Max-Planck-Institut für Mathematik
Bonn

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Max-Planck-Institut für Mathematik
Preprint Series 2015 (11)

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MPIM 15-11

ANALYSIS ON SINGULAR SPACES: LIE MANIFOLDS AND OPERATOR ALGEBRAS

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Abstract. We discuss and develop some connections between analysis on singular spaces and operator algebras, as presented in my sequence of four lectures at the conference *Noncommutative geometry and applications*, Frascati, Italy, June 16-21, 2014. Thus this is mostly a review paper, but the presentation is new, and we have included some new results as well. In particular, we provide a complete short introduction to analysis on noncompact manifolds (especially Lie manifolds). The link between the analysis on singular spaces and operator algebras is provided by Lie manifolds. The groupoids integrating Lie manifolds play an important background role in establishing this link because they provide operator algebras whose structure is well understood. The initial motivation for the work reviewed here was to develop the index theory on stratified singular spaces, but several other applications have emerged as well, including applications to Partial Differential Equations and Numerical Methods. These will be mentioned only briefly, however, due to the lack of space. Instead, we shall concentrate on the connections with the Index Theory.

Contents

Date: February 19, 2015.
The author was partially supported by ANR-14-CE25-0012-01.
Manuscripts available from **http://www.math.univ-metz.fr/˜nistor/**
AMS Subject classification (2010): 58J40 (primary) 58H05, 46L80, 46L87, 47L80.

Introduction

We review some connections between analysis on singular spaces and operator algebras. The paper follows rather closely my sequence of four lectures at the conference *Noncommutative geometry and applications*, Frescoes, Italy, June 16-21, 2014. From a technical point of view, the paper mostly sets up the analysis tools needed for developing a certain approach to the index theory of singular and non-compact spaces. There are also several new results tying the different concepts together. Also, the third and fourth sections are to a large extend self-contained. They include most of the proofs, and thus can be regarded as a very short introduction to analysis on non-compact manifolds, especially on Lie manifolds.

The main story told by this paper is, briefly, as follows. Some of the classical analysis and index theory results deal with the index of Fredholm operators. This is rather well understood in the case of smooth, compact manifolds and in the case of smooth, bounded domains. The non-smooth and non-compact cases, however, are much less understood. Moreover, it has become clear that the index theorems in these frameworks require non-local invariants and (hence) cyclic homology. The full implementation of this program, however, requires further algebraic and analytic developments. More specifically, one important auxiliary question that needs to be answered is which operators on non-smooth or non-compact spaces are Fredholm. A convenient answer to this question involves Lie manifolds and the Lie groupoids that integrate them. The techniques that were developed for this purpose have then proved to be useful also in other mathematical areas, such as spectral theory and the Finite Element Method.

Fredholm operators play a central role in this paper for the following reasons. First of all, the (Fredholm) index is defined only for Fredholm operators, thus, in order to state an index theorem, one needs to have examples of Fredholm operators. In fact, the data that is needed to decide that a given operator is Fredholm (principal symbol, boundary–or indicial–symbols) are also the data that is used for actually computing the index of these operators. Why are we interested in Fredholm operators and their index? First of all, many interesting quantities (such as the signature of a compact manifold) identify with the index of certain operators. Second, Fredholm operators have been widely used in partial differential equations (PDEs). For instance, non-linear maps whose linearization is Fredholm play a central role in the study of non-linear PDEs. Also, Fredholm operators are useful in determining the essential spectra of Hamiltonians. Finally, the Fredholm index is the first obstruction for an operator to be invertible. This is exploited in our approach to the Neumann problem on polygonal domains (Theorem 5.14), whose proof is to compute the index of an auxiliary operator, then to show that this operator is injective, and then to augment its domain so that it becomes an isomorphism [74].

A certain point on the analysis on singular and non-compact spaces is worth insisting upon. A typical approach to analysis on singular spaces, used also in this paper, is that the analysis on a singular space happens on the *smooth* part of the space, with the singularities playing the important role of providing the behavior "at infinity." Thus, from this point of view, the analysis on non-compact spaces is more general than the analysis on singular spaces. However, for the simplicity of the presentation, we shall typically discuss only singular spaces, with the understanding that the results also extend to non-compact manifolds.

Here are the contents of the paper. The first section is devoted to describing the motivation for the results presented in this paper coming from index theory. The approach to index theory used in this paper is based on exact sequences of operator algebras. Thus, in the first section, we discuss the exact sequences appearing in the Atiyah-Singer index theorem, in Connes' index theorem for foliations, and in the Atiyah-Patodi-Singer index theorem. The second section is devoted to explaining the motivation for the results presented in this paper coming from degenerate (or singular) elliptic partial differential equations. In that section, we just present some typical examples of degenerate elliptic operators that suggest how ubiquitous they are and point out some common structures that lead us to Lie manifolds, which are discussed in the third section. In this third section we include the definition of Lie manifolds, a discussion of manifolds with cylindrical ends (the simplest non-trivial example that leads to the APS framework), a discussion of Lie algebroids and of their relation to Lie manifolds, and a discussion of the natural metric and connection on a Lie manifold. The fourth section is a basic introduction to analysis on Lie manifolds. It begins with discussions of the needed functions spaces, of the comparison algebras, and of Fredholm conditions. The last section is devoted to applications, including the formulation of and index problem for Lie manifolds in periodic cyclic cohomology, an application to essential spectra, an index theorem for Callias-type operators, and the Hadamard well posedness for the Poisson problem with Dirichlet boundary conditions on polyhedral domains.

The four lectures of my presentation at the above mentioned conference were devoted to the following subjects: *Index theory*, *Lie manifolds*, *Pseudodifferential operators on groupoids,* and *Applications* and are based mostly on my joint works with Bernd Ammann (Regensburg), Catarina Carvalho (Lisbon), Alexandru Ionescu (Princeton), and Robert Lauter (Mainz), Anna Mazzucato (Penn State) and Bertrand Monthubert (Toulouse). Nevertheless, I made an effort to put the results in context by quoting and explaining other relevant results. I have also included significant background results and definitions to make the paper easier to read for non-specialists. I have also tried to summarize some of the more recent developments. Unfortunately, the size of the paper has prevented me from including more information. Also, it was unpractical to provide all the related references, and I apologize to the authors whose work was not mentioned enough.

I would like to thank the Max Planck Institute for Mathematics in Bonn, where part of this work was completed. Also, I would like to thank Bernd Ammann, Karsten Bohlen, Claire Debord, Vladimir Georgescu, Jean Renault, Elmar Schrohe, and Georges Skandalis for useful discussions.

1. Motivation I: Index Theory

This paper is devoted in large part to explaining some applications of Lie manifolds and their associated operator algebras to analysis on singular and non-compact spaces. The initial motivation of this author for studying analysis on singular and non-compact spaces (and hence also for styging Lie manifolds) comes from index theory. In this section, I will describe this initial motivation, while in the next section I will provide further motivation coming from degenerate partial differential equations. Thus, I will not attempt here to provide a comprehensive introduction to Index Theory, but rather motivate the results and constructions introduced in this paper using it. In particular, I will stress the important role that Fredholm conditions play for index theorems. In fact, in our approach, both the index theorem studied and the associated Fredholm conditions rely on the *same* exact sequence discussed in general in the next section.

1.1. **An abstract index theorem.** An approach to index theory is based on *exact sequences* of algebras of operators. We shall thus consider an abstract exact sequence

$$0 \to I \to A \to \mathrm{Symb} \to 0\,, \qquad (1)$$

in which the algebras involved will be specified in each particular application. The same exact sequence will be used to establish the corresponding Fredholm conditions. Typically, A will be a suitable *algebra of operators* that describes the analysis on a given (class of) singular space(s). In our presentation, the algebra A will be constructed using Lie algebroids and Lie groupoids. The choice of the *ideal* I also depends on the particular application at hand and is not necessarily determined by A. In fact, the analysis on singular spaces distinguishes itself from the analysis on compact, smooth manifolds in that there will be several reasonable choices for the ideal I.

Often, in problems related to classical analysis (such as the ones that involve the Fredholm index of operators), the ideal I will be contained in the ideal of compact operators $\mathcal{K}$ (on some separable Hilbert space). In fact, in most applications in this presentation, we will have $I := A \cap \mathcal{K}$. We insist, however, that this is not the only legitimate choice, even if it is the most frequently used one. An important other example is provided by taking I to be the kernel of the principal symbol map. As we will see below, in the case of singular and non-compact spaces, the kernel of the principal symbol map *does not* consist generally of compact operators. This is the case in the analysis on covering spaces and on foliations, which also lead naturally to von Neumann algebras [29, 60, 125].

If $I := A \cap \mathcal{K}$ and $P \in A$ has an invertible image in A/I (that is, it is invertible modulo I), then it is Fredholm and we can ask what is its Fredholm index (whose definition we recall below). In any case, we see that in order to formulate an index problem, we need criteria for the relevant operators to be Fredholm. This is also related to the structure of the exact sequence (1).

When the algebra A is defined using groupoids (as is the case in this presentation), then the structure of the quotient algebra $\mathrm{Symb} := A/I$ is related to the representation theory of the underlying groupoid. Unfortunately, we will not have time to treat this important subject in detail, but we will provide several references in the appropriate places.

The exact sequence (1) provides us with a boundary (or index) map

$$\partial \,:\, K_1(\mathrm{Symb}) \,\to\, K_0(I)\,, \tag{2}$$

whose calculation will be regarded as an index formula for the reasons explained in the following subsections (see, for instance, Remark 1.4). In case A and I are C^*-algebras, then we obtain also a map $\partial' : K_0(\mathrm{Symb}) \to K_1(I)$, which together with ∂ and the maps obtained from the functoriality of K-groups, give rises to a six-term exact sequence of K-groups [111, 126]. Unfortunately, often the K-groups are difficult to compute, so we need to consider subalgebras of C^*-algebras and cyclic homology (see, for instance, Subsection 1.5).

We begin with a quick introduction to differential and pseudodifferential operators needed to fix the notation and introduce some basic concepts. It is written to be accessible to graduate students. We then discuss three basic index theorems and their associated analysis (or exact sequences). These three index theorems are: the Atiyah-Singer (AS) index theorem, Connes' index theorem for foliations, and the Atiyah-Patodi-Singer (APS) index theorem. We will see that, at least from the point of view adopted in this presentation, Connes' and APS' frameworks extend the AS' framework in complementary directions.

1.2. **Differential and pseudodifferential operators.** We now fix some notation and recall a few basic concepts. On $\mathbb{R}^n$ we consider the derivations (or, which is the same thing, *vector fields*) $\partial_j = \frac{\partial}{\partial x_j}$, $j = 1, \ldots, n$ and form the differential monomials $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\ldots\partial_n^{\alpha_n}, \alpha \in \mathbb{Z}_+^n$. We denote by $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n \in \mathbb{Z}_+$. A *differential operator* P of order m on $\mathbb{R}^n$ is then an operator $P : \mathcal{C}_\mathrm{c}^\infty(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ of the form

$$Pu \,=\, \sum_{|\alpha|\le m} a_\alpha \partial^\alpha u\,, \tag{3}$$

with m minimal with this property. Typically m will denote the order of the operator being studied. The functions a_α are the *coefficients* of the operator P.

It is easy, but important, to extend the above constructions to systems of differential operators, such as: vector Laplacians, elasticity, signature, Maxwell, and many others. Then $u = (u_1, \ldots, u_k) \in \mathcal{C}_\mathrm{c}^\infty(\mathbb{R}^n)^k = \mathcal{C}_\mathrm{c}^\infty(\mathbb{R}^n; \mathbb{R}^k)$ is a smooth, compactly supported section of the trivial vector bundle $\underline{\mathbb{R}^k} = \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ on $\mathbb{R}^n$ and hence $a_\alpha \in \mathcal{C}^\infty(\mathbb{R}^n; M_k(\mathbb{R}))$ is a matrix valued function. It is *an endomorphism* of the trivial vector bundle $\underline{\mathbb{R}^k}$. Then P maps $\mathcal{C}_\mathrm{c}^\infty(\mathbb{R}^n; \mathbb{R}^k)$ to $\mathcal{C}_\mathrm{c}^\infty(\mathbb{R}^n; \mathbb{R}^k)$. Let $\Delta = -\partial_1^2 - \ldots - \partial_n^2 \ge 0$ and $s \in \mathbb{Z}_+$. We denote as usual

$$H^s(\mathbb{R}^n) \,:=\, \{u : \mathbb{R}^n \to \mathbb{C},\ \partial^\alpha u \in L^2(\mathbb{R}^n),\, |\alpha| \le s\,\} \,=\, \mathcal{D}(\Delta^{s/2})\,.$$

As we will see below, both definitions above of Sobolev spaces extend to the case of "Lie manifolds." These definitions of Sobolev spaces also extend immediately to vector valued functions and, if the coefficients a_α of P are bounded (together with enough derivatives, more precisely, if $P \in W^{s,\infty}(\mathbb{R}^n)$), then we obtain that P maps $H^s(\mathbb{R}^n)$ to $H^{s-m}(\mathbb{R}^n)$. In this presentation, we shall assume also that $a_\alpha \in \mathcal{C}^\infty(\mathbb{R}^n; M_k(\mathbb{R}))$, and hence P will map $\mathcal{C}_\mathrm{c}^\infty(\mathbb{R}^n; \mathbb{R}^k)$ to $\mathcal{C}_\mathrm{c}^\infty(\mathbb{R}^n; \mathbb{R}^k)$.

For P a differential operator of order $\leq m$ as in Equation (3) (that is, $P = \sum_{|\alpha|\leq m} a_\alpha \partial^\alpha : \mathcal{C}_c^\infty(\mathbb{R}^n)^k \to \mathcal{C}_c^\infty(\mathbb{R}^n)^k$), we introduce the function

$$\sigma_m(P)(x,\xi) \;=\; \sum_{|\alpha|=m} a_\alpha(x)(\imath\xi)^\alpha \;\in\; \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n; M_k)\,, \tag{4}$$

called the *principal symbol* of P. In particular, we have $\sigma_{m+1}(P) = 0$. Here $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$ is the *dual* variable.

The fact that ξ is a dual variable to $x \in \mathbb{R}^n$ is seen when performing transformations of coordinates. The principal symbol is thus seen to be a function on $T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$. It turns out that the principal symbol $\sigma_m(P)$ of P has a much simpler transformation formula than the *(full) symbol* of P defined by

$$\overline{\sigma}(P)(x,\xi) \;=\; \sum_{|\alpha|\leq m} a_\alpha(x)(\imath\xi)^\alpha \;\in\; \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n; M_k)\,. \tag{5}$$

The full symbol $p(x,\xi) := \overline{\sigma}(P)(x,\xi)$ of P because, if we define

$$p(x,D)u(x) \;:=\; (2\pi)^{-n}\int_{\mathbb{R}^n} e^{\imath x\cdot\xi} p(x,\xi)\hat{u}(\xi)d\xi\,, \tag{6}$$

then the operator P of Equation (3) becomes $P = p(x,D)$.

There exist more general classes of functions (or symbols) p for which $P := p(x,D)$ can still be defined by the above formula (6). The resulting operator will be a *pseudodifferential operator* with symbol p. Let us recall the definition of the two most basic classes of symbols for which the formula (6) defining $p(x,D)$ still makes sense. For simplicity, we shall consider in the beginning only scalar symbols, although matrix valued symbols can be handled in a completely similar way. For instance, we define $S^m_{1,0}(\mathbb{R}^k \times \mathbb{R}^N)$, $m \in \mathbb{R}$, to be the space of functions $a : \mathbb{R}^{k+N} \to \mathbb{C}$ that satisfy, for any $i, j \in \mathbb{Z}_+$, the estimate

$$|\partial_x^i \partial_\xi^j a(x,\xi)| \;\leq\; C_{i,j}(1+|\xi|)^{m-j}\,,$$

for a constant $C_{i,j} > 0$ independent of x and ξ.

We now introduce *classical symbols.* A function $a : \mathbb{R}^k \times \mathbb{R}^N \to \mathbb{C}$ is called *eventually homogeneous* of order s if there exists $M > 0$ such that $a(x,t\xi) = t^s a(x,\xi)$ for $|\xi| \geq M$ and $t \geq 1$. A very useful subclass of symbols is $S^m_{\text{cl}}(\mathbb{R}^{2n})$, defined as the space of symbols $a \in S^m_{1,0}(\mathbb{R}^{2n})$ that can be written as asymptotic series $a \sim \sum_{j=0}^\infty a_{m-j}$, meaning

$$a - \sum_{j=0}^{N} a_{m-j} \;\in\; S^{m-N-1}_{1,0}(\mathbb{R}^{2n})\,,$$

with $a_k \in S^k_{1,0}(\mathbb{R}^{2n})$ *eventually homogeneous* of order k. If $S^m_{\text{cl}}(\mathbb{R}^{2n})$, the pseudodifferential operator $a(x,D)$ is called a *classical pseudodifferential operator* and its principal symbol is given by

$$\sigma_m(a(x,D)) \;:=\; a_m\,, \tag{7}$$

and is regarded as a smooth, order m homogeneous function on

$$T^*\mathbb{R}^n \smallsetminus \text{“zero section”} = \mathbb{R}^{2n} \smallsetminus (\mathbb{R}^n \times \{0\})\,.$$

For index theory, it is generally enough to consider classical pseudodifferential operators. In particular, we see that if $p(x,\xi) := \sum_{|\alpha|\leq m} p_\alpha(\imath\xi)^\alpha$ and $P = \sum_{|\alpha|\leq m} p_\alpha\partial^\alpha$, then $P = p(x,D)$ is a *classical pseudodifferential operator* of order m.

The definition of a (pseudo)differential operator P (of order $\leq m$) and of its principal symbol $\sigma_m(P)$, then extend to manifolds and vector bundles by using local coordinate charts. To fix notation, if $E \to M$ is a smooth vector bundle over a manifold M, we shall denote by $\Gamma(M; E)$ the space of its smooth sections $\{s : M \to E, s(x) \in E_x\}$ and by $\Gamma_c(M; E) \subset \Gamma(M; E)$ the subspace of compactly supported smooth sections. Sometimes, when no confusion can arise, we denote $\Gamma(E) = \Gamma(M; E)$ and, similarly, $\Gamma_c(E) = \Gamma_c(M; E)$. Getting back to our extension of pseudodifferential operators to manifolds, we thus replace as follows

$$\begin{aligned} \mathbb{R}^n \quad &\leftrightarrow \quad M \;=\; \text{a smooth manifold} \\ \mathcal{C}_c^\infty(\mathbb{R}^n)^k \quad &\leftrightarrow \quad \text{sections of a vector bundle,} \end{aligned}$$

which gives for an operator P acting between (usually smooth, compactly supported) sections of E and F:

$$\begin{aligned} P : \Gamma_c(M; E) \;&\to\; \Gamma_c(M; F) \\ \sigma_m(P) \;&\in\; \Gamma(T^*M \smallsetminus \{0\}; \mathrm{Hom}(E, F)) \,. \end{aligned}$$

Of course, $\sigma_m(P)$ is homogeneous of order m. Thus, if $m = 0$ and if we denote $S^*M := (T^*M \smallsetminus \{0\})/\mathbb{R}_+^*$ the *(unit) cosphere bundle*, then $\sigma_0(P)$ identifies with a smooth function on S^*M. (The name "cosphere bundle" is due to the fact that, if we choose a metric on M, then the cosphere bundle identifies with the set of vectors of length one in T^*M.)

In order to extend the definition of Sobolev spaces to manifolds, we need also a *metric* g on our manifold M (or a Lipschitz equivalence class of such metrics) [10, 54]. Then for a complete manifold M, the Sobolev space is given by the domains of the powers of the (positive) Laplacian. In general, this will depend on the choice of the metric g.

The main property of the principal symbol is the *multiplicative property*

$$\sigma_{m+m'}(PP') = \sigma_m(P)\sigma_{m'}(P') \,, \tag{8}$$

a property that is enjoyed by its extension to *pseudodifferential operators* (which are allowed to have negative and non-integer orders as well).

Definition 1.1. A (classical, pseudo)differential operator P is called *elliptic* if its principal symbol is invertible away from the zero section of T^*M.

See [57, 118] for a more complete discussion of various classes of symbols and of pseudodifferential operators. See also [6, 14, 51, 70, 85, 100].

1.3. **The Fredholm index.** Let now M be a *compact*, smooth manifold, so the Sobolev spaces $H^s(M)$ are uniquely defined. Let also P be a (classical, pseudo) differential operator of order $\leq m$ acting between smooth sections of the hermitian vector bundles E and F. We denote by $H^s(M; E)$ and $H^s(M; F)$ the corresponding Sobolev spaces of sections of these bundles.

Recall that a continuous, linear operator $T : X \to Y$ acting between topological vector spaces is Fredholm, if and only if, the vector spaces $\ker(P) := \{u \in X, Tu = 0\}$ and $\mathrm{coker}(P) := Y/TX$ are finite dimensional. One of our *model results* is then the following classical theorem:

Theorem 1.2. *Let P be an order m pseudodifferential operator acting between sections of the bundles E and F on the smooth, compact manifold M and $s \in \mathbb{R}$.*

Then

$$P : H^s(M; E) \to H^{s-m}(M; F) \text{ is Fredholm } \Leftrightarrow P \text{ is elliptic.}$$

Fredholm operators appear all the time in applications (because elliptic operators are so fundamental). For instance, the theorem mentioned above is one of the crucial ingredients in the "Hodge theory" for smooth compact manifolds, which is quite useful in Gauge theory.

By the Open Mapping theorem, the invertibility of a continuous, linear operator $P : X_1 \to X_2$ acting between two Banach spaces is equivalent to the condition $\dim \ker(P) = \dim \operatorname{coker}(P) = 0$. It is important then to calculate the *Fredholm index* $\operatorname{ind}(P)$ of P, defined by

$$\operatorname{ind}(P) := \dim \ker(P) - \dim \operatorname{coker}(P) . \tag{9}$$

The reason for looking at the Fredholm index rather than simply at the numbers $\dim \ker(P)$ and $\dim \operatorname{coker}(P)$ is that $\operatorname{ind}(P)$ has better stability properties than these numbers. For instance, the Fredholm index is homotopy invariant and depends only on the principal symbol of P.

1.4. **The Atiyah-Singer index formula.** The index of elliptic operators on smooth, compact manifolds is computed by the Atiyah-Singer index formula [9]:

Theorem 1.3 (Atiyah-Singer)**.** *Let M be a compact, smooth manifold and let P be elliptic, classical (pseudo)differential operator acting on sections of smooth vector bundles on M. Then*

$$\operatorname{ind}(P) = \langle\, ch[\sigma_m(P)]\mathcal{T}(M) \,,\, [T^*M] \,\rangle \,.$$

There are many accounts of this theorem, and we refer the reader to some of them [51, 99, 121] for more details. See [26] for an approach using non-commutative geometry. Let us nevertheless mention some of the main ingredients appearing in the statement of this theorem, because they are being generalized (or need to be generalized) to the non-smooth case. This generalization is in part achieved by non-commutative geometry and by analysis on singular spaces. Returning to Theorem 1.3, the meaning of the undefined terms in Theorem 1.3 is as follows:

(i) The principal symbol $\sigma_m(P)$ of P defines a K-theory class in $K^0(T^*M)$ (with compact supports) by the ellipticity of P [9] and $ch[\sigma_m(P)] \in H_c^{even}(T^*M)$ is the Chern character of this class.
(ii) $\mathcal{T}(M) \in H^{even}(M) \simeq H^{even}(T^*M)$ is the Todd class of M, so the product $ch[\sigma_m(P)]\mathcal{T}(M)$ is in $H_c^{even}(T^*M)$.
(iii) $[T^*M] \in H_c^{even}(T^*M)'$ is the fundamental class of T^*M and is chosen such that no sign appears in the index formula.

The AS index formula was much studied and has found a number of applications. It is based on earlier work of Grothendieck and Hierzebruch and answers to a question of Gelfand. One of the main motivations for the work presented here is the desire to extend the index formula for compact manifolds (the AS index formula) to the noncompact and singular cases. To this end, it will be convenient to use the exact sequence formalism described in Subsection 1.1. Namely, the exact sequence (1) corresponding to the AS index formula is

$$0 \to \Psi^{-1}(M) \to \Psi^0(M) \to \mathcal{C}^\infty(S^*M) \to 0 \,, \tag{10}$$

where S^*M is the cosphere bundle of M, as before, (that is, the set of vectors of length one of the cotangent space T^*M of M). (So $I = \Psi^{-1}(M)$, $A = \Psi^0(M)$, and $\mathrm{Symb} := A/I \simeq \mathcal{C}^\infty(S^*M)$.)

It is interesting to point out that both the AS index formula and the Fredholm condition of Theorem 1.2 are based on the exact sequence (10). Of course, to actually determine the index, one has to do additional work, but the information needed is contained in the exact sequence. This remains true for most of the other index theorems.

Remark 1.4. Let us see now how the exact sequence (10) and Theorem 1.3 are related. Recall the boundary map $\partial : K_1(\mathrm{Symb}) \to K_0(I)$ in K-theory associated to the exact sequence (1), see Equation (2), and let us assume that the ideal I of that exact sequence consists of compact operators (i.e. $I \subset \mathcal{K}$). We first consider the natural map

$$Tr_* \, : \, K_0(I) \, \to \, \mathbb{Z} \, , \tag{11}$$

where the trace refers to the trace (or dimension) of a projection. Of course, $Tr_* : K_0(\mathcal{K}) \to \mathbb{Z}$ is the usual isomorphism. Then $Tr_* \circ \partial$ *computes the usual (Fredholm) index*, that is, we have the equality of the morphisms

$$\mathrm{ind} \, = \, Tr_* \circ \partial \, : \; K_1(\mathrm{Symb}) \xrightarrow{\partial} K_0(I) \xrightarrow{Tr_*} \mathbb{C} \, . \tag{12}$$

Indeed, if $P \in A$ is invertible in Symb, then, on the one hand, P defines a class $[P] \in K_1(\mathrm{Symb})$, and, on the other hand, P is *Fredholm* and its Fredholm index is given by $\mathrm{ind}(P) = Tr_* \circ \partial[P]$.

We thus see that computing the index of a Fredholm (pseudo)differential operator on M is equivalent to computing the composite map $Tr_* \circ \partial \, : \, K_1(\mathrm{Symb}) \to \mathbb{C}$. This observation is the starting point of the approach to index theorems described in this paper.

Remark 1.5. Let us discuss now shortly the role of the Chern character in the AS index formula. First, let us recall that the Chern character establishes an isomorphism $ch : K^*(M_1) \otimes \mathbb{C} \to H^*(M_1) \otimes \mathbb{C}$ for any compact, smooth manifold M_1. Moreover, in the case of the commutative algebra $\mathcal{C}^\infty(M_1)$, we have that $K_*(\mathcal{C}^\infty(M_1)) \simeq K^*(M_1)$ and hence any group morphism $K_*(\mathcal{C}^\infty(M_1)) \to \mathbb{C}$ factors through the Chern character $ch : K^*(M_1) \to H^*(M_1)$. Returning to the AS index formula, we have that $\mathrm{Symb} = \mathcal{C}^\infty(S^*M)$ and hence the index map $\mathrm{ind} = Tr_* \circ \partial : K_1(\mathrm{Symb}) \simeq K^1(S^*M) \to \mathbb{C}$ can be expressed solely in terms of the Chern character. It is therefore possible to express the AS Index Formula purely *in classical terms* (vector bundles and cohomology) because the quotient $A/I := \mathrm{Symb} \simeq \mathcal{C}^\infty(S^*M)$ *is commutative.*

Remark 1.6. Technically, one may have to replace the algebra A with $M_n(A)$ and take $P \in M_n(A)$, but this is not an issue since the K-groups are invariant for the replacement by A with its matrix algebras. However, the approach to the index of elliptic (pseudo)differential operators using exact sequences can be used to deal with operators P acting between sections of *isomorphic* bundles. For non-compact manifolds (and hence also for singular spaces), this is enough. For the AS index formula, however, one may have to replace first M with $M \times S^1$.

1.5. **Cyclic homology and Connes' index formula for foliations.** The map Tr_* of the basic equation $\text{ind}(P) = Tr_* \circ \partial[P]$, Equation (12), valid when $I \subset \mathcal{K}$, is a particular instance of the pairing between *cyclic cohomology* and K-theory [26]. See also [25, 61, 77, 78, 122]. This pairing is even more important when $I \not\subset \mathcal{K}$. Let then denote by $\mathrm{HP}^*(B)$ the *periodic cyclic cohomology* groups of an algebra B (for topological algebras, suitable topological versions of these groups have to be considered).

Let us look again at the general exact sequence of Equation (1) and let ϕ be a cyclic cocycle on I, that is, $\phi \in \mathrm{HP}^0(I)$. A more general (higher) index theorem is then to compute

$$\phi_* \circ \partial \,:\, K_1(\mathrm{Symb}) \to \mathbb{C}\,.$$

It is known that $\phi_* \circ \partial = (\partial\phi)_*$, and hence the map $\phi_* \circ \partial$ is also given by a cyclic cocycle [94].

The map ϕ_* and, in general, the approach to index theory using cyclic homology is especially useful for *foliations* for the reasons that we are explaining now. We regard a foliation $(M, \mathcal{F})$ of a smooth, compact manifold M as a sub-bundle $\mathcal{F} \subset TM$ that is *integrable* (that is, its space of smooth sections, denoted $\Gamma(\mathcal{F})$, is closed under the Lie bracket). Connes' construction of pseudodifferential operators along the leaves of a foliation [29] then yields the exact sequence of algebras

$$0 \to \Psi_{\mathcal{F}}^{-1}(M) \to \Psi_{\mathcal{F}}^{0}(M) \xrightarrow{\sigma_0} \mathcal{C}^\infty(S^*\mathcal{F}) \to 0\,, \tag{13}$$

where σ_0 is again the principal symbol, defined essentially in the same manner as for the case of smooth manifolds. In fact, for $\mathcal{F} = TM$, with M a smooth, compact manifold, this exact sequence reduces to the earlier exact sequence (10). It also yields a boundary (or index) map

$$\partial \,:\, K_1(\mathcal{C}^\infty(S^*\mathcal{F})) = K^1(S^*\mathcal{F}) \,\to\, K_0(\Psi_{\mathcal{F}}^{-1}(M)) \simeq K_0(\mathcal{C}_{\mathrm{c}}^\infty(\mathcal{F}))\,,$$

where $\mathcal{C}_{\mathrm{c}}^\infty(\mathcal{F})$ is the convolution algebra of the groupoid associated to $\mathcal{F}$ and where topological K-groups were used. A main difficulty here is that there are few calculations of $K_0(\Psi_{\mathcal{F}}^{-1}(M))$. These calculations are related to the Baum-Connes conjecture, which is however known not to be true for general foliations, see [56] and the references therein. See also [106, 107]. Another feature of the foliation case is that, unlike our other examples to follow, $\Psi_{\mathcal{F}}^{-1}(M))$ has no canonical proper ideals, so there are no other index maps.

Unlike its K-theory, the cyclic homology of $\Psi_{\mathcal{F}}^{-1}(M))$ is much better understood, in particular, it contains as a direct summand the twisted cohomology of the classifying space of the groupoid (graph) of the foliaton [19]. We thus have a large set of linearly independent cyclic cocycles and hence many linear maps $\phi_* : K_0(\mathcal{C}_{\mathrm{c}}^\infty(\mathcal{F})) \to \mathbb{C}$, each of which defines an index map

$$\phi_* \circ \partial : K_0(\mathcal{C}^\infty(S^*\mathcal{F})) \to \mathbb{C}\,.$$

We will not pursue further the determination of $\phi_* \circ \partial$, but we note Connes' results in [29, 26], the results of Benameur-Heitsch for Haefliger homology [16], and of myself for foliated bundles [93].

We stress that in the case of foliations, it is the ideal I that causes difficulties, whereas the quotient $\mathrm{Symb} := A/I$ is commutative and, hence, relatively easy to deal with. The opposite will be true in the following example. See also [28].

1.6. **The Atiyah-Patodi-Singer index formula.** A related but different type of example is provided by the Atiyah-Patodi-Singer (APS) index formulas [8]. Let $\overline{M}$ be a compact manifold with *smooth boundary* $\partial\overline{M}$. By definition, this means that $\overline{M}$ is locally diffeomorphic to an open subset of $[0,1) \times \mathbb{R}^{n-1}$. The transition functions for a manifold with boundary will be assumed be smooth. To $\overline{M}$ we attach the semi-infinite cylinder

$$\partial\overline{M} \times (-\infty, 0\,]\,,$$

yielding a *manifold with cylindrical ends.* The metric is taken to be a product metric $g = g_{\partial\overline{M}} + dt^2$ far on the end. Kondratiev's transform $r = e^t$ then maps the cylindrical end to a tubular neighborhood of the boundary, such that the cylindrical end metric becomes $g = g_{\partial\overline{M}} + (r^{-1}dr)^2$ near the boundary ($r^{-1}dr = e^{-t}d(e^t) = dt$). Thus on $\overline{M}$ we consider two metrics: first, the initial smooth everywhere (including up to the boundary) metric and, second, the singular metric g that corresponds to the compactification of the cylindrical end manifold.

We have thus obtained one of the simplest examples of a non-compact manifold, that of a manifold with cylindrical ends. We will consider on this non-compact manifold only differential operators with coefficients that extend to smooth functions even at infinity. Because of this, it will be more convenient to work on $\overline{M}$ than on $\overline{M} \cup \partial\overline{M} \times (-\infty, 0]$. This is achieved by the Kondratiev transform $r = e^t$. The Kondratiev transform is such that ∂_t becomes $r\partial_r$. On $\overline{M}$, we then take the coefficients to be smooth functions up to the boundary. Therefore, in local coordinates $(r, x') \in [0, \epsilon) \times \partial\overline{M}$ on the distinguished tubular neighborhood of $\partial\overline{M}$, we obtain the following form for our differential operators (here $n = \dim(\overline{M})$):

$$P = \sum_{|\alpha| \le m} a_\alpha(r, x')(r\partial_r)^{\alpha_1}\partial_{x_2'}^{\alpha_2} \ldots \partial_{x_n'}^{\alpha_n} = \sum_{|\alpha| \le m} a_\alpha (r\partial_r)^{\alpha_1}\partial^{\alpha'}\,. \tag{14}$$

Operators of this form are called *totally characteristic differential operators.*

Away from the boundary, the definition of the principal symbol for a totally characteristic differential operator is unchanged. However, in the same local coordinates near the boundary as in Equation (14), the *principal symbol* for the totally characteristic differential operator of Equation (14) is

$$\sigma_m(P) := \sum_{|\alpha| = m} a_\alpha \xi^\alpha\,. \tag{15}$$

Thus the principal symbols is *not* $\sum_{|\alpha|=m} a_\alpha r^{\alpha_1}\xi^\alpha$ as one might first think! Other than the fact that this definition of the principal symbol gives the "right" results, it can be motivated by considering the original coordinates $(t, x') \in (-\infty, 0] \times \partial\overline{M}$ on the cylindrical end.

The principal symbol is something that was encountered in the classical case of the AS-index formula as well as in the case of foliations, so it is not something significantly new in the case of manifolds with cylindrical ends–even if in that case the definition is slightly different. However, in the case of cylindrical ends, there is another significant new ingredient, which will turn out to be both crucial and typical in the analysis on singular spaces. This significant new ingredient is the *indicial family* of a totally characteristic differential operator. Let then P be as in Equation (14) and consider the same local coordinates near the boundary as in that equation. The definition of the indicial family $\widehat{P}$ of P is then as follows (we underline the most significant new ingredients of the definition):

$$\widehat{P}(\tau) := \sum_{|\alpha|\le m} a_\alpha(\underline{0}, x')\underline{(\imath\tau)}^{\alpha_1}\,\partial^{\alpha'} . \tag{16}$$

Note that $\widehat{P}(\tau)$ is a family of differential operators on $\partial\overline{M}$ that depends on the coefficients of P only through their restrictions to the boundary. Moreover, we see that the indicial family $\widehat{P}(\tau)$ of P is the Fourier transform of the operator

$$\mathcal{I}(P) := \sum_{|\alpha|\le m} a_\alpha(\underline{0}, x')\underline{\partial_t}^{\alpha_1}\,\partial^{\alpha'} , \tag{17}$$

which is a translation invariant operator on $\partial\overline{M}\times\mathbb{R}$. The operator $\mathcal{I}(P)$ is called the *indicial operator* of P [71, 92].

We are interested in Fredholm conditions for totally characteristic differential operators, so let us introduce the last ingredient for the Fredholm conditions. Let us endow $M := \overline{M} \smallsetminus \partial\overline{M}$ with a cylindrical end metric. Since any cylindrical end metric is complete, the Laplacian Δ is self-adjoint, and hence we can define the Sobolev space $H^s(M) := \mathcal{D}(\Delta^{s/2})$, that is, the domain of $\Delta^{s/2}$, which turns out to be independent of the choice of the cylindrical end metric. We then have a characterization of Fredholm totally characteristic differential operators similar to the compact case (the differences to the compact case are underlined).

Theorem 1.7. *Assume M has cylindrical ends and P is a totally characteristic differential operator of order m acting between the sections of the bundles E and F. Then, for any fixed $s\in\mathbb{R}$, we have that*

$$P : H^s(M;E)\to H^{s-m}(M;F) \text{ is Fredholm} \quad \Leftrightarrow \quad P \text{ is elliptic } \underline{\text{and } \widehat{P}(\tau) \text{ is invertible } \forall\tau\in\mathbb{R}}\,.$$

This result has a long history and related theorems are due to many people, too many to mention them all here. Nevertheless, one has to mention the pioneering work of Lockhart-Owen on differential operators [76] and the work of Melrose-Mendoza for totally characteristic pseudodifferential operators [84]. A closely related theorem for differential operators and domains with conical points has appeared in a landmark paper by Kondratiev in 1967 [62]. Other important results in this direction were obtained by Mazya [63] and Schrohe–Schultze [113, 114]. See the books of Schulze [115] and Lesch [71] for introductions and more information on the topics and results of this subsection.

One can easily show that $\mathcal{I}(P)$ is invertible if, and only if, $\hat{P}(\tau)$ is invertible for all $\tau\in\mathbb{R}$. Thus the Fredholmness criterion of Theorem 1.7 can also be given the following formulation that is closer to our more general result of Theorem 4.14.

Theorem 1.8. *Let M and P be as in Theorem 1.7 and $s\in\mathbb{R}$. Then*

$$P : H^s(M;E)\to H^{s-m}(M;F) \text{ is Fredholm} \quad \Leftrightarrow \quad P \text{ is elliptic } \underline{\text{and } \mathcal{I}(P) \text{ is invertible.}}$$

Let us consider now a totally characteristic, twisted Dirac operator P. In case P is Fredholm, its Fredholm index is given by the Atiyah-Patodi-Singer (APS) formula [8], which expresses $\mathrm{ind}(P)$ as the sum of two terms:

(i) the integral over $\overline{M}$ of an explicit form, which is a local term that depends only on the principal symbol of the operator P, as in the case of the AS formula, and
(ii) a boundary contribution that depends only on the indicial family $\widehat{P}(\tau)$, namely the η-invariant, which is this time a *non-local* invariant. It can be expressed in terms of $\mathcal{I}(P)$.

We thus see that even to be able to formulate the APS-index formula, we need to know which totally characteristic operators will be Fredholm. Moreover, the ingredients needed to compute the index of such an operator P (its principal symbol and $\mathcal{I}(P)$) are exactly the ingredients needed to decide that the given operator P is Fredholm. See [17, 112, 66, 82] for further results.

Let r be a defining function of the boundary $\partial\overline{M}$ of $\overline{M}$, as before. The exact sequence of the APS index formula is then the following. First of all, $A := \Psi_b^0(\overline{M})$ is the algebra of *totally characteristic* pseudodifferential operators on $\overline{M}$. One of its main properties is that the differential operators in A are exactly the totally characteristic differential operators. See [71, 115] for a definition of $\Psi_b^\infty(\overline{M})$. A definition using groupoids (of a slightly different algebra) will be given in Subsection 4.5 in a more general setting. Next, the ideal is

$$I \,:=\, r\Psi_b^{-1}(\overline{M}) \,=\, \Psi_b^0(\overline{M}) \cap \mathcal{K}\,.$$

Then the symbol algebra Symb $:= A/I$ is the fibered product

$$\text{Symb} = \mathcal{C}^\infty(S^*\overline{M}) \oplus_\partial \Psi^0(\partial\overline{M} \times \mathbb{R})^{\mathbb{R}}\,, \tag{18}$$

more precisely, Symb consists of pairs (f, Q) such that the principal symbol of the $\mathbb{R}$ invariant pseudodifferential operator Q matches the restriction of $f \in \mathcal{C}^\infty(S^*\overline{M})$ to the boundary. Recalling the definition of $\mathcal{I}$ in Equation (17) (and extending it to totally characteristic pseudodifferential operators), we obtain the exact sequence

$$0 \to r\Psi_b^{-1}(\overline{M}) \to \Psi_b^0(\overline{M}) \xrightarrow{\sigma_0 \oplus \mathcal{I}} \mathcal{C}^\infty(S^*\overline{M}) \oplus_\partial \Psi^0(\partial\overline{M} \times \mathbb{R})^{\mathbb{R}} \to 0\,, \tag{19}$$

(The exact sequence $0 \to \Psi_b^{-1}(\overline{M}) \to \Psi_b^0(\overline{M}) \xrightarrow{\sigma_0} \mathcal{C}^\infty(S^*\overline{M}) \to 0$ is, by contrast, less interesting.)

The exact sequence (19) in particular gives that P is Fredholm if, and only if, the pair $(\sigma_0(P), \mathcal{I}(P)) \in \text{Symb} := \mathcal{C}^\infty(S^*M) \oplus_\partial \Psi^0(\partial\overline{M} \times \mathbb{R})^{\mathbb{R}}$ is invertible if, and only if, P is elliptic and $\mathcal{I}(P)$ is invertible. Thus the exact sequence (19) implies Theorem 1.8.

As before, composing $\partial : K_1(\text{Symb}) \to K_0(I)$, $I = r\Psi^{-1}(\overline{M}))$, with the boundary map gives us the Fredholm index

$$\text{ind} \,=\, Tr_* \circ \partial : K_1(\text{Symb}) \,\to\, \mathbb{C}\,.$$

Since $Tr_* \circ \partial = (\partial Tr)_*$ [25] (in general by [94]), we see that the APS index formula is also equivalent to the calculation of the class of the cyclic cocycle $\partial Tr \in \text{HP}^1(\text{Symb})$. This was the approach undertaken in [82, 90].

Remark 1.9. It is important to stress here first the role of cyclic homology, which is to define natural morphisms $K_1(\text{Symb}) \to \mathbb{C}$, morphisms that are otherwise difficult to come by. Also, it is important to stress that it is the noncommutativity of the algebra of symbols Symb that explains the fact that the APS index formula is non-local.

We stress that in the case of the APS framework, it is the symbol algebra Symb $:= A/I$ that causes difficulties, in large part because it is non-commutative (so the classical Chern character is not defined), whereas the ideal $I \subset \mathcal{K}$ is easy to deal with. This is an opposite situation to the one encountered for foliations. It is for this reason that the foliation framework and the APS framework extend the AS framework in different directions.

The approach to index theory explained in this last subsection extends to more complicated singular spaces, and this has provided the author of this presentation the motivation to study analysis on singular spaces.

2. Motivation II: Degeneration and singularity

The totally characteristic differential operators studied in the previous subsection appear not only in index problems, but actually arise in many practical applications. We shall now examine how the totally characteristic differential operators and other related operators appear in practice. In a nut-shell, they can be used to model degeneration and singularities. In this section, we introduce several examples. We begin with the ones related to the APS index theorem (the totally characteristic ones, called "rank one" by analogy with locally symmetric spaces) and then we continue with other examples.

2.1. **APS-type examples: rank one.**

Let us denote by ρ the distance to the origin in $\mathbb{R}^d$. Here is a list of examples of totally characteristic operators.

Example 2.1. In our three examples below, the first one is a true totally characteristic operator, whereas the other two require us to remove the factor ρ^{-2} first.

(1) The first example is $\partial_t - L$ in one dimension and is given by the elliptic generator L of the Black-Scholes equation [116], which is a parabolic partial differential equation with generator

$$Lu := \frac{\sigma^2}{2} x^2 \partial_x^2 u + rx\partial_x u - ru\,. \tag{20}$$

It is the backward Kolmogorov equation associated to a stochastic ordinary differential equation.

(2) The second example is that of the Laplacian in polar coordinates (ρ, θ) in two dimensions

$$\Delta u = \rho^{-2}\big(\rho^2\partial_\rho^2 u + \rho\partial_\rho u + \partial_\theta^2 u\big). \tag{21}$$

This writing of the Laplace equation is especially useful when studying boundary value problems on polygonal domains.

(3) A related example is that of a Schrödinger operator in three dimensions, obtained by writing the three dimensional Laplace operator in spherical coordinates (ρ, x'), $\rho > 0$, $x' \in S^2$:

$$-(\Delta + \frac{Z}{\rho})u = -\rho^{-2}\big(\rho^2\partial_\rho^2 u + 2\rho\partial_\rho u + \Delta_{S^2} u + Z\rho u\big). \tag{22}$$

The advantage of this writing is that, in this way, the Coulomb potential becomes a negligible singularity near the origin.

A similar expansion is valid for elliptic operators in generalized spherical coordinates in arbitrary dimensions and was used by Kondratiev in [62] to study domains with conical points. Kondratiev's paper is widely used since it provides the needed analysis facts to deal with polygonal domains, the main testing ground for numerical methods.

2.2. **Manifolds with corners.** For more complicated examples we will need manifolds with corners. Recall that $\overline{M}$ is a *manifold with corners* if, and only if, $\overline{M}$ is locally diffeomorphic to an open subset of $[0,1)^n$. The transition functions of $\overline{M}$ are supposed to be smooth, as in the case of manifolds with smooth boundary. A manifold with boundary is a particular case of a manifold with corners, but we agree in this paper that a *smooth* manifold does not have boundary (or corners), since we regard the corners (or boundary) as some sort of singularity.

A point $p \in \overline{M}$ is called of *depth* k if it has a neighborhood V_p diffeomorphic to $[0,1)^k \times (-1,1)^{n-k}$ by a diffeomorphism $\phi_p : V_p \to [0,1)^k \times (-1,1)^{n-k}$ mapping p to the origin: $\phi_p(p) = 0$. A connected component F of the set of points of depth k will be called an open face (of codimension k) of $\overline{M}$. The set of points of depth 0 of $\overline{M}$ is called the *interior* of $\overline{M}$ and is also considered to be an open face of $\overline{M}$. The closure in $\overline{M}$ of an open face F of $\overline{M}$ will be called a *closed* face of $\overline{M}$. The closed faces of $\overline{M}$ may not be manifolds with corners in their own. The union of the proper faces of $\overline{M}$ is denoted by $\partial\overline{M}$ and is called the *boundary* of $\overline{M}$. The complement $M := \overline{M} \smallsetminus \partial\overline{M}$ of the boundary is *the interior* of $\overline{M}$.

The following set of vector fields will be useful when defining Lie manifolds:

$$\mathcal{V}_b := \{\, X \in \Gamma(\overline{M}; T\overline{M}),\ X \text{ tangent to all boundary faces of } \overline{M}\,\}. \tag{23}$$

Let us notice that in the case of manifolds with boundary, the totally characteristic differential operators on $\overline{M}$, see Equation (14), are generated by $\mathcal{C}^\infty(\overline{M})$ and the vector fields $X \in \mathcal{V}_b$.

2.3. **Higher rank examples.** We now continue with more complicated examples, which we call "higher rank" examples, again by analogy with locally symmetric spaces. In general, the natural domains for these higher rank examples will be manifolds with corners.

Example 2.2. There are no "higher rank" example in dimension one, so we begin with an example in dimension two.

(1) The simplest non-trivial example is the Laplacian

$$\Delta_{\mathbb{H}} = y^2(\partial_x^2 + \partial_y^2) \tag{24}$$

on the hyperbolic plane $\mathbb{H} = \mathbb{R} \times [0,\infty)$, whose metric is $y^{-2}(dx^2 + dy^2)$.

(2) The Laplacian on the hyperbolic plane is closely related to the SABR Partial Differential Equation (PDE) due to Lesniewsky and collaborators [53]. The SABR PDE is also a parabolic PDE $\partial_t - L$ associated to a stochastic differential equation, with

$$2L := y^2\big(x^2\partial_x^2 + 2\rho\nu x\partial_x\partial_y + \nu^2\partial_y^2\big)\,, \tag{25}$$

with ρ and ν parameters. Stochastic differential equations provide many interesting and non-trivial examples of degenerate parabolic PDEs that can be treated using Lie manifolds.

(3) A related example is that of the Laplacian in cylindrical coordinates (ρ,θ,z) in three dimensions:

$$\Delta u = \rho^{-2}\big((\rho\partial_\rho)^2 u + \partial_\theta^2 u + (\rho\partial_z)^2\big)\,. \tag{26}$$

Ignoring the factor ρ^{-2}, which amounts to a conformal change of metric, we see that our differential operator is generated by the vector fields

$$\rho\partial_\rho,\ \partial_\theta, \text{ and } \rho\partial_z ,$$

and that the linear span of these vector fields is a *Lie algebra*. The resulting partial differential operators are usually called *edge differential operators*. This example can be used to treat the behavior near edges of polyhedral domains of elliptic PDEs. This behavior is more difficult to treat than the behavior near vertices. For boundary value problems in three dimensions wedge, the natural domain is $[0, \alpha] \times [0, \infty) \times \mathbb{R}$, a manifold with corners of codimension two.

We thus again see that Lie algebras of vector fields are one of the main ingredients in the definition of the differential operators that we are interested in. More related examples will be provided below as examples of Lie manifolds.

Degenerate elliptic equations have many applications in Numerical Analysis, see [11, 33, 36, 75, 74], for example.

3. Lie manifolds: definition and geometry

Motivated by the previous two sections, we now introduce Lie manifolds largely following [5]. In fact, we slightly extend the definition in [5] by allowing the manifold with corners to be noncompact. We also slightly simplify it based on a comment of Skandalis. We thus define our Lie manifolds using Lie algebroids and then we recover the usual definition in terms of Lie algebras of vector fields. I have tried to make this section as self-contained as possible, thus including most of the proofs.

3.1. **Lie algebroids and Lie manifolds.** We have found it convenient to introduce Lie manifolds and "open manifolds with a Lie structure at infinity" in terms of Lie algebroids, which we recall now. Recall that we use the following notation, if $E \to X$ is a smooth vector bundle, we denote by $\Gamma(X; E)$ (respectively, by $\Gamma_c(X; E)$) the space of smooth (respectively, smooth, compactly supported) sections of E. Sometimes, when no confusion can arise, we simply write $\Gamma(E)$, or, respectively, $\Gamma_c(E)$. We now introduce Lie algebroids.

Definition 3.1. A *Lie algebroid* $A \to \overline{M}$ is a real vector bundle over a manifold with corners $\overline{M}$ together with a *Lie algebra* structure on $\Gamma(\overline{M}; A)$ (with bracket $[\ ,\ ]$) and a vector bundle map $\varrho : A \to T\overline{M}$, called *anchor*, such that the induced map $\varrho_* : \Gamma(\overline{M}; A) \to \Gamma(\overline{M}; T\overline{M})$ satisfies the following two conditions:

(i) $\varrho_*([X, Y]) = [\varrho_*(X), \varrho_*(Y)]$ and
(ii) $[X, fY] = f[X, Y] + (\varrho_*(X)f)Y$, for all $X, Y \in \Gamma(\overline{M}; A)$ and $f \in \mathcal{C}^\infty(\overline{M})$.

For further reference, let us introduce here the *isotropy* of a Lie algebroid.

Definition 3.2. Let $\varrho : A \to T\overline{M}$ be a Lie algebroid on $\overline{M}$ with anchor ϱ. Then the kernel $\ker(\varrho_x : A_x \to T_x\overline{M})$ of the anchor is the *isotropy* of A at $x \in \overline{M}$.

The isotropy at any point can be shown to be a Lie algebra. See [7] for generalizations. Recall that we denote by $\partial\overline{M}$ the boundary $\overline{M}$, that is, the union of its proper faces, and by $M := \overline{M} \smallsetminus \partial\overline{M}$ its interior.

Definition 3.3. A smooth manifold M is called an *open manifold with a Lie structure at infinity* if it is the interior of a manifold with corners $\overline{M}$ and on $\overline{M}$ there is given a Lie algebroid A with anchor $\varrho : A \to T\overline{M}$ satisfying the following properties:

(i) $\varrho : A_x \to T_x\overline{M}$ is an isomorphism for all $x \in M := \overline{M} \smallsetminus \partial M$ and
(ii) $\mathcal{V} := \varrho_*(\Gamma(\overline{M}; A)) \subset \mathcal{V}_b$.

If $\overline{M}$ is compact, then the pair $(\overline{M}, A)$ will be called a *Lie manifold*.

Condition (ii) means that the Lie algebra of vector fields $\mathcal{V} := \varrho_*(\Gamma(A))$ consists of vector fields tangent to all faces of $\overline{M}$ (we write $\varrho_*(\Gamma(A))$ instead of $\Gamma(\overline{M}; A)$ when no confusion can arise, also, we shall usually write $\varrho_*(\Gamma(A))$ instead of $\varrho_*(\Gamma(A))$.). Lie manifolds were introduced in [5]. One of the main reason for introducing open manifolds with a Lie structure at infinity is in order to be able to localize Lie manifolds. Thus, if $(\overline{M}, A)$ is a Lie manifold and $V \subset \overline{M}$ is an open subset, then $(V, A|_V)$ will not be a Lie manifold, in general, but will be an open manifold with a Lie structure at infinity. We also have the following trivial example.

Example 3.4. The "example zero" of a Lie manifold is that of a smooth, compact manifold $M = \overline{M}$ (no boundary or corners) by taking $A = TM$, thus $\mathcal{V} = \Gamma(M; TM)$. Then $(M, \mathcal{V})$ is a (trivial) example of a Lie manifold. This example of a Lie manifold provides the framework for the AS Index Theorem. Similarly, every smooth manifold M is an open manifold with a Lie structure at infinity by taking $\overline{M} = M$ and $A = TM$.

Example 3.5. Let $\overline{M}$ be a manifold with corners such that its interior $M := \overline{M} \smallsetminus \partial\overline{M}$ identifies with the quotient of a *Lie group* G by a discrete subgroup Γ and the action of G on G/Γ by left multiplication extends to an action of G on $\overline{M}$. Let $\mathfrak{g}$ be the Lie algebra of G. Then $A := \overline{M} \times \mathfrak{g}$ with anchor given by the infinitesimal action of G. Note that the action of the Lie algebra $\mathfrak{g}$ preserves the structure of faces of $\overline{M}$ and hence $\varrho_*(\Gamma(A)) \subset \mathcal{V}_b$. We call the corresponding manifold with a Lie structure at infinity *a group enlargement*. The simplest example is that of that of $G = M = \mathbb{R}_+^*$ acting on $[0, \infty]$. Many interesting Lie manifolds arising in practice are, locally, group enlargements, see for instance [50, 49] for some examples coming from quantum mechanics.

Let $(\overline{M}, A)$ be an open manifold with a Lie structure at infinity. In applications, it is easier to work with the vector fields $\mathcal{V} := \varrho_*(\Gamma(A))$ associated to a Lie manifold than with the Lie manifold itself. We shall then use the following alternative definition of Lie manifolds.

Proposition 3.6. *Let us consider a pair $(\overline{M}, \mathcal{V})$ consisting of a compact manifold with corners $\overline{M}$ and a subset $\mathcal{V} \subset \Gamma(\overline{M}; T\overline{M})$ of vector fields on $\overline{M}$ that satisfy:*

(i) $\mathcal{V}$ is closed under the Lie bracket $[\ ,\ ]$;
(ii) $\Gamma_c(M; TM) \subset \mathcal{V} \subset \mathcal{V}_b$;
(iii) $\mathcal{C}^\infty(\overline{M})\mathcal{V} = \mathcal{V}$ and $\mathcal{V}$ is a finitely-generated $\mathcal{C}^\infty(\overline{M})$–module;
(iv) $\mathcal{V}$ is projective (as a $\mathcal{C}^\infty(\overline{M})$–module).

Then there exists a Lie manifold $(\overline{M}, A)$ with anchor ϱ such that $\varrho_*(\Gamma(\overline{M}; A)) = \mathcal{V}$.

Conversely, if $(\overline{M}, A)$ is a Lie manifold, then $\mathcal{V} := \varrho_*(\Gamma(\overline{M}; A))$ satisfies conditions (i)–(iv) above.

Proof. Let $(\overline{M}, \mathcal{V})$ be as in the statement. Since $\mathcal{V}$ is a finitely generated, projective $\mathcal{C}^\infty(\overline{M})$–module, the Serre–Swan Theorem implies then that there exists a finite dimensional vector bundle $A_\mathcal{V} \to \overline{M}$, uniquely defined up to isomorphism, such that

$$\mathcal{V} \simeq \Gamma(\overline{M}; A_\mathcal{V}) \,, \tag{27}$$

as $\mathcal{C}^\infty(\overline{M})$–modules. Let $I_x := \{\phi \in \mathcal{C}^\infty(\overline{M}),\, \phi(x) = 0\}$ be the maximal ideal corresponding to $x \in \overline{M}$. The fibers $(A_\mathcal{V})_x$, $x \in \overline{M}$, of the vector bundle $A_\mathcal{V} \to \overline{M}$ are given by $(A_\mathcal{V})_x = \mathcal{V}/I_x\mathcal{V}$. Since $\Gamma(\overline{M}; A_\mathcal{V}) \simeq \mathcal{V} \subset \Gamma(\overline{M}; T\overline{M})$, we automatically obtain for each $x \in \overline{M}$ a map

$$(A_\mathcal{V})_x \,:=\, \mathcal{V}/I_x\mathcal{V} \,\to\, \Gamma(\overline{M}; T\overline{M})/I_x\Gamma(\overline{M}; T\overline{M}) \,=\, T_x\overline{M} \,.$$

These maps piece together to yield a bundle map (anchor) $\varrho : A_\mathcal{V} \to T\overline{M}$ that makes $A_\mathcal{V} \to \overline{M}$ a Lie algebroid. The anchor map ϱ is an isomorphism over the interior M of $\overline{M}$ since $\Gamma_c(M; TM) \subset \mathcal{V}$, which is part of Assumption (ii). Since $\mathcal{V} \subset \mathcal{V}_b$, again by Assumption (ii), we obtain that $(\overline{M}, A_\mathcal{V})$ is indeed a Lie manifold.

Conversely, let $(\overline{M}, A)$ be a Lie manifold with anchor $\varrho : A \to T\overline{M}$. We need to check that $\mathcal{V} := \varrho_*(\Gamma(\overline{M}; A))$ satisfies conditions (i)–(iv) of the statement. Indeed, $\mathcal{V} := \varrho_*(\Gamma(\overline{M}; A))$ is a Lie algebra because $\Gamma(\overline{M}; A)$ is a Lie algebra and $\varrho_* : \Gamma(\overline{M}; A) \to \Gamma(\overline{M}; T\overline{M})$ is an injective Lie algebra morphism. So Condition (i) is satisfied. To check the second conditions, we notice that Definition 3.3(i) (isomorphism over the interior) gives that $\Gamma_c(M; T\overline{M}) \subset \mathcal{V}$. Since we have by assumption $\mathcal{V} \subset \mathcal{V}_b$, we see that Condition (ii) is also satisfied. Finally, Conditions (iii) and (iv) are satisfied since the space of smooth sections of a finite dimensional vector bundle is a projective module over the algebra of smooth functions on the base, again by the Serre-Swan theorem. □

Let $(\overline{M}, \mathcal{V})$ as in the statement of the above proposition, Proposition 3.6. We call $\mathcal{V}$ its *structural Lie algebra of vector fields* and we call the Lie algebroid $A_\mathcal{V} \to \overline{M}$ introduced in Equation (27) the *the Lie algebroid associated to* $(\overline{M}, \mathcal{V})$. The alternative characterization of Lie manifolds in Proposition 3.6 is the one that will be used in our examples.

Remark 3.7. It is worthwhile pointing out that the condition that $\mathcal{V}$ be a finitely generated, projective $\mathcal{C}^\infty(\overline{M})$–module in Proposition 3.6 together with the fact that the anchor ϱ is an isomorphism over the interior of $\overline{M}$ are equivalent to the following condition, where $n = \dim(\overline{M})$:

> For every point $p \in \overline{M}$, there exist a neighborhood V_p of p in $\overline{M}$ and n-vector fields $X_1, X_2, \ldots, X_n \in \mathcal{V}$ such that, for any vector field $Y \in \mathcal{V}$, there exist smooth functions $\phi_1, \phi_2, \ldots, \phi_n \in \mathcal{C}^\infty(\overline{M})$ such that

$$Y = \phi_1 X_1 + \phi_2 X_2 + \ldots + \phi_n X_n \,, \quad \phi_i|_{V_p} \text{ uniquely determined.} \tag{28}$$

The vector fields $X_1, X_2, \ldots, X_n$ are then called a *local basis* of $\mathcal{V}$ around p. (This is the analog in our case of the well known fact from commutative algebra that a module is projective if, and only if, it is locally free.)

The simplest example of a non-compact Lie manifold is that of a manifold with cylindrical ends. The following example generalizes this example to the higher rank case. It is a basic example to which we will come back later. To introduce this example, however, we need also to introduce the defining functions of a hyperface. A *hyperface* is a proper face $H \subset \overline{M}$ of maximal dimension (dimension $\dim(H) = \dim(M) - 1$). Recall that a *defining function* of a hyperface H of $\overline{M}$ is a function x such that $H = \{x = 0\}$ and $dx \neq 0$ on H. The hyperface $H \subset \overline{M}$ is called *embedded* if it has a defining function. The existence of a defining function is a global property, because locally one can always find defining functions, a fact that will be needed in the example below.

Example 3.8. Let $\overline{M}$ a compact manifold with corners and $\mathcal{V} = \mathcal{V}_b$. Let us check that $(\overline{M}, \mathcal{V}_b)$ is a Lie manifold. We shall use Proposition 3.6. Condition (i) is easily verified since the Lie bracket of two vector fields tangent to a submanifold is again tangent to that submanifold. Condition (ii) in the Proposition 3.6 is even easier since by definition vector fields that are zero near the boundary $\partial\overline{M}$ are contained in $\mathcal{V}_b$. Clearly, $\mathcal{V}$ is a $\mathcal{C}^\infty(\overline{M})$ module. The only non-trivial fact to check is that $\mathcal{V}$ is finitely generated and projective as an $\mathcal{C}^\infty(\overline{M})$ module. This is in fact the only fact that we still need to check. To verify it, let us fix a corner point p of codimension k (that is, p belongs to an open face F of codimension k). Then, in a neighborhood of p, we can find k defining functions $r_1, r_2, \ldots, r_k$ of the hyperfaces containing p such that a local basis of $\mathcal{V}$ around p (see Remark 3.7) is given by

$$r_1\partial_{r_1},\, r_2\partial_{r_2},\, \ldots,\, r_k\partial_{r_k},\, \partial_{y_{k+1}},\, \ldots,\, \partial_{y_n} \tag{29}$$

where $y_{k+1}, \ldots, y_n$ are local coordinates on the open face F of dimension k containing p, so that $(r_1, r_2, \ldots, r_k, y_{k+1}, \ldots, y_n)$ provide a local coordinate system in a neighborhood of p in $\overline{M}$.

If $\overline{M}$ has a smooth boundary, then $\mathcal{V}_b$ generates the totally characteristic differential operators, which were introduced in Equation (14), and hence this example corresponds to a manifold with cylindrical ends. In fact, we will see that the natural Riemannian metric of a manifold with (asymptotically) cylindrical ends. This example was studied also by Debord and Lescure [40, 37], Melrose and Piazza [83], Monthubert [87], and Schulze [115].

3.2. **The Androulidakis-Skandalis framework.** Androulidakis and Skandalis have studied recently structures similar to ours but in which Condition (iv) in Proposition 3.6 is dropped and have obtained extensions of several of our results. It would be quite worthwhile to see which of the known results for Lie manifolds do not require Condition (iv).

By [5], every vector field $X \in \mathcal{V}$ that has compact support in $\overline{M}$ gives rise to a one parameter group of diffeomorphisms $\exp(tX) : \overline{M} \to \overline{M}$, $t \in \mathbb{R}$. We denote by $\exp(\mathcal{V})$ the subgroup of diffeomorphisms generated by all $\exp(X)$ with $X \in \mathcal{V}$ and compact support in $\overline{M}$. The results in [7] show that $\exp(\mathcal{V})$ acts by Lie automorphisms of $\mathcal{V}$ (the condition (iv) of Proposition 3.6 that $\mathcal{V}$ be a projective module is not necessary).

3.3. **The metric on Lie manifolds.** As seen in the example of manifolds with cylindrical ends, Lie manifolds have an intrinsic geometry. We now discuss some results in this direction following [5] and we extend them to open manifolds with a Lie structure at infinity (this extension is straightforward but needed). Thus, from now on, $(\overline{M}, A)$ will be an open manifold with a Lie structure at infinity. (Thus we will not assume $(\overline{M}, A)$ to be a Lie manifold, unless explicitly stated.)

Definition 3.9. Let $(\overline{M}, A)$ be an open manifold with a Lie structure at infinity. A metric on TM is called *compatible* (with the structure at infinity) if it extends to a metric on $A \to \overline{M}$.

We shall need the following lemma.

Lemma 3.10. *Let $(\overline{M}, A)$ be an open manifold with a Lie structure at infinity with compatible metric g. Assume $\overline{M}$ to be paracompact. Then there exists a smooth metric h on $T\overline{M}$ such that $h \le g$.*

Proof. Away from the boundary, we may take $h = g$. It is enough then to define h in a neighborhood of the boundary $\partial\overline{M}$. Let us choose an arbitrary metric h_0 on $\overline{M}$ (or, more precisely, on $T\overline{M}$). For each $p \in \overline{M}$, let $U_p \subset V_p$ be open neighborhoods of p in $\overline{M}$ such that V_p has compact closure and contains the closure of U_p. Since V_p has compact closure and ϱ is continuous, we obtain that there exists $M_p > 0$ such that $h_0(\xi) \le M_p g(\xi)$ for every $\xi \in A|_{V_p}$. Let us choose a locally finite covering $(U_p)_{p\in I}$ of the boundary $\partial\overline{M}$ with sets of the form U_p, with $p \in I$ the set of chosen points. Let V_0 be the complement of $\cup_{p\in I}\overline{U_p}$ and let $(\phi_p)_{p\in I\cup\{0\}}$ be a smooth, locally finite partition of unity on $\overline{M}$ subordinated to the covering $(V_p)_{p\in I\cup\{0\}}$. Then, if we define

$$h = \sum_{p\in I} \phi_p M_p^{-1} h_0 + \phi_0 g \,,$$

the metric h will satisfy $h \le g$ everywhere, as desired. □

Let us fix from now on a metric g on A. Since $TM \subset A$, the metric g restricts a compatible Riemannian metric on TM and hence also to a compatible metric on M. We shall denote all these metrics with the same symbol g, since there is no danger of confusion. The inner product of two vectors (or vector fields) $X, Y \in \Gamma(M; TM)$ in this metric will be denoted $(X, Y) \in \mathcal{C}^\infty(M)$ and the associated volume form $d\,\mathrm{vol}_g$. Of course, if $X, Y \in \mathcal{V} := \varrho_*(\Gamma(\overline{M}; A))$, then $(X, Y) \in \mathcal{C}^\infty(\overline{M})$. We now want to investigate some properties of this compatible metric g. For simplicity, we write $\Gamma(TM) = \Gamma(M; TM)$ and $\varrho_*(\Gamma(A)) = \Gamma(\overline{M}; A)$. Let us consider the Levi-Civita connection

$$\nabla^{LC} : \Gamma(TM) \to \Gamma(TM \otimes T^*M) \,. \tag{30}$$

Recall that an A–connection on a vector bundle $E \to \overline{M}$ (see [5] and the references therein) is given by a differential operator $\nabla \in \mathrm{Diff}(\mathcal{V}; E, E \otimes A^*)$ such that

$$\nabla_{fX}(g\xi) = f\big(f_1 \nabla_X(\xi) + X(g)\xi\big) \tag{31}$$

for all $f, f_1 \in \mathcal{C}^\infty(\overline{M})$ and $\xi \in \Gamma(\overline{M}; E)$. The following proposition from [5] gives that the Levi-Civita connection extends to an "A-connection."

Proposition 3.11. *The Levi-Civita connection extends to a linear differential operator $\nabla : \varrho_*(\Gamma(A)) \to \Gamma(A \otimes A^*)$, satisfying*

(i) $\nabla_X(fY) = X(f)Y + f\nabla_X(Y)$,
(ii) $X(Y, Z) = (\nabla_X Y, Z) + (Y, \nabla_X Z)$, *and*
(iii) $\nabla_X Y - \nabla_Y X = [X, Y]$,

for all $X, Y, Z \in \mathcal{V} = \varrho_*(\Gamma(A))$.

Proof. We recall the proof for the benefit of the reader. We fix a compatible metric g on $M := \overline{M} \smallsetminus \partial\overline{M}$. Since the metric g actually comes from an metric on A by restriction to $TM \subset A$, we see that

$$\phi(Z) := ([X,Y],Z)-([Y,Z],X)+([Z,X],Y)+X(Y,Z)+Y(Z,X)-Z(X,Y)\,, \tag{32}$$

defines a smooth function on $\overline{M}$ for any $Z \in \mathcal{V}$ and that this smooth function depends linearly on Z. Hence there exists a smooth section $V \in \mathcal{V}$ such that $\phi(Z) = (V,Z)$ for all $Z \in \mathcal{V}$. We then define $\nabla_X Y := V$. By the definition of ∇ and by the classical definition of the Levi-Civita connection, ∇ extends the Levi-Civita connection. Since the Levi-Civita connection satisfies the properties that we need to prove (on M), by the density of M in $\overline{M}$, we obtain that ∇ satisfies the same properties. □

We continue with some remarks

Remark 3.12. An important consequence of the above proposition is that all the covariant derivatives $\nabla^k R$ of the curvature R extend to suitable tensors involving A and A^* on $\overline{M}$. If $(\overline{M}, A)$ is a Lie manifold, it follows that the curvature and all its covariant derivatives are *bounded*. It turns out also that the radius of injectivity of M is positive [4, 38], and hence M has *bounded geometry*.

We next discuss the divergence of a vector field, which is needed to define adjoints.

Remark 3.13. Another important consequence of the extension of the Levi-Civita connection (called *the Levi-Civita connection on* $\overline{M}$) is the definition of the divergence of a vector field. Indeed, let us fix a point $p \in \overline{M}$ and a local orthonormal basis $X_1, \ldots, X_n$ of A on some neighborhood of p in $\overline{M}$ ($n = \dim(\overline{M})$). We then write $\nabla_{X_i} X = \sum_{j=1}^n c_{ij}(X) X_j$ and define

$$\operatorname{div}(X) := -\sum_{j=1}^{n} c_{jj}(X)\,, \tag{33}$$

which is a smooth function on the given neighborhood of p that does not depend on the choice of the local orthonormal basis (X_i) used to define it. Consequently, this formula defines a global function $\operatorname{div}(X) \in \mathcal{C}^\infty(\overline{M})$.

We now introduce differential operators on open manifolds with a Lie structure at infinity, which is the main reason why we are interested in Lie manifolds.

Definition 3.14. We define the algebra $\operatorname{Diff}(\mathcal{V})$ as the algebra of differential operators on $\overline{M}$ generated by the operators of multiplication with functions in $\mathcal{C}_c^\infty(\overline{M})$ and by the directional derivatives with respect to vector fields $X \in \mathcal{V}$ with compact support in $\overline{M}$.

Clearly, in our first example, Example 3.8, the resulting algebra of differential operators, namely $\operatorname{Diff}(\mathcal{V}_b)$, is the algebra of totally characteristic differential operators. See [7] for more general examples.

The differential operators in $\operatorname{Diff}(\mathcal{V})$ can be regarded as acting either on functions on $\overline{M}$ or on functions on $M := \overline{M} \smallsetminus \partial\overline{M}$. When it comes to classes of measurable functions–say Sobolev spaces–this makes no difference. However, the fact that $\operatorname{Diff}(\mathcal{V})$ maps $\mathcal{C}^\infty(\overline{M})$ to $\mathcal{C}^\infty(\overline{M})$ is a non-trivial property that does not follow from the mapping properties of $\operatorname{Diff}(\mathcal{V})$ on M. We have the following simple remark on the structure of $\operatorname{Diff}(\mathcal{V})$.

Remark 3.15. Every $P \in \mathrm{Diff}(\mathcal{V})$ of order at most m can be written as a sum of differential monomials of the form $X_1^{\alpha_1} X_2^{\alpha_2} \dots X_k^{\alpha_k}$, where $X_i \in \mathcal{V}$, $k \le m$, and α is a multi-index. If $Y_1, Y_2, \dots, Y_n$ are vector fields in $\mathcal{V}$ forming a local basis around $p \in \overline{M}$ (so $\dim(M) = n$), then every $P \in \mathrm{Diff}(\mathcal{V})$ of order at most m can be written in a neighborhood of p in $\overline{M}$ as

$$P = \sum_{|\alpha| \le m} a_\alpha Y_1^{\alpha_1} Y_2^{\alpha_2} \dots Y_n^{\alpha_n} .$$

This follows from the Poincaré-Birkhoff-Witt theorem of [98].

We shall denote the inner product on $L^2(M; \mathrm{vol}_g)$ by $(\, , \,)_{L^2}$. Let $P \in \mathrm{Diff}(\mathcal{V})$. The formal adjoint $P^\sharp$ of P is then defined by

$$(Pf_1, f_2)_{L^2} = (f_1, P^\sharp f_2)_{L^2}, \qquad f_1, f_2 \in \mathcal{C}_{\mathrm{c}}^\infty(M_0). \tag{34}$$

Let $X \in \mathcal{V} := \varrho_*(\Gamma(A))$. Since $\mathrm{div}(X) \in \mathcal{C}^\infty(\overline{M})$ of Equation (33) extends the classical definition on M, we have that

$$\int_M X(f) \, d\,\mathrm{vol}_g = \int_M f \,\mathrm{div}(X) \, d\,\mathrm{vol}_g .$$

In particular, the formal adjoint of X is

$$X^\sharp = -X + \mathrm{div}(X) \in \mathrm{Diff}(\mathcal{V}) . \tag{35}$$

and hence $\mathrm{Diff}(\mathcal{V})$ is closed under formal adjoints.

We can extend the definition of $\mathrm{Diff}(\mathcal{V})$ to include operators $\mathrm{Diff}(\mathcal{V}; E, F)$ acting between vector bundles $E, F \to \overline{M}$. This can be done either by embedding the vector bundles E and F into trivial bundles or by looking at a local basis. We shall write $\mathrm{Diff}(\mathcal{V}; A) = \mathrm{Diff}(\mathcal{V}; A, A)$. The formal adjoint of $P \in \mathrm{Diff}(\mathcal{V}; E, F)$ is then an operator $P^\sharp \in \mathrm{Diff}(\mathcal{V}; F^*, E^*)$. Typically E and F will have hermitian metrics and then we identify E^* with E and F^* with F. In particular, if E is a Hermitian bundle, then $\mathrm{Diff}(\mathcal{V}; E)$ is an algebra closed under formal adjoints.

We are ready now to prove that all geometric operators on M that are associated to a compatible metric g are generated by $\mathcal{V} := \varrho_*(\Gamma(A))$ [5]. (Recall that a compatible metric on M is a metric coming from a metric on the Lie algebroid A of our Lie manifold $(\overline{M}, A)$ by restriction to TM.) In particular, we have the following result [5].

Proposition 3.16. *We have that the de Rham differential d on M extends to a differential operator $d \in \mathrm{Diff}(\mathcal{V}; \Lambda^q A^*, \Lambda^{q+1} A^*)$. Similarly, the extension ∇ of the Levi-Civita connection to an A-valued connection defines a differential operator $\nabla \in \mathrm{Diff}(\mathcal{V}; A, A \otimes A^*)$.*

Proof. The proof of this theorem is to see that the classical formulas for these geometric operators extend to $\overline{M}$, provided that TM is replaced by A. For instance, for the de Rham differential, let $\omega \in \Gamma(\overline{M}; \Lambda^k A^*)$ and $X_0, \dots, X_k \in \mathcal{V}$, and use the formula

$$(d\omega)(X_0, \dots, X_k) = \sum_{j=0}^{q} (-1)^j X_j(\omega(X_0, \dots, \hat{X}_j, \dots, X_k)) + \\ \sum_{0 \le i < j \le q} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k) \in \mathcal{C}^\infty(\overline{M}) .$$

By choosing $X_0, \dots, X_k$ among a local basis of $\mathcal{V} := \Gamma(\overline{M}; A)$ and using the fact that $\mathcal{V}$ is closed under the Lie bracket, we obtain that $d \in \mathrm{Diff}(\mathcal{V}; \Lambda^q A^*, \Lambda^{q+1} A^*)$, as claimed.

For the Levi-Civita connection, it suffices to show that $\nabla_X \in \mathrm{Diff}(\mathcal{V}; A)$ for all $X \in \mathcal{V}$ (we could even restrict X to a local basis, but this is not really necessary). We will show that, in a local basis of A, the differential operator ∇_X is given by an operator involving only derivatives in $\mathcal{V}$. To this end, we shall use the formula (32) defining ∇_X, but choose $Y = fY_0$ for $f \in \mathcal{C}^\infty(\overline{M})$ and Y_0 and Z in a local basis in some neighborhood V of an arbitrary point $p \in \overline{M}$. Then we see from formula (32), using also the linearity of ϕ in Z, that $(\nabla_X(Y), Z) = fz_0 + X(f)z_1$, with z_0 and z_1 smooth functions on the given neighborhood V. Since the only derivative in this formula is X and $X \in \mathcal{V}$, this proves the desired statement for ∇. □

Let us consider a vector bundle $E \to \overline{M}$. If E has a metric, then an A–connection $\nabla \in \mathrm{Diff}(\mathcal{V}; E, E \otimes A^*)$ is said to *preserve the metric* if

$$(\nabla_X(\xi), \zeta)_E + (\xi, \nabla_X(\zeta))_E \;=\; X(\xi, \zeta)_E \tag{36}$$

for all $X \in \mathcal{V}$ and $\xi, \zeta \in \Gamma(\overline{M}; E)$. In particular, it follows from Proposition 3.11 that the extension of the Levi-Civita connection to an A–connection on A preserves the metric used to define it. We then have the following theorem

Theorem 3.17. *We continue to consider the fixed metric on A and its associated compatible metric g on M. Let $E \to \overline{M}$ be a hermitian bundle with a metric preserving A-connection. Then $\Delta_E := \nabla^* \nabla \in \mathrm{Diff}(\mathcal{V}; E)$. Similarly,*

$$\Delta_g \;:=\; d^* d \;\in\; \mathrm{Diff}(\mathcal{V}).$$

Proof. This follows from the fact that $\mathrm{Diff}(\mathcal{V})$ is closed under formal adjoints (as well as its vector bundle analogues) and from Proposition 3.16. □

Thus, in order to study geometric operators on a Lie manifold, it is enough to study the properties of differential operators generated by $\mathcal{V}$. It should be noted, however, that a Riemannian manifold may have different compactifications to a Lie manifold. An example is $\mathbb{R}^n$, which can either be compactified to $([-1, 1]^n, \mathcal{V}_b)$ (a product of manifolds with cylindrical ends) or it can be radially compactified to yield an asymptotically euclidean manifold. (See Example 5.3.)

Theorem 3.17 also gives the following.

Remark 3.18. The proof of Theorem 3.17 uses among other things the fact that the adjoint of an operator $P \in \mathrm{Diff}(\mathcal{V}; E, F)$ is also generated by $\mathcal{V}$. This gives that the Hodge operator $d + d^* \in \mathrm{Diff}(\mathcal{V}; \Lambda^* A^*)$. More generally, let $W \to \overline{M}$ be a Clifford bundle with admissible connection $\Gamma(\overline{M}; W) \to \Gamma(\overline{M}; W \otimes A^*)$. Then the associated Dirac operators are also generated by $\mathcal{V}$, see [5, 68]. All these statements seem to be more difficult to prove directly in local coordinates.

3.4. **Anisotropic structures.** It is very important in applications to extend the previous frameworks to include anisotropic structures [11]. We introduce them now for the purpose of later use.

Definition 3.19. An *anisotropic structure* on an open manifold $(\overline{M}, A)$ with a Lie structure at infinity is an open manifold manifold with a Lie structure at infinity $(\overline{M}, B)$ (same underlying compactification) together with a vector bundle map

$A \to B$ that is the identity over M and makes $\Gamma(\overline{M}; A)$ an ideal (in Lie algebra sense) of $\Gamma(\overline{M}; B)$.

We shall denote $\mathcal{W} := \Gamma(B)$, so $\mathcal{V} := \varrho_*(\Gamma(A))$ satisfies $[X, Y] \subset \mathcal{V}$ for all $X \in \mathcal{W}$ and $Y \in \mathcal{V}$. Recall the groups $\exp(\mathcal{V})$ and $\exp(\mathcal{W})$ introduced in Subsection 3.2 (and recall that they are generated by compactly supported vector fields). Then $\exp(\mathcal{W})$ acts on A, on $\mathcal{V} := \varrho_*(\Gamma(A))$, and on $\mathrm{Diff}(\mathcal{V})$. Moreover,

$$\exp(\mathcal{V}) \subset \exp(\mathcal{W})$$

is a normal subgroup.

4. Analysis on Lie manifolds

Our main interest is in the analytic properties of the differential operators in $\mathrm{Diff}(\mathcal{V})$. In this section, we introduce our function spaces following [4] and discuss Fredholm conditions. Throughout this section, $(\overline{M}, A)$ will denote an open manifold with a Lie structure at infinity and Lie algebroid A, anchor ϱ, and $\mathcal{V} := \varrho_*(\Gamma(A))$ the structural Lie algebra of vector fields on $\overline{M}$.

4.1. **Function spaces.** Let $(\overline{M}, A)$ be our given open manifold with a Lie structure at infinity and let g be a compatible metric on the interior M of $\overline{M}$ (that is, coming from a metric on A denoted with the same letter, see Definition 3.9). Let ∇ be the Levi-Civita connection acting on the tensor powers of the bundles A and A^*. We then define, for $m \in \mathbb{Z}_+$, the Sobolev spaces as in [4]

$$H^m(M) \ = \ \{u : M \to \mathbb{C},\ \nabla^k u \in L^2(M; A^{*\otimes k}),\ 0 \le k \le m\},\tag{37}$$

See also [55, 54].

Remark 4.1. In general, the Sobolev spaces $H^m(M)$ will depend on the choice of the metric g, but if $\overline{M}$ is compact (that is, if $(\overline{M}, A)$ is a Lie manifold), then they are independent of the choice of the metric, as we shall see below. It is interesting to notice that if denote by $d\,\mathrm{vol}_g$ the volume form (1-density) associated to g. If h is another such compatible metric, then $d\,\mathrm{vol}_h / d\,\mathrm{vol}_g$ and $d\,\mathrm{vol}_g / d\,\mathrm{vol}_h$ extend to smooth, bounded functions on $\overline{M}$. Hence the space $L^2(M) := L^2(M; \mathrm{vol}_g)$ is independent of the choice of the compatible metric g.

The spaces $H^m(M)$ behave well with respect to anisotropic structures.

Proposition 4.2. *Let $(\overline{M}, A)$ be an open manifold with a Lie structure at infinity and with an anisotropic structure $(\overline{M}, B)$, such that $\mathcal{W} := \Gamma(B) \supset \Gamma(A)$. Then $\exp \mathcal{W}$ acts by bounded operators on $H^m(M)$.*

Proof. This follows from the fact that $\exp(\mathcal{W})$ is generated by vector fields with compact support in $\overline{M}$. □

We now consider some alternative definitions of these Sobolev spaces in particular cases. We first consider the case of complete manifolds.

Remark 4.3. Let us assume that $(\overline{M}, A)$ and the compatible metric g are such that M is complete and let Δ_g be (positive) Laplacian associated to the metric g. Then $H^s(M)$ coincides with the domain of $(1 + \Delta_g)^{s/2}$.

In the bounded geometry case we can consider partitions of unity [117, 5].

Remark 4.4. Let us assume that $(\overline{M}, A)$ and the compatible metric g are such that M is of bounded geometry. Then the definition of the Sobolev spaces on M can be given using a choice of partition of unity with bounded derivatives [4] to patch the locally defined classical Sobolev spaces.

Finally, if $\overline{M}$ is an open subset of a Lie manifold, we have yet the following definition.

Remark 4.5. If $\overline{M}$ is an open subset of a Lie manifold such that $\overline{M}$ has the induced metric and structural Lie algebra of vector fields, then [5]

$$H^m(M) := \{u : M \to \mathbb{C},\ X_1 X_2 \dots X_k u \in L^2(M),\ k \le m,\ X_j \in \mathcal{V}\}, \tag{38}$$

so the Sobolev spaces $H^m(M)$ will be independent of the chosen compatible metric on the Lie manifold.

We define the *anisotropic* Sobolev spaces in a similar way.

Definition 4.6. Let $(\overline{M}, A)$ be a Lie manifold with an anisotropic structure $(\overline{M}, B)$, and $\mathcal{W} := \Gamma(B) \supset \Gamma(A)$. Then we define

$$H^m_{\mathcal{W}}(M) := \{u : M \to \mathbb{C},\ X_1 X_2 \dots X_k u \in L^2(M),\ k \le m,\ X_j \in \mathcal{W}\}, \tag{39}$$

The spaces $H^m_{\mathcal{W}}(M)$ are again independent of the chosen compatible metric on the Lie manifold.

We let $H^{-s}(M) := (H^s(M))^*$ and then we extend the definition of Sobolev space to s non-integer by interpolation. Each of these definitions has its own advantages and disadvantages. For instance, the definition (38) has the advantage that it immediately gives that, for any $P \in \mathrm{Diff}(\mathcal{V})$ of order $\mathrm{ord}(P) \le m$, the map

$$P : H^s(M) \to H^{s-m}(M)$$

is bounded for all $s \in \mathbb{R}$. In fact, we have the following lemma. Let us denote by $(E)_r$ the set of vectors of length $< r$, where E is a real or complex vector bundle endowed with a metric.

Lemma 4.7. *Let us assume that A is endowed with a metric such that the resulting metric g on $TM \subset A$ is of bounded geometry. Let $P \in \mathrm{Diff}(\mathcal{V})$ of order $\mathrm{ord}(P) \le m$. Then the map $P : H^s(M) \to H^{s-m}(M)$ is bounded for all $s \in \mathbb{R}$.*

Proof. Let $K \subset \overline{M}$ be a compact subset such that the coefficients of P are zero outside K. Let us choose a compact neighborhood L of K in $\overline{M}$ and let r_0 be the distance from K to the complement of L in a metric h on $\overline{M}$ such that $h \le g$, which exists by Lemma 3.10. Then $r_0 > 0$, because K is compact. Moreover, the distance from K to the complement of L in the metric g is $\ge r_0$ since $h \le g$. Let us fix $r_0 > r > 0$ less than the injectivity radius of M. For every $p \in M$, we then consider the exponential map $\exp : T_pM \to M$, which is a diffeomorphism from (the open ball of radius r) $(T_pM)_r$ onto its image. Thus P gives rise to a differential operator P_p on each of the open balls $(T_pM)_r$. Using the results of [117], it suffices to show that the coefficients of P in any of these balls of radius r are uniformly bounded. Indeed, this is a consequence of the following lemma, where the support of the resulting map is contained in L. □

Lemma 4.8. *Let us use the notation of the proof of the previous lemma and denote for any $p \in M$ by P_p the differential operator on $(T_pM)_r$ induced by the exponential map. Then the map $M \ni p \to P_p$ extends to a compactly supported smooth function defined on $\overline{M}$ such that P_p is a differential operator on $(A_p)_r$.*

4.2. **Pseudodifferential operators on Lie manifolds.** Let us begin by recalling the definition of a submanifold with corners from [5].

Definition 4.9. Let $\overline{M}$ be a manifold with corners and $L \subset \overline{M}$ be a submanifold. We shall say that L is a *submanifold with corners* of $\overline{M}$ if L is a manifold with corners (in its own) that intersects transversely all faces of $\overline{M}$ and each open face F of L is the open components of a set of the form $F \cap L$, where F is an open face of $\overline{M}$ of the same codimension as F.

The closed faces of a manifold with corners $\overline{M}$ are thus *not* submanifolds with corners of $\overline{M}$ even if they happen to be manifolds with corners. Also, the diagonal of the n-dimensional cube $[-1,1]^n$ is not a submanifold with corners. However, $\{0\} \times [-1,1]^{n-1}$ is a submanifold with corners of $[-1,1]^n$. In fact, all submanifolds with corners $L \subset \overline{M}$ have a tubular neighborhood [6, 4]. This tubular neighborhood allows us then to define the space $I^m(\overline{M}, L)$ of *classical conormal distributions* as in [57] or as in [98] for manifolds with corners. Similarly, $I^m_c(\overline{M}, L)$ is the space of classical conormal distributions with compact support.

Let us fix a compatible metric on M, the interior of our open manifold with a Lie structure at infinity (M, A). Also, let us fix $r > 0$ less than the injectivity radius of M. As in the proof of Lemma 4.7, the exponential map then defines a diffeomorphism from the set $(TM)_r$ of vectors of length $< r$ to an open neighborhood of the diagonal in $M \times M$. This allows us to define a natural bijection

$$\Phi : I^m((TM)_r, M) \to I^m((M \times M)_r, M)\,, \tag{40}$$

where $(M \times M)_r$ is the image of $(TM)_r$ through the exponential map. Similarly, we obtain by restriction an inclusion

$$I^m_c((A)_r, \overline{M}) \to I^m((TM)_r, M)\,. \tag{41}$$

Recall the group of diffeomorphisms $\exp(\mathcal{V})$ defined in Subsection 3.2. Then we define as in [6]

$$\Psi^m_{\mathcal{V}}(M) \;:=\; \Phi(I^m_c((A)_r, \overline{M})) \;+\; \Phi(I^{-\infty}_c((A)_r, \overline{M}))\exp(\mathcal{V})\,. \tag{42}$$

Then we have the following result [6].

Theorem 4.10. *We have $\Psi^m_{\mathcal{V}}(M)\Psi^{m'}_{\mathcal{V}}(M) = \Psi^{m+m'}_{\mathcal{V}}(M)$. The subspace $\Psi^m_{\mathcal{V}}(M)$ is closed under adjoints and the principal symbol $\sigma_m : \Psi^m_{\mathcal{V}}(M) \to S^m_{cl}(A^*)/S^{m-1}_{cl}(A^*)$ is surjective with kernel $\Psi^{m-1}_{\mathcal{V}}(M)$. These algebras do not depend on the choice of the parameter $r > 0$ used to define them. Moreover, $P \in \Psi^m_{\mathcal{V}}(M)$ defines a bounded operator $H^s(M) \to H^{s-m}(M)$. If an anisotropic structure is given, then the group $\exp(\mathcal{W})$ acts by degree preserving automorphisms on $\Psi^m_{\mathcal{V}}(M)$.*

The proof of the above theorem is too long to include here. Let us just say that it is obtained by realizing $\Psi^{m+m'}_{\mathcal{V}}(M)$ as the image of a groupoid pseudodifferential operator algebra [6, 87, 89, 98] for *any* Lie groupoid integrating the Lie algebroid A defining the Lie manifold $(\overline{M}, A)$ [38, 39, 97].

The algebra $\Psi^*_{\mathcal{V}}(M)$ has the property that its subset of differential operators coincides with $\mathrm{Diff}(\mathcal{V})$. It also has the good symbolic properties that answer to a question of Melrose [6, 85].

4.3. **Comparison algebras.** We continue to denote by $(\overline{M}, A)$ an open manifold with a Lie structure at infinity. For simplicity, we shall assume that $\overline{M}$ is connected. We now recall from [88] the *comparison* C^*-algebra $\mathfrak{A}(U, \mathcal{V})$ associated to the given Lie manifold $(\overline{M}, A)$, where $\mathcal{V} := \varrho_*(\Gamma(A))$ and $U \subset \overline{M}$ is an open subset. Its definition extends to open manifolds with a Lie structure at infinity by Lemma 4.7 that justifies the following definition.

Definition 4.11. Let us assume that A has a metric g such that the induced metric on M is of bounded geometry and let $U \subset \overline{M}$ be an open subset. Then $\mathfrak{A}(U; \mathcal{V})$ is the norm closed subalgebra of the algebra $\mathcal{B}(L^2(M; \mathrm{vol}_g))$ of bounded operators on $L^2(M; \mathrm{vol}_g)$ generated by all the operators of the form $\phi_1 P(1+\Delta)^{-k}\phi_2$, where $\phi_i \in \mathcal{C}^\infty_{\mathrm{c}}(U)$ and $P \in \mathrm{Diff}(\mathcal{V})$ is a differential operator of order $\leq 2k$.

In case an anisotropic structure is given, the group $\exp(\mathcal{W})$ acts by automorphisms on the comparison algebra $\mathfrak{A}(\overline{M}; \mathcal{V})$. We shall need the following lemma that follows right away from the results in [5] and [68].

Lemma 4.12. *Let us use the notation of Definition 4.11 and let $T := \phi_1 P(1+\Delta)^{-k}\phi_2$. Then T is contained in the norm closure of $\Psi^0_{\mathcal{V}}(M)$ and is a pseudodifferential operator of order ≤ 0 with principal symbol*

$$\sigma_0(T) = \phi_1 \sigma_{2k}(P)(1+|\xi|^*)^{-k}\phi_2\,.$$

Moreover, the principal symbol depends continuously on T, and hence extends to a continuous, surjective morphism $\sigma_0 : \mathfrak{A}(U; \mathcal{V}) \to \mathcal{C}_0(S^*A|_U)$.

As in [88], we obtain the following result.

Theorem 4.13. *Let $(\overline{M}, A)$ be a connected open manifold with a Lie structure at infinity. Then $\mathfrak{A}(M; \mathcal{V})$ contains the algebra $\mathcal{K}(L^2(M))$ of all compact operators on $L^2(M)$ and is contained in the norm closure of $\Psi^0_{\mathcal{V}}(M)$.*

Proof. We recall the proof for the benefit of the reader. The inclusion of $\mathfrak{A}(\overline{M}; \mathcal{V})$ in the norm closure of $\Psi^0_{\mathcal{V}}(M)$ follows from Lemma 4.12.

Let $\phi_1, \phi_2 \in \mathcal{C}^\infty_{\mathrm{c}}(M)$, let $\psi \in \exp(\mathcal{V})$, and let $T \in \Phi(I^{-\infty}_c(A; \overline{M}))$, then the composition operator $L := \phi_1 T \psi \phi_2$ is a compact operator and belongs to $\mathfrak{A}(\overline{M}; \mathcal{V})$, by the definition. This shows that $\mathfrak{A}(\overline{M}; \mathcal{V})$ contains the compact operators. Let $\xi_1, \xi_2 \in L^2(M)$ be nonzero. Then we can find ϕ_1, ϕ_2, ψ, and T as above such that $L := \phi_1 T \psi \phi_2$ satisfies $(T\xi_1, \xi_2) \neq 0$. Hence $\mathfrak{A}(\overline{M}; \mathcal{V})$ has no non-trivial invariant subspace. Hence $\mathfrak{A}(\overline{M}; \mathcal{V})$ contains all compact operators because any proper subalgebra of the algebra of compact operators has an invariant subspace. □

4.4. **Fredholm conditions.** Theorem 4.13 allows us, in principle, to study the Fredholm property of operators in $\mathfrak{A}(\overline{M}; \mathcal{V})$. Let us denote by $\mathcal{K} = \mathcal{K}(L^2(M)$ the ideal of compact operators in $\mathfrak{A}(\overline{M}; \mathcal{V})$. Recall then Atkinson's classical result [45] that states that $T \in \mathfrak{A}(\overline{M}; \mathcal{V})$ is Fredholm if, and only if, its image $T + \mathcal{K}$ in $\mathfrak{A}(\overline{M}; \mathcal{V})/\mathcal{K}$ is invertible.

Usually it is difficult to check directly that $T + \mathcal{K}$ is invertible in $\mathfrak{A}(\overline{M}; \mathcal{V})/\mathcal{K}$, and, instead, one checks the invertibility of operators of the form $\pi(T)$, where π ranges through a suitable family of irreducible representations of $\mathfrak{A}(\overline{M}; \mathcal{V})/\mathcal{K}$. Exactly what are the needed properties of the family of representations of $\mathfrak{A}(\overline{M}; \mathcal{V})/\mathcal{K}$ was studied in [96]. Let us recall the main conclusions of that paper. Let us consider a family of representations $\mathcal{F}$ and $\pi \in \mathcal{F}$. It is not enough for the family $\mathcal{F}$

to be faithful. The necessary condition is that the family $\mathcal{F}$ be *full* [96], in the sense that every irreducible representation of $\mathfrak{A}(\overline{M};\mathcal{V})/\mathcal{K}$ is contained in one of the representations $\pi \in \mathcal{F}$.

This approach was used in [35, 41, 48, 49, 50, 67, 68, 86, 113], and in many other papers. However, in order for this approach to be effective, we need to have a structure theorem for the quotient $\mathfrak{A}(\overline{M};\mathcal{V})/\mathcal{K}$. This seems to be difficult in general, at least without using groupoids. Thus we shall replace the comparison algebra $\mathfrak{A}(\overline{M};\mathcal{V})$ with the norm closure of the algebra $\Psi^0_{\mathcal{V}}(M)$. The algebra $\Psi^0_{\mathcal{V}}(M)$ is defined in the next section. We shall also make some the following assumptions on the Lie manifold $(\overline{M}, A)$

(a) We denote by $(Z_\alpha)_{\alpha\in I}$ the family of *orbits* $Z_\alpha = \exp(\mathcal{V})p$ of $\mathcal{V}$ on $\partial\overline{M}$. Then, we assume that for each open face $F_0 \subset F = \overline{F_0}$ of $\overline{M}$, there exists a submersion $p_F : F_0 \to B_F$ of smooth manifolds (without boundary or corners) whose fibers are the orbits of $\mathcal{G}$ on F_0.

   (Note that each open face of $\overline{M}$ is invariant for $\exp\mathcal{V}$, and hence, if an orbit Z_α intersects F_0, then it is completely contained in F_0. In particular, the set of orbits I identifies with the disjoint union of the sets B_F for F ranging through the set of faces of $\overline{M}$: $I = \cup B_F$.)

(b) We assume that, for each open face $F_0 \subset F = \overline{F_0}$ of $\overline{M}$, there exists a Lie algebroid $A_F \to B_F$ with zero anchor map such that $p_F^*(A_F)$, the pull-back of A_F by the submersion p_F, satisfies

$$A|_{F_0} \simeq \ker(p_F)_* \oplus p_F^*(A_F)\,. \tag{43}$$

   That is, the restriction of A to the invariant subset F_0 is isomorphic to the direct sum of the pull-back of A_F with the set of vertical vector fields with respect to the submersion p_F. (Note however that the sections of $\ker(p_F)_*$ act by derivation on the sections of $p_F^*(A_F)$. Also, it follows that for any $p \in F_0$, the isotropy Lie algebra $\ker(\varrho_p)$ is canonically isomorphic to the Lie algebra $(A_F)_{p_F(p)}$, see Definition 3.2.)

(c) Let us denote by G_α the simply-connected Lie group that integrates the isotropy Lie algebra $\ker(\varrho_p) \simeq (A_F)_{p_F(p)}$ for any $p \in Z_\alpha$. Also, let us denote by $\mathcal{G}$ the disjoint union $\cup_{\alpha\in I} Z_\alpha \times Z_\alpha \times G_\alpha$ with the induced groupoid structure. Then we assume that the groupoid exponential map makes $\mathcal{G}$ an amenable, Hausdorff Lie groupoid [97]. In particular, $\mathcal{G}$ is a manifold (possibly with corners).

Under the above assumptions, the results in [13] and [67] give the following theorem.

Theorem 4.14. *We can associate to each $P \in \mathrm{Diff}(\mathcal{V}; E, F)$ a family of G_α-invariant operators P_α on $Z_\alpha \times G_\alpha$ such that:*

$$P \text{ is Fredholm} \;\Leftrightarrow\; P \text{ is elliptic and all } P_\alpha \text{ are invertible.}$$

This Theorem is closely related to the representations of Lie gropoids, see [20, 24, 23, 46, 47, 123, 58, 59, 108]. More general Fredholm conditions can be obtained along the same lines [13, 42, 67], but the result mentioned here, although having a rather long list of assumptions, is easy to prove and to use.

We continue with a few remarks. If $\overline{M}$ is compact and smooth (so without corners), then $I = \emptyset$, and we recover Theorem 1.2. As we will explain below, we also recover Theorem 1.8. Each operator P_α is "of the same kind" as P (Laplace, Dirac, ...) and can be recovered by "freezing the coefficients" at the orbit Z_α.

The theorem allows us to reduce some questions on M to questions on P_α and G_α. Because of the G_α-invariance of our operators, we can use results on harmonic analysis on G_α to obtain an inductive procedure to study geometric operators on M. References to earlier results will be given in the next section when discussing examples. We note that our assumptions on $(\overline{M}, A)$ imply that the groupoid $\mathcal{G}$ considered in our assumptions must coincide with the one introduced by Claire Debord [38, 39].

4.5. **Pseudodifferential operators on groupoids.** Let us recall, for the benefit of the reader, the definition of pseudodifferential operators on a Lie groupoid $\mathcal{G}$, because they were implicit in the proofs of some of the previous results. Namely, to a Lie groupoid $\mathcal{G}$ with units $\overline{M}$ there is associated an algebra $\Psi^\infty(\mathcal{G})$, whose operators of order m form a linear space denoted by $\Psi^m(\mathcal{G})$, $m \in \mathbb{R}$, see [89, 98]. Let $d : \mathcal{G} \to \overline{M}$ be the domain map and $\mathcal{G}_x = d^{-1}(x)$. Then $\Psi^m(\mathcal{G})$, $m \in \mathbb{R}$, consists of smooth families $(P_x)_{x\in\overline{M}}$ of classical, order m pseudodifferential operators ($P_x \in \Psi^m(\mathcal{G}_x)$) that are right invariant with respect to multiplication by elements of $\mathcal{G}$ and are "uniformly supported." To define what uniformly supported means, let us observe that the right invariance of the operators P_x implies that their distribution kernels K_{P_x} descend to a distribution $k_P \in I^m(\mathcal{G}, \overline{M})$ [86, 98]. The family $P = (P_x)$ is called *uniformly supported* if, by definition, k_P has compact support in $\mathcal{G}$.

Groupoids simplify the study of pseudodifferential operators on singular and non-compact spaces. For instance, one obtains a straightforward definition of the "generalized indicial operators" as restrictions to invariant subsets [67]. More precisely, let $N \subset \overline{M}$ be an invariant subset for $\mathcal{G}$, that is, $d^{-1}(N) = r^{-1}(N)$ and let $\mathcal{G}_N := d^{-1}(N)$. Let us now assume that $P \in \Psi^m(\mathcal{G})$ is given by the family $(P_x)_{x\in\overline{M}}$, then the N–indicial family $\mathcal{I}_N(P) := (P_x)_{x\in N}$ is defined simply as the restriction of P to N and is in $\Psi^m(\mathcal{G}_N)$. See [43] for an extension of these results in relation to the adiabatic groupoid. See also [1, 18, 81] for results on the Boutet-de-Montvel calculus in the framework of groupoids.

There are many works dealing with pseudodifferential operators on groupoids, on singular spaces, or with the related C^*-algebras, see for example [2, 15, 32, 52, 73, 91, 101, 79, 119, 120, 124].

5. Examples and applications

We now discuss some applications. They are included just to give an idea of the many possible applications of Lie manifolds, so we will be short, but we refer to the existing literature for more details. We begin with some examples.

5.1. **Examples of Lie manifolds and Fredholm conditions.** We now include examples of Lie manifolds and show how to use Theorem 4.14. The following examples cover many of the examples appearing in practice.

Example 5.1. We now review our first, basic example, Example 3.8, in view of the new results. Recall that $\mathcal{V} = \mathcal{V}_b :=$ the space of vector fields on $\overline{M}$ that are tangent to $\partial\overline{M}$. Near the boundary, a local basis is given by Equation (29) of Example 3.8, and hence $\mathrm{Diff}(\mathcal{V}_b)$ is the algebra of totally characteristic differential operators. If $\overline{M}$ has a smooth boundary and we denote by r the distance to the boundary (in some metric smooth everywhere), then a typical compatible metric on M is given near the boundary by $(r^{-1}dr)^2 + h$, where h is a metric smooth up to the boundary. Hence the geometry is that of a manifold with cylindrical ends.

We have that the orbits Z_α are the open faces of $\overline{M}$, except M itself. The group $G_\alpha \simeq \mathbb{R}^k$, where k is the codimension of the corresponding face (so it is a commutative Lie group). In the case of a smooth boundary, the Z_α's are the connected components of the boundary, $G_\alpha = \mathbb{R}$, and P_α is the restriction of $\mathcal{I}(P)$ to a translation invariant operator on $Z_\alpha \times \mathbb{R}$. See also [87, 89, 83, 84, 71, 114] for just a sample of the many papers on this particular class of manifolds.

In the following examples, $\overline{M}$ will be a compact manifold with smooth boundary $\partial\overline{M}$. The following example is that of an asymptotically hyperbolic space and has the feature that it leads to non-commutative groups G_α.

Example 5.2. Let $\overline{M}$ be a compact manifold with smooth boundary $\partial\overline{M}$ and defining function r. We proceed as in Example (3.8). The structural Lie algebra of vector fields is $\mathcal{V} = r\Gamma(T\overline{M})$ = the space of vector fields on $\overline{M}$ that vanish on the boundary. Using the same notation as in the Example (3.8), near a point of the boundary $\partial\overline{M} = \{r = 0\}$, a local basis is given by

$$r\partial_r,\, r\partial_{y_2},\, \ldots,\, r\partial_{y_n}\,, \tag{44}$$

so $\mathcal{V}$ is a finitely generated, projective $\mathcal{C}^\infty(\overline{M})$–module. Since $\mathcal{V}$ is also closed under the Lie bracket and $\Gamma_c(M; TM) \subset \mathcal{V} \subset \mathcal{V}_b$, we have that $(\overline{M}, \mathcal{V})$ defines indeed a Lie manifold.

The orbits $Z_\alpha \subset \partial\overline{M}$ are reduced to points, so $\alpha \in I := \partial\overline{M}$, and $G_\alpha = T_\alpha\partial\overline{M} \rtimes \mathbb{R}$ is the semi-direct product with $\mathbb{R}$ acting by dilations on the vector space $T_\alpha\partial\overline{M}$. The pseudodifferential calculus $\Psi^*_{\mathcal{V}}(M)$for this example was defined by Lauter [64], Lauter-Moroianu [65], Mazzeo [80], and Schulze [115]. The metric is *asymptotically hyperbolic.*

The following example covers, in particular, $\mathbb{R}^n$ with the usual Euclidean metric and with the radial compactification.

Example 5.3. As in the previous example, $\overline{M}$ is a compact manifold with smooth boundary $\partial\overline{M} = \{r = 0\}$. We shall take now $\mathcal{V} = r\mathcal{V}_b$ = the space of vector fields on $\overline{M}$ that vanish on the boundary $\partial\overline{M}$ and whose normal covariant derivative to the boundary also vanishes. Using the same notation as in the previous two examples, at the boundary $\partial\overline{M}$, a local basis is given by

$$r^2\partial_r,\, r\partial_{y_2},\, \ldots,\, r\partial_{y_n}\,. \tag{45}$$

Again the orbits Z_α are reduced to points, so $\alpha \in I := \partial\overline{M}$, but this time $G_\alpha = T_\alpha\overline{M} = T_\alpha\partial\overline{M} \times \mathbb{R}$ is commutative. The pseudodifferential calculus goes back to Parenti [100]. See also [85, 113]. If $\partial\overline{M} = S^{n-1}$, the resulting geometry is that of an asymptotically Euclidean manifold. In particular, $\mathbb{R}^n$ with the radial compactification fits into the framework of this example.

Example 5.4. As in the previous two examples, $\overline{M}$ is a compact manifold with smooth boundary $\partial\overline{M} = \{r = 0\}$. To construct our Lie algebra of vector fields $\mathcal{V} = \mathcal{V}_e$, we assume that we are given a smooth fibration $\pi : \partial M \to B$, and we let $\mathcal{V}_e$ to be the space of vector fields on $\overline{M}$ that *are tangent* to the fibers of $\pi : \partial\overline{M} \to B$. By choosing a product coordinate system on a small open subset of the boundary, a local basis is then given by

$$r\partial_r,\, r\partial_{y_2}, \ldots,\, r\partial_{y_k},\, \partial_{y_{k+1}},\, \ldots,\, \partial_{y_n}\,. \tag{46}$$

Here k is such that the fibers of $\pi : \partial\overline{M} \to B$ have dimension $n-k$. Thus, when $k=1$ (so the fibration is over a point, that is, $\pi : \partial\overline{M} \to pt$), we recover our first example, Example 5.1. On the other hand, when $k=n$ (so the fibration is $\pi : \partial\overline{M} \to \partial\overline{M}$), we recover our second example, Example 5.2. For $n=3$ and $k=2$, we recover the edge differential operators of Example 2.2 (3) (see Equation (26)). We note that $\mathcal{V} := \mathcal{V}_e \subset \mathcal{V}_b =: \mathcal{W}$ yields a typical example of an anisotropic stucture.

In general, in this example, the set of orbits is $I = \{\alpha\} = B$, $Z_\alpha = \pi^{-1}(\alpha)$, and $G_\alpha = T_\alpha B \rtimes \mathbb{R}$ is a solvable Lie group with $\mathbb{R}$ acting by dilations. The geometry is related to that of locally symmetric spaces. Differential operators of this kind appear in the study of behavior at the edge of boundary value problems. This example generalizes the second example (Example 5.2) and the same references are valid for this example as well.

We conclude with some less standard examples.

Example 5.5. Let us assume that we are in the same framework as in the previous example, Example, 5.4, but we replace the fibration of $\partial\overline{M}$ with a foliation. Then the resulting Lie manifold may fail to satisfy Theorem 4.14. See however [109]. It is interesting to notice that in this case, the resulting class of Riemann manifolds lead naturally to the study of foliation algebras.

Our last example in this subsection is on a manifold with corners.

Example 5.6. Let $A \to \overline{M}$ be a Lie algebroid (we do not assume $\Gamma(\overline{M}; A) \subset \mathcal{V}_b$) and let $\phi : \overline{M} \to [0,\infty)$ be a smooth function such that $\{\phi = 0\} = \partial\overline{M}$. We define $\mathcal{V} := \phi\Gamma(\overline{M}; A)$. Then $(\overline{M}, \mathcal{V})$ defines a Lie manifold.

5.2. **Index theory.** Let now $(\overline{M}, A)$ be a Lie manifold and let f be the product of the defining functions of all its faces. We consider then the exact sequence

$$0 \to f\Psi^{-1}(\mathcal{G}) \to \Psi^0(\mathcal{G}) \to \text{Symb} \to 0\,, \tag{47}$$

which gives rise as before to the map $\partial : K_1(\text{Symb}) \to K_0(I)$. The **Fredholm index problem** is in this case to compute

$$Tr_* \circ \partial : K_1(\text{Symb}) \to \mathbb{Z}\,.$$

Since $\phi_* \circ \partial = \psi_*$, where $\psi = \partial\phi \in \text{HP}^1(\text{Symb})$, by Connes' results, the *Fredholm index problem* is equivalent to computing the class of ψ in periodic cyclic homology. This is a difficult problem that is still largely unsolved. Undoubtly, excision in cyclic theory will play an important role [34]. See also [25, 26, 93, 94, 72, 27, 104, 105, 102, 103, 110, 122]. Instead of this general problem, we shall look now at a particular, but relevant case [21, 22].

Definition 5.7. We say that a be a Lie manifold $(\overline{M}, \mathcal{W})$ is *asymptotically commutative* if all vectors in $\mathcal{W}$ vanish on $\partial\overline{M}$ and all isotropy Lie algebras $\ker(q_x)$ are commutative.

Let $x_1, x_2, \ldots, x_k$ be the defining functions of all the hyperfaces of $\overline{M}$ and $f = x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k}$ for some positive integers a_j. Then, for any Lie manifold $(\overline{M}, \mathcal{V})$ $\mathcal{W} := f\mathcal{V}$ defines an asymptotically commutative Lie manifold $(\overline{M}, \mathcal{W})$.

If $(\overline{M}, \mathcal{W})$ is asymptotically commutative, then the algebra Symb is commutative. Its completion will be of the form $C(\Omega)$, as in the work of Cordes [31, 30].

Since the algebra Symb is commutative, it is possible then to compute the index of Fredholm operators using classical invariants [22]. As an application, one obtains also the index of Dirac operators coupled with potentials of the form $f^{-1}V_0$, where V_0 is invertible at infinity on any Lie manifold (not just asymptotically commutative) [22].

5.3. **Essential spectrum.** We now present some applications to essential spectra. We use the notation introduced in subsection 4.3. The applications to essential spectra of operators are based on the fact that for a self-adjoint operator D affiliated to $\mathfrak{A}(M, \mathcal{V})$ (i.e. $(D+\imath)^{-1} \in \mathfrak{A}(M, \mathcal{V})$ [35, 48]), we have that

$$\lambda \in \sigma_{ess}(D) \ \Leftrightarrow \ D - \lambda \text{ is not Fredholm.} \tag{48}$$

Then we can use Theorem 4.14 to study when $D - \lambda$ is (or is not) Fredholm.

We shall use these ideas for $\overline{M}$ a manifold with corners and $(\overline{M}, \mathcal{V}_b)$ the Lie manifold of Example (5.1). Let Δ_M be the associated positive Laplacian, then [69]

Theorem 5.8. *Let M be the interior of $(\overline{M}, \mathcal{V}_b)$ with the induced compatible metric. Then*

$$\sigma(\Delta_M) \,=\, [0, \infty)$$

A complete characterization of the spectrum (multiplicity of the spectral measure, discreteness of the point spectrum, absence of continuous singular spectrum) is wide open, in spite of its importance.

Similarly, let $\not{D}$ be the Dirac operator associated to a $Cliff(A)$-bundle over $\overline{M}$. Then [95]

Theorem 5.9. *The Dirac operator $\not{D}$ on $M = \overline{M} \smallsetminus \partial\overline{M}$ is invertible if, and only if, the Dirac operator $\not{D}_F$ associated to any open face F of $\overline{M}$ (including M), has no harmonic spinors (that is, it has zero kernel).*

The proof uses Theorem 4.14 and the fact that the resulting operators P_α are also Dirac operators.

Many similar results were obtained in Quantum Mechanics by Georgescu and his collaborators [35, 48, 50]. In fact, certain problems related to the N–body problem can be formulated in terms of a suitable compactifications of $X := \mathbb{R}^{3n}$ to a manifold with corners $\overline{M}$ on which X still acts and such that the Lie algebra of vector fields $\mathcal{V}$ is obtained from the action of X [49]. See also [44].

5.4. **Hadamard well posedness on polyhedral domains.** This type of application [12] is of a different nature and does not use pseudodifferential operators or other operator algebras. It uses only Lie manifold and their geometry. Let then $\Omega \subset \mathbb{R}^n$ be an *open, bounded* subset of with boundary $\partial\Omega$. We shall consider the "simplest" boundary value problem on Ω, the *Poisson problem* with *Dirichlet* boundary conditions:

$$\begin{cases} -\Delta u = f \\ \ \ u|_{\partial\Omega} = 0. \end{cases} \tag{49}$$

We refer to [12] for further references and details not included here. Recall then the following classical result, which we shall refer to as the *basic well-posedness theorem (for Δ on smooth domains)*

Theorem 5.10. *Let us assume that $\partial\Omega$ is smooth. Then the Laplacian Δ defines an isomorphism*

$$\Delta : H^{s+1}(\Omega) \cap \{u|_{\partial\Omega} = 0\} \to H^{s-1}(\Omega), \quad s \geq 0\,.$$

A useful consequence (easy to contradict for non-smooth domains) is:

Corollary 5.11. *If f and $\partial\Omega$ are smooth, then the solution u of the Poisson problem with Dirichlet boundary conditions is also smooth.*

It has been known for a very long time that the basic well posedness theorem *does not extend* to the case when $\partial\Omega$ is *non-smooth*. This can be immediately seen from the following example.

Example 5.12. Let us assume that Ω is the unit square, that is $\Omega = (0,1)^2$. If u is smooth, then $\partial_x^2 u(0,0) = 0 = \partial_y^2 u(0,0)$, and hence $f(0,0) = \Delta u(0,0) = 0$. By choosing $f(0,0) \neq 0$, we will thus obtain a solution u that is not smooth.

In view of the many practical applications of the basic well-posedness theorem, we want to extend it in some form to non-smooth domains. Assume now $\Omega \subset \mathbb{R}^n$ is a *polyhedral domain.* Exactly what a polyhedral domain means in three dimensions is subject to debate. In this presentation, we shall use the definition in [12] in terms of stratified spaces (we refer to that paper for the exact definition). Note that a version of that paper was first circulated in 2004 as an IMA preprint. The key technical point in that paper is to replace the classical Sobolev spaces $H^m(\Omega)$, introduced in Equation (37) with weighted versions as in Kondratiev's paper [62]. Let us then denote by ρ the distance function to the singular part of the boundary and define

$$\mathcal{K}_a^m(\Omega) := \{u,\, \rho^{|\alpha|-a}\partial^\alpha u \in L^2(\Omega),\ |\alpha| \leq m\}\,.$$

(Notice the appearance of the factor ρ!) Thus, in two dimensions, $\rho(x)$ is the distance from $x \in \Omega$ to the vertices of Ω, whereas in three dimensions, $\rho(x)$ is the distance from $x \in \Omega$ to the set of edges of Ω.

Theorem 5.13. *Let $\Omega \subset \mathbb{R}^n$ be a bounded polyhedral domain and $m \in \mathbb{Z}_+$. Then there exists $\eta > 0$ such that*

$$\Delta\, :\, \mathcal{K}_{a+1}^{m+1}(\Omega) \cap \{u|_{\partial\Omega} = 0\} \ \to\ \mathcal{K}_{a-1}^{m-1}(\Omega)\,,$$

is an isomorphism for all $|a| < \eta$.

In two dimensions, this result is due to Kondratiev [62].

The proof of Theorem 5.13 is based on a study of the properties of a *Lie manifold with boundary* $\Sigma(\Omega)$ canonically associated to Ω by a *blow-up* procedure. The weighted Sobolev spaces $\mathcal{K}_a^m(\Omega)$ can be shown to coincide with the usual Sobolev spaces associated to $\Sigma(\Omega)$. See [4] for the definition of Lie manifolds with boundary. General blow-up procedures for Lie manifolds were studied in [3]. It can be shown that the class of Lie manifolds satisfying Theorem 4.14 is closed under blow-ups with respect to tame Lie submanifolds. Since most practical applications deal with Lie manifolds that are obtained by such a blow-up procedure from a smooth manifold, that establishes Theorem 4.14 in most cases of interest.

The blow-up procedure is an inductive procedure that consists in successively replacing cones of the form $CL := [0,\epsilon) \times L/(\{0\} \times L)$ with their associated cylinders $[0,\epsilon) \times L$.

No well posedness result similar to Theorem 5.13 holds for the *Neumann problem* (normal derivative at the boundary is zero):

$$\begin{cases} -\Delta u = f \\ \partial_\nu u = 0\,, \end{cases} \tag{50}$$

where ν is a continuous unit normal vector field at the boundary. In fact, in three dimensions, the above problem is never Fredholm.

Here is however a variant of Theorem 5.13 that has been proved proved useful in practice. Let us consider a polygonal domain Ω and a function $\chi_P \in \mathcal{C}^\infty(\mathbb{R}^2)$ that is equal to 1 around the vertex P, depends only on the distance to P, and has small support. Let W_s be the linear span of the functions χ_P, where P ranges through the set of vertices of Ω. Let $\{1\}^\perp$ be the space of functions with integral zero. Then we have the following result [74].

Theorem 5.14. *Let $\Omega \subset \mathbb{R}^2$ be a connected, bounded polygonal domain and $m \in \mathbb{Z}_+$. Then there exists $\eta > 0$ such that*

$$\Delta\,:\,\Big(\mathcal{K}^{m+1}_{a+1}(\Omega) \cap \{\partial_\nu u|_{\partial\Omega} = 0\}\,+\,W_s\Big) \cap \{1\}^\perp \,\to\, \mathcal{K}^{m-1}_{a-1}(\Omega) \cap \{1\}^\perp\,,$$

is an isomorphism for all $0 < a < \eta$.

The proof of this theorem is based on an index theorem on polygonal domains, more precisely, a relative index theorem as follows.

Proof. (Sketch) Let us denote by Δ_a the operator for the fixed value of the weight a. Then one knows by [62] (or an analysis similar to the one needed for the APS index formula), that Δ_a is Fredholm if, and only if, $a \neq k\pi/\alpha$, where $k \in \mathbb{Z}$ and α ranges through the values of the angles of our domain Ω. (For the Dirichlet problem one has a similar condition for a, except that $k \neq 0$.) One sees that Δ_0 is not Fredholm, but one can compute the relative index $\mathrm{ind}(\Delta_a) - \mathrm{ind}(\Delta_{-a}) = -2n$, $a > 0$ small, where n is the number of vertices of Ω (anyone familiar with the APS theory will have no problem proving this crucial fact). By definition $\Delta_a^* = \Delta_{-a}$, and hence this gives $\mathrm{ind}(\Delta_a) = -n$. A standard energy estimate shows that $\ker(\Delta_a) = 1$ for $a > 0$, with the kernel given by the constants. This is enough to complete the proof. □

Theorems 5.13 and 5.14 have found applications to optimal rates of convergence for the Finite Element Method in two and three dimensions [11], where optimal rates of convergence in three dimensions were obtained for the first time.

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