GEODESIC FLOWS

ON RATIONAL POLYHEDRA

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<u>Introduction</u> Geodesic flows on Euclidean polyhedra is an old subject that goes as far back as 1906 (see [4] and a related paper [3]). An example of such flow is the motion of a billiard ball inside a polygon. If the angles of the polygon are rational multiples of π, the direction of any geodesic takes only a finite number of values as time varies. Fixing these values one obtains invariant "surfaces" of the geodesic flow ([6]).

Billiards in rational polygons are interesting because they are so close to integrable Hamiltonian systems (with two degrees of freedom). In fact when the numerators of angles of the polygon P are equal to one the billiard flow is integrable. The Dirichlet problem in P corresponds to the quantum billiard in the polygon. When P is as above eigenfunctions of the Dirichlet problem are explicitly known, so the quantum billiard is integrable as well (unfortunately such polygons are scarce).

It is natural to ask how much of what is known for billiards in integrable polygons remains true for general rational polygons. The reader can find some results in this direction in [6] and [2] for classical and quantum billiards respectively.

The purpose of this paper is to clarify the structure of the phase space, how it splits into invariant surfaces and the structure of invariant surfaces themselves for general rational polyhedra. Those are Euclidean polyhedra which are topological surfaces (possibly with boundary) and whose vertex angles are rational multiples of π (Definition 3). A rational polyhedron S naturally defines an orbifold S_O (see [5], Ch. 13 for the definition of orbifold). The geodesic flow on S naturally lives in the unit circle bundle $T(S_O)$. The Eudlidean metric on S defines parallel translations in $T(S_O)$. The holonomy group of S is a subgroup of $O(2)$. Since S is rational the restricted holonomy group (corresponding to contractible loops) is a finite subgroup H of $O(2)$. If $\pi_1(S) = 0$ there is a natural mapping $f: T(S_O) \to C/H$ (C = unit circle) invariant under the geodesic flow (Theorem 1). The level sets of f are the invariant surfaces R_θ (their existence in case $\partial S = \emptyset$ was noticed in [1]).

Obviously the most general assumption for Theorem 1 would be that the holonomy group of S is finite. This will certainly hold if $\pi_1(S)$ is finite which means if $\pi_1(S) \neq 0$ that S is homeomorphic to the real projective plane. All results of this paper can be easily extended to this case.

The topology of invariant surfaces R_θ is the subject of Theorem 1, Corollary 2 to Proposition 2 and Theorem 4. If $\partial S = \emptyset$ then all R_θ are homeomorphic. If $\partial S \neq \emptyset$ then R_θ are homeomorphic for interior points θ of the parameter space $[0, \pi/n]$. The typical level surface R is orientable, closed and its genus is determined by the vertex angles of S (formula 3). There is a natural complex structure on R and S such that the holonomy group H acts on R by conformal and anticonformal transformations and $S = R/H$. If $\partial S \neq \emptyset$ then H is the dihedral group D_n and the reflections of D_n act anticonformally on R. The two exceptional invariant surfaces R_0 and R_1 are the quotients of R by two basic reflections s_0 and s_1 respectively. Theorem 4 determines the topology of R_0 and R_1 as far as it can be done from the vertex angles of S.

It is interesting to remark that if S is a polygon $T(S_0)$ is homeomorphic to the 3-sphere S^3 and the topology of decompositions $S^3 = \cup R_\theta$ $0 \leq \theta \leq 1$ is well known to knot-theorists.

If the genus of R is greater than one, by uniformization theorem there is a discrete group Γ acting freely on the Poincare plane H such that $R = H/\Gamma$. In particular R has two conformal metrics. One is pulled back from S, its curvature is the δ-function supported at the vertices of R (formula 3 is essentially the Gauss-Bonnet theorem for this singular metric). The other is Poincare metric of curvature minus one. Theorem 3 shows that there is a Γ-equivariant developing map $\varphi: H \to \mathbb{C}$ which provides a relation between the two metrics.

In section 3 I classify rational polyhedra for which the geodesic flow is integrable (flat polyhedra). Tetrahedra with vertex angles π provide an interesting family of flat polyhedra (Theorem 7). The last section consists of examples.

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1. Invariant surfaces.

Let $\mathbb{C}$ be the complex plane with the standard metric $dzd\bar{z}$. A Euclidean polygon is a closed simple polygon in $\mathbb{C}$.

Definition 1: A Euclidean polyhedron S (of dimension 2) is a finite collection of Euclidean polygons, where some polygons are glued together along certain sides by isometries.

For example any Euclidean polygon is a Euclidean polyhedron. Two copies of the same polygon glued along corresponding sides make a polyhedron which is a topological sphere. Any Euclidean polyhedron S defines a topological space which is denoted by S_T. In this paper we consider polyhedra which are connected topological surfaces, possibly with boundary. Henceforth we simply call them polyhedra or polyhedral surfaces.

If S is a polyhedron made of polygons P_i we call P_i the faces of S, their edges and vertices are called edges and vertices of S.

The unit tangent bundle $T(P)$ of a polygon is the set of unit vectors in $\mathbb{C}$ with the base points in P and pointing into P. The unit tangent bundle $T(S)$ of a polyhedron is made out of $T(P_i)$ where P_i are faces of S, with obvious identifications.

Definition 2: Let S be a polyhedral surface. The geodesic flow on S takes place in T(S) and is modeled on the motion of the billiard ball on S. The ball goes straight when inside a face. When it hits an interior edge the ball passes to the adjacent face and it bounces off boundary edges. We temporarily agree to stop the ball when it hits a vertex of S.

The set T(S) has a natural topology and a natural measure. According to our definition of the geodesic flow on S the lifetime of some trajectories is finite. It can be shown that elements $v \in T(S)$ that generate finite lifetime trajectories form a set of measure zero.

The angle a of a vertex A of S is the sum $\alpha = \alpha_1 + \ldots + \alpha_k$ of plane angles of faces containing A.

<u>Definition 3</u>: A polyhedral surface S is called rational if the angles at interior (boundary) vertices of S are of the for $2\pi r$ (πr) where r is a rational number depending on the vertex.

Let S be a rational polyhedron. We associate to S an orbifold S_O ([5], ch. 13) which coincides with S as a topological space. The isotropy group G_x is nontrivial if $x \in \partial S$ or if x is a vertex. The isotropy group of a boundary edge b is generated by the reflection about b. If x is a bounda vertex with the angle $\pi\, m/n$ (m,n are relatively prime) then G_x i the dihedral group D_n naturally acting in $\mathbb{C}$. If x is an interior vertex with the angle $2\pi m/n$ then $G_x = \mathbb{Z}/_n$ acting by rotations in $\mathbb{C}$.

The tangent unit sphere bundle $T(S_0)$ is a topological 3-fold and there is a natural continuous mapping $\varphi: T(S) \to T(S_0)$ so that the diagram below is commutative.

$$\begin{array}{ccc} T(S) & \xrightarrow{\varphi} & T(S_0) \\ p\downarrow & & \downarrow p_0 \\ S & \xrightarrow{id} & S_0 \end{array} \tag{1}$$

φ maps fibers into fibers and it is not an isomorphism only at the vertices with angles $2\pi m/n$ ($\pi m/n$) where it is m-to-1.

<u>Theorem 1:</u> Let S be a rational connected simply connected polyhedral surface and let n be the least common multiple of the denominators n_i of the vertex angles of S. We set $H = D_n$ if $\partial S \neq \emptyset$ and $H = \mathbb{Z}/_n$ otherwise and we consider H as a subgroup of the group $O(2)$ of isometrics of the circle C. Let A_i $i = 1,\ldots,\nu$ be the vertices of S.

1. There exists a function f on $T(S \setminus \{A_i \; i = 1,\ldots,\nu\})$ with values in C/H which is invariant with respect to the geodesic flow on $T(S)$ and uniquely extends to a continuous function on $T(S_0)$.

2. For any $\theta \in C/H$ the level set $R_\theta \subset T(S_0)$ given by the equation $f(v) = \theta$ is a compact connected topological

surface. If $\partial S = \emptyset$ then all the surfaces R_θ are homeomorphic. If $\partial S \neq \emptyset$ then $C/H = [0,\pi/n]$ and S_θ are homeomorphic to each other for $0 < \theta < \pi/n$.

3. Denote by R that typical level surface in $T(S_0)$. R is a closed orientable surface and the restriction of the projection $p:T(S_0) \to S$ to R is a ramified regular covering with the group H of deck transformations. If $\partial S \neq \emptyset$ the exceptional surfaces $R_0 (\theta = 0)$ and $R_1 (\theta = \pi/n)$ may have boundary. The projections $p:R_i \to S$ $i = 0,1$ are ramified coverings of order n.

Proof: 1. Choose a reference point x_0 inside a face of S and identify the fiber $T(S)_{x_0}$ with C. For any piecewise smooth curve γ on S going from x_0 to x and avoiding vertices we will define the parallel tranlation $L_\gamma: C \to T(S)_x$ along γ. Parametrize γ by $[0,1]$ and let $0 < t_1 < \ldots < t_n < 1$ be times when γ hits an edge of S. Denote by $v(t)$ the vector we translate along γ ($v(0) = v$ is the initial condition). For $t_{i-1} < t < t_i$, $v(t)$ is sitting inside a face $P_i \subset \mathbb{C}$ which determines the translation. Let $v(t_i)$ be on an edge b. If $b \in \partial S$ then $v(t_i)$ bounces off by the reflection in b. If b is an interior edge, we unfold P_i and the adjacent polygon P_{i+1} along b on $\mathbb{C}$ and continue $v(t_i)$ into P_{i+1} parallel to itself. This defines $v(t)$ for all t and we set $L_\gamma v = v(1)$.

If we change γ by a homotopy (leaving the ends fixed) L_γ changes only when γ crosses a vertex A or contacts (looses contact) with a boundary edge b. At these moments $L_\gamma v$ instanteneously rotates by the angle a at A or reflects about b respectively. Hence for $v \in T(S)_x$, $L_\gamma^{-1} v \in C$ changes by the action of H under any homotopy of γ. Thus the image $L_\gamma^{-1} v \bmod H \in C/H$ does not depend on the choice of γ and we set $f(v) = L_\gamma^{-1} v \bmod H$. It is obvious from the definition that f is invariant under the geodesic flow.

The function f extends by continuity to all of $T(S)$. Let A be an interior vertex of S with the angle $2\pi p/q$. The angular function on $T(S)_A$ defined as a limit from $T(S)_x$ as $x \to A$ will take values in $[0,2\pi/q) = C/(\mathbb{Z}/q)$, so it can be pushed down to $T(S_0)_A$. Thus f which is obtained from the angular function by further moding out $C/(\mathbb{Z}/q) \to C/(\mathbb{Z}/n)$ is well defined on $T(S_0)_A$. Analogous argument goes for a boundary vertex. The pushed down function f on $T(S_0)$ is continuous by construction and it is invariant under the geodesic flow.

2 and 3. The level sets $R_\theta \subset T(S_0)$ are closed hence compact, invariant under the geodesic flow and we have the projection $p: R_\theta \to S$ onto S for each θ. Assume first that $\partial S = \emptyset$. If $x \in S$ and x is not a vertex the parallel translation provides an isomorphism of $T(S)_x$ onto C which is unique

modulo the action of $\mathbb{Z}/n$ on C. Thus $R_\theta \cap T(S)_x$ consists of n points. Let P be a face of S. We can choose isomorphisms $\varphi_x : T(S)_x \to C$ for $x \in P \setminus \{\text{vertices}\}$ in a coherent way. Thus $p^{-1}(P \setminus \{\text{vertices}\}) \cap R_\theta$ is a disjoint union of n copies of $P \setminus \{\text{vertices}\}$ and $p: R_\theta \to S$ is an unramified n-sheeted covering over the complement to the vertices of S.

Let A be a vertex of S with the angle $2\pi i/j$. When we translate $v \in T(S)$ along a small loop around A it rotates by $2\pi i/j$ which corresponds to $2\pi/j$ in $T(S_0)$. Thus we have to go around A j times to close the loop in R_θ. So $p^{-1}(A) \cap R_\theta$ consists of n/j points and at each of them p is ramified with the branching number j. If D is a small disc around A then $p^{-1}(D) \cap R_\theta$ is a disjoint union of n/j discs thus R_θ is a closed surface which is obviously connected and orientable.

The rotation group SO(2) naturally acts on C and on $C/(\mathbb{Z}/n)$. Since SO(2) is abelian, by parallel translation we make SO(2) act on $T(S_0)$. If $a \in SO(2)$ and ρ_α is the corresponding rotation of $T(S_0)$ then by construction $\rho_\alpha : R_\theta \to R_{\theta+\alpha}$ is a homeomorphism. In particular $\mathbb{Z}/n$ acts on R_θ by deck transformations and the quotient is clearly S.

Let now $\partial S \neq \emptyset$, so S is homeomorphic to the closed disc. Let mS denote the polyhedral surface which is obtained by taking two copies of S and glueing them naturally along the boundary. This operation is called the doubling of S. Let σ be the natural orientation reversing involution of mS, let

φ:mS $\to$ S be the natural projection and let $\varphi_*:T(mS) \to T(S)$,

$\sigma_*:T(mS) \to T(mS)$ be the induced mappings.

The action ρ_α of SO(2) on T(mS) is normalized by σ_* and $\sigma_*\rho_\alpha\sigma_* = \rho_{-\alpha}$. The group O(2) is generated by SO(2) and the reflection s about π/n (or any other reflection). The homomorphism $\alpha \to \rho_\alpha$, $s \to \sigma_*$ defines an action of O(2) on T(mS) which preserves the fibration of T(mS) into invariant surfaces $\tilde{R}_\theta$. The induced action on the parameter space $C/(\mathbb{Z}/n)$ is isomorphic to the natural action of O(2) on $C/(\mathbb{Z}/n)$. In particular the subgroup D_n preserves $\tilde{R}_\theta \cup \tilde{R}_{2\pi/n-\theta}$ which for $\theta \neq 0$, π/n is a disjoint union and $\varphi_*^{-1}(R_\theta)$. Hence $R = \tilde{R}$. For $\theta = 0$, π/n the surfaces in the union coincide and we obtain actions of D_n on $\tilde{R}_0 = \varphi_*^{-1}(R_0)$ and $\tilde{R}_{\pi/n} = \varphi_*^{-1}(R_{\pi/n})$ respectively. The actions are isomorphic by the rotation $\rho_{\pi/n}:\tilde{R}_0 \to \tilde{R}_{\pi/n}$ and thus define the action of D_n on R such that $R/D_n = S$. Denote by s_0, s_1 two basic reflections that generate D_n. By our construction $R_i = R/\{s_i\}$ $i = 0,1$ so $R_i \to S$ are n-sheeted (not regular) coverings.

Corollary: If $\partial S = \emptyset$ the covering $p:R \to S$ is ramified above the vertices of S. The branching number at a vertex A with angle $2\pi i/j$ is j. The surface R has a natural structure of a Euclidean polyhedron so that $p:R \to S$ is a covering of polyhedral surfaces. The angle at a vertex $\tilde{A}$ of R above

$A \in S$ with angle $2\pi i/j$ is $2\pi i$. The geodesic flow on $T(R)$ goes along invariant surfaces which are isomorphic to R_θ via the projection $p_*: T(R) \to T(S)$. Analogous statements hold when $\partial S \neq \emptyset$.

The corollary has been proven in the course of proof of Theorem 1.

Definition 4: A singular conformal metric g on a Riemann surface S is called almost flat if

1. g is nonsingular and flat on S with a finite number of punctures.

2. At any singular point A the metric has the form $g = C|z|^{\alpha-1}dz d\bar{z}$ (α and C can depend on A).

Proposition 2: Let S be a compact connected polyhedral surface with interior vertices A_i $i = 1,\ldots,\mu$ with angles α_i and boundary vertices A_j $j = \mu+1,\ldots,\mu+\nu$ with angles α_j. Then the Euler characteristic of S is given by

$$(2) \qquad \chi(S) = \Sigma_{i=1}^{\mu}\left(1 - \alpha_i/2\pi\right) + (1/2)\Sigma_{j=\mu+1}^{\mu+\nu}\left(1 - \alpha_j/\pi\right)$$

If S is oriented there is a canonical complex structure on S which makes S a Riemann surface (with boundary if $\partial S \neq \emptyset$) and there is a canonical almost flat metric g on S whose singular support is contained in the set of vertices of S.

Proof: By definition, $\chi(S) = \Sigma_\sigma(-1)^{\dim\sigma}$ where the summation is over the cells of S. Let P be a face of S with p

sides. The contribution to $\chi(S)$ from the edges of P is $-p = -\pi^{-1}\Sigma_{k=1}^{p}\beta_k - 2$ where β_k are the plane angles of P. Let $\partial S = \emptyset$. Summing up over all faces and noticing that each edge will appear twice we get for the edge portion of $\chi(S)$

$-(2\pi)^{-1}\Sigma\beta_k - |\text{faces}|$ where the summation is over all plane angles of S. Since $\Sigma\beta_k = \Sigma_{i=1}^{\mu}\alpha_i$ we obtain (2). If $\partial S \neq \emptyset$ consider the doubling mS and notice that $\chi(mS) = 2\chi(S)$. Comparing vertices of S and mS we immediately get the general formula (2).

Let S be orientable and let $\partial S = \emptyset$. We will define a complex coordinate patch $\{(U_\sigma, z_\sigma)\}$ of S where σ runs through the set of cells of S. For any σ we define U_σ to be the interior of the union of polygons containing σ. If $\dim \sigma = 1$ or 2, U_σ is an open subset of $\mathbb{C}$ which defines z_σ. Let σ be a vertex with angle $2\pi\alpha$ and let $P_1, \ldots, P_n$ be polygons containing σ. Cut U_σ along an edge and unfold it into $\mathbb{C}$ around σ. Choose coordinate z in $\mathbb{C}$ so that σ corresponds to $z = 0$ and the cut goes along the positive real axis. The function $z_\sigma = z^{1/\alpha}$ is well defined on U_σ and gives an embedding $U_\sigma \to \mathbb{C}$. It is straightforward to check that the transition functions of this covering are complex-analytic.

If $\partial S \neq \emptyset$ then the doubled surface mS has a complex structure. The canonical involution s of mS is an orientation reversing automorphism of the polyhedral surface mS hence s is anticonformal in the canonical complex structure on mS. Thus $S = mS/\{s\}$ is a Riemann surface with boundary.

Define the conformal metric g_σ in U_σ by $g_\sigma = dz_\sigma d\bar{z}_\sigma$ if $\dim \sigma > 0$ and by $g_\sigma = |\alpha|^2 (z_\sigma \bar{z}_\sigma)^{\alpha-1} dz_\sigma d\bar{z}_\sigma$ if σ is a vertex wi[th] the angle $2\pi\alpha$. We will show that local metrics g_σ coincide in the overlappings. Let (U_1, z_1) correspond to a vertex with angle α and let (U_2, z_2) correspond to a polygon containing the vertex. Then $z_1 = z_2^{1/\alpha}$ and $dz_1 = \alpha^{-1} z_2^{(1/\alpha)-1} dz_2 = \alpha^{-1}(z_1/z_2) dz_2 = \alpha^{-1} z_1^{1-\alpha} dz_2$. Therefore $|\alpha|^2 (z_1 \bar{z}_1)^{\alpha-1} dz_1 d\bar{z}_1 = dz_2 d\bar{z}_2$, i.e. $g_1 = g_2$ in $U_1 \cap U_2$. Other cases are verified analogously. Thus (g_σ) define the canonical almost flat metric g on S.

Corollary 1: Any orientation preserving automorphism φ of a polyhedral surface S preserves the canonical complex structure and the metric on S. If φ is orientation reversing then it changes the complex structure to its conjugate and preserves the canonical metric.

Proof: The complex structure on S is uniquely defined by the Euclidean polyhedral structure and an orientation on S. Reversing the orientation conjugates the complex structure. The metric g is determined by the Euclidean structure alone. The corollary follows.

Corollary 2: Let S be a rational connected simply-connected polyhedral surface and let R be the closed surface constructed in Theorem 1. Let A_i $i = 1,\ldots,\mu$ be the interior vertices of S with angles $2\pi m_i/n_i$ and let A_j $j = \mu+1,\ldots,\nu$ be the boundary vertices of S. Let n be the least common multiple of the denominators.

1. The genus of R is given by

$$(3) \qquad g(R) = 1 + (n/2)\Sigma_{i=1}^{\nu}(m_i-1)/n_i.$$

2. Let $\partial S = \emptyset$. The group $H = \mathbb{Z}/n$ preserves the canonical complex structure and the almost flat metric on R. $p:R \to S$ is a regular covering of Riemann surfaces. If $\partial S \neq \emptyset$ the group $H = D_n$ acts by conformal and anticonformal transformations and preserves the canonical metric on R. We have $S = R/H$ as a Riemann surface with boundary.

Proof: 1. follows directly from (2). 2. follows from Corollary 1 and the proof of Theorem 1.

Let S, R, A_i $i = 1,\ldots,\nu$, m_i/n_i, n, H and $p:R \to S$ be as in Theorem 1. By (3), $g(R) \geq 1$ so $R = D/\Gamma$ where $D = \mathbb{C}$ if $g(R) = 1$ and $D = \mathbb{H}$ (the hyperbolic plane) if $g(R) > 1$. Let $SO(D)$ $(O(D))$ denote the group of conformal (and anticonformal transformations) of D. We have $R = D/\Gamma$ where $\Gamma = \pi_1(R)$ is a discrete torsion-free subgroup of $SO(D)$. Denote by $q:D \to R$ the projection and let $G \subset O(D)$ be the group of all liftings of $h \in H$ to transformations of D. Then G acts in D properly discontinuously, Γ is normal in G, $H = G/\Gamma$, $S = D/G$ and $pq: D \to S$ is the natural projection.

<u>Theorem 3</u>: 1. The preimage $(pq)^{-1}b$ of any boundary edge b of S is a union of geodesic segments in D and $(pq)^{-1}\partial S$ is a union of geodesics.

2. Let A_i be an interior vertex of S with angle $2\pi m_i/n_i$ and denote by ρ_A the rotation of D around $A \in (pq)^{-1}A_i$ by $2\pi/n_i$. The group G is generated by rotations ρ_A and geodesic reflections r_ℓ $(\ell \in (pq)^{-1}\partial S)$,

3. There is a holomorphic developing map $\varphi: D \to \mathbb{C}$ and a homomorphism $\chi: G \to O(\mathbb{C})$ $(SO(\mathbb{C})$ if $\partial S = \emptyset)$ such that $\varphi \circ g = \chi(g) \circ \varphi$ for any $g \in G$. Let $V \subset D$ be the preimage of the set of vertices of S. Let $A \in V$ be such that $A_i = pqA$ has angle $2\pi m_i/n_i$ $(\pi m_i/n_i$ if $A_i \in \partial S)$. The mapping φ is a covering, the branching locus belongs to V, the branching number at A is m_i. For any $A \in V$ and any geodesic $\ell \subset (pq)^{-1}\partial S$, $\chi(\rho_A)$ is the rotation around $\varphi(A)$ by the angle $2\pi m_i/n_i$ and $\chi(r_\ell)$ is the reflection about $\varphi(\ell)$. Let $\bar{\chi}$ be the composition of χ and the projection $O(\mathbb{C}) \to O(2)$. Then $\mathrm{Ker}\,\bar{\chi} = \Gamma$, $\bar{\chi}(G) = \mathbb{Z}/n$ if $\partial S = \emptyset$ and $\bar{\chi}(G) = D_n$ if $\partial S \neq \emptyset$. The developing map φ and the homomorphism χ are unique up to a conjugation by any element of $O(\mathbb{C})$.

Proof: 1. Let b be a boundary edge of S, let $b' \subset R$ be an edge above b and let $\tilde{b}$ be a connected curve in D such that $q:\tilde{b} \to b'$ is one-to-one. There is a reflection $h \in H$ that fixes b'. Lift h to $\tilde{h} \in G$ that fixes $\tilde{b}$. Since $\tilde{h}$ fixes $\tilde{b}$ it is the geodesic reflection r_ℓ where $\tilde{b} \subset \ell$, is a geodesic segment. Since r_ℓ projects onto the canonical reflection of mS about ∂S, the geodesic ℓ projects onto ∂S. Thus $(pq)^{-1}\partial S$ is a union of geodesics and for any $\ell \subset (pq)^{-1}\partial S$, $r_\ell \in G$.

2. Let G' be the minimal normal subgroup of G containing all rotations ρ_A and reflections r_ℓ. Then G' is the kernel of the natural homomorphism $G \to \pi_1(S) \to 1$. Since S is simply connected, $G' = G$.

3. Let $\partial S = \emptyset$. One can order the vertices $A_1,\ldots,A_\nu$ of S so that G has a presentation

$$G = \langle g_1,\ldots,g_\nu : g_1 \cdots g_\nu = 1, g_i^{n_i} = 1 \; i = 1,\ldots,\nu\rangle.$$

Choose a reference polygon $P_0 \subset S$. A polygonal path on S starting at P_0 is a sequence $P_0,\ldots,P_n$ of faces where P_{i-1}, P_i are adjacent for $i = 1,\ldots,n$. Each polygonal loop $\gamma = (P_0,\ldots,P_n = P_0)$ defines a loop on $S \setminus \{A_1,\ldots,A_\nu\}$ thus defines an element of $\pi_1(S\setminus\{A_1,\ldots,A_\nu\})$. The loop γ is called trivial if this element is identity. Two polygonal paths $\gamma = (P_0,\ldots,P)$ and $\gamma' = (P_0,\ldots,P)$ are equivalent if their difference is a trivial loop. The set of equivalence classes of polygonal loops with the natural composition law is a group which is called the fundamental group of the polyhedron S and denoted by $\pi_1(S_P)$. Obviously $\pi_1(S_P) = \pi_1(S\setminus\{A_1,\ldots,A_\nu\})$. By virtue of the assumption $\pi_1(S) = 0$ the group $\pi_1(S_P)$ has a presentation $\langle g_1,\ldots,g_\nu : g_1\cdots g_\nu = 1\rangle$ where g_i is a simple loop around A_i.

Definition 5: Let S be an arbitrary closed polyhedral surface. The universal covering polyhedron $\hat{S}$ of S is the set of pairs (x,γ) where $x \in P \subset S$ and γ is an equivalence class of polygonal paths $(P_0,\ldots,P)$. $\hat{S}$ has an obvious polyhedral structure and the natural projection $\hat{p}:\hat{S} \to S$ is a covering of Euclidean polyhedra. The group $\pi_1(S_P)$ naturally acts on $\hat{S}$ by isometries and $S = \hat{S}/\pi_1(S_P)$.

It is obvious that the covering $\hat{p}:\hat{S} \to S$ is universal in the usual sense, that is if $p: R \to S$ is a covering of polyhedra there exists a covering $p':\hat{S} \to R$ such that $pp' = \hat{p}$.

One can define in a similar way the universal covering polyhedral surface $\hat{S}$ without the assumption $\partial S = \emptyset$ but we will not need this here.

Let φ_0 be an isometric embedding of P_0 into $\mathbb{C}$. The identification $P_0 = (P_0,1)$ defines φ on $(P_0,1) \subset \hat{S}$. If $\gamma = (P_0,\ldots,P_n)$ is a polygonal path we successively develop polygons P_i $i = 0,\ldots,n$ into $\mathbb{C}$ starting with $\varphi_0(P_0)$. This defines $\varphi: P \to \mathbb{C}$ for any polygonal path from P_0 to P. Equivalent paths define the same embedding so φ is well defined on $\hat{S}$. By definition φ is an isomorphism on any face of $\hat{S}$.

<u>Definition 6</u>: Let S be a closed rational connected simply-connected surface. Let A_i $i = 1,\dots,\nu$ be the vertices of S with angles $2\pi m_i/n_i$. Denote by $G_0 \subset \pi_1(S_p)$ the minimal normal subgroup containing $g_i^{n_i}$ $i = 1,\dots,\nu$.

The universal rational covering $\tilde{S}$ of S is the quotient $\tilde{S} = \hat{S}/G_0$. By definition $\tilde{p}: \tilde{S} \to S$ is a regular covering with the group $\pi_1(S_0) = \pi_1(S_p)/G_0$ of deck transformations.

Let $\varphi: \hat{S} \to \mathbb{C}$ be the developing map defined above. Let g_i be a generator of $\pi_1(S_p)$ and develop the corresponding sequence $P_0,\dots,P_n = P_0$ of faces into $\mathbb{C}$. The vertex A_i will go into a point $\varphi(A_i) \in \mathbb{C}$. The polygon P_n will go into $\varphi(P_0)$ rotated by $2\pi m_i/n_i$ around $\varphi(A_i)$. Thus if we develop along g_i n_i times we come back to $\varphi(P_0)$. Therefore $\varphi: \hat{S} \to \mathbb{C}$ is invariant under the action of G_0 and we can push φ down to a mapping of $\tilde{S}$. Denote the pushed down mapping by φ also and call it the developing map of $\tilde{S}$. Obviously φ is uniquely defined by the initial embedding $\varphi_0: P_0 \to \mathbb{C}$. Such φ_0 is uniquely defined up to the conjugation by any $g \in O(\mathbb{C})$. When we conjugate φ_0 by g, φ obviously changes by the same conjugation. By construction φ is equivariant with respect to the homomorphism $\chi: \pi_1(S_0) \to SO(\mathbb{C})$

where $\chi(g_i)$ is the rotation by $2\pi m_i/n_i$ around $\varphi(A_i)$.

Both $pq:D \to S$ and $\tilde{p}:\tilde{S} \to S$ are universal coverings in the sense of orbifolds therefore they are isomorphic regular coverings and the construction above gives the promised equivariant developing map $\varphi:D \to \mathbb{C}$.

Let now $\partial S \neq \emptyset$, let $\tilde{S}$ be the universal rational covering of mS and let $\varphi:\tilde{S} \to \mathbb{C}$ be a developing map. The canonical reflection s of mS lifts to a reflection $\tilde{s}$ of $\tilde{S}$ and the composition $\varphi\circ\tilde{s}:\tilde{S} \to \mathbb{C}$ is another developing map thus there is $\chi(\tilde{s}) \in O(\mathbb{C})$ such that $\varphi\circ\tilde{s} = \chi(\tilde{s})\circ\varphi$. If P is a face of $\tilde{S}$ whose edge b is fixed by $\tilde{s}$ then $\varphi\circ\tilde{s}(P)$ is the mirror image of $\varphi(P)$ with respect to $\varphi(b)$. Thus $\varphi(b) \subset \ell$ = straight line and $\chi(\tilde{s}) = r_\ell$. The isomorphism $\tilde{S} = D$ takes $\tilde{s}$ into geodesic reflection r_λ where $\lambda \subset (pq)^{-1}\partial S$.

It remains to check a few properties of the developing map φ. It is obviously holomorphic and ramified only at the vertices of $\tilde{S}$. Let A be such and let the angle at $A_i =: (pq)A$ be $2\pi m_i/n_i$. Then the Euclidean angle at A is $2\pi m_i$ and it is preserved by φ. Thus the branching number at A is m_i.

The group $\chi(G)$ is generated by rotations by $2\pi m_i/n_i$ about $\varphi(A_i)$ and by reflections about $\varphi(b)$, $b \in \partial S$. Thus $\bar{\chi}(G) = Z_n$ if $\partial S = \emptyset$ and $\bar{\chi}(G) = D_n$ otherwise. The composition $G \to \bar{\chi}(G)$ is isomorphic to the projection $G \to H$, so Ker $\bar{\chi} = \Gamma$. This completes the proof of the Theorem.

2. Exceptional level surfaces

Throughout this section S is a polyhedral surface satisfying assumptions of Theorem 1 and $\partial S \neq \emptyset$. We recall that $T(S)$ is foliated into invariant level surfaces R_θ $0 \leq \theta \leq \pi/n$ and for $\theta \in (0,\pi/n)$ the surfaces R_θ are homeomorphic to a closed surface R whose genus is given by formula (3). Thus the topology of R is determined by the vertex angles of S. The dihedral group D_n with the presentation $\langle s_0, s_1 : s_0^2 = s_1^2 = (s_0 s_1)^n = 1\rangle$ acts on R by conformal and anticonformal transformations and exceptional surfaces R_0, R_1 are the quotients R/s_0, R/s_1 respectively.

__Theorem 4__: 1. If n is odd then R_0 and R_1 are homeomorphic and $p:\partial R_i \to \partial S$ are homeomorphisms.

2. Let n be even. Let $A_1,\ldots,A_\nu$ be consecutive vertices of ∂S such that the denominators n_i of their angles $\pi m_i/n_i$ are even. For $i = 1,\ldots,\nu$ denote by I_i the closed interval of ∂S between A_i and A_{i+1} ($A_{\nu+1} = A_1$). Then $C_i = p^{-1}(I_i) \cap \partial(R_0 \cup R_1)$ is a circle and $p:C_i \to I_i$ is the natural 2-fold covering. The circles C_i $i = 1,\ldots,\nu$ exhaust the boundary $\partial(R_0 \cup R_1)$.

Let I_i, I_{i+1} be two consecutive intervals and let $\pi m_i/n_i$ be the angle at the vertex A_i between them. Then C_i and C_{i+1} belong to the same exceptional surface if and only if n/n_i is even.

3. If n is odd and $g(R)$ is odd then R_0 and R_1 are not orientable. If n is even and ν is odd then R_0 and R_1 are not homeomorphic and at least one of them is not orientable. If n is even and $\nu = 0$ then R_0 is closed and nonorientable and R_1 has two boundary components. If n is even and $\nu = 1$ then R_0 is closed and nonorientable and R_1 has one boundary component.

Proof: When n is odd all n reflections of D_n are conjugate to each other, therefore R/s_0 and R/s_1 are homeomorphic.

For any boundary segment $b \subset \partial S$ only two segments in the preimage $p^{-1}(b) \cap (R_0 \cup R_1)$ belong to the boundary $\partial R_0 \cup \partial R_1$. They correspond to the ball going along b in the positive or negative directions. Let b_1 and b_2 be two consecutive boundary segments and let $\pi m_1/n_1$ be the angle between them. Assume first that $m_1 = 1$. The ball going along b_1 in the positive direction will continue along b_2 if n_1 is odd and will bounce back along b_1 if n_1 is even. Thus in the first case the segment above b_1 corresponding to the positive direction on b_1 is joint to the positive segment above b_2. In the second case the two segments are disjoint and the positive b_1 is joint to the negative b_1 instead. Since the topology of the projection $p:R_A \to S$ does not depend on the numerators m_i the same holds when $m_1 > 1$.

1. If n is odd then all denominators n_i are odd. Thus when we go along ∂S in the positive direction the loop lifts uniquely to ∂R_0. The opposite loop lifts to ∂R_1.

2. According to the previous argument the path formed by going along I_i in the positive direction and returning in the negative direction lifts to a circle C_i in $R_0 \cup R_1$. The circles C_i and C_{i+1} belong to the same component of $R_0 \cup R_1$ if and only if $m_i/n_i = 2k/n$ for some k. Since m_i is odd this happens if and only if n/n_i is even.

3. It suffices to show that R_0 is not orientable. Assume that R_0 is orientable. We have $R_0 = R/s$ and since s is orientation reversing the number of connected components of ∂R_0 must be of opposite pairity with $g(R)$. Since ∂R_0 is a circle we come to a contradiction.

If n is even and ν is odd then the number $\pi_0(\partial R_0 \cup \partial R_1)$ of connected components is odd so $\pi_0(\partial R_0)$ and $\pi_0(\partial R_1)$ have opposite pairities. Thus one of them has the same pairity as $g(R)$, so that surface can not be orientable. If $\nu = 0$ then $\partial R_0 \cup \partial R_1$ consists of two circles corresponding to the positive and negative directions along the boundary. Since n is even these directions belong to the same surface, say R_1. If $\nu = 1$ then one of the surfaces (call it R_0) has no boundary, thus it can not be orientable.

<u>Remark</u>: If S is a polygon then $T(S_0)$ is homeomorphic to the three-sphere S^3. In this case Theorem 1 and Theorem 4 provide decompositions $S^3 = R \times (0,1) \cup R_0 \cup R_1$. Representations of this kind have to do with knots and links in S^3 and seem to be well known to topologists.

3. Flat polyhedral surfaces

<u>Proposition 5</u>: Let S be a polyhedral surface satisfying assumptions of Theorem 1. The following conditions are equivalent.

1. $g(R) = 1$
2. All vertex angles of S are $2\pi/n_i$ and π/n_j.
3. The canonical almost flat metric on R is flat.
4. The developing map $\varphi: D \to \mathbb{C}$ of Theorem 3 is a conformal equivalence.
5. The natural mapping $T(S) \to T(S_0)$ is an isomorphism.
6. The geodesic flow on $T(S_0)$ has a unique continuous extension to all times.

<u>Proof:</u> If $g(R) = 1$ then by formula (3) all numerators $m_i = 1$. Singularities of the canonical almost flat metric on R (see Theorem 3) correspond to vertices with $m_i > 1$. If all $m_i = 1$ the developing map $\varphi: D \to \mathbb{C}$ is conformal and since $\mathbb{C}$ is simply-connected it is one-to-one. The natural mapping $T(S) \to T(S_0)$ is one-to-one everywhere except at the vertices where it is m_i-to-one. This proves the equivalence of 1 through 5.

Around any nonvertex point of R_θ the geodesic flow in the natural coordinates is $z \to z + te^{i\theta}$. Around a vertex A with angle $2\pi m_j$ the flow takes form $z \to (z^{m_j} + te^{i\theta})^{1/m_j}$. The vertex A has m_j incoming and m_j outgoing separatrices (see fig. 1). Thus the flow "extends through A" if and only if $m_j = 1$.

Definition 7: A polyhedral surface S is called flat if any of the conditions of Proposition 5 is satisfied.

We will classify flat polyhedral surfaces. Obviously we don't tell between isometric surfaces. We also don't tell between S and S' if one is a dilation of another. Given a polyhedral surface S we can subdivide some of its faces into smaller ones without changing any of relevant properties of S. Such operation might create vertices with angle 2π inside S or π on ∂S. We call them fictitious vertices.

Definition 8: 1. Polyhedral surfaces S and S' are equivalent if one can be obtained from another by dilation and adding and erasing some vertices and edges.

2. Rational polyhedral surfaces S and S' are weakly equivalent if there exist a conformal equivalence $f: R \to R'$ commuting with the action of H and H'.

3. Rational polyhedral surfaces S and S' are equivalent in the sense of orbifolds if S_0 and S_0' are isomorphic orbifolds.

Obviously 1 implies 2 implies 3.

<u>Theorem 6:</u> Let S be a flat polyhedron satisfying assumptions of Theorem 1.

1. $\partial S = \emptyset$. Then either S is equivalent to a tetrahedron with vertex angles π or S is equivalent to a doubled triangle. Possible triangles are 90°, 45°, 45°; 90°, 60°, 30° and 60°, 60°, 60°.

2. $\partial S \neq \emptyset$. Either S is equivalent to a doubled rectangle with one slit side or to the doubled 90°, 45°, 45° triangle slit along the hypothemuse or a side or S is equivalent to the double 90°, 60°, 30° triangle slit along the longer side or S is a polygon. Possible polygons are rectangles and the three triangles of 1.

<u>Proof:</u> A flat polyhedral surface has the form $S = \mathbb{C}/G$ where G is a discrete subgroup of $O(\mathbb{C})$. There are 17 such groups up to isomorphism which correspond to 17 types of flat orbifolds. The condition $\pi_1(S) = 0$ singles 12 out of 17. A case by case verification shows that these are the twelve possibilities listed in the Theorem. Thus the Theorem gives a complete list of flat polyhedral surfaces up to weak equivalence. An elementary argument which we omit shows that any flat polyhedral surface is in fact equivalent to one in the list.

Only rectangles, double slit rectangles and the tetrahedra have moduli. For rectangles and double rectangles equivalence coincides with the weak equivalence. Equivalence classes are determined by the ratio r of the sides, for rectangles $1 \leq r$, for double rectangles $0 < r$.

The case of tetrahedra is more interesting. The surface R of a tetrahedron S is of course a topological torus. The group $H = \mathbb{Z}/2$ is generated by a 180° rotation of the torus. As a Riemann surface R is the quotient of $\mathbb{C}$ by a lattice, i.e. R is an elliptic curve.

<u>Theorem 7</u>: Equivalence classes of tetrahedra with vertex angles π are in one-to-one correspondence with triangles (up to similarity). Tetrahedra S and S' are weakly equivalent if and only if the corresponding elliptic curves R and R' are conformally equivalent.

The projection $q: R \to S$ is identified with the canonical 2-fold covering of Riemann sphere given by the Weierstrass $\wp$-function.

Let $\mathbb{H}$ be the upper half-plane and let $\mathbb{Z}/3$ be the subgroup of $PSL(2,\mathbb{Z})$ generated by $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. The mapping $f:\{S\} \to \{R\}$ from the tetrahedra into elliptic curves is naturally identified with the covering $f: \mathbb{H}/(\mathbb{Z}/3) \to \mathbb{H}/PSL(2,\mathbb{Z})$.

<u>Proof:</u> Let A, B, C, D, be the vertices of S. Slit S along the edges AB, AC, AD and let the faces ABC, ACD, ABD fall down

on the plane of BCD. The polygon that we get there is in fact a triangle AA'A". The triangle BCD is formed by the midpoints of AA'A". It follows that all four faces of S are equal triangles and AA'A" is similar to the face of S. The construction goes the other way too and establishes a one-to-one correspondence between tetrahedra and triangles.

Let AA'A"A"' be a parallelogramm built on AA'A" and let L be the lattice generated by the vectors AA' and AA". The group G corresponding to S is generated by 180° rotations in the vertices of L/2, thus G is completely determined by L. Therefore the weak equivalence class of S is determined by the elliptic curve $R = \mathbb{C}/L$. The projection $\varphi : R \to S$ is identified with $\mathbb{C}/L \xrightarrow{\mathbb{Z}/2} \mathbb{C}/G$, the latter being the canonical covering of elliptic curve onto the Riemann sphere given by the Weierstrass p-function.

The covering $\mathbb{H}/\mathbb{Z}/3 \to \mathbb{H}/PSL(2\mathbb{Z})$ comes from the correspondence: tetrahedron S → triangle AA'A" → parallelogramm AA'A"A"' → lattice L. The group $\mathbb{Z}/3$ accounts for 3 possible parallelogramms that can be built on a triangle.

In view of Theorem 7 it is interesting to study the correspondence between equivalence classes of polyhedral surfaces S and Riemann surfaces R. We will return to these questions in a future publication.

4. Examples

Using Theorem 4 we determine the exceptional surfaces for eight types of flat polyhedral surfaces with boundary.

1. Slit double rectangle. Two boundary vertices with $n_1 = n_2 = 1$. Thus $\nu = 0$ and $n = 2$. By Theorem 4, 3) R_0 is closed and nonorientable and R_1 has two circles on the boundary. Therefore R_0 is the Klein bottle and R_1 is the cylinder.

2. Double 90°, 45°, 45° triangle slit along the hypothenuse. We have $n_1 = n_2 = n = \nu = 2$. By Theorem 4 both R_0 and R_1 have one circle on the boundary, thus $R_0 = R_1$ = Möbius band.

3. Same double triangle slit along a side. We have $n_1 = 1$, $n_2 = 2$, $n = 4$, $\nu = 1$. By Theorem 4, 3) R_0 = Klein bottle, R_1 = Mobius band.

4. Double 90°, 60°, 30° rectangle slit along the long side. Here $n_1 = 1$, $n_2 = n = 3$, Since n is odd, $R_0 = R_1$ = Möbius band.

5. Rectangle. Here $n_1 = \ldots = n_4 = n = 2$. Four segments of the boundary correspond to 4 circles of $\partial R_0 \cup \partial R_1$. Consecutive circles belong to different surfaces, hence $R_0 = R_1$ = cylinder.

6. 90°, 45°, 45° triangle. Here $n_1 = 2$, $n_2 = n_3 = n = 4$, $\nu = 3$. Two circles corresponding to the sides belong to the same surface, thus R_0 = cylinder and R_1 = Mobius band.

7. 90°, 60°, 30° triangle. Here $n_1 = \nu = 2$, $n_2 = 3$, $n_3 = n = 6$. Two circles belong to different surfaces, thus R_0 and R_1 are Mobius bands.

8. Equilateral triangle. Here $n = 3$ therefore $R_0 = R_1 =$ Mobius band.

Information that serves as data in Theorem 4 does not suffice in general to determine R_0 and R_1. The simplest example is diamond ($120^\circ - 60^\circ$ parallelogramm) and the $120^\circ - 60^\circ$ trapezoid. For both, $n = 3$ and $g = 2$. By Theorem 4, R_0 and R_1 are homeomorphic and have one circle on the boundary. A direct verification shows that for the diamond R_0 = handle (torus with a hole) and for the trapezoid R_0 = Klein bottle with a hole.

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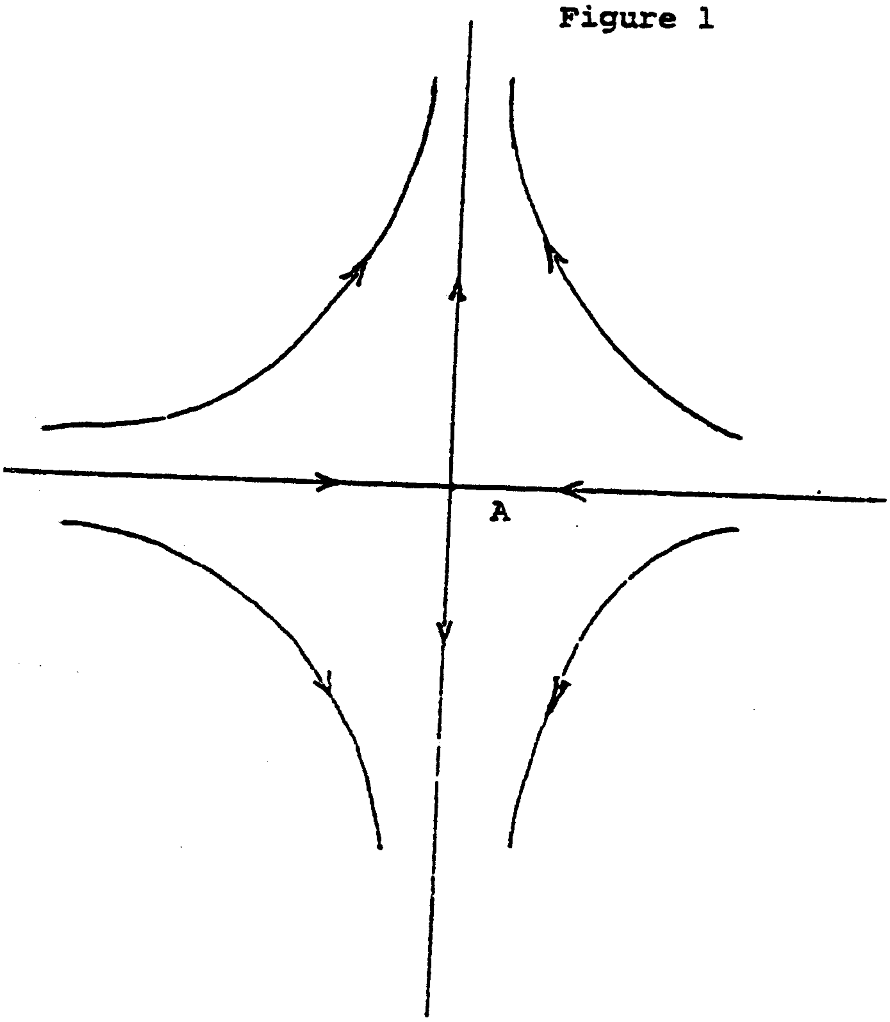
Figure 1



The flow around a vertex with $m_i = 2$.