to see that the restriction $\Sigma(f_y)| \times \mathrm{id}\colon E(\widetilde{\mu}_{\widetilde{N}}) \times I \to E(\widetilde{\mu}) \times I$ is transverse to $E(\widetilde{\nu}) \times I$ in the sense above.

Next we look at the embedding $C_\pm \times I \hookrightarrow D_\pm \times I$. Observe that the composition $C_\pm \times I \hookrightarrow D_\pm \times I \to D_\pm$ is a homeomorphism. Therefore we can consider the embedding $C_\pm \times I \cong D_\pm \hookrightarrow D_\pm \times I$ as an embedding of a graph of some function $\omega\colon D_\pm \to I$ into $D_\pm \times I$. The restriction of g can be expressed as $(x,t) \mapsto t - \omega(x)$. Again it is not difficult to see that the restriction $\Sigma(f_y)|\times\mathrm{id}\colon (D_\pm(V\oplus\mathbb{R}^2)\smallsetminus\widetilde{N})\times I \to D_\pm \times I$ is transverse to $C_\pm \times I$ in the sense above.

The details are left to the reader. □

In the next lemma we identify the homotopy type of $\widetilde{N}' := (f'_y)^{-1}(S(V\oplus\mathbb{R}) \times I)$.

Lemma 4. *Let $\tau\colon \widetilde{N}' \to \mathbb{R}$ be the composition $\widetilde{N}' \hookrightarrow S(V\oplus\mathbb{R}^2) \times I \to I \hookrightarrow \mathbb{R}$. Then τ is transverse to $s \in \mathbb{R}$ for all $s \neq 0$. Hence $\widetilde{N}' \simeq \tau^{-1}(0) \cong \Sigma\widetilde{N}$ where $\Sigma\widetilde{N}$ means the unreduced suspension of $\widetilde{N}$.*

Proof. A key observation here is that the manifold (with boundary) $\widetilde{N}'$ has a decomposition

$$\widetilde{N}' = E(\widetilde{\nu}_{\widetilde{N}}) \times I \cup (D_\pm(V\oplus\mathbb{R}^2) \smallsetminus \widetilde{N}) \tag{14}$$

into codimension 0 submanifolds which corresponds to the previous decomposition $S(V\oplus\mathbb{R}) \times I = (E(\widetilde{\nu}) \times I) \cup D_\pm$. This can be seen as follows.

We can think of $\widetilde{N}'$ as a pullback of $S(\zeta) \circ j$ along $\Sigma(f_y) \times \mathrm{id}$:

$$\begin{tikzcd}
\widetilde{N}' \arrow[r] \arrow[d] & S(V\oplus\mathbb{R}) \times I \arrow[d, "S(\zeta)\circ j"] \\
S(V\oplus\mathbb{R}^2) \times I \arrow[r, "\Sigma(f_y)\times\mathrm{id}"] & S(V\oplus\mathbb{R}^2) \times I.
\end{tikzcd}$$

Look first at the following restriction of this diagram:

$$\begin{tikzcd}
? \arrow[r] \arrow[d] & E(\widetilde{\nu}) \times I \arrow[d, "S(\zeta)\circ j"] \\
E(\widetilde{\mu}_{\widetilde{N}}) \times I \arrow[r, "\Sigma(f_y)|\times\mathrm{id}"] & E(\widetilde{\mu}) \times I.
\end{tikzcd}$$

In this diagram the lower horizontal arrow is a bundle map over the restriction $f_y| \times \mathrm{id}\colon \widetilde{N} \times I \to S(V) \times I$. Here $E(\widetilde{\mu}_{\widetilde{N}}) \times I$ is a 2-dimensional vector bundle over $\widetilde{N} \times I$, $E(\widetilde{\mu}) \times I$ is a 2-dimensional vector bundle over $S(V) \times I$ and $E(\widetilde{\nu}) \times I$ is its codimension 1 subbundle. It follows that the top row $? \to E(\widetilde{\nu}) \times I$ is a bundle map and it also follows from the construction that ? can be identified with $E(\widetilde{\nu}_{\widetilde{N}}) \times I$.

Secondly, look at the restriction of the diagram to $C_\pm \times I \cong D_\pm$. Here, because the map $S(\zeta) \circ j$ is a map over I, we can look at the following pullback diagrams

$$\begin{tikzcd}
?? \arrow[r] \arrow[d] & C_\pm \times I \arrow[d, "\mathrm{pr}_1\circ S(\zeta)\circ j"] \\
S(V\oplus\mathbb{R}^2) \arrow[r, "\Sigma(f_y)"] & S(V\oplus\mathbb{R}^2).
\end{tikzcd}$$

where pr_1 is the projection onto the first factor. As noted before the composition $(\mathrm{pr}_1 \circ S(\zeta) \circ j)\colon C_\pm \times I \to D_\pm$ is a homeomorphism and thus we obtain that $?? \cong \Sigma(f_y)^{-1}(D_\pm) = (D_\pm(V\oplus\mathbb{R}^2) \smallsetminus \widetilde{N})$. This yields the decomposition (14).

We describe the map τ on each submanifold in the decomposition. The restriction of τ to the submanifold $E(\widetilde{\nu}_{\widetilde{N}}) \times I$ is just the projection:

$$\tau| : E(\widetilde{\nu}_{\widetilde{N}}) \times I \to I \hookrightarrow \mathbb{R}$$

which is clearly a submersion.

On the two submanifolds $D_{\pm}(V \oplus \mathbb{R}^2) \smallsetminus \widetilde{N}$ the restriction of the map τ can be described as follows. Observe first that $D_{\pm}(V \oplus \mathbb{R}^2) \smallsetminus \widetilde{N} \subset D_{\pm}(V \oplus \mathbb{R}^2)$, and $D_{\pm} \subset D_{\pm}(V \oplus \mathbb{R}^2)$ and that $D_{\pm}(V \oplus \mathbb{R}^2)$ can be seen as a cone on $S(V \oplus \mathbb{R})$. The restriction of τ is the following composition:

$$\tau| : D_{\pm}(V \oplus \mathbb{R}^2) \smallsetminus \widetilde{N} \xrightarrow{C(f_y)} D_{\pm} \cong C_{\pm} \times I \to I \hookrightarrow \mathbb{R}$$

where $C(f_y)$ denotes the canonical extension of f_y to the cones suitably restricted. The homeomorphism $D_{\pm} \cong C_{\pm} \times I$ can be described as follows. The fiber $C_{\pm} \times \{s\}$ is embedded in $D_{\pm}$ as an open 'meridian' through the point $(0, s, \pm\sqrt{1-s^2})$. It is not difficult to see that the cone map $C(f_y)$ is transverse to all such meridians except the one corresponding to $s = 0$ (details left to the reader).

The identification $\tau^{-1}(0) \cong \Sigma\widetilde{N}$ follows from the construction of the map f'_y by formula (11). $\square$

Lemma 5. *Let $2m = -1 - n$ and let $\theta(f_y) \in L_{-1-n}(1)$ be represented by a simple $(-1)^m$-hermitian form (H^+, λ_0, μ_0). Then $\theta(f'_y) \in L_{-1-n}(1)$ is represented by the same simple $(-1)^m$-hermitian form.*

Proof. Recall that $f_y \colon S(V \oplus \mathbb{R}) \to S(V \oplus \mathbb{R})$ is a homotopy equivalence about which we can assume that it is transverse to $S(V)$ and also that it is obtained from a homotopy equivalence $E(\widetilde{\nu}) \to E(\widetilde{\nu})$ by 'coning off' (by Proposition 4). Here $E(\widetilde{\nu})$ denotes the total space of the normal disk bundle of $S(V)$ in $S(V \oplus \mathbb{R})$. The splitting obstruction $\theta(f_y)$ is represented by a simple $(-1)^m$-hermitian form (H^+, λ_0, μ_0) obtained as an equivariant ambient surgery obstruction associated to the embedding $\widetilde{i} \colon \widetilde{N} = (f_y)^{-1}(S(V)) \hookrightarrow E(\widetilde{\nu})$ (see Construction 8).

The homotopy equivalence $f'_y \colon S(V \oplus \mathbb{R}^2) \times I \to S(V \oplus \mathbb{R}^2) \times I$ is constructed by formula (11). Let $E(\widetilde{\nu}')$ be the total space of the normal disk bundle of $S(V \oplus \mathbb{R})$ in $S(V \oplus \mathbb{R}^2)$ and note that f'_y already is in the form of a homotopy equivalence obtained from another homotopy equivalence $E(\widetilde{\nu}') \times I \to E(\widetilde{\nu}') \times I$ by 'coning off' so there is no need to adjust it. The splitting obstruction $\theta(f'_y)$ should now be read off as the equivariant ambient surgery obstruction from the embedding $\widetilde{i}' \colon \widetilde{N}' \hookrightarrow E(\widetilde{\nu}') \times I$ by the recipe given in Construction 8.

In previous Lemma 4 we have made a first step by identifying the homotopy type of $\widetilde{N}'$. Moreover it follows from the construction of the map f'_y by formula (11) that the embedding $\widetilde{i}'$ when restricted to $\tau^{-1}(0)$ is just the unreduced suspension map $\Sigma\widetilde{i}$. So we have a commutative diagram

$$\begin{tikzcd}
\Sigma\widetilde{N} \arrow[r, "\Sigma\widetilde{i}"] \arrow[d, "\simeq"] & E(\widetilde{\nu}') \times \{0\} \arrow[d, "\simeq"] \\
\widetilde{N}' \arrow[r, "\widetilde{i}'"] & E(\widetilde{\nu}') \times I
\end{tikzcd}$$

where all maps are embeddings.

Let $2l = n - 1$. From the diagram above it follows that $\widetilde{i}'$ is $(l+1)$-connected, and also that $\pi_{l+2}(\widetilde{i}') \cong \pi_{l+1}(\widetilde{i}) \cong H$ and that the corresponding splitting of H is the same $H^+ \oplus H^-$ as before. It only remains to find the embedded spheres representing the generators of H and to show that their intersection numbers are preserved.

Let $\{e_i, te_i\}$ be the set of generators of H as a $\mathbb{Z}$-module, where generators are represented by embedded spheres $S_i, TS_i \hookrightarrow \widetilde{N}$ as in Construction 8. Denote the images under suspension map by $S_i', TS_i' \hookrightarrow \Sigma\widetilde{N} \hookrightarrow \widetilde{N}'$. We need to put them in a general position.

First we describe the topology of $\widetilde{N}'$ more. The manifold $\widetilde{N}'$ is a special manifold 3-ad with two distinguished parts of the boundary $\partial_\pm \widetilde{N}' = S(V \oplus \mathbb{R}) \times \{\pm 1\}$. From the decomposition (14) we also see that the product $\widetilde{N} \times I$ is a separating submanifold of $\widetilde{N}'$. By taking its normal disk bundle (using collars) we obtain the codimension 0 embedding $\widetilde{N} \times I^2 \hookrightarrow \widetilde{N}'$, which when restricted to $\widetilde{N} \times \{\pm 1\} \times I \hookrightarrow \partial_\pm \widetilde{N}'$ coincides with the embedding of the closed normal bundle of $\widetilde{N} \times \{\pm 1\} \hookrightarrow \partial_\pm \widetilde{N}'$. In particular there are disks $D_i^\pm, TD_i^\pm$ attached to the normal bundle $\widetilde{N} \times I$ as described in Construction 8. See Figure 4.

Denote now by $d_\pm \colon I \to I^2$ the maps $d_\pm(s) = (\pm s, s)$. Using the isotopy induced by the embedding of $\Sigma D_i^\pm$, $\Sigma TD_i^\pm$ into $\widetilde{N}'$ (we omit the details) we can change the embedded spheres S_i', TS_i' to embedded spheres in a general position given by the two formulas $S_i'' = (D_i^+ \times 1) \cup (S_i \times d_+ I) \cup (D_i^- \times -1)$ and $TS_i'' = (TD_i^- \times -1) \cup (TS_i \times d_- I) \cup (TD_i^+ \times 1)$. It follows that the associated intersection forms are λ', μ', since the only intersections are the intersections in $\widetilde{N}$.

The situation is depicted in Figure 4. The thicker parts correspond to the spheres S_i'', TS_i'' in general position. The only intersections are those in $\widetilde{N} \times I$. ☐

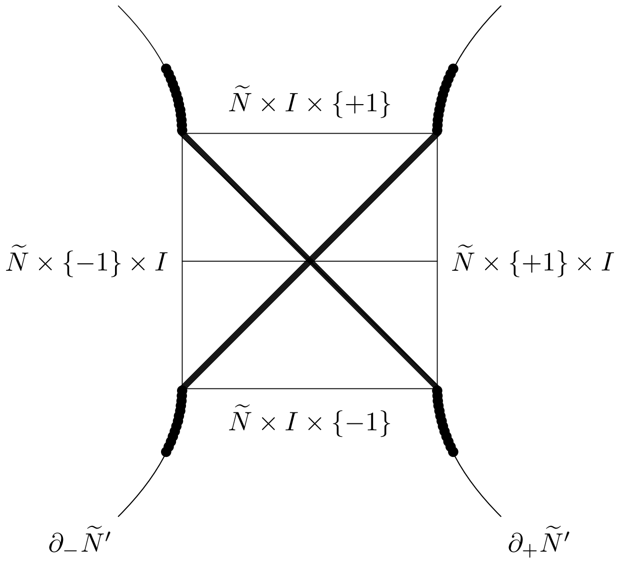

FIGURE 4. Intersections of the generators of (H, λ, μ) in $\widetilde{N}'$.

Proposition 8. *If $F^{(1)}(V, y)$ is not empty then the map $\tilde{\sigma}_1 \colon F^{(1)}(V, y) \to X_y$ is a homotopy equivalence.*

Proof. This is proved by showing that the map $\tilde{\sigma}_1$ induces isomorphisms on homotopy groups. Notice that we have

$$\pi_k(F^{(1)}(V, y)) \cong \pi_k(X_y) \cong L_{k-n}(1).$$

Because the surgery obstruction map θ is an isomorphism we can realize any element of the group $L_{k-n}(1)$ by a k-simplex in $F^{(1)}(V, y)$, that means by an equivariant splitting problem $f\colon S(V\oplus\mathbb{R})\times\Delta^{k+1}\to S(V\oplus\mathbb{R})\times\Delta^{k+1}$ along $S(V)\times\Delta^{k+1}$ satisfying certain conditions on the boundary.

By Construction 9 the image $f'=\tilde{\sigma}_1(f)$ is a k-simplex in X_y, that means an equivariant splitting problem $f\colon S(V\oplus\mathbb{R}^2)\times\Delta^{k+1}\times I\to S(V\oplus\mathbb{R}^2)\times\Delta^{k+1}\times I$ along $S(V\oplus\mathbb{R})\times\Delta^{k+1}\times I$ again with certain conditions on the boundary.

In order to show that the map $\tilde{\sigma}_1$ induces an isomorphism on homotopy groups it is enough to show that if the splitting obstruction $\theta(f)$ is represented by a simple $(-1)^m$-hermitian form (H^+,λ_0,μ_0), where $2m=k-n$, then the splitting obstruction $\theta(f')$ is represented by the same simple $(-1)^m$-hermitian form. But this is just a version of the statement that was proved in a sequence of Lemmas 3, 4, 5 parametrized over Δ^{k+1}. The proofs carry over to the parametrized case as well. $\square$

Theorem 7. *We have*

$$\pi_k(\mathbf{\Theta}F^{(1)})\cong L_k(1)\ \text{ for }\ k\in\mathbb{Z},$$

where the groups $L_k(1)$ are the 4-periodic L-groups from surgery theory associated to the trivial group.

Proof. Notice that the proof of Proposition 7 tells us in the special case $y=*$ that modulo some low-dimensional deviations the spectrum $\mathbf{\Theta}F^{(1)}$ is an Ω-spectrum and we know its homotopy groups. Namely

$$\pi_k(\mathbf{\Theta}F^{(1)})\cong\pi_{k+|k|+8}\big(F^{(1)}(\mathbb{R}^{|k|+8})\big)\cong L_k(1)$$

for all $k\in\mathbb{Z}$. $\square$

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Tibor Macko, Max Planck Institut für Mathematik, Vivatsgasse 7, Bonn, D-53111, Germany,

E-mail address: `macko@mpim-bonn.mpg.de`