K-theory of stratified vector bundles

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Abstract

We show that the Atiyah-Hirzebruch K-theory of spaces admits a canonical generalization for stratified spaces. For this we study algebraic constructions on stratified vector bundles. In particular the tangent bundle of a stratified manifold is such a stratified vector bundle.

1 Families of vector spaces

Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}^n = \mathbb{R} \oplus \cdots \oplus \mathbb{R}$ be the standard n-dimensional $\mathbb{R}$-vector space. Let **Vect** denote the category of finite dimensional $\mathbb{R}$-vector spaces and linear maps and $\mathbf{V}$ a subcategory of **Vect**, termed the *structure category*. For example let $\mathbf{V}$ be the subcategory of the surjective maps in **Vect**. Or let G be a subgroup of the automorphism group $GL_n(\mathbb{R})$ of $\mathbb{R}^n$. Then G yields the subcategory $G \subset \mathbf{Vect}$ consisting of one object $\mathbb{R}^n$ and morphisms given by elements in G.

The category **Vect** is a topological category. We say that $\mathbf{V}$ is a *closed* subcategory if for objects V, $W \in \mathbf{V}$ the inclusion of morphisms sets

$$\hom_{\mathbf{V}}(V, W) \subset \hom_{\mathbf{Vect}}(V, W)$$

is a closed subspace. Here $\hom_{\mathbf{Vect}}(V, W)$ is homeomorphic to $\mathbb{R}^N$ while $\hom_{\mathbf{V}}(V, W)$ needs not to be a vector space.

Following Atiyah [1] a *family of vector spaces* in $\mathbf{V}$ is a topological space X together with a map $p_X : X \to \bar{X}$ and for every $b \in \bar{X}$ a homeomorphism, named *chart*, $\Phi_b : p_X^{-1}(b) \approx X_b$ where X_b is a suitable object of $\mathbf{V}$ depending upon $b \in \bar{X}$. We denote the family $(p_X : X \to \bar{X}, X_b, \Phi_b, b \in \bar{X})$ simply by X (it will be clear from the context whether X is the underlying space or the family). The map p_X is termed the *projection*, the space X the *total space* of the family of vector spaces, the space $\bar{X}$ the *base space*, and for every b the vector space X_b is termed *fiber* over b. A family X is also termed $\mathbf{V}$-family, to make explicit the choice of $\mathbf{V}$. The $\mathbf{V}$-family X is *discrete* if $\bar{X}$

is a discrete space. In this case X is a disjoint union of objects in $\mathbf{V}$ and we call X a $\mathbf{V}$-*set.*

Given two $\mathbf{V}$-families X and Y a $\mathbf{V}$-*map* from X to Y (equivalently, a *homomorphism* or an *admissible map controlled by* $\mathbf{V}$) is a pair of maps $(f, \bar{f})$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p_X"'] & Y \arrow[d, "p_Y"] \\ \bar{X} \arrow[r, "\bar{f}"] & \bar{Y}, \end{tikzcd} \tag{1.1}$$

and such that for every $b \in \bar{X}$ the composition given by the dashed arrow of the diagram

$$\begin{tikzcd} p_X^{-1}(b) \arrow[r, "f|p^{-1}(b)"] & p_Y^{-1}(\bar{f}b) \arrow[d, "\Phi_{\bar{f}b}"] \\ X_b \arrow[u, "\Phi_b^{-1}"] \arrow[r, dashed] & Y_{\bar{f}b} \end{tikzcd}$$

is a morphism of $\mathbf{V}$. A $\mathbf{V}$-isomorphism $(f, \bar{f})\colon X \to Y$ induces homeomorphisms $f\colon X \to Y$, $\bar{f}\colon \bar{X} \to \bar{Y}$, and $f|X_b\colon X_b \to Y_{\bar{f}b}$ for every $b \in \bar{X}$.

If V is a vector space in $\mathbf{V}$ and $\bar{X}$ is a topological space, then the projection onto the first factor $p_1\colon X = \bar{X} \times V \to \bar{X}$ is the projection map of the *product family* with fiber V; the charts $\Phi_b : \{b\} \times V \to V = X_b$ are given by projection and $X_b = V$ for all $b \in \bar{X}$.

If Y is a family $\mathbf{V}$-isomorphic to a product family then Y is said to be a *trivial* vector bundle.

Given a family Y and a map $\bar{f}\colon \bar{X} \to \bar{Y}$, the pull-back $X = \bar{f}^*Y$ is defined by the following pull-back diagram.

$$\begin{tikzcd} X = \bar{f}^*Y \arrow[r, dashed] \arrow[d, dashed] & Y \arrow[d, "p"] \\ \bar{X} \arrow[r, "\bar{f}"] & \bar{Y}. \end{tikzcd} \tag{1.2}$$

If $\bar{f} = i\colon \bar{X} \subset \bar{Y}$ is an inclusion, then i^*Y is called the *restriction* of Y to $\bar{X}$ and is denoted by $Y|\bar{X}$.

A $\mathbf{V}$-*vector bundle* or a $\mathbf{V}$-bundle is a locally trivial family of vector spaces in $\mathbf{V}$, i.e. a family X over $\bar{X}$ such that every $b \in \bar{X}$ admits a neighborhood U for which $X|U$ is trivial.

If X is a $\mathbf{V}$-family and Z is a topological space, then the cartesian product $X \times Z$ is a $\mathbf{V}$-family with projection $p_{X\times Z} = p \times 1_Z\colon X \times Z \to \bar{X} \times Z$. The fiber over a point $(b, z) \in \bar{X} \times Z$ is equal to $p_X^{-1}(b) \times \{z\}$ and the chart $\Phi_{(b,z)}$ is defined by $(x, z) \mapsto \Phi_b(x) \in X_b$, where of course we set $(X \times Z)_{(b,z)} = X_b$ and Φ_b is the chart $\Phi_b : p_X^{-1}(b) \to X_b$.

In particular, if Z is the unit interval I one obtains the cylinder object $X \times I$; this leads to the definition of *homotopy*: two **V**-maps $f_0, f_1 \colon X \to Y$ are **V**-homotopic (in symbols $f_0 \sim f_1$) if there is a **V**-map $F \colon X \times I \to Y$ such that $f_0 = Fi_0$ and $f_1 = Fi_1$. Here i_0 and i_1 are the inclusions $X \to X \times I$ to $X \times \{0\}$ and $X \times \{1\}$ respectively.

Let **Top** be the category of topological spaces and (continuous) maps. Then **V-Top** is the category of **V**-families $p_X : X \to \bar{X}$ in **Top** and **V**-maps. Homotopy of **V**-maps yields the homotopy relation $\sim$ which is a natural equivalence relation on **V-Top** so that the homotopy category $(\mathbf{V}\text{-}\mathbf{Top})/_{\sim}$ is defined.

A **V**-map $i : A \to Y$ is termed a *closed inclusion* if $\bar{i} : \bar{A} \to \bar{Y}$ is an inclusion, $\bar{i}\bar{A}$ is closed in $\bar{Y}$ and $\bar{i}^* Y = A$. A closed inclusion $i : A \to Y$ induces isomorphisms on fibers.

The push-out construction can be extended to the category **V-Top**, provided the push-out is defined by a closed inclusion.

(1.3) Lemma. *Given* **V***-families* A*,* X*,* Y *and* **V***-maps* $f \colon A \to X$*,* $i \colon A \to Y$ *with* i *a closed inclusion, the push-out diagram in* **V-Top**

(1.4)
$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

exists and $X \to Z$ *is a closed inclusion.*

Proof. See [3], lemma (2.5). *q.e.d.*

2 Stratified vector bundles

We introduce the notion of stratified space and stratified vector bundle. They are particular cases of stratified bundles as defined in [3], definition (4.1). In the next section we describe as an example the stratified tangent bundle.

Let A be a closed subset of a space X. We say that (X, A) is a *CW-pair* if there exists a homeomorphism $(X, A) \approx (X', A')$ of pairs where X' is a CW-complex and A' a subcomplex of X'. A *CW-space* X is a space homeomorphic to a CW-complex.

(2.1) Definition. A *stratified* space is a space $\bar{X}$ together with a filtration

$$\bar{X}_0 \subset \bar{X}_1 \subset \cdots \subset \bar{X}_n \subset \cdots \subset \lim_{n\to\infty} \bar{X}_n = \bar{X}$$

such that for every $i \geq 1$ there is a CW-pair $(\bar{M}_i, \bar{A}_i)$ and an attaching map $\bar{h}_i \colon \bar{A}_i \to \bar{X}_{i-1}$ with the property that the subspace $\bar{X}_i$ is obtained by attaching $\bar{M}_i$ to $\bar{X}_{i-1}$ via the attaching map $\bar{h}_i$, i.e. $\bar{X}_i = \bar{M}_i \cup_{\bar{h}_i} \bar{X}_{i-1}$.

The complements $\bar{X}_i \smallsetminus \bar{X}_{i-1}$ are termed *strata*, while the filtration $\bar{X}_0 \subset \bar{X}_1 \subset \cdots \subset \bar{X}_n \subset \dots$ is called *stratification* of $\bar{X}$. In definition (2.1) the strata $\bar{X}_i \smallsetminus \bar{X}_{i-1}$ coincide with the complements $\bar{M}_i \smallsetminus \bar{A}_i$. The pairs $(\bar{M}_i, \bar{A}_i)$ are the *attached spaces* of $\bar{X}$. A stratified space $\bar{X}$ is Hausdorff and regular and if all the attaching maps $\bar{h}_i$ are cellular then $\bar{X}$ is a CW-complex. If the number of non-empty strata of $\bar{X}$ is finite, then $\bar{X}$ is said *finite*. We always assume that a stratified space which is not finite has cellular attaching maps. A finite stratified space $\bar{X}$ is *compactly stratified* if all the attached spaces $\bar{M}_i$ and $\bar{X}_0$ are compact.

A map $\bar{f} \colon \bar{X} \to \bar{X}'$ between stratified spaces is a filtration preserving map $\bar{f} = \{\bar{f}_n\}_{n\geq 0}$ together with commutative diagrams

$$\begin{tikzcd}
\bar{X}_{i-1} \arrow[d, "\bar{f}_{i-1}"] & \bar{A}_i \arrow[l] \arrow[r, tail] \arrow[d] & \bar{M}_i \arrow[d, "\bar{g}_i"] \\
\bar{X}'_{i-1} & \bar{A}'_i \arrow[l] \arrow[r, tail] & \bar{M}'_i
\end{tikzcd}$$

such that $\bar{g}_i \cup \bar{f}_{i-1} = \bar{f}_i$ for $i \geq 1$. A map is termed *stratum-preserving* if for every i

$$\bar{f}_i(\bar{X}_i \smallsetminus \bar{X}_{i-1}) \subset \bar{X}'_i \smallsetminus \bar{X}'_{i-1}.$$

Let **Stra** be the category of finite stratified spaces and stratum-preserving maps. Two maps f_0 and f_1 are homotopic in **Stra** if there exists a homotopy f_t such that the map f_t is a map in **Stra** for every $t \in I$. With this notion of homotopy the homotopy category $\mathbf{Stra}/_\sim$ is defined.

(2.2) Definition. A $\mathbf{V}$*-stratified bundle* is a stratified space $\bar{X}$ together with a $\mathbf{V}$-family

$$X \to \bar{X}$$

with the following properties. For $i \geq 1$ the restriction $X_i = X|\bar{X}_i$ is the push-out of $\mathbf{V}$-maps

$$\begin{tikzcd}
M_i \arrow[d] & A_i \arrow[l, tail] \arrow[r, "h_i"] \arrow[d] & X_{i-1} \arrow[d] \\
\bar{M}_i & \bar{A}_i \arrow[l, tail] \arrow[r, "\bar{h}_i"] & \bar{X}_{i-1}
\end{tikzcd}$$

where $M_i \to \bar{M}_i$ is a **V**-vector bundle and $A_i = M_i|\bar{A}_i$. Moreover, $X_0 \to \bar{X}_0$ is a **V**-vector bundle and $X = \lim_{i\to\infty} X_i$. If $(\bar{M}_i, \bar{A}_i)$ is a disjoint union of i-dimensional cells (D^i, S^{i-1}) then $\bar{X}$ is a CW-complex and X is termed a ***V**-complex*.

A ***V**-stratified map* $f\colon X \to X'$ between **V**-stratified bundles is given by sequences $\{f_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 1}$ of **V**-maps such that given the commutative diagrams

$$\begin{tikzcd} X_{i-1} \arrow[d, "f_{i-1}"] & A_i \arrow[l] \arrow[r, tail] \arrow[d] & M_i \arrow[d, "g_i"] \\ X'_{i-1} & A'_i \arrow[l] \arrow[r, tail] & M'_i \end{tikzcd}$$

for every i we have $g_i \cup f_{i-1} = f_i$ for $i \geq 1$. The **V**-map f is termed *stratum-preserving* if for every i

$$f(X_i \smallsetminus X_{i-1}) \subset X'_i \smallsetminus X'_{i-1}.$$

Let **V**-**Stra** denote the category of **V**-stratified bundles and stratum-preserving **V**-stratified maps. Of course a **V**-isomorphism in **V**-**Stra** is a stratum-preserving **V**-stratified map f with an inverse f'.

The following theorems are proved in [3] (theorem (4.7) and theorem (5.1)).

(2.3) Pull back theorem. *Let $\bar{X}$ and $\bar{X}'$ be finite stratified spaces with attached spaces which are locally finite and countable CW-complexes. Let $\bar{f}\colon \bar{X} \to \bar{X}'$ be a stratum-preserving map (in* **Stra***) and $X' \to \bar{X}'$ a* **V***-stratified bundle. Then the pull-back $\bar{f}^*X' \to \bar{X}$ is a* **V***-stratified bundle.*

(2.4) Bundle theorem. *Let* **V** $\subset$ **Vect** *be a subcategory such that all morphisms in* **V** *are isomorphisms (i.e.* **V** *is a groupoid). Then a* **V***-stratified bundle $X \to \bar{X}$ is a* **V***-bundle over $\bar{X}$. Conversely, given a stratified space $\bar{X}$ and a* **V***-bundle X over $\bar{X}$ then X is a* **V***-stratified bundle.*

In particular, a **V**-complex $X \to \bar{X}$ is a **V**-bundle over $\bar{X}$ and, given a CW-complex $\bar{X}$ and a **V**-bundle X over $\bar{X}$, then X is **V**-isomorphic to a **V**-complex.

3 Stratified manifolds

In this section we describe special families of vector spaces which arise as "stratified tangent bundles" of stratified manifolds or of manifolds with singularities. Such stratified tangent bundles are motivating examples of stratified vector bundles above. Concerning the general theory of stratification of manifolds we refer the reader for example to [5, 9, 12, 6, 10]. We here introduce stratified manifolds only since they show that stratified vector bundles are natural generalizations of vector bundles in a similar way as manifolds with singularities or stratifolds [5] generalize manifolds.

If M is a manifold with boundary ∂M, N a closed manifold, and $h\colon \partial M \to N$ a submersion, then the push-out of h and the inclusion $\partial M \subset M$ is a stratified space $X = M \cup_h N$ with stratification

$$X_0 = N \subset X_1 = X.$$

This is a stratifold in the sense of [5] and generalizes "manifolds with singularities". See Rudyak [7], Baas [2], Botvinnik [4], Sullivan [8], Vershinin [11]. Below we introduce differentiable stratified spaces which we consider as the general form of stratified manifolds

If M is a differentiable manifold then the tangent bundle $TM \to M$ restricted to the boundary $TM_{|\partial M}$ is isomorphic to the sum of the tangent bundle $T\partial M$ of the boundary of M and the normal bundle $\nu\partial M = \partial M \times \mathbb{R}$ as in the diagram

$$\begin{tikzcd} TM_{|\partial M} \arrow[rr, "\cong"] \arrow[dr] & & T\partial M \oplus \nu\partial M \arrow[dl] \\ & \partial M & \end{tikzcd} \tag{3.1}$$

The sum decomposition of $TM|\partial M$ for example is obtained by a tubular neighborhood N of the boundary and an explicit homeomorphism $N \approx \partial M \times \mathbb{R}_+$. The sum decomposition yields the projection

$$p_1\colon TM_{|\partial M} \cong T\partial M \oplus \nu\partial M \to T\partial M. \tag{3.2}$$

Now we define a stratified manifold X and the stratified tangent bundle $\mathbb{T}X$ by induction in the number of strata.

(3.3) Definition. A stratified space X with finite stratification $X_0 \subset \cdots \subset X_n = X$ and attached spaces (M_i, A_i) is termed *differentiable* if the following conditions hold. The spaces M_i are compact differentiable manifolds with boundary $A_i = \partial M_i$. Also X_0 is a compact differentiable manifold with $\partial X_0 = \emptyset$. Moreover for $i = 1, \ldots, n$ the restriction f_i° as in the diagram

$$\begin{tikzcd} & M_{i-1} \setminus \partial M_{i-1} \arrow[d, equal] \\ f_i^{-1}(X_{i-1} \setminus X_{i-2}) \arrow[r, "f_i^\circ"] \arrow[d, tail] & X_{i-1} \setminus X_{i-2} \arrow[d, tail] \\ \partial M_i \arrow[r, "f_i"] & X_{i-1} \end{tikzcd}$$

is differentiable and a submersion. In particular, $f_1 : \partial M_1 \to X_0$ is a submersion. We assume that $f_i^{-1}(X_{i-1} \smallsetminus X_{i-2})$ is dense in ∂M_i so that the map f_i is uniquely determined by f_i°.

We obtain inductively the tangent family $\mathbb{T}X_i$ in **Vect-Top** together with inclusions

$$T(X_i \smallsetminus X_{i-1}) \subset \mathbb{T}X_i$$

as follows. For $i = 0$ we have $\mathbb{T}X_0 = TX_0$. Assume $T(X_{i-1} \smallsetminus X_{i-2}) \subset \mathbb{T}X_{i-1}$ is defined. Then we consider the diagram in **Vect-Top**

$$\begin{tikzcd}
& T(M_{i-1} \smallsetminus \partial M_{i-1}) \arrow[d, equal] \\
Tf_i^{-1}(X_{i-1} \smallsetminus X_{i-2}) \arrow[r, "df_i^\circ"] \arrow[d, tail] & T(X_{i-1} \smallsetminus X_{i-2}) \arrow[d, tail] \\
T\partial M_i \arrow[r, dotted, "df_i"] & \mathbb{T}X_{i-1}
\end{tikzcd} \tag{3.4}$$

Since the inclusion in the first column of the diagram is dense there exists at most one **Vect**-map df_i for which the diagram commutes. In addition to the differentiability in (3.3) we assume inductively that df_i exists for $i \geq 2$ and that df_i is surjective. Then $\mathbb{T}X_i$ is defined by the following push-out diagram, which exists by lemma (1.3).

$$\begin{tikzcd}
& TM_i|\partial M_i \arrow[rrr, tail] \arrow[ddl, bend right, "p_1"'] \arrow[dddd] \arrow[dr] & & & TM_i \arrow[dl] \arrow[dddd] \\
& & \partial M_i \arrow[r, tail] \arrow[dd, "f_i"'] & M_i \arrow[dd] & \\
T\partial M_i \arrow[ddr, dotted, "df_i"'] & & & & \\
& & X_{i-1} \arrow[r] & X_i & \\
& \mathbb{T}X_{i-1} \arrow[ur] \arrow[rrr] & & & TM \cup_{df_i \circ p_1} \mathbb{T}X_{i-1} = \mathbb{T}X_i \arrow[ul, dotted, "p_i"']
\end{tikzcd}$$

A differentiable stratified space is termed a *stratified manifold.*

We call $\mathbb{T}X = \mathbb{T}X_n$ the *stratified tangent bundle* of the differentiable stratified space X. As a set $\mathbb{T}X$ is the disjoint union of $T(X_i \smallsetminus X_{i-1})$, that is of the tangent bundles $T(M_i \smallsetminus \partial M_i)$ for $i = 0, \dots, n$. Moreover, for every i, the pull-back of $\mathbb{T}X$ over the inclusion of the stratum $X_i \smallsetminus X_{i-1} \subset X$ is the tangent bundle of the stratum itself. But the topology of $\mathbb{T}X$ is not the topology of the disjoint union.

(3.5) Lemma. *Different choices of projections p_1 in the definition of $\mathbb{T}X$ yield isomorphic tangent families.*

Proof. We prove it by induction. For $n = 0$ it is true, because $\mathbb{T}X_0 = TX_0$. So assume that $\mathbb{T}X_{i-1}$ does not depend upon the choice of p_1. Let p_1 and p_1' be two projections $p_1, p_1' : TM_i|\partial M_i \to T\partial M_i$ corresponding to two choices of the normal bundle

$$\nu\partial M_i, \nu'\partial M_i \subset T\partial M_i|\partial M_i.$$

The two normal bundles are equivalent and there is a homotopy of bundles $H_t : TM_i|\partial M_i$ covering the identity of ∂M_i such that H_1 is the identity, H_t is the identity whenever restricted to $T\partial M_i$ and an isomorphism of bundles for every $t \in I$, and H_0 sends $\nu\partial M_i$ to $\nu'\partial M_i$. The fat homotopy $(H_t, t)\colon (TM_i|\partial M_i) \times I \to (TM_i|\partial M_i) \times I$ defined by $(x, t) \mapsto (H_t(x), t)$ is therefore an isomorphism of $\mathbf{V}$-families covering the identity of $\partial M_i \times I$ extending the $\mathbf{V}$-isomorphism $F = H_0$. Moreover, there is a tubular neighborhood $N \approx \partial M_i \times I$ of ∂M in M_i, where we identify ∂M_i with $\partial M_i \times \{0\}$ in N. Because $TM_i|N = (TM_i|\partial M_i) \times I$, the fat homotopy (H_t, t) induces a $\mathbf{V}$-isomorphism on $TM_i|N$ covering the identity of N, with the further property that it is the identity on $TM_i|(\partial M_i \times \{1\})$. Thus it is possible to extend $F = H_0$ to a $\mathbf{V}$-isomorphism $\tilde{F}$ on TM_i. By construction, $p_1'F = p_1$, and therefore the following diagram commutes and the $\mathbf{V}$-equivalence G exists.

$$\begin{tikzcd}
TM_i|\partial M_i \arrow[rrr] \arrow[dd, "p_1'"'] & & & TM_i \arrow[dddd] \\
& TM_i|\partial M_i \arrow[ul, "F"'] \arrow[r] \arrow[d, "p_1"] & TM_i \arrow[ur, "\tilde{F}"] \arrow[dd] & \\
T\partial M_i \arrow[dd, "df_i"] & T\partial M_i \arrow[l, "1"] \arrow[d, "df_i"] & & \\
& \mathbb{T}X_{i-1} \arrow[r] \arrow[dl, "1"] & \mathbb{T}X_i \arrow[dr, dotted, "G"] & \\
\mathbb{T}X_{i-1} \arrow[rrr] & & & \mathbb{T}X_i
\end{tikzcd}$$

Thus the isomorphism type of $\mathbb{T}X_i$ does not depend on the choice of the projection p_i, as claimed. *q.e.d.*

(3.6) Remark. The map f_i is proper since ∂M_i is compact and X_{i-1} is Hausdorff. Therefore the restriction map f_i°

$$f_i^\circ : f_i^{-1}(X_{i-1} \smallsetminus X_{i-2}) \to X_{i-1} \smallsetminus X_{i-2} = M_{i-1} \smallsetminus \partial M_{i-1}$$

is a proper surjective submersion and hence by Ehresmann's theorem it is a fiber bundle.

(3.7) Example. Let M be a differentiable manifold with boundary; let $N = \partial M$ and $h = 1_{\partial M} : \partial M \to \partial M = N$. Then, the stratified space $M \cup_h \partial M$ coincides with M. The tangent family $p : \mathbb{T}M \to M$ defined above is *not* isomorphic to the tangent bundle $TM \to M$ (the fibers over ∂M in $\mathbb{T}M$ are vector spaces of dimension $\dim \partial M$ and not $\dim M$). Actually the stratified tangent bundle $\mathbb{T}M$ has the quotient topology under the map $\phi : TM \to \mathbb{T}M$ induced by the push-out. For example let $M = D^n$ the closed n-dimensional unit disc in $\mathbb{R}^n$, and N its boundary $\partial M = S^{n-1}$. M is a smooth manifold and the rays through the origin give rise to a normal bundle of ∂M in M. The identity map $h = 1 : \partial M \to S^{n-1}$ is of course differentiable, and the map $dh \circ p_1$ is nothing but the projection $TM_{|\partial M} = S^{n-1} \times \mathbb{R}^n \to TS^{n-1}$ defined by $(x, v) \mapsto (x, v - \frac{v \cdot x}{x \cdot x} x)$, where (x, v), with $x \in S^{n-1}$ and $v \in \mathbb{R}^n$, orthogonal to x, are the usual coordinates for the tangent bundle of a sphere. Thus the stratified tangent bundle $\mathbb{T}M$ is defined by the quotient map $\phi : TM \to \mathbb{T}M$ which is the inclusion on $TM|(M \smallsetminus \partial M)$. Moreover two points (x, v) and (y, w) of $TM|\partial M = S^n \times \mathbb{R}^n$ have the same image in $\mathbb{T}M$ if and only if $v - \frac{v \cdot x}{x \cdot x} x = w - \frac{w \cdot y}{y \cdot y} y$.

(3.8) Example. Let X be the standard cube in $\mathbb{R}^3$. Let X_0 be the set of vertices of X, X_1 the union of the edges, X_2 the faces and $X_3 = X$. Thus the filtration

$$X_0 \subset X_1 \subset X_2 \subset X_3 = X$$

yields a decomposition into strata of dimension 0,1, 2 and 3 respectively. It is not difficult to see that for every $i = 1 \dots 3$ there is a disjoint union of discs $M_i = \cup D^i$ and a surjective map $f_i : \partial M_i = \cup S^{i-1} \to X_{i-1}$ such that X_i is obtained as attachment of M_i to X_{i-1} via f_i. The stratum X_0 consists of the 8 vertices, the attaching manifold M_1 of 12 edges (and the boundary of M_1 consists of 24 disjoint points); in a similar way, M_2 is the union of 6 discs, with boundary 6 circles S^1, and M_3 is a unique disc with boundary S^2. The cube X is a differentiable stratified space.

4 V-function spaces

Let Y be a $\mathbf{V}$-family. For $V \in \mathbf{V}$ we denote by Y^V the space of all the $\mathbf{V}$-maps $V \to Y$, endowed with the compact-open topology. A sub-basis of the topology in V^Y is given by all the sets of the form $N_{K,U} = \{f \in Y^V : fK \subset U\}$ with $K \subset V$ compact and $U \subset Y$ open. For every $\mathbf{V}$-family Y there is a standard projection $Y^V \to \bar{Y}$ sending a $\mathbf{V}$-map $a \colon V \to Y$ to the point $p_Y a(0) \in \bar{Y}$, where $0 \in V$ is the zero of the vector space. The pre-image of a point $b \in \bar{Y}$ under this projection is homeomorphic to $\hom_{\mathbf{V}}(V, Y_b)$.

Given a $\mathbf{V}$-map $f \colon X \to Y$, let f^V denote the map $f^V \colon X^V \to Y^V$ defined by composing with f the $\mathbf{V}$-maps $V \to X$. Moreover for $\varphi \colon V \to W$ let $X^\varphi \colon X^W \to X^V$ be the map induced by φ. Given a $\mathbf{V}$-map $f \colon Z \times V \to X$, its adjoint $\hat{f}$ is the function $\hat{f} \colon Z \to X^V$ defined by $\hat{f}(z)(v) = f(z,v)$ for every $z \in Z$ and $v \in V$. Lemma (6.10) of [3] states that for every space Z a $\mathbf{V}$-function $f \colon Z \times V \to X$ is continuous if and only if the adjoint $\hat{f} \colon Z \to X^V$ is continuous. As a consequence, the evaluation map $X^V \times V \to X$ which sends (g, v) to $g(v)$ is continuous and hence if $f \colon X \to Y$ is continuous then the induced function $f^V \colon X^V \to Y^V$ is continuous, and in the same way given $\varphi \colon V \to W$ the induced map $X^\varphi \colon X^W \to X^V$ is continuous.

(4.1) Theorem. *Let (X, D) be a relative $\mathbf{V}$-complex, where $\mathbf{V}$ is a closed subcategory of* **Vect**. *Let $\Phi : Z_n \times D^n \to X$ be its characteristic map of n-cells and let h_n denote the n-attaching map of X, i.e. the restriction of Φ to $Z_n \times S^{n-1}$. For every $n \geq 0$ and every $V \in \mathbf{V}$ the diagram*

$$\begin{tikzcd}
(Z_n \times S^{n-1})^V \arrow[r, "h_n^V"] \arrow[d, tail] & X_{n-1}^V \arrow[d] \\
(Z_n \times D^n)^V \arrow[r, "\Phi^V"] & X_n^V
\end{tikzcd}$$

is a push-out (in **Top***) with $X_{-1} = D$. Moreover, $X^V = \lim_{n\to\infty} X_n^V$.*

Proof. See corollary (10.16) of [3]. For this we need the following lemma. *q.e.d.*

A family $\mathcal{K}$ of compact sets of V is termed *generating* if for every $\mathbf{V}$-family Y the subsets

$$N_{K,U} = \{f \in Y^V, f(K) \subset U\}$$

with $K \in \mathcal{K}$ and U open in Y are a sub-basis for the topology of the function space Y^V.

We say that a structure category $\mathbf{V}$ has the NKC-*property* (see [3], definition (6.5)) if for every object $V \in \mathbf{V}$ there is a generating family of compact sets $\mathcal{K}$ such that for every $K \in \mathcal{K}$, every $W \in \mathbf{V}$ and every compact subset $C \subset W$ the subspace

$$N_{K,C} \subset \hom_{\mathbf{V}}(V, W) = W^V$$

is compact.

(4.2) Lemma. *Let* $\mathbf{V}$ *be a closed subcategory of* $\mathbf{Vect}$*. Then* $\mathbf{V}$ *is a NKC category.*

Since a closed subcategory of a NKC category is again a NKC category, it suffices to consider the case $\mathbf{V} = \mathbf{Vect}$. Thus lemma (4.2) is a consequence of lemma (4.7) below.

(4.3) Definition. A compact $K \subset V$ containing a linear basis of the vector space V is termed *spanning* compact subset.

(4.4) Lemma. *Let* Y *be a* $\mathbf{V}$*-family,* $f\colon V \to Y$ *a* $\mathbf{V}$*-map,* $U \subset Y$ *an open subset,* $K \subset V$ *a compact subset of* V *and* $f \in N_{K,U}$*. Then there is a spanning compact set* $K' \subset V$ *such that* $K' \supset K$*,* K' *and* $f \in N_{K',U} \subset N_{K,U}$*. Thus the family* $\mathcal{K}$ *of spanning compact subsets of* V *is generating.*

Proof. Being f continuous, $f^{-1}U$ is open in V, and therefore there exists a basis $B = \{b_1, b_2, \dots, b_d\}$ of V contained in $f^{-1}U$. Then take $K' = K \cup B$. By definition $fB \subset U$ and hence $f \in N_{K',U}$. Because $K' \supset K$, $N_{K',U} \subset N_{K,U}$. *q.e.d.*

(4.5) Lemma. *Let* V *and* W *be finitely dimensional normed vector spaces,* $B = \{b_1, b_2, \dots, b_d\}$ *a basis of* V *and* R *a constant. Let* $\beta\colon \mathbb{R}^d \to V$ *be an isomorphism sending the standard basis of* $\mathbb{R}^d$ *to the basis* B *of* V*, and* $|\beta^{-1}|$ *the norm of its inverse. If* $f\colon V \to W$ *is a linear map such that* $|f(b_i)| \leq R$ *for* $i = 1, \dots, d$*, then* $|f| \leq \sqrt{d} R |\beta^{-1}|$.

Proof. The composition $f\beta\colon \mathbb{R}^d \to W$ has norm

$$|f\beta| = \sup_{x \in \mathbb{R}^d} \frac{|f\beta(x)|}{|x|}.$$

Write $x \in \mathbb{R}^d$ as $\sum_j x_j e_j$, where e_j are the elements of the standard basis of $\mathbb{R}^d$; by assumption $\beta(e_j) = b_j$ for every $j = 1, \dots, d$. Now consider the inequalities

$$\frac{|\sum_j x_j f(b_j)|}{|x|} \leq \frac{\sum_j |x_j||f(b_j)|}{\sqrt{\sum_j x_j^2}} \leq \sqrt{d} R.$$

This implies that $|f\beta| \le \sqrt{d}R$, and hence $|f| \le |f\beta\beta^{-1}| \le \sqrt{d}R|\beta^{-1}|$, as claimed. *q.e.d.*

(4.6) Lemma. *If X is a trivial* **Vect***-family $X = W \times Y$ over an Hausdorff space Y then X^V is a closed subspace of* $\mathrm{Map}(V,X)$. *In particular, by taking $Y = *$, we obtain that for every V, $W \in \mathbf{V}$, the function space $W^V =$* $\mathbf{Vect}(V,W)$ *is a closed subspace of* $\mathrm{Map}(V,W)$.

Proof. Let p_X be the projection $W \times Y \to Y$ and $\{b_1, \dots, b_n\}$ a basis of the vector space V. First, assume that $f_0 \in \mathrm{Map}(V,X)$ is a map not in X^V such that $p_X f(V) \ne p_X f(0) = y_0$. Then there is $y \in Y$ such that $y \ne y_0$ and $p_X f(V) \ni y$. Since Y is Hausdorff, there are neighborhoods $\bar{U}_0$ and $\bar{U}_1$ in Y such that $y_0 \in \bar{U}_0$, $y \in \bar{U}_1$ and $\bar{U}_0 \cap \bar{U}_1 = \emptyset$. Let use define $U_i = W \times \bar{U}_i \subset X$, $K_0 = \{0\} \subset V$ and $K_1 = \{v\}$, with some $v \in (p_X f)^{-1} y$. We have $fK_i \subset U_i$ and

$$N_{K_0,U_0} \bigcap N_{K_1,U_1} \bigcap X^V = \emptyset,$$

hence $N_{K_0,U_0} \bigcap N_{K_1,U_1}$ is an open neighborhood of f contained in $\mathrm{Map}(V,X) \smallsetminus X^V$.

On the other hand, assume that $p_X f(V) = p_X f(0) = y_0$. Thus $f \colon V \to p_X^{-1} y_0 = W$. Since $f \notin X^V$, f is not linear, there exist a vector $v \in V$ and n scalars λ_i in the field $\mathbb{K}$ such that

$$f(\sum_{i=1}^n \lambda_i b_i) \ne \sum_{i=1}^n \lambda_i f(b_i).$$

Since $W \times Y$ is Hausdorff, there are disjoint open sets U_0 and U_1 in $W \times Y$ such that $f(v) \in U_0$ and $\sum_{i=1}^n \lambda_i f(b_i) \in U_1$. Now, the map $W^n \to W$ with sends n vectors v_i' to the linear combination $\sum_{i=1}^n \lambda_i v_i'$ is continuous, hence there are n open sets $U_i' \subset X$, neighborhoods of $f(b_i)$, such that if $x_i \in U_i'$ for every $i = 1, \dots, n$ and $p_X(x_i) = p_X(x_j)$ for every i, j then $\sum_{i=1}^n \lambda_i x_i \subset U_1$. Consider the compact set $K_0 = \{v\}$, $K_i = \{b_i\} \subset V$, $i = 1, \dots, n$. Consider the open subset $A \subset \mathbf{Map}(V,X)$ defined by

$$A = N_{K_0,U_0} \bigcap N_{K_1,U_1'} \bigcap \cdots \bigcap N_{K_n,U_n'}.$$

It consists of all maps $g \colon V \to X$ such that $g(v) \subset U_0$ and $g(b_i) \subset U_i'$. By construction $f \in A$. Moreover, assume that $g \in A \cap X^V$. Then

$$g(v) = \sum_{i=1}^n \lambda_i g(b_i).$$

But the left hand side of the equation belongs to U_0, while the right hand side belongs to U_1, hence g cannot belong to $A \cap X^V$. Thus $A \cap X^V = \emptyset$. *q.e.d.*

(4.7) Lemma. *If K is a spanning compact subset of V, W is an object of $\mathbf{V} = \mathbf{Vect}$ and C a compact subset of W then $N_{K,C}$ is compact.*

Proof. As in the proof of lemma (10.6) of [3] it is easy to see that $N_{K,C}$ is closed. Now we prove that it is equicontinuous. By assumption C is compact and therefore bounded, and by assumption K contains a basis $B = \{b_1, \ldots, b_d\}$; so there exists a constant $R \geq 0$, $R = |C|$, such that if $f \in N_{K,C}$ then $|f(b_j)| \leq R$ for every $j = 1, \ldots, d$. But because of lemma (4.5) this implies that there exists a constant R' (depending only on B and R) such that $|f| \leq R'$. This implies that for each $v_0 \in V$ there is a constant α such that for every $f \in N_{K,C}$ and for every $v \in V$

$$|f(v) - f(v_0)| \leq \alpha |v - v_0|,$$

hence $N_{K,C}$ is equicontinuous. Now consider for every $v_0 \in V$ the space $\omega_{v_0} = \{f(v_0)\}$. By the same reason it is bounded in W, hence it has compact closure in W. Thus, by the Ascoli-Arzelà theorem, the closure of $N_{K,C}$ in $\mathbf{Map}(V, W)$ is compact. But since $N_{K,C}$ is a closed subset of $\mathbf{Vect}(V, W)$ which in turn is closed in $\mathbf{Map}(V, W)$ by lemma (4.6), it coincides with its closure in $\mathbf{Map}(V, W)$ and therefore is compact. *q.e.d.*

With lemma (4.7) we have actually proved lemma (4.2).

(4.8) Remark. The principal bundle theorem (6.17) in [3] shows that if $\mathbf{V}$ is a NKC category then the diagram $X^\circ \colon \mathbf{V}^{\mathrm{op}} \to \mathbf{Top}$ given by $X^\circ(V) = X^V$ is the "stratified principal bundle" associated to X. Moreover X is determined by X° since $X = X^\circ \otimes_{\mathbf{V}} F$ where $F \colon \mathbf{V} \to \mathbf{Top}$ is the inclusion functor.

5 Algebraic constructions

In the following we will apply theorem (4.1).

Let $\mathbf{V}$ and $\mathbf{W}$ be two closed subcategories of $\mathbf{Vect}$ and let $\psi \colon \mathbf{V} \to \mathbf{W}$ be a continuous functor (that is, for every $V_1, V_2 \in \mathbf{V}$ the induced map on hom-spaces $\psi \colon \hom_{\mathbf{V}}(V_1, V_2) \to \hom_{\mathbf{W}}(\psi V_1, \psi V_2)$ is continuous). Given a $\mathbf{V}$-complex X we want to build an associated $\mathbf{W}$-complex ψX over $\bar{X}$ with fibers given by the images ψX_b of the fibers of X under ψ. At a set level, we know that X is the disjoint union

$$X = \coprod (Z_n \times e^n),$$

where $e^n = D^n \smallsetminus S^{n-1}$ is the open cell and Z_n is the discrete $\mathbf{V}$-family of n-cells of X. Therefore it is possible to define ψX as a set by

$$\psi X = \coprod (\psi(Z_n) \times e^n).$$

We now define the topology of ψX inductively. Furthermore, we show that for every $V \in \mathbf{V}$ the map $\psi^V \colon X^V \to (\psi X)^{\psi V}$ defined by sending $a \colon V \to X_b \subset X$ in X^V to $\psi(a) \in \hom(\psi(V), \psi(X_b))$, $\psi(a) \colon \psi(V) \to \psi(X_b) = \psi(X)_b$, is continuous. The topology of ψX_0 is the topology of the $\mathbf{W}$-set $\psi X_0 = \psi Z_0$. Moreover, since ψ is a continuous functor we see that for every $V \in \mathbf{V}$ the induced map $\psi_0^V \colon X_0^V \to (\psi X_0)^{\psi V}$ is continuous. Now, assume that the topology of ψX_{n-1} is defined and that the map $\psi_{n-1}^V \colon X_{n-1}^V \to (\psi X_{n-1})^{\psi V}$ is continuous. The adjoint of the attaching map $h_n : Z_n \times S^{n-1} \to X_{n-1}$ is a continuous function

$$\hat{h}_n \colon S^{n-1} \to X_{n-1}^{Z_n} = \prod_{V \in Z_n} X_{n-1}^V$$

where the right hand side is the product space. By inductive assumption $\psi_{n-1}^{Z_n} \colon X_{n-1}^{Z_n} \to (\psi X_{n-1})^{\psi Z_n}$ is continuous because it is continuous on the factors of $X_{n-1}^{Z_n} = \prod_{V \in Z_n} X_{n-1}^V$. Therefore the composition

$$S^{n-1} \xrightarrow{\hat{h}_n} X_{n-1}^{Z_n} \xrightarrow{\psi^{Z_n}} (\psi X_{n-1})^{\psi Z_n}$$

is continuous and yields a $\mathbf{W}$-map $\psi(h_n) \colon \psi Z_n \times S^{n-1} \to \psi X_{n-1}$. We can therefore endow ψX_n with the topology of the push-out

$$\begin{tikzcd}
\psi Z_n \times S^{n-1} \arrow[r, "\psi(h_n)"] \arrow[d, tail] & \psi X_{n-1} \arrow[d] \\
\psi Z_n \times D^n \arrow[r] & \psi X_n
\end{tikzcd}$$

Now we need to show that $\psi_n^V : X_n^V \to (\psi X_n)^{\psi V}$ is continuous for every $V \in \mathbf{V}$. Consider the diagram (5.1).

(5.1)
$$\begin{tikzcd}
(\psi Z_n \times S^{n-1})^V \arrow[rrr, "\psi(h_n)^{\psi V}"] \arrow[ddd] & & & \psi X_{n-1}^{\psi V} \arrow[ddd] \\
& (Z_n \times S^{n-1})^V \arrow[ul, "\hat{\psi}"] \arrow[r, "h_n^V"] \arrow[d] & X_{n-1}^V \arrow[ur, "\hat{\psi}_{n-1}^V"] \arrow[d] & \\
& (Z_n \times D^n)^V \arrow[r] \arrow[dl] & X_n^V \arrow[dr, dotted, "\hat{\psi}_n^V"] & \\
(\psi Z_n \times D^n)^V \arrow[rrr] & & & \psi X_n^{\psi V}
\end{tikzcd}$$

Because of theorem (4.1) the two square diagrams are push-out diagrams and therefore the induced map $\hat{\psi}_n^V$ is continuous.

This inductive process yields a topology for every ψX_n. Let $\psi X = \lim_{n \geq 0} \psi X_n$. Because $\hat{\psi}_n^V \colon X_n^V \to (\psi X_n)^{\psi V} \subset (\psi X)^{\psi X}$ is continuous for every V, the induced map $\hat{\psi} \colon X^V \to (\psi X)^{\psi V}$ is continuous for every $V \in \mathbf{V}$; here we use again theorem (4.1). Let $\mathbf{V}$-CW be the full subcategory of $\mathbf{V}$-**Top** consisting of $\mathbf{V}$-complexes.

(5.2) Lemma. *Let ψ be a continuous functor $\mathbf{V} \to \mathbf{W}$ where $\mathbf{V}$ and $\mathbf{W}$ are closed subcategories of* **Vect**. *The operation that sends each $\mathbf{V}$-complex X to the $\mathbf{W}$-complex ψX defined above yields a well-defined functor $\psi \colon \mathbf{V}\text{-}CW \to \mathbf{W}\text{-}CW$.*

This, in fact, is an example of the functor $\varphi_\#$ in [3] which carries a stratified bundle to an associated stratified bundle.

Proof. We only need to define the image $\psi(f) \colon \psi X \to \psi Y$ under ψ of a $\mathbf{V}$-map $f \colon X \to Y$. We define $\psi(f)$ inductively on the skeleta of ψX. Let $f_n \colon X_n \to Y$ denote the restriction of f to the n-skeleton of X. On the zero-skeleton ψX_0, $\psi(f) = \psi(f_0)$ is defined. Assume that $\psi(f_{n-1})$ is defined. Let $\Phi_n : Z_n \times D^n \to X_n$ be the n-characteristic map of X and consider the adjoint of the composition

$$(f_n \Phi_n) \colon Z_n \times D^n \to X \to Y$$

$$\widehat{(f_n \Phi_n)} \colon D^n \to Y^{Z_n}.$$

By construction the map $\hat{\psi} \colon Y^{Z_n} \to (\psi Y)^{\psi Z_n}$ is continuous. Therefore the composition $\hat{\psi}\widehat{(f_n \Phi_n)}$ is the adjoint of a continuous $\mathbf{W}$-map $\psi Z_n \times D^n \to \psi Y$. Furthermore, it is possible to check that this map fits into the outside square of the following diagram making it commutative.

$$\begin{tikzcd}
\psi Z_n \times S^{n-1} \arrow[r] \arrow[d] & \psi X_{n-1} \arrow[d] \arrow[ddr, bend left, "\psi(f_{n-1})"] & \\
\psi Z_n \times D^n \arrow[r] \arrow[drr, bend right] & \psi X_n \arrow[dr, dotted, "\psi(f_n)"'] & \\
& & \psi Y
\end{tikzcd}$$

Hence the $\mathbf{W}$-map $\psi(f_n)$ exists for every $n \geq 0$. By taking the limit, we obtain the desired $\mathbf{W}$-map $\psi(f) \colon \psi X \to \psi Y$. *q.e.d.*

Now consider a continuous bifunctor $\psi\colon \mathbf{V}\times\mathbf{V}'\to\mathbf{W}$, a $\mathbf{V}$-complex X over $\bar{X}=B$ and a $\mathbf{V}'$-complex X' over the same base space $\bar{X}'=B$; assume that the cellular decompositions on B induced by X and X' are the same and that $\mathbf{V}$, $\mathbf{V}'$ and $\mathbf{W}$ are closed. We are now going to define a $\mathbf{W}$-complex $\psi(X,X')$ with base space B. If $X=X_0$ and $X'=X_0'$ are a $\mathbf{V}$-set and a $\mathbf{V}'$-set respectively over the same base set B, then $\psi(X,X')$ is defined pointwise as $\psi(V,V')$, whenever V and V' are fibers over the same point in B. In general, let Φ and Φ' be the characteristic maps of X and X'. Using the decomposition into open cells of X and X' it is possible to define $\psi(X,X')$ as a set as follows:

$$X=\coprod_{n\geq 0} Z_n\times e^n,\quad X'=\coprod_{n\geq 0} Z_n'\times e^n,$$

$$\psi(X,X')=\coprod_{n\geq 0}\psi(Z_n,Z_n')\times e^n. \tag{5.3}$$

We need to endow $\psi(X,X')$ with a suitable topology.

Consider a space Y and a space Y' which are sources of maps to the space B. We denote by $Y\times_B Y'$ the pull-back

$$\begin{tikzcd} Y\times_B Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ Y \arrow[r] & B. \end{tikzcd}$$

For every $V\in\mathbf{V}$ and $V'\in\mathbf{V}'$, consider the pull-back $X^V\times_B {X'}^{V'}$ with respect to the maps $X^V\to\bar{X}$ and ${X'}^{V'}\to\bar{X}$. The pre-image of a point $b\in\bar{X}=\bar{X}'$ under the natural projection $X^V\times_B {X'}^{V'}\to\bar{X}$ is equal to $\hom(V,X_b)\times\hom(V',X_b')$. Therefore we obtain a function

$$\hat{\psi}\colon X^V\times_B {X'}^{V'}\to\psi(X,X')^{\psi(V,V')}$$

which is continuous on fibers (by continuity of ψ) defined by

$$\hat{\psi}\colon\ \hom(V,X_b)\times\hom(V',X_b')\to\hom(\psi(V,V'),\psi(X_b,X_b')).$$

We want to give a topology to $\psi(X,X')$ so that $\hat{\psi}$ is continuous. We can do it inductively on the skeleta of X and X'. If $n=0$, then $\psi(X_0,X_0')$ is defined as above, and it is trivial to see that $\hat{\psi}_0\colon X_0^V\sqcap {X_0'}^{V'}\to\psi(X_0,X_0')^{\psi(V,V')}$ is continuous. More directly, we can obtain $\psi(X,X')$ by

$$\psi(X,X')=\psi_{\#}(X\times_B X')$$

where the right hand side is defined by the associated bundle in [3].

(5.4) Remark. If X and X' are $\mathbf{V}$-vector bundles over a CW-complex $\bar{X}$, then by the bundle theorem (5.1) of [3] X and X' have a structure of $\mathbf{V}$-complexes. In this case it is not difficult to show that the bundles $\psi(X)$, $\psi(X, X')$ coincide with the corresponding constructions on vector bundles (see e.g. Atiyah [1], page 6).

6 Algebraic constructions on stratified vector bundles

Consider the closure $\bar{\mathbf{V}}$ in $\mathbf{Vect}$ of a subcategory $\mathbf{V} \subset \mathbf{Vect}$. Since composition maps are continuous, the closure $\bar{\mathbf{V}}$ is a closed subcategory of $\mathbf{Vect}$. We say that a continuous functor $\psi\colon \mathbf{V} \to \mathbf{W}$ is *regular* if there is a continuous functor $\bar{\psi}\colon \bar{\mathbf{V}} \to \bar{\mathbf{W}}$ such that ψ is the restriction of $\bar{\psi}$. A similar definition holds for continuous bifunctors. As a slight generalization of lemma (5.2) we get.

(6.1) Lemma. *Let ψ be a regular continuous functor $\mathbf{V} \to \mathbf{W}$ where $\mathbf{V}$ and $\mathbf{W}$ are subcategories of $\mathbf{Vect}$. Then the operation that sends each $\mathbf{V}$-complex X to the $\mathbf{W}$-complex ψX defined above yields a well-defined functor $\psi\colon \mathbf{V}\text{-}CW \to \mathbf{W}\text{-}CW$.*

Proof. The embeddings of categories $\mathbf{V} \subset \bar{\mathbf{V}}$ and $\mathbf{W} \subset \bar{\mathbf{W}}$ induce embeddings $i_{\mathbf{V}}\colon \mathbf{V}\text{-CW} \subset \bar{\mathbf{V}}\text{-CW}$ and $i_{\mathbf{W}}\colon \mathbf{W}\text{-CW} \subset \bar{\mathbf{W}}\text{-CW}$ By lemma (5.2), the continuous functor $\bar{\psi}$ yields a functor $\bar{\psi}\colon \bar{\mathbf{V}}\text{-CW} \to \bar{\mathbf{W}}\text{-CW}$. Since the image of the composition of functors $\bar{\psi} i_{\mathbf{V}}$ is actually in $\mathbf{W}$-CW, we obtain a functor $\psi\colon \mathbf{V}\text{-CW} \to \mathbf{W}\text{-CW}$ as claimed. *q.e.d.*

Let $\mathbf{V}\text{-CW} \sqcap \mathbf{V}'\text{-CW}$ denote the category with as objects pairs (X, X') where X is a $\mathbf{V}$-complex and X' a V'-complex, such that $\bar{X}$ and $\bar{X}'$ coincide. A morphism in $\mathbf{V}\text{-CW} \sqcap \mathbf{V}'\text{-CW}$ from (X, X') to (Y, Y') is a pair of maps (f, f'), where $f\colon X \to Y$ is a $\mathbf{V}$-map and $f'\colon X' \to Y'$ is a $\mathbf{V}'$-map, such that the induced maps on the base spaces coincide: $\bar{f} = \bar{f}'\colon \bar{X} \to \bar{Y}$. Given an object (X, X') in $\mathbf{V}\text{-CW} \sqcap \mathbf{V}'\text{-CW}$ the $\mathbf{W}$-complex $\psi(X, X')$ has been defined above. Consider a morphism $(f, f')\colon (X, X') \to (Y, Y')$. This yields a map $f \sqcap f'\colon X \sqcap X' \to Y \sqcap Y'$.

More generally than (5.3) we get:

(6.2) Lemma. *Let $\mathbf{V}$, $\mathbf{V}'$ and $\mathbf{W}$ be subcategories of $\mathbf{Vect}$. For every regular continuous bifunctor ψ $\psi\colon \mathbf{V} \times \mathbf{V}' \to \mathbf{W}$ the operation that sends (X, X') to $\psi(X, X')$ and $(f, f')\colon (X, X') \to (Y, Y')$ to $\psi(f, f')$ is a well-defined functor*

$$\psi\colon \mathbf{V}\text{-}CW \sqcap \mathbf{V}'\text{-}CW \to \mathbf{W}\text{-}CW.$$

Proof. The proof is identical as the proof of lemma (6.1). *q.e.d.*

(6.3) Theorem. *Let* $\psi\colon \mathbf{V} \to \mathbf{W}$ *be a regular continuous functor, where* $\mathbf{V}$ *and* $\mathbf{W}$ *are subcategories of* **Vect**. *There is an induced functor* $\psi\colon \mathbf{V}\text{-}\mathbf{Stra} \to \mathbf{W}\text{-}\mathbf{Stra}$ *which sends a* $\mathbf{V}$*-stratified vector bundle* X *to the space* ψX *over* $\bar{X}$ *with fibers* ψX_b*, for* $b \in \bar{X}$.

Proof. By lemma (6.1) such functor is already defined for $\mathbf{V}$-complexes. Using the same argument as in the proof of lemma (6.1), it suffices to consider the case of closed (and hence NKC) subcategories of **Vect**. Consider the subcategories $\mathbf{V}\text{-}\mathbf{Stra}_n$ and $\mathbf{W}\text{-}\mathbf{Stra}_n$ of $\mathbf{V}$-stratified (resp. $\mathbf{W}$-stratified) vector bundles of length at most n (i.e. with at most n strata). If X is an object in $\mathbf{V}\text{-}\mathbf{Stra}_1$ then it is a vector bundle over a CW-complex, hence it is a $\mathbf{V}$-complex by remark (5.4). Thus the objects of $\mathbf{V}\text{-}\mathbf{Stra}_1$ and $\mathbf{W}\text{-}\mathbf{Stra}_1$ are in $\mathbf{V}$-CW and $\mathbf{W}$-CW respectively, and the maps are of course $\mathbf{V}$-maps and $\mathbf{W}$-maps. This implies that

$$\psi\colon \mathbf{V}\text{-}\mathbf{Stra}_1 \to \mathbf{W}\text{-}\mathbf{Stra}_1$$

is well-defined. Assume by induction that ψ is defined on $\mathbf{V}\text{-}\mathbf{Stra}_n$, for some $n \geq 2$. Let X be an object in $\mathbf{V}\text{-}\mathbf{Stra}_{n+1}$. Thus there are CW-pairs $(\bar{M}_n, \bar{A}_n)$ a $\mathbf{V}$-vector bundle M_n and a $\mathbf{V}$-map $h_n\colon A_n = M_n|\bar{A}_n \to X_{n-1}$ such that $X = X_{n-1} \cup_{h_n} M_n$. Since X_{n-1} and A_n are both objects of $\mathbf{V}\text{-}\mathbf{Stra}_n$ we have that $\psi(h_n)\colon \psi A_n \to \psi X_{n-1}$ is a well-defined $\mathbf{W}$-map. Thus we can define ψX as the push-out $\psi X = \psi X_{n-1} \cup_{\psi(h_n)} M_n$. The same argument can be applied to a stratified map $f\colon X \to X'$, where X and X' are objects in $\mathbf{V}\text{-}\mathbf{Stra}_n$. Hence ψ is well-defined on every $\mathbf{V}\text{-}\mathbf{Stra}_n$. Now, since every object in $\mathbf{V}\text{-}\mathbf{Stra}$ belongs to $\mathbf{V}\text{-}\mathbf{Stra}_n$ for some n, this implies that ψ is a well-defined functor from $\mathbf{V}\text{-}\mathbf{Stra}$ to $\mathbf{W}\text{-}\mathbf{Stra}$. *q.e.d.*

Consider now a regular continuous bifunctor $\psi\colon \mathbf{V} \times \mathbf{V}' \to \mathbf{W}$. Let $\mathbf{V}\text{-}\mathbf{Stra} \sqcap \mathbf{V}'\text{-}\mathbf{Stra}$ denote the category of pairs (X, X'), where X is an object in $\mathbf{V}\text{-}\mathbf{Stra}$ and X' an object in $\mathbf{V}'\text{-}\mathbf{Stra}$ with the same base space $\bar{X} = \bar{X}'$. A morphism in $\mathbf{V}\text{-}\mathbf{Stra} \sqcap \mathbf{V}'\text{-}\mathbf{Stra}$ is a pair of maps (f, f') such that f is a morphism in $\mathbf{V}\text{-}\mathbf{Stra}$ and f' a morphism in $\mathbf{V}'\text{-}\mathbf{Stra}$ which induce the same map on the base space $\bar{f} = \bar{f}'\colon \bar{X} \to \bar{X}$.

(6.4) Theorem. *Every regular continuous bifunctor* $\psi\colon \mathbf{V} \times \mathbf{V}' \to \mathbf{W}$ *induces a well-defined functor*

$$\psi\colon \mathbf{V}\text{-}\mathbf{Stra} \sqcap \mathbf{V}'\text{-}\mathbf{Stra} \to \mathbf{W}\text{-}\mathbf{Stra},$$

which sends a pair of $\mathbf{V}$*-stratified vector bundles* X, X' *over* $\bar{X}$ *to the space* $\psi(X, X')$ *over* $\bar{X}$ *with fibers* $\psi(p_X^{-1}b, p_{X'}^{-1}b)$*, for* $b \in \bar{X}$.

Proof. First, assume that the structure categories are closed in **Vect**. As in the proof of theorem (6.3), let $\mathbf{V}\text{-}\mathbf{Stra}_n \sqcap \mathbf{V}'\text{-}\mathbf{Stra}_n$ denote the subcategory of pairs of **V**-stratified vector bundles of length at most n. The same for $\mathbf{W}\text{-}\mathbf{Stra}_n$. We are going to show that ψ is defined on $\mathbf{V}\text{-}\mathbf{Stra}_n \sqcap \mathbf{V}'\text{-}\mathbf{Stra}_n$ for every $n \geq 1$. If $n = 1$ this is true as a consequence of lemma (6.2). So, assume that ψ is well-defined on $\mathbf{V}\text{-}\mathbf{Stra}_{n-1} \sqcap \mathbf{V}'\text{-}\mathbf{Stra}_{n-1}$. Let (M_n, A_n) be the attached pair for X, with attaching map $h\colon A_n \to X_{n-1}$. Let (M'_n, A'_n) and h'_n the corresponding for X'. Since $\bar{X} = \bar{X}'$, without loss of generality we can assume that $\bar{M}_n = \bar{M}'_n$, $\bar{A}_n = \bar{A}'_n$ and that the induced maps $\bar{h}_n = \bar{h}'_n$ coincide. Thus the induced map

$$\psi(h_n, h'_n)\colon \psi(A_n, A'_n) \to \psi(X_{n-1}, X'_{n-1})$$

is well-defined, since (X_{n-1}, X'_{n-1}) and (A_n, A'_m) belong to $\mathbf{V}\text{-}\mathbf{Stra}_{n-1} \sqcap \mathbf{V}'\text{-}\mathbf{Stra}_{n-1}$. The same for the inclusions $(i, i')\colon (A_n, A'_n) \to (M_n, M'_n)$. Thus it is possible to define $\psi(X, X')$ as the push-out

$$\psi(X, X') = \psi(X_{n-1}, X'_{n-1}) \cup_{\psi(h_n, h'_n)} \psi(M_n, M'_n).$$

In the same way, given a pair of stratified maps $(f, f')\colon (X, X') \to (Y, Y')$ in $\mathbf{V}\text{-}\mathbf{Stra}_n \sqcap \mathbf{V}'\text{-}\mathbf{Stra}_n$, it is possible to define a **W**-map

$$\psi(f, f')\colon \psi(X, X') \to \psi(Y, Y').$$

Now, if the categories are not close, it suffices to follow the same argument as in the proof of lemma (6.1). *q.e.d.*

7 K-theory of stratified spaces

Assume that the structure category **V** is closed under direct sum $\oplus$ of vector spaces. Then, if $\bar{X}$ is any space, the set $\mathbf{V}(\bar{X})$ of isomorphism classes of **V**-vector bundles over $\bar{X}$ is an abelian semigroup where the sum is the sum induced as in lemma (6.2) by the bifunctor $\oplus$. Let $K_0^{\mathbf{V}}(\bar{X}) = K_0(\mathbf{V}(\bar{X}))$ be the Grothendieck group of the abelian semigroup $(\mathbf{V}(\bar{X}),\oplus)$. If **V** is also closed under tensor product $\otimes$ then $K_0^{\mathbf{V}}(\bar{X})$ is a ring. This is the Atiyah-Hirzebruch K-theory of a space $\bar{X}$.

We can do the same for the category $\mathbf{V}\text{-}\mathbf{Stra}(\bar{X})$ of **V**-stratified bundles over the stratified base space $\bar{X}$. Let $\bar{X}$ be a stratified space in **Stra**. Then by theorem (6.4) the category $\mathbf{V}\text{-}\mathbf{Stra}(\bar{X})$ of **V**-stratified vector bundles over $\bar{X}$ and stratified **V**-maps over $1_{\bar{X}}$ is an additive category with sum $\oplus$.

(7.1) Definition. Let $\mathcal{K}_0^{\mathbf{V}}(\bar{X})$ denote the Grothendieck group $K_0(\mathbf{V}\text{-}\mathbf{Stra}(\bar{X}))$.

If $\mathbf{V}$ is closed under the tensor product $\otimes$ then $\mathcal{K}_0^{\mathbf{V}}(\bar{X})$ is a ring. Now consider the full subcategory $\mathbf{Stra}_0 \subset \mathbf{Stra}$ of finite stratified spaces with attached spaces which are locally finite and countable CW-complexes.

(7.2) Theorem. *For every subcategory* $\mathbf{V}$ *of* $\mathbf{Vect}$ *which is closed under direct sum* $\oplus$ *there is a functor*

$$\mathcal{K}_0^{\mathbf{V}} : \mathbf{Stra}_0^{\mathrm{op}} \to \mathbf{Ab},$$

where $\mathbf{Ab}$ *denotes the category of Abelian groups. If* $\mathbf{V}$ *is closed under the tensor product* $\otimes$ *then the functor has its image in the category of rings. Moreover there are natural homomorphisms*

$$\begin{tikzcd} K_0^{\mathbf{V}}(\bar{X}) \arrow[r] & \mathcal{K}_0^{\mathbf{V}}(\bar{X}) \arrow[r] \arrow[d] & K_0^{\mathbf{V}}(\bar{M}_i) \\ & K_0^{\mathbf{V}}(\bar{X}_0) & \end{tikzcd}$$

where $\bar{M}_i$, $\bar{X}_0$ *are the attached spaces of* $\bar{X} \in \mathbf{Stra}_0$.

Proof. Let $\bar{X}$, $\bar{X}'$ be two stratified spaces in $\mathbf{Stra}_0$ and $X' \to \bar{X}'$ an object of $\mathbf{V}$-$\mathbf{Stra}(\bar{X}')$. By theorem (2.3), given a stratum-preserving map $\bar{f}\colon \bar{X} \to \bar{X}'$ in $\mathbf{Stra}_0$, the pull-back $X = \bar{f}^*X'$ is a well-defined object of $\mathbf{V}$-$\mathbf{Stra}(\bar{X})$. Moreover, a morphism α in $\mathbf{V}$-$\mathbf{Stra}(\bar{X}')$ (that is, a self-map $\alpha\colon X' \to X'$ over the identity of $\bar{X}'$) induces a morphism $\alpha^*\colon X \to X$ over the identity of $\bar{X}$. Hence $\bar{f}^*\colon \mathbf{V}$-$\mathbf{Stra}(\bar{X}') \to \mathbf{V}$-$\mathbf{Stra}(\bar{X})$ is a functor. To show that this construction yields a homomorphism $\mathcal{K}_0^{\mathbf{V}}(\bar{f})\colon \mathcal{K}_0^{\mathbf{V}}(\bar{X}') \to \mathcal{K}_0^{\mathbf{V}}(\bar{X})$ is sufficient to show that $\bar{f}^*$, as a functor, preserves the sum $\oplus$, i.e. that Y and Y' are in $\mathbf{V}$-$\mathbf{Stra}(\bar{X}')$ then

$$\bar{f}^*(Y \oplus Y') = \bar{f}^*Y \oplus \bar{f}^*Y'. \tag{7.3}$$

Let $Y_0 \subset Y_1 \subset \dots$ and $Y_0' \subset Y_1' \subset \dots$ be the filtrations of Y and Y'. We prove by induction that equation (7.3) holds for every stratified space $\bar{X}'$ with at most n strata. If $n = 1$ then it is true, since in this case the $\mathbf{V}$-stratified bundles are $\mathbf{V}$-vector bundles. So assume that (7.3) is true for spaces with at most $n-1$ strata. Let $(\bar{M}_n', \bar{A}_n')$ be the pair attached via the map $\bar{h}'$ to $\bar{X}_{n-1}'$ to obtain $\bar{X}'$, where $\bar{X}'$ has n exactly strata; in the same way, let $(\bar{M}_n, \bar{A}_n)$ the corresponding CW-pair for $\bar{X}$ and $\bar{h}\colon \bar{A}_n \to \bar{X}_{n-1}$ the attaching map. Let (M_n', A_n'), (M_n'', A_n'') be the attached bundles to Y_{n-1} and Y_{n-1}'. Since $\bar{f}$ is a map in $\mathbf{Stra}_0$, there is a map $\bar{g}\colon \bar{M} \to \bar{M}'$ such that $\bar{f} = \bar{f}_{n-1} \cup \bar{g}$. By proposition (7.4) of [3] we have the equality

$$\bar{f}^*(Y \oplus Y') = \bar{f}^*(Y_{n-1} \oplus Y_{n-1}) \cup_h \bar{g}^*(M_n \oplus M_n'), \tag{7.4}$$

where $h\colon \bar g^*(A_n \oplus A'_n) \to \bar f^*(Y_{n-1} \oplus Y'_{n-1})$ is induced by the pull-back. Now, by induction

$$\bar f^*(Y_{n-1} \oplus Y_{n-1}) = \bar f^* Y_{n-1} \oplus \bar f^* Y_{n-1}$$

$$\bar g^*(M_n \oplus M'_n) = \bar g^* M_n \oplus \bar g^* M'_n,$$

$$\bar g^*(A_n \oplus A'_n) = \bar g^* A_n \oplus \bar g^* A'_n.$$

But by definition the sum of the pull-backs $\bar f^* Y \oplus \bar f^* Y'$ is equal to the push-out

$$\begin{aligned} \bar f^* Y \oplus \bar f^* Y' &= \bar f^* Y_{n-1} \oplus \bar f^* Y_{n-1} \cup_h \bar g^* M_n \oplus \bar g^* M'_n \\ &= \bar f^*(Y_{n-1} \oplus Y_{n-1}) \cup_h \bar g^*(M_n \oplus M'_n). \end{aligned} \tag{7.5}$$

Comparing equations (7.4) and (7.5) yields the desired result. *q.e.d.*

(7.6) Theorem. *Consider a* $\mathbf{V}$*-stratified vector bundle* X'*, a compactly stratified space* $\bar X$ *and a homotopy of (stratum-preserving) maps* $\bar f_t\colon \bar X \to \bar X'$*,* $t \in I$*. Then*

$$\bar f_0^* X' \cong \bar f_1^* X'.$$

Proof. We prove it by induction on the number of strata of $\bar X'$. If $n = 1$ then X' is a $\mathbf{V}$-vector bundle and hence the result is classical. Assume that it is true for $\bar X'$ with at most $n-1$ strata. Then if $\bar X$ has less than n strata the claim is true. So let $(\bar M_n, \bar A_n)$ be the CW-pair attached to $\bar X_{n-1}$, (M'_n, A'_n) the CW-pair attached to $\bar X'_{n-1}$ and $\bar g_t\colon (\bar M_n, \bar A_n) \to (\bar M'_n, \bar A'_n)$ the homotopy such that $\bar f_t = \bar f_{n-1,t} \cup \bar g_t$, with $t \in I$. Since $\bar X_0$ and $\bar M_i$ are compact for every $i \geq 1$, we know that $\bar X_i$ is compact for every i. This implies that, by induction, $\bar f^*_{0,n-1} X'_{n-1} \cong f^*_{1,n-1} X'_{n-1}$, $\bar g_0^* A'_n \cong g_1^* A'_n$ and $\bar g_0^* M'_n \cong g_1^* M'_n$. But by proposition (7.4) of [3]

$$\bar f_0^* X' = \bar f^*_{0,n-1} X'_{n-1} \cup \bar g_0^* M'_n,$$

$$\bar f_1^* X' = \bar f^*_{1,n-1} X'_{n-1} \cup \bar g_1^* M'_n,$$

and as a consequence $f_0^* X' \cong f_1^* X'$. *q.e.d.*

(7.7) Corollary. *If* $\mathbf{Stra}_c^{\mathrm{op}}$ *denotes the full subcategory of* $\mathbf{Stra}^{\mathrm{op}}$ *consisting of compactly stratified spaces and stratum-preserving maps, the functor*

$$\mathcal{K}_0^{\mathbf{V}} : \mathbf{Stra}_c^{\mathrm{op}} \to \mathbf{Ab}$$

induces a functor on the homotopy category

$$\mathcal{K}_0^{\mathbf{V}} : \mathbf{Stra}_c^{\mathrm{op}}/_\sim \to \mathbf{Ab}.$$

Hence if $\bar X$ *is homotopy equivalent (in* $\mathbf{Stra}_c$*) to* $\bar Y$*, then*

$$\mathcal{K}_0^{\mathbf{V}}(\bar X) = \mathcal{K}_0^{\mathbf{V}}(\bar Y).$$

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