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Abstract

Our aim is to find the irreducibility criteria for the Koopman representation, when the group acts on some space with a measure (Conjecture 1.5). Some general necessary conditions of the irreducibility of this representation are established. In the particular case of the group $\mathrm{GL}_0(2\infty,\mathbb{R}) = \varinjlim_n \mathrm{GL}(2n-1,\mathbb{R})$, the inductive limit of the general linear groups we prove that these conditions are also the necessary ones. The corresponding measure is infinite tensor products of one-dimensional arbitrary Gaussian non-centered measures. The corresponding G-space X_m is a subspace of the space $\mathrm{Mat}(2\infty,\mathbb{R})$ of infinite in both directions real matrices. In fact, X_m is a collection of m infinite in both directions rows. This result was announced in [20]. We give the proof only for $m \leq 2$. The general case will be studied later.

Keywords: infinite-dimensional groups, irreducible representation, Koopman's representation, Ismagilov's conjecture, Schur-Weyl duality, quasi-invariant, ergodic measure
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Contents

1. Introduction

1.1. Description of the dual for locally compact groups

The main problem in the representation theory for a locally compact group G is to find the set of *all unitary irreducible representations* of G up to unitary equivalence and to decompose reducible representations into a direct sum or direct integral of irreducible. This set is called the *unitary dual* of G and is denoted by $\hat{G}$. For many locally compact groups this problem has been solved, but for some particular cases it remains open, for example, for the group $\mathrm{SO}(p,q)$. To find the dual for locally compact groups G, one can use *regular, quasiregular or induced representations.* In the case of locally compact groups all these constructions are based on the existence of the invariant *Haar measure* on the initial group G or some G-quasi-invariant measure on the corresponding homogeneous space $H\setminus G$, where H is a closed subgroup of G or on some general G-space X.

1.2. Regular, quasiregular and induced representations for infinite-dimensional groups

It is well known that there is no general method to describe $\hat{G}$ for infinite-dimensional groups G. Our aim is to start the development of the harmonic analysis on infinite-dimensional groups.

In the previous articles we have generalized the notions of the regular, quasiregular and induced representations for infinite-dimensional groups by constructing G-quasi-invariant measures on suitable completions of the corresponding objects (groups, homogeneous spaces and G-spaces). In addition, we study the irreducibility of the constructed representations in the framework of the Ismagilov conjecture (see 1.1).

In this article we consider the case when the infinite-dimensional group G, the *inductive limit of the general linear groups*, acts on the space of m infinite rows equipped with the Gaussian measure. We establish the *criteria of irreducibility* of constructed representations (see Theorem 2.1) in terms of the corresponding measure and express some general conjectures dealing with the irreducibility. These conjectures are natural generalization of the *Ismagilov conjecture* (see Conjecture 1.5).

Recall some previous constructions. *Regular representations* for infinite-dimensional groups were defined and studied in [14, 15, 16]. Due to the result of A.Weil [30], there is no invariant measure on non locally compact groups. Therefore, to construct an analogue of a regular representation of an infinite-dimensional group G we can, for example, construct a G-quasi-invariant measure on a suitable completion $\tilde{G}$ of the initial group G. The regular representation of an infinite-dimensional group can be irreducible, which never happens for a locally compact group, except for the trivial one!

To define a *quasiregular representation* we should construct a G-quasi-invariant measure on a suitable completion $\tilde{H}\setminus\tilde{G}$ of the homogeneous space $H\setminus G$ [17, 18, 19].

To construct the induced representation for infinite-dimensional groups we need to extend by continuity the representation of the subgroup H to the corresponding completion $\tilde{H}$. The general construction of the *induced representations* and the beginning of the *orbit methods* for infinite-dimensional group of upper triangular matrices were done in [22].

To construct the regular representation for an infinite-dimensional group G, first we should find some larger topological group $\widetilde{G}$ and a measure μ on $\widetilde{G}$ such that G is a dense subgroup in $\widetilde{G}$, and $\mu^{R_t} \sim \mu$ for all $t \in G$, (or $\mu^{L_t} \sim \mu$ for all $t \in G$), here $\sim$ means *equivalence*. The right and left representations $T^{R,\mu}, T^{L,\mu} : G \to U(L^2(\tilde{G}, \mu))$ are naturally defined in the Hilbert space $L^2(\tilde{G}, \mu)$ by the following formulas:

$$(T_t^{R,\mu} f)(x) = (d\mu(xt)/d\mu(x))^{1/2} f(xt),$$
$$(T_s^{L,\mu} f)(x) = (d\mu(s^{-1}x)/d\mu(x))^{1/2} f(s^{-1}x).$$

The right regular representation of infinite-dimensional groups can be irreducible if no left actions are *admissible* for the measure μ, i.e., when $\mu^{L_t} \perp \mu$ for all $t \in G\backslash\{e\}$. In this case a von Neumann algebra $\mathfrak{A}^{T^{L,\mu}}$ generated by the left regular representation $T^{L,\mu}$ is trivial. More precisely:

Conjecture 1.1 (Ismagilov, 1985). *The right regular representation*

$$T^{R,\mu} : G \to U(L^2(\tilde{G}, \mu))$$

is irreducible if and only if

1) $\mu^{L_t} \perp \mu \ \forall t \in G\backslash\{e\}$, *(where* $\perp$ *stands for singular),*

2) the measure μ *is* G*-ergodic.*

This conjecture was verified for a lot of particular cases. In the general case, it is an open problem. In the case of a finite field $\mathbb{F}_p$ we need some additional conditions for the irreducibility [21].

1.3. Koopman representation

Let $\alpha : G \to \mathrm{Aut}(X)$ be a measurable action of a group G on a measurable space (X, μ) with G-quasi-invariant measure μ, i.e, $\mu^{\alpha_t} \sim \mu$ for all $t \in G$. With these date one can associate the representation $\pi^{\alpha,\mu,X} : G \to U(L^2(X, d\mu))$, by the following formula:

$$(\pi_t^{\alpha,\mu,X} f)(x) = (d\mu(\alpha_{t^{-1}}(x))/d\mu(x))^{1/2} f(\alpha_{t^{-1}}(x)), \quad f \in L^2(X, \mu). \quad (1.1)$$

In the case of an invariant measure this representation called *Koopman's representation*, see [13]. We would like to solve the following problems:

Problem 1.2. *Find criteria of irreducibility of the representation $\pi^{\alpha,\mu,X}$ defined by (1.1).*

Problem 1.3. *Find the description of the commutant of the von Neumann algebra generated by representation $\pi^{\alpha,\mu,X}$ when representation is reducible.*

To study properties of the Koopman representation, in particular, the irreducibility, *we need some conjectures to describe the commutant of the von Neumann* algebras generated by this representation. The Schur–Weyl duality and the Dixmier commutation theorem below give us a very good hint for such a conjecture, see Conjecture 1.6 in a general context.

1.4. Schur–Weyl duality

Schur–Weyl duality [25, 26, 31] is a typical situation in representation theory involving two kinds of symmetry that determine each other.

From [32]: "If V is a finite-dimensional complex vector space, then the symmetric group S_n naturally acts on the tensor power $V^{\otimes n}$ by permuting the factors. This action of S_n commutes with the action of $\mathrm{GL}(V)$, so all permutations $\sigma : V^{\otimes n} \to V^{\otimes n}$ are morphisms of $\mathrm{GL}(V)$-representations. This defines a morphism $\mathbb{C}[S_n] \to \mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes n})$, and a natural question to ask is whether this map is surjective.

Part of Schur–Weyl duality asserts that the answer is yes. The *double commutant theorem* plays an important role in the proof and also highlights an important corollary, namely that $V^{\otimes n}$ admits a canonical decomposition

$$V^{\otimes n} = \bigoplus_{\lambda} V_\lambda \otimes S_\lambda$$

where λ runs over partitions, V_λ are some irreducible representations of $\mathrm{GL}(V)$, and S_λ are the *Specht modules*, which describe all irreducible representations of S_n. This gives a fundamental relationship between the representation theories of the general linear and symmetric groups; in particular, the assignment $V \mapsto V_\lambda$ can be upgraded to a functor called a *Schur functor*, generalizing the construction of the exterior and symmetric products."

Let $\dim V = m$ then $\mathrm{GL}(V) = \mathrm{GL}(m, \mathbb{C})$. The abstract form of the Schur–Weyl duality asserts that two algebras of operators on the tensor space generated by the actions of $\mathrm{GL}(m, \mathbb{C})$ and S_n are the full mutual centralizers in the algebra of the endomorphisms $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^m \otimes \mathbb{C}^m \otimes \cdots \otimes \mathbb{C}^m)$.

Denote by α and β the corresponding actions of S_n and $\mathrm{GL}(m,\mathbb{C})$ in the group of all automorphisms $\mathrm{Aut}(\mathbb{C}^m\otimes\mathbb{C}^m\otimes\cdots\otimes\mathbb{C}^m)$:

$$\alpha: S_n\to \mathrm{Aut}(X),\quad \beta:\mathrm{GL}(m,\mathbb{C})\to\mathrm{Aut}(X).$$

Let M' be the commutant of the subset M in the von Neumann algebra $B(H)$ of all bounded operators in a Hilbert space H:

$$M'=\{B\in B(H)\mid [B,a]=0\ \forall a\in M\}\ \text{where}\ [B,a]=Ba-aB. \tag{1.2}$$

Set $M_1=(\alpha(S_n))''$ and $M_2=(\beta(\mathrm{GL}(m,\mathbb{C})))''$ then the Schur–Weyl duality states that $M_1'=M_2$ hence, $M_2'=M_1$.

In [27] the authors extend the classical Schur–Weyl duality between representations of the groups $\mathrm{SL}(m,\mathbb{C})$ and S_n to the case of $\mathrm{SL}(m,\mathbb{C})$ and the infinite symmetric group S_∞. In [24] the authors extend Weyl results to the classical infinite-dimensional locally finite algebras $\mathfrak{g}l_\infty$, $\mathfrak{s}l_\infty$, $\mathfrak{s}p_\infty$, $\mathfrak{s}o_\infty$.

1.5. The Dixmier commutation theorem, locally compact groups

Let G be a locally compact group and let h be the right invariant Haar measure on G, i.e., $h^{R_t}=h$ for all $t\in G$. Consider the left L and the right R action of the group G on itself:

$$R_t(x)=xt^{-1},\ L_s(x)=sx,\ x,t,s\in G.$$

The right and the left regular representations of the group G are defined in the Hilbert space $L^2(G,h)$ by

$$(\rho_tf)(x)=f(xt),\quad (\lambda_sf)(x)=\big(dh(s^{-1}x)/dh(x)\big)^{-1/2}f(s^{-1}x),\ f\in L^2(G,h),$$

where $dh(s^{-1}x)/dh(x)$ is the Radon-Nikodim derivative.

Theorem 1.4 (Dixmier's commutation theorem [5]). *The commutant of the von-Neumann algebra generated by the right regular representation is generated by the left regular representation. More precisely, let* $\rho,\lambda: G\to U(L^2(G,h))$ *be the right and the left regular representations of the group* G, *and let* $\mathfrak{A}^\rho=(\rho_t\mid t\in G)''$ *and* $\mathfrak{A}^\lambda=(\lambda_s\mid s\in G)''$ *be the corresponding von Neumann algebras. Then*

$$(\mathfrak{A}^\rho)'=\mathfrak{A}^\lambda\quad \textit{and}\quad (\mathfrak{A}^\lambda)'=\mathfrak{A}^\rho. \tag{1.3}$$

1.6. G-action and irreducibility of the Koopman representation

In both examples we have two commuting actions of the group G_1 and G_2 on the same space X. Let $Z_G(H)$ be a *centralizer* of the subgroup H in the group G:

$$Z_G(H) = \{g \in G \mid \{g, a\} = e \ \forall a \in H\},$$

where $\{g, a\} = gag^{-1}a^{-1}$. In the first example, we have two commuting actions α and β of the groups $G_1 = S_n$ and $G_2 = \mathrm{GL}(n, \mathbb{C})$ on the space X such that $Z_{\mathrm{Aut}(X)}(\alpha(G_1)) \supseteq \beta(G_2)$. In the second example, we have two commuting actions R and L of the same group G in the space $X = G$. In this case we have $\{R(G), L(G)\} = e$ or $Z_{\mathrm{Aut}(G)}(R(G)) \supseteq L(G)$. In the general case, if we have only one group G acting via α on the space X, the second group should be the *centralizer* of the group $\alpha(G)$ in the group $\mathrm{Aut}(X)$, i.e., it is natural to consider $G_2 = Z_{\mathrm{Aut}(X)}(\alpha(G))$.

Come back to the Koopman representation (1.1). Consider the centralizer $Z_{\mathrm{Aut}(X)}(\alpha(G))$ of the subgroup $\alpha(G) = \{\alpha_t \mid t \in G\}$ in the group $\mathrm{Aut}(X)$ and its subgroup G_2 defined as follows:

$$G_2 := Z^{\mu}_{\mathrm{Aut}(X)}(\alpha(G)) := \left\{g \in Z_{\mathrm{Aut}(X)}(\alpha(G)) \mid \mu^g \sim \mu\right\}.$$

Define the representation T of the group G_2 as follows:

$$(T_g f)(x) = (d\mu(gx)/d\mu(x))^{1/2} f(gx). \tag{1.4}$$

Consider two von Neumann algebras

$$\mathfrak{A}^{\pi}(G) = (\pi_t \mid t \in G)'', \quad \mathfrak{A}^{T}(G_2) = (T_g \mid g \in G_2)''.$$

The conditions 1) and 2) below are necessary conditions of the irreducibility of the representation $\pi^{\alpha,\mu,X}$. It would be interesting to know when they are sufficient, i.e., when the following conjecture is true

Conjecture 1.5 (Kosyak, [16, 18]). *The representation*

$$\pi^{\alpha,\mu,X} : G \to U(L^2(X, \mu))$$

is irreducible if and only if

1) $\mu^g \perp \mu \ \forall g \in Z_{\mathrm{Aut}(X)}(\alpha(G)) \backslash \{e\}$,

2) the measure μ *is* G*-ergodic.*

Recall that a measure μ is G-ergodic if $f(\alpha_t(x)) = f(x)$ μ a.e. for all $t \in G$ implies $f(x) = const$ μ a.e.(almost everywhere) for all functions $f \in L^1(X, \mu)$.

Conjecture 1.6. *The commutant of the von Neumann algebra generated by representation π (1.1) of the group G coincides with the von Neumann algebra generated by the representation T (1.4) of the subgroup G_2 in the centralizer $Z_{\mathrm{Aut}(X)}(\alpha(G))$:*

$$(\mathfrak{A}^{\pi}(G))' = \mathfrak{A}^{T}(G_2).$$

For a lot of particular cases Conjecture 1.6 holds, but in general it fails. Below we give several example for which Conjecture 1.6 fails.

1.7. Counterexample to Conjecture 1.6

1.7.1. Case $X = S_{n-1} \setminus S_n$

Example 1.1. Consider the group S_n acting on the homogeneous space $X = S_{n-1}\setminus S_n$. For corresponding right quasiregular representation of S_n in $L^2(X)$ Conjecture 1.6 fails.

Proof. To simplify details set $n = 3$. For general n the proof is the same. Let σ_1, σ_2 be two generators of the group S_3:

$$S_3 = \Big(\sigma_1, \sigma_2 \mid \sigma_1^2 = e, \sigma_2^2 = e,\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\Big). \tag{1.5}$$

Let the group S_2 is generated by σ_1, then the space X consists of three classes $x_0 = \{e, \sigma_1\},\ x_1 = \{\sigma_2, \sigma_1\sigma_2\},\ x_3 = \{\sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\}$. The right action of S_3 on the space X is as follows:

$$x_0\sigma_1 = x_0, \quad x_1\sigma_1 = x_2, \quad x_2\sigma_1 = x_1,$$
$$x_0\sigma_2 = x_1, \quad x_1\sigma_2 = x_0, \quad x_2\sigma_2 = x_2.$$

Therefore, in $L^2(X)$ the corresponding representations for T_{σ_1} and T_{σ_2} are as follows:

$$T_{\sigma_1} = \begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix}, \quad T_{\sigma_2} = \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix}.$$

The representation T is reducible, since the vector $e_0 + e_1 + e_2$ is invariant. It splits into one-dimensional and two- dimensional irreducible representations. But the group S_3 acts on X by permutations so, its centralizer is trivial. □

1.7.2. Case $X = O(3)\setminus O(3)$

Example 1.2. Consider the group $O(3)$ acting on the homegeneous space $O(3)\setminus O(3) \simeq S^2$. The centralizer of $O(3)$ in the group of all automorphisms $\mathrm{Aut}(S^2)$ consists of two elements I and $-I$ by Lemma 1.7 but the representation of $O(3)$ in $L^2(X)$ is an infinite direct sum of irreducible representations generated by eigenvectors of the Laplace operator on S^2, see [8, Chapter I,§3]. Therefore, Conjecture 1.6 fails.

1.7.3. Centralizer of $SO(2k+1)$

Let $n \geq 0$, and $SO(n)$ be the group of all real orthogonal $n \times n$-matrices with determinant 1. This group effectively and transitively acts $n-1$-dimensional sphere S^{n-1}, and so it can be regarded as a subgroup of the group $\mathcal{H}(S^{n-1})$ of all homeomorphisms of S^{n-1}.

Let I be the unit matrix. Then $-I$ is an "antipodal" map, that is $-I(p) = -p$ for all $p \in S^{n-1}$. Evidently, I and $-I$ commute with all elements from $SO(n)$, and so $\{\pm I\}$ belongs to the centralizer of $SO(n)$ in $O(n)$.

Lemma 1.7. *Suppose $n = 2k+1$ is odd. Then the group $\{\pm I\}$ is the centralizer of $SO(2k+1)$ in all the group $\mathcal{H}(S^{2k})$.*

Proof. (given by S. Maximenko.) Suppose $h \in \mathcal{H}(S^{n-1})$ commutes with all matrices $A \in SO(n)$, that is $h \circ A(x) = A \circ h(x)$ for all $x \in S^{n-1}$. We should prove that then $h = \pm I$.

First we claim that $h(x) \in \{\pm x\}$ for each $x \in S^{n-1}$. Indeed, since n is odd, for each $x \in S^{n-1}$ there exists $A \in SO(n)$ such that $\{\pm x\}$ is the set of all fixed points for A. Hence

$$h(x) = h \circ A(x) = A \circ h(x),$$

that is $h(x)$ is a fixed point for A, and so $h(x) = \pm x$.

Now, suppose $h(x) = \varepsilon x$ for some $\varepsilon = \pm 1$. We claim that then $h = \varepsilon I$. Let $F = \{x \in S^{n-1} \mid h(x) = \varepsilon x\}$ be the set of points where h coincides with εI. We will show that F is a non-empty open-closed subset of S^{n-1}, which will imply that F coincides with all of S^{n-1}.

As shown above $x \in F$, so $F \neq \varnothing$. Moreover, as h and $-I$ are continuous, F is closed. It remains to show that F is open. Let U be a small neighbourhood of x such that $U \cap -U = \varnothing$, that is U does not contain antipodal pairs. Since h is continuous and $h(x) = \varepsilon x \in \varepsilon U$, there exists a neighbourhood V of x such that $h(V) = \varepsilon U$. Then for each $y \in V$ we have that $h(y) \in \{\pm y\} \cap \varepsilon U = \varepsilon y$. In other words, $h = \varepsilon I$ on V, and so $V \subset F$. This proves that $F = S^{n-1}$. □

2. Representations of the inductive limit of the general linear groups $\mathbf{GL_0(2\infty, \mathbb{R})}$

2.1. Finite-dimensional case

Consider the space $X_{m,n} = \Big\{x = \sum_{1\leq k\leq m} \sum_{-n\leq r\leq n} x_{kr} E_{kr},\ x_{kr} \in \mathbb{R}\Big\}$, with the measure (see (2.4)) $\mu^{m,n}_{(b,a)}(x) = \otimes_{k=1}^{m} \otimes_{-n\leq r\leq n} \mu_{(b_{kr},a_{kr})}(x_{kr})$. On the space $X_{m,n}$ acts two groups $\mathrm{GL}(m,\mathbb{R})$ from the left and $\mathrm{GL}(2n+1,\mathbb{R})$ from the right and these actions commute. Therefore, two von Neumann algebras $\mathfrak{A}_1$ and $\mathfrak{A}_{2,n}$ in the Hilbert space $L^2(X_{m,n},\mu^{m,n}_{(b,a)})$ generated respectively by the left and the right actions of the corresponding groups have the property that $\mathfrak{A}_1'\subseteq\mathfrak{A}_{2,n}$. We study what happens when $n\to\infty$. As the limit we obtain some unitary representation of the group $\mathrm{GL}_0(2\infty,\mathbb{R})=\varinjlim_{n,i^s}\mathrm{GL}(2n-1,\mathbb{R})$ (see below). In generic case, this representation is reducible, namely, if there exists a non trivial element $s\in\mathrm{GL}(m,\mathbb{R})$ such the the left action is admissible for the measure $\mu^m_{(b,a)}$, i.e., $(\mu^m_{(b,a)})^{L_s}\sim\mu^m_{(b,a)}$. But when no non-trivial left actions are admissible, i.e., when $(\mu^m_{(b,a)})^{L_s}\perp\mu^m_{(b,a)}$ for all $s\in\mathrm{GL}(m,\mathbb{R})\backslash\{e\}$ we prove that this representation is irreducible Theorem 2.1. Here, as in the case of the regular [14, 15] and quasiregular [17, 18] representations of the group $B_0^{\mathbb{N}}$ we obtain the remarkable fact that the irreducible representations can be obtained as the inductive limit of reducible representations!

2.2. Infinite-dimensional case

Let us denote by $\mathrm{Mat}(2\infty,\mathbb{R})$ the space of all real matrices infinite in both directions:

$$\mathrm{Mat}(2\infty,\mathbb{R})=\Big\{x=\sum_{k,n\in\mathbb{Z}}x_{kn}E_{kn},\ x_{kn}\in\mathbb{R}\Big\},\tag{2.1}$$

where $E_{kn},\ k,n\in\mathbb{Z}$ are infinite matrix unities.

The group $\mathrm{GL}_0(2\infty,\mathbb{R})=\varinjlim_{n,i^s}\mathrm{GL}(2n-1,\mathbb{R})$ is defined as the inductive limit of the general linear groups $G_n=\mathrm{GL}(2n-1,\mathbb{R})$ with respect to the symmetric embedding i^s (2.2):

$$\mathrm{GL}(2n-1,\mathbb{R})\ni x\mapsto i^s_{n+1}(x)=x+E_{-(n+1),-(n+1)}+E_{n+1,n+1}\in\mathrm{GL}(2n+1,\mathbb{R}).\tag{2.2}$$

We consider a G-space $X_m,\ m\in\mathbb{N}$ as the following subspace of the space $\mathrm{Mat}(2\infty,\mathbb{R})$:

$$X_m=\Big\{x\in\mathrm{Mat}(2\infty,\mathbb{R})\,|\,x=\sum_{k=1}^{m}\sum_{n\in\mathbb{Z}}x_{kn}E_{kn}\Big\}.\tag{2.3}$$

The group $\mathrm{GL}_0(2\infty,\mathbb{R})$ acts from the right on the space X_m. Namely, the right action of the group $G=\mathrm{GL}_0(2\infty,\mathbb{R})$ is correctly defined on the space

X_m by the formula $R_t(x) = xt^{-1},\ t \in G,\ x \in X_m$. We define a Gaussian noncentered product measure $\mu^m = \mu^m_{(b,a)}$ on the space X_m :

$$\mu^m_{(b,a)}(x) = \otimes_{k=1}^m \otimes_{n\in\mathbb{Z}} \mu_{(b_{kn},a_{kn})}(x_{kn}), \tag{2.4}$$

where

$$d\mu_{(b_{kn},a_{kn})}(x_{kn}) = (b_{kn}/\pi)^{1/2} \exp(-b_{kn}(x_{kn}-a_{kn})^2)dx_{kn}$$

and $b = (b_{kn})_{k,n},\ b_{kn} > 0,\ a = (a_{kn})_{k,n},\ a_{kn} \in \mathbb{R},\ 1 \le k \le m,\ n \in \mathbb{Z}$. Define the representation $T^{R,\mu,m}$ of the group $\mathrm{GL}_0(2\infty,\mathbb{R})$ in the space $L^2(X_m, \mu^m_{(b,a)})$ by the formula:

$$(T^{R,\mu,m}_t f)(x) = \left(d\mu^m_{(b,a)}(xt)/d\mu^m_{(b,a)}(x)\right)^{1/2} f(xt),\ \ f \in L^2(X_m, \mu^m_{(b,a)}).$$

Obviously, the centralizer $Z_{\mathrm{Aut}(X_m)}(\alpha(G)) \subset \mathrm{Aut}(X_m)$ contains the group $L(\mathrm{GL}(m,\mathbb{R}))$, i.e., the image of the group $\mathrm{GL}(m,\mathbb{R})$ with respect to the left action $L : \mathrm{GL}(m,\mathbb{R}) \to \mathrm{Aut}(X_m)$, $L_s(x) = sx,\ s \in \mathrm{GL}(m,\mathbb{R}),\ x \in X_m$. We prove the following theorem for $m \le 2$.

Theorem 2.1. *The representation* $T^{R,\mu,m} : \mathrm{GL}_0(2\infty,\mathbb{R}) \to U(L^2(X_m, \mu^m_{(b,a)}))$ *is irreducible if and only if* $(\mu^m_{(b,a)})^{L_s} \perp \mu^m_{(b,a)}\ \forall s \in \mathrm{GL}(m,\mathbb{R})\backslash\{e\}$.

Remark 2.1. Any Gaussian product-measure $\mu^m_{(b,a)}$ on X_m is $\mathrm{GL}_0(2\infty,\mathbb{R})$-right-ergodic [28, §3, Corollary 1]. For non-product-measures this is not true in general.

To study the condition $(\mu^m_{(b,a)})^{L_t} \perp \mu^m_{(b,a)}$ for $t \in \mathrm{GL}(m,\mathbb{R})$ set

$$t = (t_{rs})_{r,s=1}^m \in \mathrm{GL}(m,\mathbb{R}),\ \ B_n = \mathrm{diag}(b_{1n}, b_{2n}, ..., b_{mn}),\ \ X_n(t) = B_n^{1/2} t B_n^{-1/2}. \tag{2.5}$$

Let $M^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_r}(t)$ be the minors of the matrix t with $i_1, i_2, ..., i_r$ rows and $j_1, j_2, ..., j_r$ columns, $1 \le r \le m$. Let δ_{rs} be the Kronecker symbols.

Lemma 2.2. *For the measures* $\mu^m_{(b,a)}$, $m \in \mathbb{N}$ *the relation* $(\mu^m_{(b,a)})^{L_t} \perp \mu^m_{(b,a)}$ $\forall t \in \mathrm{GL}(m,\mathbb{R})\backslash\{e\}$ *holds if and only if*

$$\prod_{n\in\mathbb{Z}} \frac{1}{2^m|\det t|} \det\left(I + X_n^*(t)X_n(t)\right) + \sum_{n\in\mathbb{Z}}\sum_{r=1}^m b_{rn}\left(\sum_{s=1}^m (t_{rs}-\delta_{rs})a_{sn}\right)^2 = \infty,$$

$$\text{where}\quad \det\left(I + X_n^*(t)X_n(t)\right) =$$

$$1 + \sum_{r=1}^m \sum_{1\le i_1<i_2<\dots<i_r\le m; 1\le j_1<j_2<\dots<j_r\le m} \left(M^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_r}(X_n(t))\right)^2.$$

This lemma will be proved in Section 4.3.

Remark 2.2. The idea of the proof of the irreducibility. Let us denote by $\mathfrak{A}^m$ the von Neumann algebra generated by the representation $T^{R,\mu,m}$: $\mathfrak{A}^m = (T_t^{R,\mu,m} \mid t \in G)''$. For $\alpha = (\alpha_k) \in \{0,1\}^m$ define the von Neumann algebra $L^\infty_\alpha(X_m, \mu^m)$ as follows:

$$L^\infty_\alpha(X_m,\mu^m)=\big(\exp(itB^\alpha_{kn}) \mid 1\le k\le m,\ t\in\mathbb{R},\ n\in\mathbb{Z}\big)'',$$

where $B^\alpha_{kn}=\begin{cases} x_{kn}, & \text{if} \quad \alpha_k=0\\ D_{kn}, & \text{if} \quad \alpha_k=1\end{cases}$.

The proof of the irreducibility is based on three facts:

1) using the orthogonality condition $(\mu^m)^{L_t} \perp \mu^m$ for all $t \in \mathrm{GL}(m,\mathbb{R})\backslash\{e\}$ we can approximate by generators $A_{kn} = A^{R,m}_{kn} = \frac{d}{dt}T^{R,\mu,m}_{I+tE_{kn}}|_{t=0}$ the set of operators $(B^\alpha_{kn})^m_{k=1}$, $n\in\mathbb{Z}$ for some $\alpha\in\{0,1\}^m$ depending on the measure μ^m,

2) the subalgebra $L^\infty_\alpha(X_m,\mu^m) \subset \mathfrak{A}^m$ is a maximal abelian subalgebra in $\mathfrak{A}^m$,

3) the measure μ^m is G-ergodic.

Here the generators $A^{R,m}_{kn}$ are given by the formulas:

$$A^{R,m}_{kn}=\sum_{r=1}^m x_{rk}D_{rn}, \quad k,n\in\mathbb{Z}, \quad \text{where} \quad D_{kn}=\partial/\partial x_{kn}-b_{kn}(x_{kn}-a_{kn}).$$

Remark 2.3. The fact that conditions $(\mu^m)^{L_t} \perp \mu^m$ for all $t \in \mathrm{GL}(m,\mathbb{R})\backslash\{e\}$ implies the possibility of the approximation of x_{kn} and D_{kn} is based on some completely independent statement about the properties of projections of two infinite vectors $f = (f_k)_{k\in\mathbb{N}}$ and $g = (g_k)_{k\in\mathbb{N}}$ such that $f,\ g,\ f+sg \notin l_2$ for all $s \in \mathbb{R}$ (Lemma 4.10). This lemma is a key part of the proof of the irreducibility of the representation.

Remark 2.4. Similarly, for the "nilpotent group" $B_0^{\mathbb{N}}$ and the infinite product of arbitrary Gaussian measures on $\mathbb{R}^m$ (see [2]) the proof of the irreducibility is based on another completely independent statement namely, Hadamard – Fischer's inequality, see Lemma 2.3.

Lemma 2.3 (Hadamard – Fischer's inequality [9], [10]). *For any positive definite matrix $C \in \mathrm{Mat}(m,\mathbb{R})$, $m \in \mathbb{N}$ and any two subsets α and β with $\emptyset \subseteq \alpha,\ \beta \subseteq \{1,...,m\}$ the following inequality holds:*

$$\begin{vmatrix} M(\alpha) & M(\alpha\bigcap\beta)\\ M(\alpha\bigcup\beta) & M(\beta)\end{vmatrix} = \begin{vmatrix} A(\hat\alpha) & A(\hat\alpha\bigcup\hat\beta)\\ A(\hat\alpha\bigcap\hat\beta) & A(\hat\beta)\end{vmatrix} \ge 0 \qquad (2.6)$$

where $M(\alpha)=M^\alpha_\alpha(C),\ A(\alpha)=A^\alpha_\alpha(C)$ *and* $\hat\alpha=\{1,...,m\}\setminus\alpha.$

For details see [9, p.573], [10, Chapter 2.5, problem 36].

The conditions of orthogonality $\mu^{L_t}\perp\mu$ with respect to the left action of the group $B(m,\mathbb{R})$ on X^m were expressed as the divergence of some series, $S^L_{kn}(\mu)=\infty,\ 1\le k<n\le m$. Conditions of the approximation of the variables x_{kn} by combinations of generators A_{pq} were expressed in terms of the divergence of another series Σ_{kn}. The proof of the fact that conditions $S^L_{kn}(\mu)=\infty,\ 1\le k<n\le m$ imply conditions $\Sigma_{kn}=\infty,\ 1\le k<n\le m$ is based on the Hadamard – Fischer's inequality.

3. The proof of the irreducibility

3.1. The cases $m=1$

As before, let us denote by $\langle f_n\mid n\in\mathbb{N}\rangle$ the closure of the linear space generated by the set of vectors $(f_n)_{n\in\mathbb{N}}$ in a Hilbert space H. We shall write $\mu_{(b,a)}=\mu^1_{(b,a)}$.

In the case $m=1$ the generators $A^{R,1}_{kn}$ have the form

$$A^{R,1}_{kn}=x_{1k}D_{1n},\qquad k,n\in\mathbb{Z}.$$

The following lemmas are proved in [1]

Lemma 3.1. *The following three conditions are equivalent:*
(i) $(\mu_{(b,a)})^{L_t}\perp\mu_{(b,a)}$ *for all* $t\in GL(1,\mathbb{R})\setminus\{e\}$,
(ii) $(\mu_{(b,a)})^{L_{-E_{11}}}\perp\mu_{(b,a)}$,
(iii) $S^L_{11}(\mu)=4\sum_{n\in\mathbb{Z}}b_{1n}a^2_{1n}=\infty.$

Lemma 3.2. *For* $k,m\in\mathbb{Z}$ *we have*

$$x_{1k}x_{1m}\mathbf{1}\in\langle A^{R,1}_{kn}A^{R,1}_{mn}\mathbf{1}=x_{1k}x_{1m}D^2_{1n}\mathbf{1}\mid n\in\mathbb{Z}\rangle.$$

Lemma 3.3. *For any* $k\in\mathbb{Z}$ *we have*

$$x_{1k}\mathbf{1}\in\langle x_{1k}x_{1n}\mathbf{1}\mid n\in\mathbb{Z}\rangle\Leftrightarrow S^L_{11}(\mu)=\infty.$$

So, operators $x_{1k},\ k\in\mathbb{Z}$ are affiliated (see [6]) with the von Neumann algebra $\mathfrak{A}^1$ (notation $x_{1k}\ \eta\ \mathfrak{A}^1$) which completes the proof of the irreducibility for $m=1$.

4. The proof of the irreducibility in the cases $m = 2$

In the case $m = 2$ the generators $A_{kn} := A^{R,2}_{kn} := \frac{d}{dt}T^{R,\mu,2}_{I+tE_{kn}}\mid_{t=0}$ have the form:

$$A_{kn} = x_{1k}D_{1n} + x_{2k}D_{2n}, \qquad k, n \in \mathbb{Z}.$$

Lemma 4.1. *Three following conditions (i)–(iii) are equivalent for the measure* $\mu = \mu^2_{(b,a)}$:

(i) $\mu^{L_t} \perp \mu$ *for all* $t \in GL(2, \mathbb{R}) \setminus \{e\}$, *where* $L_t(x) = tx$, $x \in X_2$;

$$(ii)\begin{cases} (a) & \mu^{L_{\exp(tE_{12})}} \perp \mu, & \forall t \in \mathbb{R}\backslash\{0\}, \\ (b) & \mu^{L_{\exp(tE_{21})}} \perp \mu, & \forall t \in \mathbb{R}\backslash\{0\}, \\ (c) & \mu^{L_{\exp(tE_{12})P_1}} \perp \mu, & \forall t \in \mathbb{R}, \\ (d) & \mu^{L_{\exp(tE_{21})P_2}} \perp \mu, & \forall t \in \mathbb{R}, \\ (e) & \mu^{L_{\tau_-(\phi,s)}} \perp \mu, & \forall \tau_-(\phi, s) \in GL(2, \mathbb{R}) \setminus \{e\}, \end{cases}$$

$$(iii)\begin{cases} (a) & S^L_{12}(\mu) = \infty, & \\ (b) & S^L_{21}(\mu) = \infty, & \\ (c) & S^{L,-}_{12}(\mu, t) = \infty, & \forall t \in \mathbb{R}, \\ (d) & S^{L,-}_{21}(\mu, t) = \infty, & \forall t \in \mathbb{R}, \\ (e) & \Sigma^-_{12}(\tau_-(\phi, s)) = \infty, & \forall s > 0, \phi \in [0, 2\pi), \end{cases}$$

where

$$S^L_{kn}(\mu) = \sum_{m\in\mathbb{Z}} \frac{b_{km}}{2}\left(\frac{1}{2b_{nm}} + a^2_{nm}\right), \quad k \neq n, \tag{4.1}$$

$$S^{L,-}_{kn}(\mu, t) = \frac{t^2}{4}\sum_{m\in\mathbb{Z}} \frac{b_{km}}{b_{nm}} + \sum_{m\in\mathbb{Z}} \frac{b_{km}}{2}(-2a_{km} + ta_{nm})^2, \tag{4.2}$$

$$\Sigma^-_{12}(\tau_-(\phi, s)) = \sin^2\phi\Sigma_1(s) + \Sigma^-_2(\tau_-(\phi, s)),$$

$$\Sigma_1(s) := \sum_{n\in\mathbb{Z}}\left(s^2\sqrt{\frac{b_{1n}}{b_{2n}}} - s^{-2}\sqrt{\frac{b_{2n}}{b_{1n}}}\right)^2, \tag{4.3}$$

$$\Sigma^-_2(\tau_-(\phi, s)) := \sum_{n\in\mathbb{Z}}\big(4\sin^2\frac{\phi}{2}b_{1n} + 4\cos^2\frac{\phi}{2}s^{-4}b_{2n}\big)\big(\sin\frac{\phi}{2}a_{1n} - s^2\cos\frac{\phi}{2}a_{2n}\big)^2, \tag{4.4}$$

$$\exp(tE_{12}) = I + tE_{12} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \exp(tE_{21}) = I + tE_{21} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

$$\exp(tE_{12})P_1 = \begin{pmatrix} -1 & t \\ 0 & 1 \end{pmatrix}, \quad \exp(tE_{21})P_2 = \begin{pmatrix} 1 & 0 \\ t & -1 \end{pmatrix},$$

$$\tau_-(\phi, s) = \begin{pmatrix} \cos\phi & s^2 \sin\phi \\ s^{-2}\sin\phi & -\cos\phi \end{pmatrix} \textit{ and } P_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \ P_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Moreover, $(ii)(\sharp) \Leftrightarrow (iii)(\sharp)$ *for* $\sharp = a, b, c, d, e$.

Remark 4.1. We observe that

$$\tau_-(\phi, s) = \begin{pmatrix} \cos\phi & s^2 \sin\phi \\ s^{-2}\sin\phi & -\cos\phi \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} s^{-1} & 0 \\ 0 & s \end{pmatrix} P_2.$$

Remark 4.2. We note [12, Chapter V ,§8 Problems, 2, p. 147] that every element of $\mathrm{SL}(2,\mathbb{R})$ is conjugate to at least one matrix of the form

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \ a \neq 0, \ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \ \begin{pmatrix} -1 & t \\ 0 & -1 \end{pmatrix}, \ \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}.$$

Remark 4.3. The three following conditions are equivalent:

(i)	$\mu^{L_{\tau_-(\phi,s)}} \perp \mu,$	$\phi \in [0, 2\pi), \ s > 0,$
(ii)	$\Sigma_{12}^-(\tau_-(\phi,s)) = \sin^2\phi\,\Sigma_1(s) + \Sigma_2^-(\tau_-(\phi, s)) = \infty,$	$\phi \in [0, 2\pi), \ s > 0,$
(iii)	$\Sigma_1(s) + \Sigma_2(C_1, C_2) = \infty,$	$s > 0, \ (C_1, C_2) \in \mathbb{R}^2 \setminus \{0\},$

where $\Sigma_1(s)$ is defined by (4.3) and

$$\Sigma_2(C_1, C_2) := \sum_{n\in\mathbb{Z}} (C_1^2 b_{1n} + C_2^2 b_{2n})(C_1 a_{1n} + C_2 a_{2n})^2.$$

Proof. In Section 4.3 we shall show that $(i) \Leftrightarrow (ii)$ (see (4.66)), i.e., that

$$\mu^{L_{\tau_-(\phi,s)}} \perp \mu \Leftrightarrow \Sigma_{12}^-(\tau_-(\phi, s)) = \sin^2\phi\,\Sigma_1(s) + \Sigma_2^-(\tau_-(\phi, s)) = \infty.$$

To prove $(ii) \Leftrightarrow (iii)$ set

$$\sin\frac{\psi}{2} = \sin\frac{\phi}{2}\left(\sin^2\frac{\phi}{2} + s^4\cos^2\frac{\phi}{2}\right)^{-1/2}, \ \cos\frac{\psi}{2} = s^2\cos\frac{\phi}{2}\left(\sin^2\frac{\phi}{2} + s^4\cos^2\frac{\phi}{2}\right)^{-1/2}$$

then using (4.4) we get

$$\Sigma_2^-(\tau_-(\phi, s)) := \left(\sin^2\frac{\psi}{2} + s^4\cos^2\frac{\psi}{2}\right)^2 4\sum_{n\in\mathbb{Z}} \left(\sin^2\frac{\psi}{2} b_{1n} + s^{-8}\cos^2\frac{\psi}{2} b_{2n}\right)\times$$

$$\big(\sin\frac{\psi}{2}a_{1n}-\cos\frac{\psi}{2}a_{2n}\big)^2\sim\Sigma_2(\psi):=\sum_{n\in\mathbb{Z}}\big(\sin^2\frac{\psi}{2}b_{1n}+\cos^2\frac{\psi}{2}b_{2n}\big)\times$$

$$\big(\sin\frac{\psi}{2}a_{1n}-\cos\frac{\psi}{2}a_{2n}\big)^2=\Sigma(C_1,C_2)=\sum_{n\in\mathbb{Z}}(C_1^2b_{1n}+C_2^2b_{2n})(C_1a_{1n}+C_2a_{2n})^2.$$

□

4.1. Some orthogonality problem in measure theory

Our aim now is to find the minimal set of conditions of the orthogonality $\mu^{L_t}\perp\mu$ for all $t\in GL(2,\mathbb{R})\setminus\{e\}$. To be more precise, consider more general situation.

Let $\alpha: G\to \mathrm{Aut}(X)$ be a measurable action of a group G on a measurable space $(X,\mathfrak{B},\mu)$ with the following property: $\mu^{\alpha_t}\perp\mu$ for all $t\in G\setminus\{e\}$. Consider a subset $G^{\perp}(\mu)$ in the group G having the following property:

$$\text{if}\quad \mu^{\alpha_t}\perp\mu\ \forall t\in G^{\perp}(\mu)\quad\text{then}\quad \mu^{\alpha_t}\perp\mu\ \forall t\in G\setminus\{e\}. \tag{4.5}$$

Problem. Find a minimal subset $G_0^{\perp}(\mu)$ having the property (4.5).

Example 4.1. Consider the nilpotent group $B(m,\mathbb{R})$ of upper triangular real $m\times m$ matrices with units on the diagonal acting on the space X_m with the Gaussian product measure $\mu=\mu^m_{(b,a)}$, where X_m and μ are defined as follows (see details in [17, 18]):

$$X_m=\{I+\sum_{1\leq k\leq m}\sum_{k<n}x_{kn}E_{kn}\},\quad \mu^m=\otimes_{1\leq k\leq m}\otimes_{k<n}\mu_{(b_{kn},a_{kn})}.$$

Using results form [17] and [18] we conclude that the three following conditions are equivalent:

$$\begin{array}{lll}(i) & \mu^{L_t}\perp\mu & \forall t\in B(m,\mathbb{R})\backslash\{e\},\\ (ii) & \mu^{L_{\exp(tE_{kn})}}\perp\mu & \forall t\in\mathbb{R}\backslash\{0\},\quad 1\leq k<n\leq m,\\ (iii) & S^L_{kn}(\mu)=\infty & 1\leq k<n\leq m,\end{array}$$

where $S^L_{kn}(\mu)$ is defined by (4.1)

$$S^L_{kn}(\mu)=\sum_{r=n+1}^{\infty}\frac{b_{kr}}{2}\Big(\frac{1}{2b_{nr}}+a^2_{nr}\Big).$$

In fact, it is sufficient to fix a nontrivial point $t_{kn}\neq 0$ on any subgroup $\exp(tE_{kn})=I+tE_{kn},\ t\in\mathbb{R},\ 1\leq k<n\leq m$. In this case the subset $G_0^{\perp}(\mu)$ is discrete and consists of $m(m-1)/2$ points:

$$G_0^{\perp}(\mu,t)=\Big(I+t_{kn}E_{kn}\mid t_{kn}\in\mathbb{R}\backslash\{0\},1\leq k<n\leq m\ \Big),$$

where $t=(t_{kn})_{kn}\in(\mathbb{R}\backslash\{0\})^{m(m-1)/2}$. For $t_1\neq t_2\in(\mathbb{R}\backslash\{0\})^{m(m-1)/2}$ we get two distinct minimal subsets $G_0^{\perp}(\mu^m,t_1)$ and $G_0^{\perp}(\mu^m,t_2)$.

Example 4.2. Consider the solvable group $Bor(m,\mathbb{R})$ of upper triangular real $m\times m$ matrices with nonzero elements on the diagonal acting on the space X_m with the Gaussian product measure $\mu=\mu^m_{(b,a)}$, where X_m and μ are defined as follows (see details in [1])

$$X_m=\{x=\sum_{1\leq k\leq m}\sum_{k\leq n}x_{kn}E_{kn}\},\quad \mu^m_{(b,a)}=\otimes_{1\leq k\leq m}\otimes_{k\leq n}\mu_{(b_{kn},a_{kn})}.$$

Using [1, Theorem 5] we conclude that the following three conditions are equivalent:

$$\begin{array}{llll}
(i) & \mu^{L_t}\perp\mu, & & \forall t\in Bor(m,\mathbb{R})\backslash\{e\},\\
(ii) & \mu^{L_{\exp(tE_{kn})}}\perp\mu\ \forall t\in\mathbb{R}\backslash\{0\}, & & 1\leq k<n\leq m,\\
 & \mu^{L_{\exp(tE_{kn})P_k}}\perp\mu\ \forall t\in\mathbb{R}, & & 1\leq k<n\leq m,\\
(iii) & S^L_{kn}(\mu)=\infty, & S^{L,-}_{kn}(\mu,t)=\infty, & 1\leq k<n\leq m,
\end{array}$$

where $S^{L,-}_{kn}(\mu,t)$ is defined by (4.2). As before, it is sufficient to fix a nontrivial point $t_{kn}\neq 0$ on any subgroup $\exp(tE_{kn})=I+tE_{kn},\ t\in\mathbb{R}$. But on the curves $\exp(tE_{kn})P_k$ we can not omit any point $t\in\mathbb{R}$. Finally, a minimal subset depending on the choice of $t=(t_{kn})_{kn}\in(\mathbb{R}\backslash\{0\})^{m(m-1)/2}$ can be chosen as follows:

$$G_0^{\perp}(\mu,t)=\Big(\exp(t_{kn}E_{kn})=I+t_{kn}E_{kn}\mid t_{kn}\in\mathbb{R}\backslash\{0\},1\leq k<n\leq m\ \Big)\bigcup$$

$$\Big(\exp(tE_{kn})P_k\mid\ \forall t\in\mathbb{R},1\leq k<n\leq m\ \Big)$$

where $P_k=I-2E_{kk}$. For example, for $m=2$ we get $P_1=\mathrm{diag}(-1,1)$ and $P_2=\mathrm{diag}(1,-1)$.

Example 4.3. In the case of the group $G = \mathrm{GL}(2,\mathbb{R})$ acting on the space X_2 defined by (2.3) with the measure $\mu^2_{(b,a)}$ defined by (2.4) using Lemmas 4.1, we conclude that the description of the set $G_0^{\perp}(\mu^2_{(b,a)})$ is as follows:

$$G_0^{\perp}(\mu^2_{(b,a)}, t_{12}, t_{21}) = \Big(\exp(t_{12}E_{12}),\ \exp(t_{21}E_{21}) \mid t_{12}, t_{21} \in \mathbb{R}\backslash\{0\}\Big) \bigcup$$

$$\Big(\exp(tE_{12})P_1,\ \exp(tE_{21})P_2 \mid \forall t \in \mathbb{R}\Big) \bigcup \Big(\tau_-(\phi, s) \mid \forall s > 0,\ \phi \in [0, 2\pi)\Big).$$

Remark 4.4. We note that except the one-parameter groups $E_{kn}(t) = I + tE_{kn}$, $t \in \mathbb{R}$ all other element from the set $G_0^{\perp}(\mu)$ for $G = \mathrm{GL}(2,\mathbb{R})$ are of order 2, i.e., if $g \in \{\exp(tE_{kn})P_k,\ \tau_-(\phi, s)\}$ then $g^2 = e$.

4.2. Approximation of x_{kn} and D_{kn}

We will formulate several lemmas, which will be useful for approximation of the independent variables x_{kn} and operators D_{kn} by combinations of the generators A_{kn}. For short, we shall write A_{kn} instead of $A^{R,2}_{kn}$.

In what follows we use the following notation for $f, g \in \mathbb{R}^m$

$$\Delta(f, g) = \frac{\Gamma(f) + \Gamma(f, g)}{\Gamma(g) + 1}. \tag{4.6}$$

Lemma 4.2. *For any $k, t \in \mathbb{Z}$ one has*

$$x_{1n}x_{1t} \in \langle A_{nk}A_{tk}\mathbf{1} \mid k \in \mathbb{Z}\rangle \ \Leftrightarrow\ \lim_m \Sigma_{1,m}(x, x) = \infty,$$

where $\Sigma_{1,m}(x, x) = \Delta(f^1_m, g^1_m)$ and

$$f^1_m = \Big(\frac{b_{1k}}{\sqrt{b^2_{1k} + 2b_{1k}b_{2k}}}\Big)^m_{k=-m}, \quad g^1_m = \Big(\frac{b_{2k}}{\sqrt{b^2_{1k} + 2b_{1k}b_{2k}}}\Big)^m_{k=-m}. \tag{4.7}$$

Lemma 4.3. *For any $k, t \in \mathbb{Z}$ we have*

$$x_{2k}x_{2t} \in \langle A_{kn}A_{tn}\mathbf{1} \mid n \in \mathbb{Z}\rangle \ \Leftrightarrow\ \lim_m \Sigma_{2,m}(x, x) = \infty,$$

where $\Sigma_{2,m}(x, x) = \Delta(f^2_m, g^2_m)$ and

$$f^2_m = \Big(\frac{b_{2k}}{\sqrt{b^2_{2k} + 2b_{1k}b_{2k}}}\Big)^m_{k=-m}, \quad g^2_m = \Big(\frac{b_{1k}}{\sqrt{b^2_{2k} + 2b_{1k}b_{2k}}}\Big)^m_{k=-m}. \tag{4.8}$$

Remark 4.5. We say that two series $\sum_n a_n$ and $\sum_n b_n$ with positive a_n, b_n are *equivalent* if they are simultaneously convergent or divergent. In this case we shall use the notaions $\sum_n a_n \sim \sum_n b_n$. Using the obvious equivalence of the following two series with positive a_n and b_n

$$\sum_{n\in\mathbb{N}} \frac{a_n}{a_n+b_n} \sim \sum_{n\in\mathbb{N}} \frac{a_n}{b_n} \tag{4.9}$$

we have the following estimation (we set $\Sigma^{12} = \sum_{k\in\mathbb{Z}} \frac{b_{1k}}{b_{2k}}$ and $\Sigma^{21} = \sum_{k\in\mathbb{Z}} \frac{b_{2k}}{b_{1k}}$)

$$\begin{aligned}
\|f^1\|^2 &= \sum_{k\in\mathbb{Z}} \frac{b_{1k}^2}{b_{1k}^2+2b_{1k}b_{2k}} \sim \sum_{k\in\mathbb{Z}} \frac{b_{1k}}{2b_{2k}} = \frac{\Sigma^{12}}{2},\\
\|f^2\|^2 &= \sum_{k\in\mathbb{Z}} \frac{b_{2k}^2}{b_{2k}^2+2b_{1k}b_{2k}} \sim \sum_{k\in\mathbb{Z}} \frac{b_{2k}}{2b_{1k}} = \frac{\Sigma^{21}}{2},\\
\|g^1\|^2 &= \sum_{k\in\mathbb{Z}} \frac{b_{2k}^2}{b_{1k}^2+2b_{1k}b_{2k}} < \sum_{k\in\mathbb{Z}} \frac{b_{2k}}{2b_{1k}} = \frac{\Sigma^{21}}{2},\\
\|g^1\|^2 &= \sum_{k\in\mathbb{Z}} \frac{b_{1k}^2}{b_{2k}^2+2b_{1k}b_{2k}} < \sum_{k\in\mathbb{Z}} \frac{b_{1k}}{2b_{2k}} = \frac{\Sigma^{12}}{2},
\end{aligned}$$

we conclude that $\lim_m \Sigma_{1,m}(x,x)=\infty$ if

$$\lim_m \Sigma'_{1,m}(x,x) := \lim_m \Big(\sum_{k=-m}^{m} \frac{b_{1k}}{b_{2k}}\Big)\Big(\sum_{k=-m}^{m} \frac{b_{2k}}{b_{1k}}\Big)^{-1} = \Sigma^{12}/\Sigma^{21} = \infty$$

and $\lim_m \Sigma_{2,m}(x,x)=\infty$ if

$$\lim_m \Sigma'_{2,m}(x,x) := \lim_m \Big(\sum_{k=-m}^{m} \frac{b_{2k}}{b_{1k}}\Big)\Big(\sum_{k=-m}^{m} \frac{b_{1k}}{b_{2k}}\Big)^{-1} = \Sigma^{21}/\Sigma^{12} = \infty.$$

Lemma 4.4. *For any $n\in\mathbb{Z}$ we have*

$$D_{1n}\mathbf{1} \in \langle A_{kn}\mathbf{1} \mid k\in\mathbb{Z}\rangle \quad \Leftrightarrow \quad \lim_m \Sigma_{1,m}(D) = \infty,$$

where $\Sigma_{1,m}(D) = \Delta(f_m, g_m)$ and

$$f_m = \Big(a_{1k}\Big(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}}\Big)^{-1/2}\Big)_{k=-m}^{m}, \quad g_m = \Big(a_{2k}\Big(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}}\Big)^{-1/2}\Big)_{k=-m}^{m}. \tag{4.10}$$

Lemma 4.5. *Set $\Sigma_{2,m}(D) = \Delta(g_m, f_m)$. For any $n \in \mathbb{Z}$ we get*

$$D_{2n}\mathbf{1} \in \langle A_{kn}\mathbf{1} \mid k \in \mathbb{Z}\rangle \quad \Leftrightarrow \quad \lim_m \Sigma_{2,m}(D) = \infty.$$

Lemma 4.6. *For any $k \in \mathbb{Z}$ we get*

$$x_{1k}\mathbf{1} \in \langle D_{1n}A_{kn}\mathbf{1} \mid n \in \mathbb{Z}\rangle \quad \Leftrightarrow \quad \sum_{n\in\mathbb{Z}} \frac{b_{1n}}{b_{2n}} = \infty.$$

Lemma 4.7. *For any $k \in \mathbb{Z}$ we have*

$$x_{2k}\mathbf{1} \in \langle D_{2n}A_{kn}\mathbf{1} \mid n \in \mathbb{Z}\rangle \quad \Leftrightarrow \quad \sum_{n\in\mathbb{Z}} \frac{b_{2n}}{b_{1n}} = \infty.$$

Our aim now is to show that some of the expressions $\Sigma_{1,m}(x, x)$, $\Sigma_{2,m}(x, x)$ and $\Sigma_{1,m}(D)$, $\Sigma_{2,m}(D)$ tend to infinity if $\mu^{L_t} \perp \mu$ for all $t \in GL(2, \mathbb{R}) \setminus \{e\}$.

Let $\Gamma(f_1, f_2, \ldots, f_n)$ be the Gramm determinant and $\gamma(f_1, f_2, \ldots, f_n)$ be the Gramm matrix of n vectors $f_1, f_2, \ldots, f_n$ in a Hilbert space (see [7]). The following lemma is trivial and well known but we need exact formulas.

Lemma 4.8. *Let f_1, f_2 be two vectors in a Hilbert space. The distance $\delta\langle f_2, f_1\rangle$ of the vector f_2 from the line $\langle f_1\rangle$ generated by f_1 is given by the following formula:*

$$\delta^2\langle f_2, f_1\rangle = \|f_2 - \frac{(f_2, f_1)}{(f_1, f_1)} f_1\|^2 = \frac{\Gamma(f_1, f_2)}{\Gamma(f_1)}. \tag{4.11}$$

Proof. Obviously, $\delta^2\langle f_2, f_1\rangle = \|f_2 - f_0\|^2$ where $f_0 = C_1 f_1$ such that $(f_2 - f_0, f_1) = 0$. We have

$$0 = (f_2 - f_0, f_1) = (f_2, f_1) - C_1(f_1, f_1) \quad \text{hence,} \quad C_1 = \frac{(f_2, f_1)}{(f_1, f_1)}.$$

Finally,

$$\delta^2\langle f_2, f_1\rangle = \|f_2 - f_0\|^2 = \|f_2 - C_1 f_1\|^2 = (f_2, f_2) - 2C_1(f_2, f_1) + C_1^2(f_1, f_1) =$$
$$(f_2, f_2) - \frac{2(f_2, f_1)(f_2, f_1)}{(f_1, f_1)} + \frac{(f_2, f_1)^2}{(f_1, f_1)^2}(f_1, f_1) =$$
$$\frac{(f_2, f_2)(f_1, f_1) - (f_2, f_1)(f_1, f_2)}{(f_1, f_1)} = \frac{\Gamma(f_1, f_2)}{\Gamma(f_1)}.$$

□

Lemma 4.9. *Let $f=(f_k)_{k\in\mathbb{N}}$ and $g=(g_k)_{k\in\mathbb{N}}$ be two real vectors such that $\|f\|^2=\infty$ where $\|f\|^2=\sum_k f_k^2$. Denote by $f_{(n)},\ g_{(n)}\in\mathbb{R}^n$ their projections to the subspace $\mathbb{R}^n$, i.e., $f_{(n)}=(f_k)_{k=1}^n,\quad g_{(n)}=(g_k)_{k=1}^n$ and set*

$$\Delta(f_{(n)},g_{(n)})=\frac{\Gamma(f_{(n)})+\Gamma(f_{(n)},g_{(n)})}{\Gamma(g_{(n)})+1}\quad \textit{then}\quad \lim_{n\to\infty}\Delta(f_{(n)},g_{(n)})=\infty \tag{4.12}$$

in the following cases:

$$\begin{array}{ll}(a) & \|g\|^2<\infty,\\ (b) & \|g\|^2=\infty,\quad \textit{and}\quad \lim_{n\to\infty}\frac{\|f_{(n)}\|}{\|g_{(n)}\|}=\infty,\\ (c) & \|f\|^2=\|g\|^2=\|f+sg\|^2=\infty,\quad \textit{for all}\quad s\in\mathbb{R}\setminus\{0\}.\end{array}$$

Proof. Obviously $\lim_{n\to\infty}\Delta(f_{(n)},g_{(n)})=\infty$ if conditions (a) or (b) hold. The implication $(c)\Rightarrow$ (4.12) is based on the following lemma. □

Lemma 4.10. *Let $f=(f_k)_{k\in\mathbb{N}}$ and $g=(g_k)_{k\in\mathbb{N}}$ be two real vectors such that*

$$\|f\|^2=\|g\|^2=\|C_1f+C_2g\|^2=\infty,\quad \textit{for all}\quad (C_1,C_2)\in\mathbb{R}^2\setminus\{0\}, \tag{4.13}$$

$$\textit{then}\quad \lim_{n\to\infty}\frac{\Gamma(f_{(n)},g_{(n)})}{\Gamma(g_{(n)})}=\infty\quad \textit{and}\quad \lim_{n\to\infty}\frac{\Gamma(f_{(n)},g_{(n)})}{\Gamma(f_{(n)})}=\infty. \tag{4.14}$$

Proof. Assume that $\frac{\|f_{(n)}\|}{\|g_{(n)}\|}\leq C_1,\ \forall n\in\mathbb{N}$. The case $\frac{\|f_{(n)}\|}{\|g_{(n)}\|}\geq C_1$ is similar. In this case $\frac{\Gamma(f_{(n)},g_{(n)})}{\Gamma(g_{(n)})}\leq C_1^2\frac{\Gamma(f_{(n)},g_{(n)})}{\Gamma(f_{(n)})}$ therefore, to prove (4.14) it is sufficient to prove that $\lim_{n\to\infty}\frac{\Gamma(f_{(n)},g_{(n)})}{\Gamma(g_{(n)})}=\infty$. Let us suppose the opposite, i.e., that for all $n\in\mathbb{N}$ holds

$$\frac{\Gamma(f_{(n)},g_{(n)})}{\Gamma(g_{(n)})}\leq C. \tag{4.15}$$

Set $t_n=\frac{\|f_{(n)}\|}{\|g_{(n)}\|}$ then by the inequality $\frac{\|f_{(n)}\|}{\|g_{(n)}\|}\leq C_1$ there exists a subsequence t_{n_k} such that the limit exists

$$\lim_{k\to\infty}t_{n_k}=t_0\in[0,C_1].$$

Let α_n be an angle between two vectors $f_{(n)},g_{(n)}\in\mathbb{R}^n$. Since $\frac{\Gamma(f,g)}{\Gamma(g)}$ is the square of the distance of the vector f from the line generated by g by Lemma 4.8, we have

$$\frac{\Gamma(f_{(n)},g_{(n)})}{\Gamma(g_{(n)})}=\|f_{(n)}\|^2\sin^2\alpha_n\leq C,\quad \text{therefore}\quad \alpha_n\sim\|f_n\|^{-1}\to 0. \tag{4.16}$$

For $k,n \in \mathbb{N}$ set $M(k,n) = \begin{vmatrix} f_k & f_n \\ g_k & g_n \end{vmatrix}$, then by the Lagrange identity for $f_{(m)} = (f_k)_{k=1}^m,\ g_{(m)} = (g_k)_{k=1}^m \in \mathbb{R}^m$ ([3, Ch.11, §6, formulae (7)]) we have

$$\Gamma(f_{(m)}, g_{(m)}) = \sum_{k<n\le m} M^2(k,n),$$

therefore, the inequality (4.15) will have the following form

$$\frac{\Gamma(f_{(m)}, g_{(m)})}{\Gamma(g_{(m)})} = \frac{\sum_{k<n\le m} M^2(k,n)}{\sum_{k=1}^m g_k^2} \le C, \quad m \in \mathbb{N}. \tag{4.17}$$

For $t \in \mathbb{R}$ and $f_{(n)}, g_{(n)} \in \mathbb{R}^n$ introduce the function

$$F_n(t) = \|f_{(n)} - tg_{(n)}\|^2 = (f_{(n)}, f_{(n)}) - 2t(f_{(n)}, g_{(n)}) + t^2(g_{(n)}, g_{(n)}).$$

The minimum of the function $F_n(t)$ is reached at $t_0^{(n)} = \frac{(f_{(n)}, g_{(n)})}{(g_{(n)}, g_{(n)})}$ therefore, we have

$$F_n(t) = (g_{(n)}, g_{(n)})(t - t_0^{(n)})^2 + \frac{\Gamma(f_{(n)}, g_{(n)})}{\Gamma(g_{(n)})}, \quad F_n(t_0^{(n)}) = \frac{\Gamma(f_{(n)}, g_{(n)})}{\Gamma(g_{(n)})},$$

hence,

$$F_n(t_0) - F_n(t_0^{(n)}) = (g_{(n)}, g_{(n)})(t_0 - t_0^{(n)})^2. \tag{4.18}$$

Since $F_n(t_0^{(n)}) = \frac{\Gamma(f_{(n)}, g_{(n)})}{\Gamma(g_{(n)})}$ is bounded by assumption and

$$\lim_{n\to\infty} F_n(t) = \lim_{n\to\infty} \|f_{(n)} - tg_{(n)}\|^2 = \infty \quad \text{for all} \quad t \in \mathbb{R},$$

by the condition (4.13), we conclude that $\lim_{n\to\infty} \big(F_n(t_0) - F_n(t_0^n)\big) = \infty$.

We show that condition (4.15) implies that $F_n(t_0) - F_n(t_0^{(n)})$ is bounded. This contradiction will prove the lemma. Indeed, we have

$$t_0^{(n+1)} - t_0^{(n)} = \frac{(f_{(n+1)}, g_{(n+1)})}{(g_{(n+1)}, g_{(n+1)})} - \frac{(f_{(n)}, g_{(n)})}{(g_{(n)}, g_{(n)})} = -\frac{\sum_{k=1}^n M(k, n+1) g_k g_{n+1}}{(g_{(n)}, g_{(n)})(g_{(n+1)}, g_{(n+1)})}$$

and

$$t_0^{(n+m)} - t_0^{(n)} = \frac{(f_{(n+m)}, g_{(n+m)})}{(g_{(n+m)}, g_{(n+m)})} - \frac{(f_{(n)}, g_{(n)})}{(g_{(n)}, g_{(n)})}$$

$$= -\frac{\sum_{k=1}^{n}\sum_{r=n+1}^{n+m} M(k,r)g_k g_r}{(g_{(n)},g_{(n)})(g_{(n+m)},g_{(n+m)})} = -\frac{(M_{n,m}g^{n,m},g_{(n)})}{(g_{(n)},g_{(n)})(g_{(n+m)},g_{(n+m)})} \qquad (4.19)$$

where the vector $g^{n,m} \in \mathbb{R}^m$ and the rectangular matrix $M_{n,m} \in \mathrm{Mat}(\mathbb{R}, n \times m)$ are defined as follows:

$$g^{n,m} = (g_k)_{k=n+1}^{n+m} \quad \text{and} \quad M_{n,m} = (M(k,r))_{k,r} \quad 1 \le k \le n, \quad n+1 \le r \le n+m.$$

We observe that $\lim_n t_0^{(n)} = \lim_n t_n = t_0$. Indeed, if $n \to \infty$ by (4.16) we have

$$t_0^{(n)} = \frac{(f_{(n)},g_{(n)})}{(g_{(n)},g_{(n)})} = \frac{\|f_{(n)}\|\|g_{(n)}\|\cos\alpha_n}{\|g_{(n)}\|^2} = t_n\cos\alpha_n \to t_0.$$

Finally, for all $n, m \in \mathbb{N}$ we get by (4.18), (4.19) and the Schwartz inequality

$$F_n(t_0^{(n+m)}) - F_n(t_0^{(n)}) = (g_{(n)},g_{(n)})(t_0^{(n+m)} - t_0^{(n)})^2 =$$

$$(g_{(n)},g_{(n)})\left[\frac{(M_{n,m}g^{n,m},g_{(n)})}{(g_{(n)},g_{(n)})(g_{(n+m)},g_{(n+m)})}\right]^2 \le$$

$$\frac{\|g_{(n)}\|^2\|M_{n,m}g^{n,m}\|^2\|g_{(n)}\|^2}{\|g_n(n)\|^4\|g_{(n+m)}\|^4} \le \frac{\|M_{n,m}\|_{\sigma_2}^2\|g^{n,m}\|^2}{\|g_{(n+m)}\|^4} \le \frac{\|M_{n+m}\|_{\sigma_2}^2}{\|g_{(n+m)}\|^2} \le C,$$

where $M_m := (M(k,r))_{k<r\le m}$ and

$$\|M_{n,m}\|_{\sigma_2}^2 = \sum_{k=1}^{n}\sum_{r=n+1}^{n+m} M^2(k,r), \quad \|M_m\|_{\sigma_2}^2 = \sum_{k<r\le m} M^2(k,r) = \Gamma(f_{(m)},g_{(m)}).$$

Fix $\varepsilon > 0$. Since $\lim_m t_0^{(m)} = t_0$ and the functions $F_n(t)$ are continuous we conclude that there exists $m_n \ge n$ such that $F_n(t_0^{(m)}) > F_n(t_0) - \varepsilon, \forall m \ge m_n$, in particular, $F_n(t_0^{(m_n)}) > F_n(t_0) - \varepsilon$. Since $\lim_n F_n(t_0) = \infty$ we conclude that

$$\lim_n F_n(t_0^{(m_n)}) \ge \lim_n (F_n(t_0) - \varepsilon) = \infty$$

that contradicts the condition $F_n(t_0^{(n+m)}) - F_n(t_0^{(n)}) \le C$ for all $m, n \in \mathbb{N}$. $\Box$

Lemma 4.11. *If $\mu^{L_t} \perp \mu$ for all $t \in GL(2,\mathbb{R}) \setminus \{e\}$, we can approximate one of the following pair of operators: (x_{1n}, x_{2n}), (x_{1n}, D_{2n}), (D_{1n}, x_{2n}), or (D_{1n}, D_{2n}).*

PROOF. For the convenience of the readers we collect the important formulas below:

$$\Sigma_{1,m}(x,x)=\frac{\Gamma(f_m^1)+\Gamma(f_m^1,g_m^1)}{\Gamma(g_m^1)+1}=\frac{\sum_{k=-m}^m\frac{b_{1k}^2}{b_{1k}^2+2b_{1k}b_{2k}}+\Gamma(f_m^1,g_m^1)}{\sum_{k=-m}^m\frac{b_{2k}^2}{b_{1k}^2+2b_{1k}b_{2k}}+1}, \tag{4.20}$$

$$\Sigma_{2,m}(x,x)=\frac{\Gamma(f_m^2)+\Gamma(f_m^2,g_m^2)}{\Gamma(g_m^2)+1}=\frac{\sum_{k=-m}^m\frac{b_{2k}^2}{b_{2k}^2+2b_{1k}b_{2k}}+\Gamma(f_m^2,g_m^2)}{\sum_{k=-m}^m\frac{b_{1k}^2}{b_{2k}^2+2b_{1k}b_{2k}}+1}, \tag{4.21}$$

$$\Sigma_{1,m}(D)=\frac{\Gamma(f_m)+\Gamma(f_m,g_m)}{\Gamma(g_m)+1}=\frac{\sum_{k=-m}^m\frac{a_{1k}^2}{\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}}+\Gamma(f_m,g_m)}{\sum_{k=-m}^m\frac{a_{2k}^2}{\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}}+1}, \tag{4.22}$$

$$\Sigma_{2,m}(D)=\frac{\Gamma(g_m)+\Gamma(g_m,f_m)}{\Gamma(f_m)+1}=\frac{\sum_{k=-m}^m\frac{a_{2k}^2}{\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}}+\Gamma(g_m,f_m)}{\sum_{k=-m}^m\frac{a_{1k}^2}{\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}}+1}, \tag{4.23}$$

$$f_m^1=\Big(\frac{b_{1k}}{\sqrt{b_{1k}^2+2b_{1k}b_{2k}}}\Big)_{k=-m}^m,\quad g_m^1=\Big(\frac{b_{2k}}{\sqrt{b_{1k}^2+2b_{1k}b_{2k}}}\Big)_{k=-m}^m, \tag{4.24}$$

$$f_m^2=\Big(\frac{b_{2k}}{\sqrt{b_{2k}^2+2b_{1k}b_{2k}}}\Big)_{k=-m}^m,\quad g_m^2=\Big(\frac{b_{1k}}{\sqrt{b_{2k}^2+2b_{1k}b_{2k}}}\Big)_{k=-m}^m, \tag{4.25}$$

$$f_m=\Big(a_{1k}\Big(\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}\Big)^{-1/2}\Big)_{k=-m}^m,\quad g_m=\Big(a_{2k}\Big(\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}\Big)^{-1/2}\Big)_{k=-m}^m. \tag{4.26}$$

To estimate $\Sigma_{1,m}(x,x)$ and $\Sigma_{2,m}(x,x)$ consider three possibilities:

$$(1)\ \Sigma^{12}:=\sum_{k\in\mathbb{Z}}\frac{b_{1k}}{b_{2k}}<\infty,\ \ (2)\ \Sigma^{21}:=\sum_{k\in\mathbb{Z}}\frac{b_{2k}}{b_{1k}}<\infty,\ \ (3)\ \sum_{k\in\mathbb{Z}}\frac{b_{1k}}{b_{2k}}=\sum_{k\in\mathbb{Z}}\frac{b_{2k}}{b_{1k}}=\infty. \tag{4.27}$$

We present the results in the table I.

table I	(1)	(2)	(3a)	(3b)	(3c)
Σ^{12}	$<\infty$		∞	∞	∞
Σ^{21}		$<\infty$	∞	∞	∞
$\Vert g^1\Vert$			$<\infty$		
$\Vert g^2\Vert$				$<\infty$	
Lemma	4.3, 4.4, 4.5, 4.10	4.2, 4.4, 4.5 4.10	4.2,	4.3,	4.2, 4.3, 4.10, 4.13
	$x_{2n},\ D_{1n}, D_{2n}$	$x_{1n},\ D_{1n},\ D_{2n}$	$x_{1n},\ x_{2n}$	$x_{1n},\ x_{2n}$	$x_{1n},\ x_{2n}$

Case (1). If $\Sigma^{12} < \infty$ then $\Sigma^{21} = \infty$ and we have $\lim_{m\to\infty} \Sigma_{2,m}(x, x) = \infty$ by Remark 4.5. Hence, $x_{2n}x_{2t}\ \eta\ \mathfrak{A}$, by Lemma 4.3 and $x_{2n}\ \eta\ \mathfrak{A}$, by Lemma 3.3. We can approximate D_{1n} and D_{2n} by Lemmas 4.4, 4.5 and Lemma 4.10:

$$D_{1n}\ \eta\ \mathfrak{A} \quad \text{if}\ \frac{\Gamma(f_m)+\Gamma(f_m,g_m)}{\Gamma(g_m)+1}\to\infty,\ \ D_{2n}\ \eta\ \mathfrak{A} \quad \text{if}\ \frac{\Gamma(g_m)+\Gamma(g_m,f_m)}{\Gamma(f_m)+1}\to\infty,$$

where f_m and g_m are defined by (4.26). Set

$$f = \Big(a_{1k}\Big(\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}\Big)^{-1/2}\Big)_{k\in\mathbb{Z}}, \quad g = \Big(a_{2k}\Big(\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}\Big)^{-1/2}\Big)_{k\in\mathbb{Z}}. \quad (4.28)$$

Since $\sum_{k\in\mathbb{Z}}\frac{b_{1k}}{b_{2k}} < \infty$, we conclude that

$$\Vert f\Vert^2 = \Vert g\Vert^2 = \Vert f - sg\Vert^2 = \infty. \quad (4.29)$$

Indeed, we have

$$\Vert f\Vert^2 = \sum_{k\in\mathbb{Z}}\frac{a_{1k}^2}{\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}} = \sum_{k\in\mathbb{Z}}\frac{b_{1k}a_{1k}^2}{\frac{1}{2}+\frac{b_{1k}}{2b_{2k}}} \sim 2\sum_{k\in\mathbb{Z}} b_{1k}a_{1k}^2 = S_{11}^L(\mu) = \infty,$$

$$\Vert g\Vert^2 = \sum_{k\in\mathbb{Z}}\frac{b_{1k}a_{2k}^2}{\frac{1}{2}+\frac{b_{1k}}{2b_{2k}}} \sim \sum_{k\in\mathbb{Z}} b_{1k}a_{2k}^2 \sim \sum_{k\in\mathbb{Z}}\frac{b_{1k}}{2}\Big(\frac{1}{2b_{2k}}+a_{2k}^2\Big) = S_{12}^L(\mu) = \infty,$$

$$\Vert f - sg\Vert^2 = \sum_{k\in\mathbb{Z}}\frac{b_{1k}(a_{1k}-sa_{2k})^2}{\frac{1}{2}+\frac{b_{1k}}{2b_{2k}}} \sim \sum_{k\in\mathbb{Z}} b_{1k}(a_{1k}-sa_{2k})^2 = \frac{1}{4}\sum_{k\in\mathbb{Z}} b_{1k}(-2a_{1k}+2sa_{2k})^2$$

$$\sim \frac{1}{2}\Big(\frac{(2s)^2}{4}\sum_{k\in\mathbb{Z}}\frac{b_{1k}}{b_{2k}} + \sum_{k\in\mathbb{Z}}\frac{b_{1k}}{2}(-2a_{1k}+2sa_{2k})^2\Big) = \frac{1}{2}S_{12}^{L,-}(\mu,t) = \infty,$$

for $t=2s$ (see (4.2)). Therefore, by Lemma 4.10 we conclude (see(4.14)) that $\lim_{n\to\infty}\frac{\Gamma(f_{(n)},g_{(n)})}{\Gamma(g_{(n)})}=\infty$ and $\lim_{n\to\infty}\frac{\Gamma(f_{(n)},g_{(n)})}{\Gamma(f_{(n)})}=\infty$, so $D_{1n},\ D_{2n}\ \eta\ \mathfrak{A}$ by Lemmas 4.4 and 4.5. Finally, $x_{2n}\ D_{1n}\ D_{2n}\ \eta\ \mathfrak{A}$. Now we get $A_{kn}-x_{2k}D_{2n}=x_{1k}D_{1n},\ k,n\in\mathbb{Z}$ and the proof is complete since we are in the case $m=1$.

Case (2). If $\Sigma^{21}<\infty$ then $\Sigma^{12}=\infty$ and we have $\lim_{m\to\infty}\Sigma_{1,m}(x,x)=\infty$, by Remark 4.5. Hence, $x_{1n}x_{1t}\,\eta\,\mathfrak{A}$, by Lemma 4.2 and $x_{1n}\,\eta\,\mathfrak{A}$, by Lemma 3.3. As in the previous case, the condition $\sum_{k\in\mathbb{Z}}\frac{b_{2k}}{b_{1k}}<\infty$ implies

$$\|f\|^2\sim S_{21}^L(\mu)=\infty,\quad \|g\|^2\sim S_{22}^L(\mu)=\infty,\quad \|f-sg\|^2\sim S_{21}^{L,-}(\mu,t)=\infty,$$

for $t=\frac{2}{s}$. Exactly, as in the case (1), we can approximate D_{1n} and D_{2n}. Finally, $x_{1n}\ D_{1n}\ D_{2n}\ \eta\ \mathfrak{A}$. Further, $A_{kn}-x_{1k}D_{1n}=x_{2k}D_{2n},\ k,n\in\mathbb{Z}$ and the proof is complete.

Case (3). Let $\sum_{k\in\mathbb{Z}}\frac{b_{1k}}{b_{2k}}=\sum_{k\in\mathbb{Z}}\frac{b_{2k}}{b_{1k}}=\infty$. Set $c_n=\frac{b_{2n}}{b_{1n}},\ n\in\mathbb{Z}$. The vectors f_m^1, g_m^1, f_m^2, g_m^2 are defined as follows (see (4.24) and (4.25)):

$$f_m^1=\Big(\frac{1}{\sqrt{1+2c_n}}\Big)_{-m}^m,\qquad g_m^1=\Big(\frac{c_n}{\sqrt{1+2c_n}}\Big)_{-m}^m,\tag{4.30}$$

$$f_m^2=\Big(\sqrt{\frac{c_n}{c_n+2}}\Big)_{-m}^m,\qquad g_m^2=\Big(\frac{1}{\sqrt{c_n^2+2c_n}}\Big)_{-m}^m.\tag{4.31}$$

We show that

$$\|f^1\|^2=\|f^2\|^2=\|g^1\|^2+\|g^2\|^2=\infty.\tag{4.32}$$

Indeed, we have

$$\|f^1\|^2=\sum_{n\in\mathbb{Z}}(1+2c_n)^{-1}\sim\sum_{n\in\mathbb{Z}}c_n^{-1}=\Sigma^{12}=\infty,$$

$$\|f^2\|^2=\sum_{n\in\mathbb{Z}}c_n(c_n+2)^{-1}\sim\sum_{n\in\mathbb{Z}}c_n=\Sigma^{21}=\infty.$$

Let us suppose that $\|g^1\|^2+\|g^2\|^2<\infty$ then

$$\infty>\|g^1\|^2+\|g^2\|^2=\sum_{n\in\mathbb{Z}}\Big(\frac{c_n^2}{1+2c_n}+\frac{1}{c_n^2+2c_n}\Big)>\sum_{n\in\mathbb{Z}}\frac{1+c_n^2}{(1+c_n)^2},$$

hence, $\sum_{n\in\mathbb{Z}}\frac{1}{(1+c_n)^2}<\infty$ and $\sum_{n\in\mathbb{Z}}\frac{c_n^2}{(1+c_n)^2}<\infty$ therefore,

$$\infty>\sum_{n\in\mathbb{Z}}\frac{(1+c_n)^2}{(1+c_n)^2}=\sum_{n\in\mathbb{Z}}1=\infty.$$

This contradiction proves that $\|g^1\|^2+\|g^2\|^2=\infty$. We shall come back to the case $I(3)$ later. *We show that in the case A (see (4.38)) we can approximate* x_{1n} *and* x_{2n}.

Now we study the possibility of the approximation of D_{1n} and D_{2n} by Lemmas 4.4, 4.5 and 4.10. Recall the notations:

$$\|f_m\|^2=\sum_{k=-m}^{m} a_{1k}^2\Big(\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}\Big)^{-1},\quad \|g_m\|^2=\sum_{k=-m}^{m} a_{2k}^2\Big(\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}\Big)^{-1}. \quad (4.33)$$

All the different cases are presented in the following tables:

table II	(1)	(2)	(3a)	(3b)	(3c)	(4)
$\|f\|^2$	∞	$<\infty$	∞	∞	∞	$<\infty$
$\|g\|^2$	$<\infty$	∞	∞	∞	∞	$<\infty$
$\frac{\|f_m\|^2}{\|g_m\|^2}$			$\to\infty$	$\to 0$	$C_1\le\frac{\|f_m\|^2}{\|g_m\|^2}\le C_2$	
Lemma	4.4 4.6	4.5 4.7	4.4 4.6	4.5 4.7	4.4 , 4.5 4.14, 4.10	
	$D_{1n},\ x_{1n}$	$D_{2n},\ x_{2n}$	$D_{1n},\ x_{1n}$	$D_{2n},\ x_{2n}$	$D_{1n},\ D_{2n}$	

Remark 4.6. We show that if $\|g\|^2<\infty$ and $S_{12}^L(\mu)=\infty$ then $\sum_n\frac{b_{1n}}{b_{2n}}=\infty$. Indeed, let us suppose that $\sum_n\frac{b_{1n}}{b_{2n}}<\infty$, then

$$\|g\|^2=\sum_{n\in\mathbb{Z}}\frac{a_{2n}^2}{\frac{1}{2b_{1n}}+\frac{1}{2b_{2n}}}\sim\sum_{n\in\mathbb{Z}}b_{1n}a_{2n}^2\sim\sum_{n\in\mathbb{Z}}\frac{b_{1n}}{2}\Big(\frac{1}{2b_{2n}}+a_{2n}^2\Big)=S_{12}^L(\mu)=\infty. \quad (4.34)$$

We explain the tables II in details. *The first two case (1) and (2) are independent of the case* I(3).

(1) If $\|g\|^2<\infty$ and $\|f\|^2=\infty$, we have $D_{1k}\ \eta\ \mathfrak{A}$ by Lemma 4.4. The condition $\|g\|^2<\infty$ implies $\sum_{k\in\mathbb{Z}}\frac{b_{1k}}{b_{2k}}=\infty$, by Remark 4.6 therefore, $x_{1k}\ \eta\ \mathfrak{A}$, by Lemma 4.6. Further, $A_{kn}-x_{1k}D_{1n}=x_{2k}D_{2n},\ k,n\in\mathbb{Z}$ and the proof is complete since we are reduced to the case $m=1$.

(2) If $\|g\|^2=\infty$ and $\|f\|^2<\infty$, we have $D_{2k}\ \eta\ \mathfrak{A}$ by Lemma 4.5. By remark similar to the Remark 4.6, we conclude that $\sum_{k\in\mathbb{Z}}\frac{b_{2k}}{b_{1k}}=\infty$ therefore, $x_{2k}\ \eta\ \mathfrak{A}$ by Lemma 4.7 and $A_{kn}-x_{2k}D_{2n}=x_{1k}D_{1n},\ k,n\in\mathbb{Z}$, case $m=1$.

(3) *Consider now the case* I(3). Let both series be divergent: $\|g\|^2=\infty$ and $\|f\|^2=\infty$. *We show that in the case* (B) *(see (4.38)) holds* $\|f+sg\|^2=\infty$

for all $s\in\mathbb{R}$, by Lemma 4.13 therefore, by Lemma 4.10, we can approximate D_{1n} and D_{2n}. To be more precise consider three possibilities:

(3a) let $\frac{\|f_m\|^2}{\|g_m\|^2}\to\infty$, then $D_{1k}\ \eta\ \mathfrak{A}$. Since $\sum_n\frac{b_{1n}}{b_{2n}}=\infty$, we have $x_{1n}\ \eta\ \mathfrak{A}$ by Lemma 4.6 and finally, $x_{1n},\ D_{1n}\ \eta\ \mathfrak{A},\ n\in\mathbb{Z}$. We are reduced to the case $m=1$.

(3b) Let $\frac{\|f_m\|^2}{\|g_m\|^2}\to 0$, then $D_{2k}\ \eta\ \mathfrak{A}$. Since $\sum_n\frac{b_{2n}}{b_{1n}}=\infty$, we get $x_{2n}\ \eta\ \mathfrak{A}$, by Lemma 4.7 and finally, $x_{2n},\ D_{2n}\ \eta\ \mathfrak{A},\ n\in\mathbb{Z}$. We are reduced to the case $m=1$.

(3c) The case when $\|f\|^2=\|g\|^2=\infty$ and $C_1{\leq}\frac{\|f_m\|^2}{\|g_m\|^2}{\leq}C_2$.

(4) The case when $\|f\|^2+\|g\|^2<\infty$.

To complete the proof of the lemma it remains to consider I(3), i.e., the last case (3) in the table I and the last two cases in the table II, i.e., II(3c) and II(4), where:

$$\text{I(3)}\quad \sum_{k\in\mathbb{Z}}\frac{b_{1k}}{b_{2k}}=\sum_{k\in\mathbb{Z}}\frac{b_{2k}}{b_{1k}}=\infty, \tag{4.35}$$

$$\text{II(c3)}\quad \sum_{k\in\mathbb{Z}}a_{1k}^2\Big(\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}\Big)^{-1}=\sum_{k\in\mathbb{Z}}a_{2k}^2\Big(\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}\Big)^{-1}=\infty, \tag{4.36}$$

$$\text{II(4)}\quad \sum_{k\in\mathbb{Z}}\Big(a_{1k}^2+a_{2k}^2\Big)\Big(\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}\Big)^{-1}<\infty. \tag{4.37}$$

Come back to the condition $\mu^{L_t}\perp\mu$. By Remark 4.3 we have

$$\mu^{L_{\tau_-(\phi,s)}}\perp\mu,\ \ \phi\in[0,2\pi),\ \ s>0\Leftrightarrow\Sigma_1(s)+\Sigma_2(C_1,C_2){=}\infty,\ \ s>0,$$

for $(C_1,C_2)\in\mathbb{R}^2\setminus\{0\}$. Recall that (see (4.4))

$$\Sigma_1(s)=\sum_{n\in\mathbb{Z}}\Big(s^2\sqrt{\frac{b_{1n}}{b_{2n}}}-s^{-2}\sqrt{\frac{b_{2n}}{b_{1n}}}\Big)^2,$$

$$\Sigma_2(C_1,C_2)=\sum_{n\in\mathbb{Z}}(C_1^2b_{1n}+C_2^2b_{2n})(C_1a_{1n}+C_2a_{2n})^2.$$

The condition $\Sigma_1(s)+\Sigma_2(C_1,C_2){=}\infty$ splits into two cases:

$$\begin{array}{ll}(A) & \Sigma_1(s)=\infty,\\ (B) & \Sigma_1(s)<\infty\quad\text{but}\quad\Sigma_2(C_1,C_2)=\infty.\end{array} \tag{4.38}$$

(**A**)&**I**(**3**). In this case independently of the conditions II(3c) and II(4) we can approximate x_{1n} and x_{2n} by Lemma 4.2 and 4.3.

(**B**)&**II**(**3c**) In this case we can approximate D_{1n} and D_{2n} by Lemmas 4.4 and 4.5 respectively. More precisely, to use Lemma 4.10 we show that conditions (4.13) are satisfied for two vectors f and g defined by (4.26) (see Lemma 4.14).
(**B**)&**II**(**4**) This case (see (4.37)) can not be realized if $\Sigma_2(C_1, C_2) = \infty$.

Case (A)&I(3). Using Lemma 4.10 we conclude that

$$\Gamma(f_m^1, g_m^1)(\Gamma(g_m^1))^{-1} \to \infty \quad \text{and} \quad \Gamma(f_m^2, g_m^2)(\Gamma(g_m^2))^{-1} \to \infty. \tag{4.39}$$

To use Lemma 4.10, it is sufficient to show that in the case (A) relations (4.13) hold for f^1, g^1 and f^2, g^2, i.e., for all $s \in \mathbb{R} \setminus \{0\}$ we have (see Lemma 4.13)

$$\|f^1\|^2 = \|g^1\|^2 = \|f^1 + sg^1\|^2 = \infty, \ \|f^2\|^2 = \|g^2\|^2 = \|f^2 + sg^2\|^2 = \infty. \tag{4.40}$$

Consider three possibilities in the case I(3):

(3a) If $\|g^1\| < \infty$ then $\|g^2\| = \infty$ therefore, we have $\|f_m^1\|/\|g_m^1\| \to \infty$ so, $x_{1n}\ \eta\ \mathfrak{A}$ by Lemma 4.9 (a). In the case (A) by Lemma 4.13 holds $\|f^2\|^2 = \|g^2\|^2 = \|f^2 + sg^2\|^2 = \infty$ therefore, $x_{2n}\ \eta\ \mathfrak{A}$ by Lemma4.10.

(3a) If $\|g^2\| < \infty$ then $\|g^1\| = \infty$ therefore, we have $\|f_m^2\|/\|g_m^2\| \to \infty$ so, $x_{2n}\ \eta\ \mathfrak{A}$ by Lemma4.9 (a). In the case (A) by Lemma 4.13 holds $\|f^1\| = \|g^1\| = \|f^1 + sg^1\| = \infty$ therefore, $x_{1n}\ \eta\ \mathfrak{A}$ by Lemma 4.10.

(3c) If $\|g^1\| = \|g^2\| = \infty$ then by Lemma 4.13 all relations (4.40) hold in the case (A) therefore, $x_{1n},\ x_{2n}\ \eta\ \mathfrak{A}$.

To prove (4.40) we need the following auxiliary lemma.

Lemma 4.12. *The following two conditions are equivalent:*

$$(i) \quad \Sigma_1(s) = \sum_{n\in\mathbb{Z}} \left(s^2\sqrt{\frac{b_{1n}}{b_{2n}}} - s^{-2}\sqrt{\frac{b_{2n}}{b_{1n}}}\right)^2 = \infty, \tag{4.41}$$

$$(ii) \quad \Sigma_2(s) = \sum_{n\in\mathbb{Z}} \left(s^4\frac{b_{1n}}{b_{2n}} - 1\right)^2 + \left(s^{-4}\frac{b_{2n}}{b_{1n}} - 1\right)^2 = \infty. \tag{4.42}$$

Proof. We show that $(i) \Rightarrow (ii)$. Indeed, we have

$$(a^2-1)^2 + (a^{-2}-1)^2 = (a^2-1)^2(1+a^{-4}) = (a-a^{-1})^2(a^2+a^{-2}).$$

Set $a = s^2(b_{1n}/b_{2n})^{1/2}$, then

$$\Sigma_2(s) = \sum_{n\in\mathbb{Z}} \left(s^2\sqrt{\frac{b_{1n}}{b_{2n}}} - s^{-2}\sqrt{\frac{b_{2n}}{b_{1n}}}\right)^2 \left(s^4\frac{b_{1n}}{b_{2n}} + s^{-4}\frac{b_{2n}}{b_{1n}}\right) \geq 2\Sigma_1(s).$$

We prove that $(ii) \Rightarrow (i)$. Denote by $s^4 \frac{b_{1n}}{b_{2n}} = 1 + a_n$, then we have

$$\Sigma_1(s) = \sum_{n\in\mathbb{Z}} \left(\sqrt{1+a_n} - \frac{1}{\sqrt{1+a_n}}\right)^2 = \sum_{n\in\mathbb{Z}} \left(\frac{a_n}{\sqrt{1+a_n}}\right)^2 = \sum_{n\in\mathbb{Z}} \frac{a_n^2}{1+a_n},$$

$$\Sigma_2(s) = \sum_{n\in\mathbb{Z}} \left(a_n^2 + \left(\frac{1}{1+a_n} - 1\right)^2\right) = \sum_{n\in\mathbb{Z}} \left(a^2 + \frac{a_n^2}{(1+a_n)^2}\right) \overset{(4.9)}{\sim} \sum_{n\in\mathbb{Z}} a_n^2 + \sum_{n\in\mathbb{Z}} \frac{a_n^2}{1+a_n}.$$

Let $\Sigma_2(s) = \infty$. If $\sum_{n\in\mathbb{Z}} \frac{a_n^2}{1+a_n} = \infty$, the proof is complete. Suppose that $\sum_{n\in\mathbb{Z}} a_n^2 = \infty$. We show that in this case $\Sigma_1(s) = \infty$. It is sufficient to prove that

$$\sum_{n\in\mathbb{N}} a_n^2 = \infty \quad \text{implies} \quad \sum_{n\in\mathbb{N}} a_n^2 (1+a_n)^{-1} = \infty.$$

Consider three cases:
(a) If $0 < \varepsilon \leq 1 + a_n \leq C < \infty$ for all $n \in \mathbb{N}$, then

$$C^{-1} \sum_{n\in\mathbb{N}} a_n^2 \leq \sum_{n\in\mathbb{N}} a_n^2 (1+a_n)^{-1} \leq \varepsilon^{-1} \sum_{n\in\mathbb{N}} a_n^2.$$

(b) If $\lim_{k\to\infty}(1+a_{n_k}) = 0$, then

$$\lim_{k\to\infty} a_{n_k}^2 (1+a_{n_k})^{-1} = \infty \quad \text{and} \quad \sum_{n\in\mathbb{N}} a_n^2 (1+a_n)^{-1} = \infty.$$

(c) If $\lim_{k\to\infty}(1+a_{n_k}) = +\infty$, then

$$\sum_{n\in\mathbb{N}} a_n^2 (1+a_n)^{-1} > \sum_{k\in\mathbb{N}} a_{n_k} (a_{n_k}^{-1} + 1)^{-1} \sim \sum_{k\in\mathbb{N}} a_{n_k} = \infty.$$

□

Lemma 4.13. *If $\Sigma_1(s) = \infty$ for any $s > 0$, then*

$$\|f^1 - Cg^1\|^2 = \infty \quad \textit{and} \quad \|f^2 - Cg^2\|^2 = \infty, \quad \textit{for any} \quad C > 0.$$

Proof. Set as before $c_n = \frac{b_{2n}}{b_{1n}}$, $n \in \mathbb{Z}$. Suppose that $\Sigma_1(s) = \infty$, then

$$\infty = \Sigma_1(s) = \sum_{n\in\mathbb{Z}} \left(\frac{s^2}{\sqrt{c_n}} - \frac{\sqrt{c_n}}{s^2}\right)^2 = \sum_{n\in\mathbb{Z}} \frac{a_n^2}{1+a_n},$$

where $s^4c_n^{-1}=1+a_n$ or $c_n=\frac{s^4}{1+a_n}$. We show that

$$\|s^4f^1-g^1\|^2=\infty \quad \text{and} \quad \|s^{-4}f^2-Cg^2\|^2=\infty.$$

Indeed, using (4.30) and (4.31) we get

$$\|s^4f^1-g^1\|^2=\sum_{k\in\mathbb{Z}}\frac{(s^4-c_k)^2}{1+2c_k}=\sum_{k\in\mathbb{Z}}\Big(\frac{s^4}{c_k}-1\Big)\Big(\frac{1}{c_k^2}+\frac{2}{c_k}\Big)^{-1}=$$

$$\sum_{k\in\mathbb{Z}}\frac{a_k^2}{\big(\frac{1+a_k}{s^4}\big)^2+2\frac{1+a_k}{s^4}}\sim\sum_{k\in\mathbb{Z}}\frac{a_k^2}{(1+a_k)^2+2(1+a_k)}=$$

$$\sum_{k\in\mathbb{Z}}\frac{a_k^2}{3+4a_k+a_k^2}\sim\sum_{k\in\mathbb{Z}}\frac{a_k^2}{1+a_k}=\infty$$

and

$$\|s^{-4}f^2-g^2\|^2=\sum_{k\in\mathbb{Z}}\frac{(s^{-4}c_k^2-1)^2}{c_k^2+2c_k}=$$

$$\sum_{k\in\mathbb{Z}}\Big(\frac{1}{1+a_k}-1\Big)^2\Big(\Big(\frac{s^4}{1+a_k}\Big)^2+2\frac{s^4}{1+a_k}\Big)^{-1}=$$

$$\sum_{k\in\mathbb{Z}}\frac{a_k^2}{s^8+2s^4(1+a_k)}\sim\sum_{k\in\mathbb{Z}}\frac{a_k^2}{1+a_k}=\infty.$$

$\square$

So, in the case (A)&I(3) we can approximate x_{1n} and x_{2n}.

Case (B)&II(3c).

Lemma 4.14. *When $\Sigma_1(s)<\infty$ and $\Sigma_2(C_1,C_2)=\infty$, we get*

$$\sigma(C_1,C_2):=\|C_1f+C_2g\|^2=\sum_{n\in\mathbb{Z}}\frac{(C_1a_{1n}+C_2a_{2n})^2}{\frac{1}{2b_{1n}}+\frac{1}{2b_{2n}}}=\infty,\ (C_1,C_2)\in\mathbb{R}^2\setminus\{0\}, \tag{4.43}$$

where f and g are defined by (4.28)

$$f=\Big(a_{1k}\Big(\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}\Big)^{-1/2}\Big)_{k\in\mathbb{Z}},\quad g=\Big(a_{2k}\Big(\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}\Big)^{-1/2}\Big)_{k\in\mathbb{Z}}.$$

Proof. Let $\Sigma_1(s) = \sum_{n\in\mathbb{Z}} \frac{a_n^2}{1+a_n} < \infty$, where $s^4\frac{b_{1n}}{b_{2n}} = 1 + a_n$ or $s^4 b_{1n} = (1+a_n)b_{2n}$. We see that $\lim_n \frac{a_n^2}{1+a_n} = 0$ hence, $\lim_n a_n = \lim_n \left(s^4\frac{b_{1n}}{b_{2n}} - 1\right) = 0$. We have

$$\sigma(C_1, C_2) = \sum_{n\in\mathbb{Z}} \frac{b_{1n}(C_1 a_{1n} + C_2 a_{2n})^2}{\frac{1}{2} + \frac{b_{1n}}{2b_{2n}}} = \sum_{n\in\mathbb{Z}} \frac{b_{1n}(C_1 a_{1n} + C_2 a_{2n})^2}{\frac{1}{2} + \frac{1}{2}\frac{1+a_n}{s^4}}$$

$$\sim \sum_{n\in\mathbb{Z}} C_1^2 b_{1n}(C_1 a_{1n} + C_2 a_{2n})^2,$$

$$\sigma(C_1, C_2) = \sum_{n\in\mathbb{Z}} \frac{b_{2n}(C_1 a_{1n} + C_2 a_{2n})^2}{\frac{b_{2n}}{2b_{1n}} + \frac{1}{2}} = \sum_{n\in\mathbb{Z}} \frac{b_{2n}(C_1 a_{1n} + C_2 a_{2n})^2}{\frac{1}{2}\frac{s^4}{1+a_n} + \frac{1}{2}}$$

$$\sim \sum_{n\in\mathbb{Z}} C_2^2 b_{2n}(C_1 a_{1n} + C_2 a_{2n})^2,$$

hence, $\sigma(C_1, C_2) \sim \sum_{n\in\mathbb{Z}} (C_1^2 b_{1n} + C_2^2 b_{2n})(C_1 a_{1n} + C_2 a_{2n})^2 = \Sigma_2(C_1, C_2)$. □

Finally, we can approximate D_{1n} and D_{2n} in the case (B)&II(3c).

Case (B)&II(4). The last case (B)&II(4) (see (4.37)) can not be realized if $\Sigma_2(C_1, C_2) = \infty$. Indeed, in this case by Lemma 4.14 $\sigma(C_1, C_2) \sim \Sigma_2(C_1, C_2) = \infty$. This contradicts (4.37):

$$\sum_{k\in\mathbb{Z}} \left(a_{1k}^2 + a_{2k}^2\right)\left(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}}\right)^{-1} < \infty.$$

This completes the proof of Lemma 4.11 for $m = 2$. □

The proof of the irreducibility for $m = 2$ follows from Remark 2.2. Depending on the measure, we can approximate four different families of commuting operators $B^\alpha = (B_{1n}^\alpha, B_{2n}^\alpha)_{n\in\mathbb{Z}}$ for $\alpha \in \{0, 1\}^2$:

$$B^{(0,0)}=(x_{1n}, x_{2n})_n,\ B^{(0,1)}=(x_{1n}, D_{2n})_n,\ B^{(1,0)}=(D_{1n}, x_{2n})_n,\ B^{(0,0)}=(D_{1n}, D_{2n})_n.$$

The von Neumann algebra $L_\alpha^\infty(X_2, \mu^2)$ consists of all essentially bounded functions $f(B^\alpha)$ in the commuting family of operators B^α (see, e.g., [4]) as, in particular, $L_{(0,0)}^\infty(X_2, \mu^2) = L^\infty(X_2, \mu^2)$. Since the von Neumann algebras $L_\alpha^\infty(X_2, \mu^2)$ are maximal abelian, the commutant $\left(\mathfrak{A}^2\right)'$ of the von Neumann algebra $\mathfrak{A}^2$ generated by the representation is contained in $L_\alpha^\infty(X_2, \mu^2)$. Hence, the bounded operator $A \in \left(\mathfrak{A}^2\right)'$ will be some function $A = a(B^\alpha) \in$

$L^\infty_\alpha(X_2,\mu^2)$. The commutation relation $[A,T_t^{R,\mu,2}]=0$ gives us the following relations: $a((B^\alpha)^{R_t})=a(B^\alpha)$ for all $t\in\mathrm{GL}_0(2\infty,\mathbb{R})$. Set $B^\alpha_r=(B^\alpha_{rn})_n$, $x_r=(x_{rn})_n$, $D_r=(x_{rn})_n$, $r=1,2$, $n\in\mathbb{Z}$ and set as before, $E_{kn}(t):=I+tE_{kn}$, $t\in\mathbb{R}$, $k,n\in\mathbb{Z}$, $k\neq n$. Then the action $(B^\alpha)^{R_s}$ is defined as follows:

$$(B^\alpha_1,B^\alpha_2)^{R_t}=((B^\alpha_1)^{R_t},(B^\alpha_2)^{R_t}),\quad (x_r)^{R_t}=x_rt,\quad (D_r)^{R_t}=D_rt^*,$$
$$a(\dots,x_{rk},\dots,x_{rn},\dots)^{R_{E_{kn}(t)}}=a(\dots,x_{rk},\dots,x_{rn}+tx_{rk},\dots),$$
$$a(\dots,D_{rk},\dots,D_{rn},\dots)^{R_{E_{kn}(t)}}=a(\dots,D_{rk}+tD_{rn},\dots,D_{rn},\dots),\ \ t\in\mathbb{R}.$$

In all the cases, by ergodicity of the measure μ^2, we conclude that a is constant.

4.3. The proof of Lemmas 2.2, 4.1

Lemma 2.2 follows from Lemmas 4.15- 4.18.

Lemma 4.15. *For $t\in\mathrm{GL}(m,\mathbb{R})\backslash\{e\}$ we have $(\mu^m_{(b,a)})^{L_t}\perp\mu^m_{(b,a)}$ if and only if*

$$(\mu^m_{(b,0)})^{L_t}\perp\mu^m_{(b,0)}\quad or\quad \mu^m_{(b,L_ta)}\perp\mu^m_{(b,a)}. \tag{4.44}$$

Let us define the following measures on the spaces $\mathbb{R}^m$ and X_m:

$$\mu_m^{(B_n,0)}=\otimes_{k=1}^m\mu_{(b_{kn},0)},\quad \mu_m^{(B_n,a_n)}=\otimes_{k=1}^m\mu_{(b_{kn},a_{kn})},$$

where $a_n=(a_{1n},...,a_{mn})\in\mathbb{R}^m$ and $B_n=\mathrm{diag}(b_{1n},...,b_{mn})\in\mathrm{Mat}(m,\mathbb{R})$. Since

$$\mu^m_{(b,a)}=\otimes_{n\in\mathbb{Z}}\mu_m^{(B_n,a_n)},\quad \mu^m_{(b,0)}=\otimes_{n\in\mathbb{Z}}\mu_m^{(B_n,0)},$$
$$\left(\mu^m_{(b,a)}\right)^{L_t}=\otimes_{n\in\mathbb{Z}}\left(\mu_m^{(B_n,a_n)}\right)^{L_t},\quad \left(\mu^m_{(b,0)}\right)^{L_t}=\otimes_{n\in\mathbb{Z}}\left(\mu_m^{(B_n,0)}\right)^{L_t},$$

and

$$\mu^m_{(b,L_ta)}=\otimes_{n\in\mathbb{Z}}\mu_m^{(B_n,L_ta_n)},$$

by Kakutani criterion [11], we have two lemmas:

Lemma 4.16. *For measures $\mu^m_{(b,0)}$, $m\in\mathbb{N}$ and $t\in\mathrm{GL}(m,\mathbb{R})\backslash\{e\}$ we obtain*

$$(\mu^m_{(b,0)})^{L_t}\perp\mu^m_{(b,0)}\ \Leftrightarrow\prod_{n\in\mathbb{Z}}H\left(\left(\mu_m^{(B_n,0)}\right)^{L_t},\mu_m^{(B_n,0)}\right)=0.$$

Lemma 4.17. *For measures* $\mu^m_{(b,0)}$, $m \in \mathbb{N}$ *and* $t \in \mathrm{GL}(m,\mathbb{R})\backslash\{e\}$ *we get*

$$\mu^m_{(b,L_t a)} \perp \mu^m_{(b,a)} \Leftrightarrow \prod_{n\in\mathbb{Z}} H\left(\mu_m^{(B_n,L_t a_n)}, \mu_m^{(B_n,a_n)}\right) = 0.$$

To prove Lemma 2.2 it is sufficient to show, due to Lemma 4.15, that

$$H\left(\left(\mu_m^{(B_n,0)}\right)^{L_t}, \mu_m^{(B_n,0)}\right) = \left(\frac{1}{2^m|\det\, t|}\det\left(I + X_n^*(t)X_n(t)\right)\right)^{-1/2}, \quad (4.45)$$

to prove the equivalence

$$\prod_{n\in\mathbb{Z}} H\left(\mu_m^{(B_n,L_t a_n)}, \mu_m^{(B_n,a_n)}\right) = 0 \Leftrightarrow \sum_{n\in\mathbb{Z}}\sum_{r=1}^m b_{rn}\Big(\sum_{s=1}^m (t_{rs}-\delta_{rs})a_{sn}\Big)^2 = \infty, \quad (4.46)$$

and to use the following lemma:

Lemma 4.18. *For* $X \in \mathrm{Mat}(m,\mathbb{R})$ *we have*

$$\det\left(I + X^*X\right) = 1 + \sum_{r=1}^m \sum_{1\leq i_1<i_2<\ldots<i_r\leq m;1\leq j_1<j_2<\ldots<j_r\leq m} \left(M^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_r}(X)\right)^2. \quad (4.47)$$

The proof of equality (4.45) is based on the exact formula of the Hellinger integral (see [23] for definition) for two Gaussian measures $\mu = \mu_m^{(B_n,0)}$ and $\nu = \mu_m^{(C_n,0)}$ in the space $\mathbb{R}^m$ (see [23])

$$H(\mu,\nu) = \int_X \sqrt{\frac{d\mu}{d\rho}\frac{d\nu}{d\rho}}d\rho = \left(\frac{\det B_n \det C_n}{\det^2 \frac{B_n+C_n}{2}}\right)^{1/4}. \quad (4.48)$$

The latter formula is based on the following formula for a positive definite operator C in the space $\mathbb{R}^m$:

$$\frac{1}{\sqrt{\pi^m}}\int_{\mathbb{R}^m} \exp(-(Cx,x))dx = \frac{1}{\sqrt{\det C}}. \quad (4.49)$$

Let, as before, $t = (t_{rs})_{r,s=1}^m \in \mathrm{GL}(m,\mathbb{R})$, $B_n = \mathrm{diag}(b_{1n}, b_{2n}, ..., b_{mn})$, $X_n(t) = B_n^{1/2} t B_n^{-1/2} \in \mathrm{Mat}(m,\mathbb{R})$. Let $M^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_r}(t)$ be the minors of the matrix t with $i_1, i_2, ..., i_r$ rows and $j_1, j_2, ..., j_r$ columns.

Let us denote by $\mu^{(B,a)}=\mu_{(C,a)}$ the Gaussian measure with the covariance operator $C=(2B)^{-1}$ on the space $\mathbb{R}^m$ defined by the formula: $\frac{d\mu^B(x)}{dx}=$

$$\sqrt{\frac{\det B}{\pi^m}}\exp\Big(-\,(Bx,x)\Big)=\frac{1}{\sqrt{(2\pi)^m\det C}}\exp\Big(-\,\frac{1}{2}\big(C^{-1}x,x)\big)\Big)=\frac{d\mu_C(x)}{dx}. \tag{4.50}$$

Recall that by definition $\mu^f(\Delta)=\mu(f^{-1}(\Delta))$. Since $L_tx=tx$, we get $\mu^{L_t^{-1}}(x)=\mu(tx)$ therefore,

$$\big(\mu_m^{(B_n,a)}\big)^{L_t^{-1}}(x)=\mu_m^{(B_n(t),t^{-1}a)}\quad\text{where}\quad B_n(t)=t^*B_nt. \tag{4.51}$$

Indeed,

$$d\left(\mu_m^{(B_n,a)}\right)^{L_t^{-1}}(x)=\sqrt{\frac{\det B_n}{\pi^m}}\exp\Big(-\big(B_nt(x-t^{-1}a),t(x-t^{-1}a)\big)\Big)dtx=$$

$$\sqrt{\frac{|\det\, t|^2\det B_n}{\pi^m}}\exp\Big(-\big(t^*B_nt(x-t^{-1}a),(x-t^{-1}a)\big)\Big)dx=d\mu_m^{(B_n(t),t^{-1}a)}(x),$$

where $B_n(t)=t^*B_nt$, $B_n=\mathrm{diag}(b_{1n},b_{2n},...,b_{mn})$, $\det B_n(t)=|\det\, t|^2\det B_n$.

Using (4.48), (4.50) and (4.51) we obtain

$$H\left(\left(\mu_m^{(B_n,0)}\right)^{L_t},\mu_m^{(B_n,0)}\right)=H\left(\mu_m^{(B_n(t),0)},\mu_m^{(B_n,0)}\right)=\left(\frac{\det B_n(t)\det B_n}{\pi^m\pi^m}\right)^{1/4}\times$$

$$\int_{\mathbb{R}^m}\exp\Big(-\Big(\frac{B_n(t)+B_n}{2}x,x\Big)\Big)dx=\left(\frac{\det B_n(t)\det B_n}{\det^2\frac{B_n(t)+B_n}{2}}\right)^{1/4}=\left(\frac{\det C_n(t)}{|\det t|\det B_n}\right)^{-1/2},$$

where $C_n(t)=\frac{B_n(t)+B_n}{2}=\frac{t^*B_nt+B_n}{2}$. Now we show that

$$\frac{\det C_n(t)}{|\det t|\det B_n}=\frac{1}{2^m|\det\, t|}\det\left(I+X_n^*(t)X_n(t)\right), \tag{4.52}$$

where $X_n(t)=B_n^{1/2}tB_n^{-1/2}$. The latter equation is equivalent to

$$\frac{\det\left(t^*B_nt+B_n\right)}{\det B_n}=\det\left(I+X_n^*(t)X_n(t)\right).$$

To complete the proof of (4.45) it is sufficient to see that

$$I+X_n^*(t)X_n(t)=I+B_n^{-1/2}t^*B_n^{1/2}B_n^{1/2}tB_n^{-1/2}=B_n^{-1/2}(B_n+t^*B_nt)B_n^{-1/2}.$$

The proof of relation (4.46) is based on the following theorem that one can find, e.g., in [29, Ch. III, §16, Theorem 2].

Theorem 4.19. *Two Gaussian measures $\mu_{B,a}$ and $\mu_{B,b}$ are equivalent if and only of $B^{-1/2}(a-b) \in H$.*

Indeed, we have

$$\|C^{-1/2}(ta-a)\|^2_H = \sum_{n\in\mathbb{Z}} \|C_n^{-1/2}(t-I)a_n\|^2_{H_n} = 2\sum_{n\in\mathbb{Z}}\sum_{r=1}^{m}\frac{b_{kn}}{d_{kn}}\Big(\sum_{s=1}^{m}(t_{rs}-\delta_{rs})a_{sn}\Big)^2 d_{kn}.$$

To explain the latter equality let us describe H and C. To find an operator C we present the measure $\mu^m_{(b,a)}$ in the canonical form $\mu_{C,a}$ defined by its Fourier transform:

$$\int_H \exp i(y,x)d\mu_{C,a}(x) = \exp\left(i(a,y) - \frac{1}{2}(Cy,y)\right),\ y \in H, \tag{4.53}$$

where C is a positive *nuclear operator* (called the *covariance operator*) on the Hilbert space H, and $a \in H$ is the *mathematical expectation* or *mean.*

Recall the *Kolmogorov zero-one law.* Let us consider in the space $\mathbb{R}^\infty = \mathbb{R}\times\mathbb{R}\times\cdots$ the infinite tensor product $\mu_b = \otimes_{n\in\mathbb{N}}\mu_{b_k}$ of one-dimensional Gaussian measures μ_{b_k} on $\mathbb{R}$ defined as follows:

$$d\mu_b(x) = \sqrt{b/\pi}\exp(-bx^2)dx. \tag{4.54}$$

Consider a Hilbert space $l_2(a)$ defined by

$$l_2(a) = \big\{x\in\mathbb{R}^\infty \,:\, \|x\|^2_{l_2(a)} = \sum_{k\in\mathbb{N}} x_k^2 a_k < \infty\big\},$$

where $a = (a_k)_{k\in\mathbb{N}}$ is an infinite sequence of positive numbers.

Theorem 4.20 (Kolmogorov's zero-one law, [28]). *We have*

$$\mu_b(l_2(a)) = \begin{cases} 0, & \textit{if}\ \ \sum_{k\in\mathbb{N}}\frac{a_k}{b_k} = \infty, \\ 1, & \textit{if}\ \ \sum_{k\in\mathbb{N}}\frac{a_k}{b_k} < \infty. \end{cases}$$

Define the Hilbert space $H \subset X_m$ as follows:

$$H = l_2(\mathbb{R}^m, d) = \big\{x = (x_{kn})_{k,n} \in X_m \mid \|x\|^2_H := \sum_{1\le k\le m, n\in\mathbb{N}} x^2_{kn}d_{kn} < \infty\big\},$$

where a sequence $d = (d_{kn})_{1\le k\le m, n\in\mathbb{Z}}$ of positive numbers is chosen such that $\sum_{1\le k\le m,n\in\mathbb{N}}\frac{d_{kn}}{b_{kn}} < \infty$. Then by the Kolmogorov zero-one law, $\mu^m_{(b,a)}(H) = 1$. We show that $C = \operatorname{diag}(c_{kn})$, where $c_{kn} = \frac{d_{kn}}{2b_{kn}}$. Indeed, we get

$$\sum_{1\le k\le m,n\in\mathbb{N}} b_{kn}x^2_{kn} = \frac{1}{2}\sum_{1\le k\le m,n\in\mathbb{N}}\frac{2b_{kn}}{d_{kn}}x^2_{kn}d_{kn} = \frac{1}{2}(C^{-1}x,x)_H.$$

Proof of Lemma 4.18. Let as recall the definition of the Gram determinant and the Gram matrix (see [7], Chap IX, §5). For vectors $x_1, x_2, ..., x_m$ in some Hilbert space H the Gram matrix $\gamma(x_1, x_2, ..., x_m)$ is defined by the formula

$$\gamma(x_1, x_2, ..., x_m) = ((x_k, x_n)_{k,n=1}^m).$$

The determinant of this matrix is called the Gram determinant for the vectors $x_1, x_2, ..., x_m$ and is denoted by $\Gamma(x_1, x_2, ..., x_m)$. Thus,

$$\Gamma(x_1, x_2, ..., x_m) := \det \gamma(x_1, x_2, ..., x_m).$$

Let

$$X = \begin{pmatrix} x_{11} & x_{12} & ... & x_{1m} \\ x_{21} & x_{22} & ... & x_{2m} \\ ... & ... & ... & ... \\ x_{m1} & x_{m2} & ... & x_{mm} \end{pmatrix}.$$

Set $x_k = (x_{1k}, x_{2k}, ..., x_{mk}) \in \mathbb{R}^m$, $1 \leq k \leq m$, then, obviously, we get

$$X^*X = \begin{pmatrix} (x_1, x_1) & (x_1, x_2) & ... & (x_1, x_m) \\ (x_2, x_1) & (x_2, x_2) & ... & (x_2, x_m) \\ ... & ... & ... & ... \\ (x_m, x_1) & (x_m, x_2) & ... & (x_m, x_m) \end{pmatrix} = \gamma(x_1, x_2, ..., x_m).$$

We would like to find an exact expression for $\det\left(I + \gamma(x_1, x_2, ..., x_m)\right)$. It is convenient to consider the following function:

$$F_{m,X}^{\lambda} = F_{m;x_1,x_2,...,x_m}^{\lambda_1,\lambda_2,...,\lambda_m} = \det\Big(\sum_{k=1}^m \lambda_k E_{kk} + \gamma(x_1, x_2, ..., x_m)\Big), \quad \lambda \in \mathbb{C}^m.$$

It is easy to see that for $m = 2$ we have

$$F_{2;x_1,x_2}^{\lambda_1,\lambda_2} = \det\begin{pmatrix} \lambda_1 + (x_1, x_1) & (x_1, x_2) \\ (x_2, x_1) & \lambda_2 + (x_2, x_2) \end{pmatrix} =$$

$$\lambda_1\lambda_2 + \lambda_1\Gamma(x_2) + \lambda_2\Gamma(x_1) + \Gamma(x_1, x_2) =$$

$$\lambda_1\lambda_2\left(1 + \lambda_1^{-1}\Gamma(x_1) + \lambda_2^{-1}\Gamma(x_2) + (\lambda_1\lambda_2)^{-1}\Gamma(x_1, x_2)\right). \tag{4.55}$$

The general formula is

$$F_{m;x_1,x_2,...,x_m}^{\lambda_1,\lambda_2,...,\lambda_m} = \det\Big(\sum_{k=1}^m \lambda_k E_{kk} + \gamma(x_1, x_2, ..., x_m)\Big) = \tag{4.56}$$

$$\prod_{k=1}^m \lambda_k\Big(1+\sum_{r=1}^m \sum_{1\le i_1<i_2<...<i_r\le m} \Big(\lambda_{i_1}\lambda_{i_2}...\lambda_{i_r}\Big)^{-1}\Gamma(x_{i_1},x_{i_2},...,x_{i_r})\Big) =$$

$$\prod_{k=1}^m \lambda_k\Big(1+\sum_{r=1}^m \sum_{1\le i_1<i_2<...<i_r\le m;1\le j_1<j_2<...<j_r\le m} \Big(\lambda_{i_1}\lambda_{i_2}...\lambda_{i_r}\Big)^{-1}\Big(M^{i_1i_2...i_r}_{j_1j_2...j_r}(X)\Big)^2\Big).$$

We have used the following formula (see [7], Chap IX, §5 formula (25)):

$$\Gamma(x_{i_1},x_{i_2},...,x_{i_r}) = \sum_{1\le j_1<j_2<...<j_r\le m} \big(M^{i_1i_2...i_r}_{j_1j_2...j_r}(X)\big)^2. \qquad (4.57)$$

Finally, using (4.56) for $(\lambda_1,\lambda_2,...,\lambda_m)=(1,1,...,1)$ we get (4.47).

We study the case $m=2$ more carefully.

Lemma 4.21. *For $t\in GL(2,\mathbb{R})$ we have, if $\det t>0$,*

$$(\mu^2_{(b,0)})^{L_t}\perp\mu^2_{(b,0)} \quad \Leftrightarrow$$

$$\sum_{n\in\mathbb{Z}}\Big[(1-\mid\det t\mid)^2+(t_{11}-t_{22})^2+\Big(t_{12}\sqrt{\frac{b_{1n}}{b_{2n}}}+t_{21}\sqrt{\frac{b_{2n}}{b_{1n}}}\Big)^2\Big]=\infty. \qquad (4.58)$$

If $\det t<0$ we have

$$(\mu^2_{(b,0)})^{L_t}\perp\mu^2_{(b,0)} \quad \Leftrightarrow$$

$$\sum_{n\in\mathbb{Z}}\Big[(1-\mid\det t\mid)^2+(t_{11}+t_{22})^2+\Big(t_{12}\sqrt{\frac{b_{1n}}{b_{2n}}}-t_{21}\sqrt{\frac{b_{2n}}{b_{1n}}}\Big)^2\Big]=\infty. \qquad (4.59)$$

Proof. Using (4.45) set

$$H_{m,n}(t)=H\left(\big(\mu_m^{(B_n,0)}\big)^{L_t^{-1}},\mu_m^{(B_n,0)}\right)=\left(\frac{1}{2^m|\det\ t|}\det\left(I+X_n^*(t)X_n(t)\right)\right)^{-1/2}.$$

For $m=2$ using (2.5) we get $X(t)=B^{1/2}tB^{-1/2}$ hence,

$$X(t)=\begin{pmatrix} b_{1n} & 0\\ 0 & b_{2n}\end{pmatrix}^{1/2}\begin{pmatrix} t_{11} & t_{12}\\ t_{21} & t_{22}\end{pmatrix}\begin{pmatrix} b_{1n} & 0\\ 0 & b_{2n}\end{pmatrix}^{-1/2}=\begin{pmatrix} t_{11} & \sqrt{\frac{b_{1n}}{b_{2n}}}t_{12}\\ \sqrt{\frac{b_{2n}}{b_{1n}}}t_{21} & t_{22}\end{pmatrix}.$$

Therefore, using (4.55) we get

$$H^{-2}_{2,n}(t)=\frac{1}{2^2\mid\det t\mid}\left(1+\mid\det t\mid^2+t_{11}^2+t_{22}^2+\frac{b_{1n}}{b_{2n}}t_{12}^2+\frac{b_{2n}}{b_{1n}}t_{21}^2\right).$$

Using Lemma 4.16 it is sufficient to calculate $H_{2,n}^{-2}(t)-1$. Indeed, for $\det t>0$ we have

$$H_{2,n}^{-2}(t)-1=\frac{1}{2^2\mid\det t\mid}\times$$
$$\Big(1-2\det t+\mid\det t\mid^2+t_{11}^2+t_{22}^2+\frac{b_{1n}}{b_{2n}}t_{12}^2+\frac{b_{2n}}{b_{1n}}t_{21}^2-2(t_{11}t_{22}-t_{12}t_{21})\Big)=$$
$$\frac{1}{2^2\mid\det t\mid}\Big[(1-\mid\det t\mid)^2+(t_{11}-t_{22})^2+\Big(t_{12}\sqrt{\frac{b_{1n}}{b_{2n}}}+t_{21}\sqrt{\frac{b_{2n}}{b_{1n}}}\Big)^2\Big].$$

For $\det t<0$ we get

$$H_{2,n}^{-2}(t)-1=\frac{1}{2^2\mid\det t\mid}\times$$
$$\Big(1+2\det t+\mid\det t\mid^2+t_{11}^2+t_{22}^2+\frac{b_{1n}}{b_{2n}}t_{12}^2+\frac{b_{2n}}{b_{1n}}t_{21}^2+2(t_{11}t_{22}-t_{12}t_{21})\Big)=$$
$$\frac{1}{2^2\mid\det t\mid}\Big[(1-\mid\det t\mid)^2+(t_{11}+t_{22})^2+\Big(t_{12}\sqrt{\frac{b_{1n}}{b_{2n}}}-t_{21}\sqrt{\frac{b_{2n}}{b_{1n}}}\Big)^2\Big].$$

□

Using Lemma 4.17, Lemma 4.21 and (4.46) we get

Lemma 4.22. *For $t\in GL(2,\mathbb{R})$ we have*

$$(\mu_{(b,a)}^2)^{L_t}\perp\mu_{(b,a)}^2\quad \textit{if}\quad \mid\det t\mid\neq 1.$$

If $\det t=1$, we have

$$(\mu_{(b,a)}^2)^{L_t}\perp\mu_{(b,a)}^2\quad\Leftrightarrow\quad\Sigma^+(t)=\Sigma_1^+(t)+\Sigma_2(t)=\infty.$$

If $\det t=-1$, we have

$$(\mu_{(b,a)}^2)^{L_t}\perp\mu_{(b,a)}^2\quad\Leftrightarrow\quad\Sigma^-(t)=\Sigma_1^-(t)+\Sigma_2(t)=\infty,$$

where

$$\Sigma_1^+(t)=\sum_{n\in\mathbb{Z}}\Big[(t_{11}-t_{22})^2+\Big(t_{12}\sqrt{\frac{b_{1n}}{b_{2n}}}+t_{21}\sqrt{\frac{b_{2n}}{b_{1n}}}\Big)^2\Big],$$

$$\Sigma_1^-(t)=\sum_{n\in\mathbb{Z}}\Big[(t_{11}+t_{22})^2+\Big(t_{12}\sqrt{\frac{b_{1n}}{b_{2n}}}-t_{21}\sqrt{\frac{b_{2n}}{b_{1n}}}\Big)^2\Big],$$

$$\Sigma_2(t^{-1})=\sum_{n\in\mathbb{Z}}\Big[b_{1n}\big[(t_{11}-1)a_{1n}+t_{12}a_{2n}\big]^2+b_{2n}\big[t_{21}a_{1n}+(t_{22}-1)a_{2n}\big]^2\Big].\quad(4.60)$$

Proof. *of Lemma 4.1.* We show that it is sufficient to consider only five particular cases:

$$\exp(tE_{12}) = I + tE_{12} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \exp(tE_{21}) = I + tE_{21} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

$$\exp(tE_{12})P_1 = \begin{pmatrix} -1 & t \\ 0 & 1 \end{pmatrix}, \quad \exp(tE_{21})P_2 = \begin{pmatrix} 1 & 0 \\ t & -1 \end{pmatrix},$$

and

$$\tau_-(\phi, s) = \begin{pmatrix} \cos\phi & s^2\sin\phi \\ s^{-2}\sin\phi & -\cos\phi \end{pmatrix},$$

where

$$P_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We note that $\tau_-(\phi, s) =$

$$\begin{pmatrix} \cos\phi & s^2\sin\phi \\ s^{-2}\sin\phi & -\cos\phi \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} s^{-1} & 0 \\ 0 & s \end{pmatrix} P_2.$$

Using Lemma 4.21 we see that we have to consider only two special cases:

$$t \in GL(2, \mathbb{R}), \quad \det t = 1, \quad t_{11} = t_{22},$$

and

$$t \in GL(2, \mathbb{R}), \quad \det t = -1, \quad t_{11} = -t_{22}.$$

In the first case we have

$$t = \begin{pmatrix} \alpha & t_{12} \\ t_{21} & \alpha \end{pmatrix}, \quad \det t = \alpha^2 - t_{12}t_{21} = 1.$$

In the second case we have

$$t = \begin{pmatrix} \alpha & t_{12} \\ t_{21} & -\alpha \end{pmatrix}, \quad \det t = -\alpha^2 - t_{12}t_{21} = -1.$$

We can see that in the first (respectively second) case, when $t_{12}t_{21} > 0$ (respectively $t_{12}t_{21} < 0$), we have $\Sigma_1^+(t) = \infty$ (respectively $\Sigma_1^-(t) = \infty$).

Indeed, if $\det t = 1$ and $t_{12}t_{21} \geq 1$, then $|t_{21}| \geq |t_{12}|^{-1}$ and we have

$$\left| t_{12}\sqrt{\frac{b_{1n}}{b_{2n}}} + t_{21}\sqrt{\frac{b_{2n}}{b_{1n}}} \right| = |t_{12}|\sqrt{\frac{b_{1n}}{b_{2n}}} + |t_{21}|\sqrt{\frac{b_{2n}}{b_{1n}}} \geq |t_{12}|\sqrt{\frac{b_{1n}}{b_{2n}}} + |t_{12}|^{-1}\sqrt{\frac{b_{2n}}{b_{1n}}} \geq 2.$$

When $\det t = 1$ and $t_{12}t_{21} \in (0,1)$, then $|t_{12}|^{-1} > |t_{21}|$ and we get

$$\left| t_{12}\sqrt{\frac{b_{1n}}{b_{2n}}} + t_{21}\sqrt{\frac{b_{2n}}{b_{1n}}} \right| = t_{12}t_{21}\left(|t_{21}|^{-1}\sqrt{\frac{b_{1n}}{b_{2n}}} + |t_{12}|^{-1}\sqrt{\frac{b_{2n}}{b_{1n}}} \right) \geq 2|t_{12}t_{21}|.$$

The same is true for the second case, i.e., when $\det t = -1$ and $t_{12}t_{21} < 0$.

When

$$\det t = \alpha^2 - t_{12}t_{21} = 1, \quad \text{and} \quad t_{12}t_{21} = 0,$$

we have four cases

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & t \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ t & -1 \end{pmatrix}, \quad t \in \mathbb{R}. \quad (4.61)$$

When

$$\det t = -\alpha^2 - t_{12}t_{21} = -1, \quad \text{and} \quad t_{12}t_{21} = 0,$$

we also have four cases:

$$\begin{pmatrix} -1 & t \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ t & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & t \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ t & 1 \end{pmatrix}, \quad t \in \mathbb{R}. \quad (4.62)$$

Thus, it remains to consider two cases:

$$\det t = \alpha^2 - t_{12}t_{21} = 1, \quad \text{and} \quad t_{12}t_{21} \in [-1,\, 0),$$

$$\det t = -\alpha^2 - t_{12}t_{21} = -1, \quad \text{and} \quad t_{12}t_{21} \in (0,\, 1].$$

Finally, we can set in the first case $\alpha = \cos\phi$ since $\alpha^2 = 1 + t_{12}t_{21} \in [0,1)$. Then $-t_{12}t_{21} = \sin^2\phi$ so, $t_{12} = -s^2\sin\phi$ and $t_{21} = s^{-2}\sin\phi$, with $s > 0$.

In the second case we can set $\alpha = \cos\phi$ since $\alpha^2 = 1 - t_{12}t_{21} \in [0,1)$. Then $t_{12}t_{21} = \sin^2\phi$ so $t_{12} = s^2\sin\phi$ and $t_{21} = s^{-2}\sin\phi$, with $s > 0$. Finally, in the first (the second) case we have to consider

$$t = \tau_+(\phi, s) = \begin{pmatrix} \cos\phi & -s^2\sin\phi \\ s^{-2}\sin\phi & \cos\phi \end{pmatrix}, \quad t = \tau_-(\phi, s) = \begin{pmatrix} \cos\phi & s^2\sin\phi \\ s^{-2}\sin\phi & -\cos\phi \end{pmatrix}. \quad (4.63)$$

We show that only the first *two cases in (4.61) and (4.62) and the second case in (4.63) are independent*. Indeed, we have for $a = \begin{pmatrix} a_{1n} \\ a_{2n} \end{pmatrix}$ (see Lemma 4.22)

$$t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \; (t^{-1} - I)a = \left(\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} - I \right) \begin{pmatrix} a_{1n} \\ a_{2n} \end{pmatrix} = \begin{pmatrix} -ta_{2n} \\ 0 \end{pmatrix},$$

$$t=\begin{pmatrix}1&0\\t&1\end{pmatrix},\quad (t^{-1}-I)a=\left(\begin{pmatrix}1&0\\-t&1\end{pmatrix}-I\right)\begin{pmatrix}a_{1n}\\a_{2n}\end{pmatrix}=\begin{pmatrix}0\\-ta_{1n}\end{pmatrix},$$

$$t=\begin{pmatrix}-1&t\\0&1\end{pmatrix}=\begin{pmatrix}1&t\\0&1\end{pmatrix}\begin{pmatrix}-1&0\\0&1\end{pmatrix},\quad t^{-1}=\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}1&-t\\0&1\end{pmatrix}=$$

$$\begin{pmatrix}-1&t\\0&1\end{pmatrix},\quad (t^{-1}-I)a=\begin{pmatrix}-2&t\\0&0\end{pmatrix}\begin{pmatrix}a_{1n}\\a_{2n}\end{pmatrix}=\begin{pmatrix}-2a_{1n}+ta_{2n}\\0\end{pmatrix},$$

$$t=\begin{pmatrix}1&0\\t&-1\end{pmatrix}=\begin{pmatrix}1&0\\t&1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad t^{-1}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\-t&1\end{pmatrix}=$$

$$\begin{pmatrix}1&0\\t&-1\end{pmatrix},\quad (t^{-1}-I)a=\begin{pmatrix}0&0\\t&-2\end{pmatrix}\begin{pmatrix}a_{1n}\\a_{2n}\end{pmatrix}=\begin{pmatrix}0\\ta_{1n}-2a_{2n}\end{pmatrix}.$$

Therefore, we get

$$\Sigma^+\begin{pmatrix}1&t\\0&1\end{pmatrix}=t^2\sum_{n\in\mathbb{Z}}b_{1n}\left(\frac{1}{b_{2n}}+a_{2n}^2\right)\simeq S_{12}^L(\mu)=\sum_{n\in\mathbb{Z}}\frac{b_{1n}}{2}\left(\frac{1}{2b_{2n}}+a_{2n}^2\right),\ t\neq 0,$$

$$\Sigma^+\begin{pmatrix}1&0\\t&1\end{pmatrix}=t^2\sum_{n\in\mathbb{Z}}b_{2n}\left(\frac{1}{b_{1n}}+a_{1n}^2\right)\simeq S_{21}^L(\mu)=\sum_{n\in\mathbb{Z}}\frac{b_{2n}}{2}\left(\frac{1}{2b_{1n}}+a_{1n}^2\right),\ t\neq 0,$$

$$\Sigma^-\begin{pmatrix}-1&t\\0&1\end{pmatrix}=t^2\sum_{n\in\mathbb{Z}}\frac{b_{1n}}{b_{2n}}+\sum_{n\in\mathbb{Z}}b_{1n}(-2a_{1n}+ta_{2n})^2=:S_{12}^{L,-}(\mu,t),$$

$$\Sigma^-\begin{pmatrix}1&0\\t&-1\end{pmatrix}=t^2\sum_{n\in\mathbb{Z}}\frac{b_{2n}}{b_{1n}}+\sum_{n\in\mathbb{Z}}b_{2n}(ta_{1n}-2a_{2n})^2=:S_{21}^{L,-}(\mu,t).$$

For the last two cases in (4.61) and (4.62) we get respectively

$$t=\begin{pmatrix}-1&t\\0&-1\end{pmatrix}=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&-t\\0&1\end{pmatrix},\quad t^{-1}=\begin{pmatrix}1&t\\0&1\end{pmatrix}\begin{pmatrix}-1&0\\0&-1\end{pmatrix}=$$

$$\begin{pmatrix}-1&-t\\0&-1\end{pmatrix},\quad (t^{-1}-I)a=\begin{pmatrix}-2&-t\\0&-2\end{pmatrix}\begin{pmatrix}a_{1n}\\a_{2n}\end{pmatrix}=-\begin{pmatrix}2a_{1n}+ta_{2n}\\2a_{2n}\end{pmatrix},$$

$$t=\begin{pmatrix}-1&0\\t&-1\end{pmatrix}=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\-t&1\end{pmatrix},\quad t^{-1}=\begin{pmatrix}1&0\\t&1\end{pmatrix}\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$$

$$=\begin{pmatrix}-1&0\\-t&-1\end{pmatrix},\quad (t^{-1}-I)a=\begin{pmatrix}-2&0\\-t&-2\end{pmatrix}\begin{pmatrix}a_{1n}\\a_{2n}\end{pmatrix}=-\begin{pmatrix}2a_{1n}\\ta_{1n}+2a_{2n}\end{pmatrix}.$$

$$t = \begin{pmatrix} 1 & t \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad t^{-1} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & t \\ 0 & -1 \end{pmatrix}, \quad (t^{-1} - I)a = \begin{pmatrix} 0 & t \\ 0 & -2 \end{pmatrix} \begin{pmatrix} a_{1n} \\ a_{2n} \end{pmatrix} = \begin{pmatrix} ta_{2n} \\ -2a_{2n} \end{pmatrix},$$

$$t = \begin{pmatrix} -1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad t^{-1} = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} -1 & 0 \\ t & 1 \end{pmatrix}, \quad (t^{-1} - I)a = \begin{pmatrix} -2 & 0 \\ t & 0 \end{pmatrix} \begin{pmatrix} a_{1n} \\ a_{2n} \end{pmatrix} = \begin{pmatrix} -2a_{1n} \\ ta_{1n} \end{pmatrix}.$$

Set

$$S_{11}^L(\mu) := S_{12}^{L,-}(\mu, 0) = 4 \sum_{n\in\mathbb{Z}} b_{1n} a_{1n}^2, \quad S_{22}^L(\mu) := S_{21}^{L,-}(\mu, 0) = 4 \sum_{n\in\mathbb{Z}} b_{2n} a_{2n}^2. \tag{4.64}$$

With this notation we see that the second two cases in (4.61) and (4.62) are dependent:

$$\Sigma^+ \begin{pmatrix} -1 & t \\ 0 & -1 \end{pmatrix} = t^2 \sum_{n\in\mathbb{Z}} \frac{b_{1n}}{b_{2n}} + \sum_{n\in\mathbb{Z}} \left[b_{1n}(-2a_{1n} - ta_{2n})^2 + b_{2n}(-2a_{2n})^2 \right]$$

$$= S_{12}^{L,-}(\mu, -t) + S_{22}^L(\mu), \quad \text{note that} \quad \begin{pmatrix} -1 & t \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & t \\ 0 & 1 \end{pmatrix}.$$

$$\Sigma^+ \begin{pmatrix} -1 & 0 \\ t & -1 \end{pmatrix} = t^2 \sum_{n\in\mathbb{Z}} \frac{b_{2n}}{b_{1n}} + \sum_{n\in\mathbb{Z}} \left[b_{1n}(-2a_{1n})^2 + b_{2n}(-ta_{1n} - 2a_{2n})^2 \right]$$

$$= S_{21}^{L,-}(\mu, -t) + S_{11}^L(\mu), \quad \text{note that} \quad \begin{pmatrix} -1 & 0 \\ t & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & -1 \end{pmatrix}.$$

$$\Sigma^- \begin{pmatrix} 1 & t \\ 0 & -1 \end{pmatrix} = t^2 \sum_{n\in\mathbb{Z}} \frac{b_{1n}}{b_{2n}} + t^2 \sum_{n\in\mathbb{Z}} b_{1n} a_{2n}^2 + 4 \sum_{n\in\mathbb{Z}} b_{2n} a_{2n}^2 \simeq t^2 S_{12}^L(\mu) + S_{22}^L(\mu),$$

$$\Sigma^- \begin{pmatrix} -1 & 0 \\ t & 1 \end{pmatrix} = t^2 \sum_{n\in\mathbb{Z}} \frac{b_{2n}}{b_{1n}} + 4 \sum_{n\in\mathbb{Z}} b_{1n} a_{1n}^2 + t^2 \sum_{n\in\mathbb{Z}} b_{2n} a_{1n}^2 \simeq t^2 S_{21}^L(\mu) + S_{11}^L(\mu).$$

To compare $(\mu_{(b,a)}^2)^{L_{\tau_\pm(\phi,s)}}$ and $\mu_{(b,a)}^2$ we calculate $\tau_+^{-1}(\phi, s)$ and $\tau_-^{-1}(\phi, s)$. Since

$$\tau_+(\phi, s) = \begin{pmatrix} \cos\phi & -s^2 \sin\phi \\ s^{-2} \sin\phi & \cos\phi \end{pmatrix}, \quad \tau_-(\phi, s) = \begin{pmatrix} \cos\phi & s^2 \sin\phi \\ s^{-2} \sin\phi & -\cos\phi \end{pmatrix},$$

we get

$$\tau_+^{-1}(\phi,s)=\begin{pmatrix}\cos\phi & s^2\sin\phi\\ -s^{-2}\sin\phi & \cos\phi\end{pmatrix},\quad \tau_-^{-1}(\phi,s)=\begin{pmatrix}\cos\phi & s^2\sin\phi\\ s^{-2}\sin\phi & -\cos\phi\end{pmatrix}$$

$=\tau_-(\phi,s)$. Since $\tau_+^{-1}(\phi,s)-I=$

$$\begin{pmatrix}\cos\phi-1 & s^2\sin\phi\\ -s^{-2}\sin\phi & \cos\phi-1\end{pmatrix}=\begin{pmatrix}-2\sin^2\frac{\phi}{2} & s^2 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\\ -s^{-2}2\sin\frac{\phi}{2}\cos\frac{\phi}{2} & -2\sin^2\frac{\phi}{2}\end{pmatrix}=$$

$$\begin{pmatrix}-2\sin\frac{\phi}{2} & 0\\ 0 & -2\sin\frac{\phi}{2}\end{pmatrix}\begin{pmatrix}\sin\frac{\phi}{2} & -s^2\cos\frac{\phi}{2}\\ s^{-2}\cos\frac{\phi}{2} & \sin\frac{\phi}{2}\end{pmatrix}$$

and $\tau_-^{-1}(\phi,s)-I=$

$$\begin{pmatrix}\cos\phi-1 & s^2\sin\phi\\ s^{-2}\sin\phi & -\cos\phi-1\end{pmatrix}=\begin{pmatrix}-2\sin^2\frac{\phi}{2} & s^2 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\\ s^{-2}2\sin\frac{\phi}{2}\cos\frac{\phi}{2} & -2\cos^2\frac{\phi}{2}\end{pmatrix}=$$

$$\begin{pmatrix}-2\sin\frac{\phi}{2} & 0\\ 0 & -2\cos\frac{\phi}{2}\end{pmatrix}\begin{pmatrix}\sin\frac{\phi}{2} & -s^2\cos\frac{\phi}{2}\\ -s^{-2}\sin\frac{\phi}{2} & \cos\frac{\phi}{2}\end{pmatrix},$$

we have (see (4.60))

$$\Sigma_2(\tau_-(\phi,s))=4\sin^2\frac{\phi}{2}\sum_{n\in\mathbb{Z}}b_{1n}\Big(\sin\frac{\phi}{2}a_{1n}-s^2\cos\frac{\phi}{2}a_{2n}\Big)^2+$$

$$4\cos^2\frac{\phi}{2}\sum_{n\in\mathbb{Z}}b_{2n}\Big(-s^{-2}\sin\frac{\phi}{2}a_{1n}+\cos\frac{\phi}{2}a_{2n}\Big)^2$$

$$\sim\sum_{n\in\mathbb{Z}}\big(4\sin^2\frac{\phi}{2}b_{1n}+4\cos^2\frac{\phi}{2}s^{-2}b_{2n}\big)\Big(\sin\frac{\phi}{2}a_{1n}-s^2\cos\frac{\phi}{2}a_{2n}\Big)^2,$$

$$\Sigma_2(\tau_-(\phi,s))=\sum_{n\in\mathbb{Z}}\big(4\sin^2\frac{\phi}{2}b_{1n}+4\cos^2\frac{\phi}{2}s^{-2}b_{2n}\big)\Big(\sin\frac{\phi}{2}a_{1n}-s^2\cos\frac{\phi}{2}a_{2n}\Big)^2. \tag{4.65}$$

Finally, for $t=\tau_-(\phi,s)$ we get

$$\mu^{L_{\tau_-(\phi,s)}}\perp\mu\Leftrightarrow\sin^2\phi\Sigma_1(s)+\Sigma_2(\tau_-(\phi,s))=\infty, \tag{4.66}$$

where

$$\Sigma_1(s)=\sum_{n\in\mathbb{Z}}\Big(s^2\sqrt{\frac{b_{1n}}{b_{2n}}}-s^{-2}\sqrt{\frac{b_{2n}}{b_{1n}}}\Big)^2.$$

We have for $t = \tau_+(\phi, s)$ (see (4.60))

$$\mu^{L_{\tau_+(\phi,s)}} \perp \mu \Leftrightarrow \sin^2 \phi \Sigma_1(s) + \Sigma_2(\tau_+(\phi, s)) = \infty, \tag{4.67}$$

where $\Sigma_2(\tau_+(\phi, s)) =$

$$4\sin^2 \frac{\phi}{2} \sum_{n\in\mathbb{Z}} \Big[b_{1n}\Big(\sin \frac{\phi}{2} a_{1n} - s^2 \cos \frac{\phi}{2} a_{2n}\Big)^2 + b_{2n}\Big(s^{-2} \cos \frac{\phi}{2} a_{1n} + \sin \frac{\phi}{2} a_{2n}\Big)^2 \Big]. \tag{4.68}$$

We show that the condition $\mu^{L_{\tau_+(\phi,s)}} \perp \mu$ depends on the previous conditions of the orthogonality. Indeed, for $t = \tau_-(\phi, s)$ we have

$$\mu^{L_{\tau_-(\phi,s)}} \perp \mu \Leftrightarrow (a)\ \Sigma_1(s) = \infty \quad \text{or} \quad (b)\ \Sigma_1(s) < \infty,\ \text{ but }\ \Sigma_2(\tau_-(\phi, s)) = \infty.$$

For $t = \tau_+(\phi, s)$ we get respectively

$$\mu^{L_{\tau_+(\phi,s)}} \perp \mu \Leftrightarrow (c)\ \Sigma_1(s) = \infty \quad \text{or} \quad (d)\ \Sigma_1(s) < \infty,\ \text{ but }\ \Sigma_2(\tau_+(\phi, s)) = \infty.$$

We see that $(c) \Leftrightarrow (a)$. To investigate the condition (d) we observe that if $\Sigma_1(s) < \infty$, then $\lim_{n\to\infty} s^2\sqrt{\frac{b_{1n}}{b_{2n}}} = 1$ therefore, we have $b_{2n} \sim s^4 b_{1n}$ hence, the following equivalence holds: $\Sigma_2(\tau_+(\phi, s)) =$

$$4\sin^2 \frac{\phi}{2} \sum_{n\in\mathbb{Z}} \Big[b_{1n}\Big(\sin \frac{\phi}{2} a_{1n} - s^2 \cos \frac{\phi}{2} a_{2n}\Big)^2 + b_{2n}\Big(s^{-2} \cos \frac{\phi}{2} a_{1n} + \sin \frac{\phi}{2} a_{2n}\Big)^2 \Big]$$

$$\sim 4\sin^2 \frac{\phi}{2} \sum_{n\in\mathbb{Z}} \Big[b_{1n}\Big(\sin \frac{\phi}{2} a_{1n} - s^2 \cos \frac{\phi}{2} a_{2n}\Big)^2 + b_{1n}\Big(\cos \frac{\phi}{2} a_{1n} + s^2 \sin \frac{\phi}{2} a_{2n}\Big)^2 \Big] =$$

$$4\sin^2 \frac{\phi}{2} \sum_{n\in\mathbb{Z}} b_{1n} \left[a_{1n}^2 + s^4 a_{2n}^2 \right] \sim 4\sin^2 \frac{\phi}{2} \sum_{n\in\mathbb{Z}} \left(b_{1n} a_{1n}^2 + b_{2n} a_{2n}^2 \right) =$$

$$\sin^2 \frac{\phi}{2} \left[S_{11}^L(\mu) + S_{22}^L(\mu) \right].$$

We see that condition (d) follows from the conditions $S_{11}^L(\mu) = S_{12}^{L,-}(\mu, 0) = \infty$ and $S_{22}^L(\mu) = S_{21}^{L,-}(\mu, 0) = \infty$. This completes the proof of Lemma 4.1. □

4.4. The explicit expression for $(D^{-1}(\lambda)\mu, \mu)$

The following lemma will be systematically used in what follows.

Lemma 4.23. *For the matrix $D(\lambda_1, \lambda_2, ..., \lambda_m)$ defined below*

$$D(\lambda_1, \lambda_2, ..., \lambda_m) = \begin{pmatrix} 1+\lambda_1 & 1 & ... & 1 \\ 1 & 1+\lambda_2 & ... & 1 \\ & & ... & \\ 1 & 1 & ... & 1+\lambda_m \end{pmatrix} \tag{4.69}$$

and $\mu = (\mu_k)_{k=1}^m \in \mathbb{R}^m$ we have

$$(D^{-1}(\lambda_1, \lambda_2, ..., \lambda_m)\mu, \mu) = \frac{\sum_{k=1}^m \frac{\mu_k^2}{\lambda_k} + \sum_{1\le k<n\le m} \frac{(\mu_k-\mu_n)^2}{\lambda_k\lambda_n}}{1+\sum_{k=1}^m \frac{1}{\lambda_k}}. \tag{4.70}$$

Proof. Let us set $d_m(\lambda_1, \lambda_2, ..., \lambda_m) = \det\left(D(\lambda_1, \lambda_2, ..., \lambda_m)\right)$. It is easy to see that

$$d_m(\lambda_1, \lambda_2, ..., \lambda_m) = \prod_{k=1}^m \lambda_k \left(1 + \sum_{k=1}^m \frac{1}{\lambda_k}\right). \tag{4.71}$$

For arbitrary m we have

$$D^{-1}(\lambda_1, \lambda_2, ..., \lambda_m) = \begin{pmatrix} 1+\lambda_1 & 1 & ... & 1 \\ 1 & 1+\lambda_2 & ... & 1 \\ & & ... & \\ 1 & 1 & ... & 1+\lambda_m \end{pmatrix}^{-1} = \left(D_{kn}^{-1}\right)_{k,n=1}^m,$$

where

$$D_{nn}^{-1} = \frac{d_{m-1}(\lambda_1, ..., \hat{\lambda}_n, ...\lambda_m)}{d_m(\lambda_1, \lambda_2, ..., \lambda_m)} = \left(1+\sum_{k=1}^m \frac{1}{\lambda_k}\right)^{-1} \frac{1}{\lambda_n}\left(1+\sum_{k=1,k\neq n}^m \frac{1}{\lambda_k}\right),$$

$$D_{kn}^{-1} = \frac{-d_{m-1}(\lambda_1, ..., \hat{\lambda}_n, ...\lambda_m)|_{\lambda_k=0}}{d_m(\lambda_1, \lambda_2, ..., \lambda_m)} = -\frac{1}{\lambda_k\lambda_n}\left(1+\sum_{k=1}^m \frac{1}{\lambda_k}\right)^{-1}, \ k \neq n,$$

since using (4.71) we have

$$d_{m-1}(\lambda_1, ..., \hat{\lambda}_n, ...\lambda_m)|_{\lambda_k=0} = \lim_{\lambda_k\to 0} \prod_{p=1,p\neq n}^m \lambda_p \left(1+\sum_{p=1,p\neq n}^m \frac{1}{\lambda_p}\right) = \frac{1}{\lambda_k\lambda_n}\prod_{p=1}^m \lambda_p.$$

Finally, we have for $\mu = (\mu_1, \mu_2, ..., \mu_m) \in \mathbb{R}^m$

$$(D^{-1}(\lambda_1, \lambda_2, ..., \lambda_m)\mu, \mu) = \sum_{k,n=1}^{m} D^{-1}_{kn}\mu_k\mu_n =$$

$$\left(1 + \sum_{k=1}^{m} \frac{1}{\lambda_k}\right)^{-1} \left[\sum_{n=1}^{m} \frac{\mu_n^2}{\lambda_n}\left(1 + \sum_{k=1, k\neq n}^{m} \frac{1}{\lambda_k}\right) - 2 \sum_{1\le k<n\le m} \frac{\mu_k\mu_n}{\lambda_k\lambda_n}\right] =$$

$$\left(1 + \sum_{k=1}^{m} \frac{1}{\lambda_k}\right)^{-1} \left[\sum_{n=1}^{m} \frac{\mu_n^2}{\lambda_n} + \sum_{1\le k<n\le m} \frac{(\mu_k - \mu_n)^2}{\lambda_k\lambda_n}\right].$$

□

Remark 4.7. Some useful observations. If we set $f_{(m)} = (f_k)_{k=1}^m$ and $g_{(m)} = (g_k)_{k=1}^m$ where $f_k = \frac{\mu_k}{\sqrt{\lambda_k}}$ and $g_k = \frac{1}{\sqrt{\lambda_k}}$ we can recognize that

$$\sum_{n=1}^{m} \frac{\mu_n^2}{\lambda_n} = \|f_{(m)}\|^2 = \Gamma(f_{(m)})$$

and

$$\sum_{1\le k<n\le m} \frac{(\mu_k - \mu_n)^2}{\lambda_k\lambda_n} = \sum_{1\le k<n\le m} \begin{vmatrix} f_k & f_n \\ g_k & g_n \end{vmatrix}^2 = \Gamma(f_{(m)}, g_{(m)})$$

since

$$\begin{vmatrix} f_k & f_n \\ g_k & g_n \end{vmatrix}^2 = \begin{vmatrix} \frac{\mu_k}{\sqrt{\lambda_k}} & \frac{\mu_n}{\sqrt{\lambda_n}} \\ \frac{1}{\sqrt{\lambda_k}} & \frac{1}{\sqrt{\lambda_n}} \end{vmatrix}^2 = \frac{(\mu_k - \mu_n)^2}{\lambda_k\lambda_n}.$$

Set $\Delta(f, g) = \frac{\Gamma(f)+\Gamma(f,g)}{\Gamma(g)+1}$ for two vectors f and g. Finally, we get

$$(D^{-1}(\lambda_1, \lambda_2, ..., \lambda_m)\mu, \mu) = \Delta(f_{(m)}, g_{(m)})) = \frac{\Gamma(f_{(m)}) + \Gamma(f_{(m)}, g_{(m)})}{\Gamma(g_{(m)}) + 1}. \quad (4.72)$$

where $\Gamma(f_1, f_2, \ldots, f_n)$ is the Gram determinant and $\gamma(f_1, f_2, \ldots, f_n)$ is the Gram matrix of n vectors $f_1, f_2, \ldots, f_n$ in a Hilbert space (see [7]).

4.5. The proof of Lemmas 4.2 – 4.7

PROOF. The proof of Lemma 4.4 is based on Lemma 4.23. We find out when the inclusion

$$D_{1n}\mathbf{1}\in\langle A_{kn}\mathbf{1}=(x_{1k}D_{1n}+x_{2k}D_{2n})\mathbf{1}\mid k\in\mathbb{Z}\rangle$$

holds. Fix $m\in\mathbb{N}$, since $Mx_{1k}=a_{1k}$, we put $\sum_{k=-m}^{m}t_ka_{1k}=(t,b)=1$, where $t=(t_k)_{k=-m}^{m}$ and $b=(a_{1k})_{k=-m}^{m}$. We have

$$\|\,[\sum_{k=-m}^{m}t_k(x_{1k}D_{1n}+x_{2k}D_{2n})-D_{1n}]\mathbf{1}\|^2=$$

$$\|\sum_{k=-m}^{m}t_k[(x_{1k}-a_{1k})D_{1n}+x_{2k}D_{2n}]\mathbf{1}\|^2=\sum_{-m\leq k,r\leq m}(f_k,f_r)t_kt_r=:(A_{2m+1}t,t),$$

where $A_{2m+1}=((f_k,f_r))_{k,r=-m}^{m}$, and $f_k=[(x_{1k}-a_{1k})D_{1n}+x_{2k}D_{2n}]\mathbf{1}$. We have

$$(f_k,f_k)=\|\,[(x_{1k}-a_{1k})D_{1n}+x_{2k}D_{2n}]\,\mathbf{1}\|^2=\frac{1}{2b_{1k}}\frac{b_{1n}}{2}+\Big(\frac{1}{2b_{2k}}+a_{2k}^2\Big)\frac{b_{2n}}{2}\sim$$

$$\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}+a_{2k}^2,$$

$$(f_k,f_r)=([(x_{1k}-a_{1k})D_{1n}+x_{2k}D_{2n}]\,\mathbf{1},[(x_{1r}-a_{1r})D_{1n}+x_{2r}D_{2n}]\,\mathbf{1})=$$

$$(x_{2k},x_{2r})(D_{2n}\mathbf{1},D_{2n}\mathbf{1})=a_{2k}a_{2r}\frac{b_{2n}}{2}\simeq a_{2k}a_{2r}.$$

Finally, we have

$$(f_k,f_k)\sim\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}+a_{2k}^2,\quad (f_k,f_r)\sim a_{2k}a_{2r},\quad k\neq r.\tag{4.73}$$

For $A_{(m)}=((f_k,f_r))_{k,r=1}^{m}$, and $b=(a_{11},a_{12},...,a_{1m})\in\mathbb{R}^m$ we have

$$A_{(m)}=\gamma(f_1,f_2,...,f_m)=\begin{pmatrix}(f_1,f_1)&(f_1,f_2)&...&(f_1,f_m)\\(f_2,f_1)&(f_2,f_2)&...&(f_2,f_m)\\&&...&\\(f_m,f_1)&(f_m,f_2)&...&(f_m,f_m)\end{pmatrix}=$$

$$\begin{pmatrix} \frac{1}{2b_{11}}+\frac{1}{2b_{21}}+a_{21}^2 & a_{21}a_{22} & ... & a_{21}a_{2m} \\ a_{22}a_{21} & \frac{1}{2b_{12}}+\frac{1}{2b_{22}}+a_{22}^2 & ... & a_{22}a_{2m} \\ & & ... & \\ a_{2m}a_{21} & a_{2m}a_{22} & ... & \frac{1}{2b_{1m}}+\frac{1}{2b_{2m}}+a_{2m}^2 \end{pmatrix} =$$

$$\begin{pmatrix} a_{21} & 0 & ... & 0 \\ 0 & a_{22} & ... & 0 \\ & & ... & \\ 0 & 0 & ... & a_{2m} \end{pmatrix}\begin{pmatrix} 1+\lambda_1 & 1 & ... & 1 \\ 1 & 1+\lambda_2 & ... & 1 \\ & & ... & \\ 1 & 1 & ... & 1+\lambda_m \end{pmatrix}\begin{pmatrix} a_{21} & 0 & ... & 0 \\ 0 & a_{22} & ... & 0 \\ & & ... & \\ 0 & 0 & ... & a_{2m} \end{pmatrix},$$

where $\lambda_k = \frac{\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}}{a_{2k}^2}$, $1 \le k \le m$. Using (4.69) we conclude that

$$A_{(m)} = \mathrm{diag}(a_{21}, a_{22}, \dots, a_{2m})D(\lambda_1, \lambda_2, \dots, \lambda_m)\mathrm{diag}(a_{21}, a_{22}, ..., a_{2m}).$$

Recall that $\mu = \mathrm{diag}(a_{21}, a_{22}, , ..., a_{2m})^{-1}b = (\frac{a_{11}}{a_{21}}, \frac{a_{12}}{a_{22}}, ..., \frac{a_{1m}}{a_{2m}})$, where $b = (a_{11}, a_{12}, ..., a_{1m}) \in \mathbb{R}^m$, then

$$(A_{(m)}^{-1}b, b) = (D^{-1}(\lambda_1, \lambda_2, ..., \lambda_m)\mu, \mu),\ \ \lambda_k = \Big(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}}\Big)a_{2k}^{-2},\ \ \mu_k = a_{1k}a_{2k}^{-1}. \tag{4.74}$$

Using Lemma 4.23 for the operator A_{2m+1}, and the vector $b \in \mathbb{R}^{2m+1}$ we obtain

$$(A_{2m+1}^{-1}b, b) = \frac{\sum_{k=-m}^{m} \frac{a_{1k}^2}{\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}} + \sum_{-m\le k<n\le m} \frac{(a_{1k}a_{2n}-a_{1n}a_{2k})^2}{\Big(\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}\Big)\Big(\frac{1}{2b_{1n}}+\frac{1}{2b_{2n}}\Big)}}{\sum_{k=-m}^{m} \frac{a_{2k}^2}{\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}} + 1}$$

$= \Delta(f_{(m)}, g_{(m)})$, where

$$f_m = \Big(a_{1k}\Big(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}}\Big)^{-1/2}\Big)_{k=-m}^{m},\ \ g_m = \Big(a_{2k}\Big(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}}\Big)^{-1/2}\Big)_{k=-m}^{m} \tag{4.75}$$

This proves Lemma 4.4 □

The proof of Lemma 4.5 is exactly the same.

The proof of Lemma 4.2 is also based on Lemma 4.23.

PROOF. We study when $x_{1n}x_{1t} \in \langle A_{nk}A_{tk}\mathbf{1} \mid k \in \mathbb{Z}\rangle$. Since

$$A_{nk}A_{tk} = (x_{1n}D_{1k} + x_{2n}D_{2k})(x_{1t}D_{1k} + x_{2t}D_{2k}) =$$

$$x_{1n}x_{1t}D_{1k}^2+(x_{1n}x_{2t}+x_{2n}x_{1t})D_{1k}D_{2k}+x_{2n}x_{2t}D_{2k}^2$$

and $MD_{1k}^2\mathbf{1}=-\frac{b_{1k}}{2}$, set $-\sum_{k=-m}^{m}t_k\frac{b_{1k}}{2}=(t,b')=1$, where $t=(t_k)_{k=-m}^{m}$ and $b'=-(\frac{b_{1k}}{2})_k\sim b=(b_{1k})_{k=-m}^{m}$. We have

$$\|\big[\sum_{k=-m}^{m}t_kA_{nk}A_{tk}-x_{1n}x_{1t}\big]\mathbf{1}\|^2=$$

$$\|\sum_{k=-m}^{m}t_k\Big[x_{1n}x_{1t}\Big(D_{1k}^2+\frac{b_{1k}}{2}\Big)+(x_{1n}x_{2t}+x_{2n}x_{1t})D_{1k}D_{2k}+x_{2n}x_{2t}D_{2k}^2\Big]\mathbf{1}\|^2$$

$$=\sum_{-m\le k,r\le m}(f_k,f_r)t_kt_r=:(A_{2m+1}t,t),$$

where $A_{2m+1}=((f_k,f_r))_{k,r=-m}^{m}$ and

$$f_k=\Big[x_{1n}x_{1t}\Big(D_{1k}^2+\frac{b_{1k}}{2}\Big)+(x_{1n}x_{2t}+x_{2n}x_{1t})D_{1k}D_{2k}+x_{2n}x_{2t}D_{2k}^2\Big]\mathbf{1}.$$

If we denote by $c_{kn}=\|x_{kn}\|^2=\frac{1}{2b_{kn}}+a_{kn}^2$, we get

$$(f_k,f_k)=\|\big[x_{1n}x_{1t}\Big(D_{1k}^2+\frac{b_{1k}}{2}\Big)+(x_{1n}x_{2t}+x_{2n}x_{1t})D_{1k}D_{2k}+x_{2n}x_{2t}D_{2k}^2\big]\mathbf{1}\|^2=$$

$$c_{1n}c_{1t}2\Big(\frac{b_{1k}}{2}\Big)^2+\big(c_{1n}c_{2t}+c_{1t}c_{2n}+2a_{1n}a_{2t}a_{1t}a_{2n}\big)\frac{b_{1k}}{2}\frac{b_{2k}}{2}+c_{2n}c_{2t}3\Big(\frac{b_{2k}}{2}\Big)^2$$

$$\sim(b_{1k}+b_{2k})^2,\quad (f_k,f_r)=$$

$$\Big(\Big(x_{1n}x_{1t}\Big(D_{1k}^2+\frac{b_{1k}}{2}\Big)+(x_{1n}x_{2t}+x_{2n}x_{1t})D_{1k}D_{2k}+x_{2n}x_{2t}D_{2k}^2\Big)\mathbf{1},$$

$$\Big(x_{1n}x_{1t}\Big(D_{1r}^2+\frac{b_{1r}}{2}\Big)+(x_{1n}x_{2t}+x_{2n}x_{1t})D_{1r}D_{2r}+x_{2n}x_{2t}D_{2r}^2\Big)\mathbf{1}\Big)=$$

$$c_{2n}c_{2t}\frac{b_{2k}}{2}\frac{b_{2r}}{2}\sim b_{2k}b_{2r}.$$

Finally, we have

$$(f_k,f_k)\sim(b_{1k}+b_{2k})^2,\quad (f_k,f_r)\sim b_{2k}b_{2r},\quad k\neq r. \tag{4.76}$$

For $A_{(m)} = ((f_k, f_r))_{k,r=1}^m$, and $b = (a_{11}, a_{12}, ..., a_{1m}) \in \mathbb{R}^m$ we have

$$A_{(m)} = \gamma(f_1, f_2, ..., f_m) = \begin{pmatrix} (f_1,f_1) & (f_1,f_2) & ... & (f_1,f_m) \\ (f_2,f_1) & (f_2,f_2) & ... & (f_2,f_m) \\ & & ... & \\ (f_m,f_1) & (f_m,f_2) & ... & (f_m,f_m) \end{pmatrix} = \tag{4.77}$$

$$\begin{pmatrix} (b_{11}+b_{21})^2 & b_{21}b_{22} & ... & b_{21}b_{2m} \\ b_{22}b_{21} & (b_{12}+b_{22})^2 & ... & b_{22}b_{2m} \\ & & ... & \\ b_{2m}b_{21} & b_{2m}b_{22} & ... & (b_{1m}+b_{2m})^2 \end{pmatrix} =$$

$$\begin{pmatrix} b_{21} & 0 & ... & 0 \\ 0 & b_{22} & ... & 0 \\ & & ... & \\ 0 & 0 & ... & b_{2m} \end{pmatrix} \begin{pmatrix} 1+\lambda_1 & 1 & ... & 1 \\ 1 & 1+\lambda_2 & ... & 1 \\ & & ... & \\ 1 & 1 & ... & 1+\lambda_m \end{pmatrix} \begin{pmatrix} b_{21} & 0 & ... & 0 \\ 0 & b_{22} & ... & 0 \\ & & ... & \\ 0 & 0 & ... & b_{2m} \end{pmatrix}.$$

At last, we have for $\mu = \text{diag}(b_{21}, b_{22}, ..., b_{2m})^{-1}b = (\frac{b_{11}}{b_{21}}, \frac{b_{12}}{b_{22}}, .., \frac{b_{1m}}{b_{2m}})$

$$(A_{(m)}^{-1}b, b) = (D^{-1}(\lambda_1, \lambda_2, ..., \lambda_m)\mu, \mu), \quad \lambda_k = \left(1 + \frac{b_{1k}}{b_{2k}}\right)^2 - 1, \quad \mu_k = \frac{b_{1k}}{b_{2k}}.$$

Using Lemma 4.23 for the operator A_{2m+1}, and the vector $b \in \mathbb{R}^{2m+1}$ we obtain

$$(A_{2m+1}^{-1}b, b) = \frac{\sum_{k=-m}^m \frac{\left(\frac{b_{1k}}{b_{2k}}\right)^2}{\left(\frac{b_{1k}}{b_{2k}}+1\right)^2 - 1} + \sum_{-m \le k < n \le m} \frac{\left(\frac{b_{1k}}{b_{2k}} - \frac{b_{1n}}{b_{2n}}\right)^2}{\left[\left(\frac{b_{1k}}{b_{2k}}+1\right)^2-1\right]\left[\left(\frac{b_{1n}}{b_{2n}}+1\right)^2-1\right]}}{\sum_{k=-m}^m \frac{1}{\left(\frac{b_{1k}}{b_{2k}}+1\right)^2-1} + 1} =$$

$\Delta(f_m^1, g_m^1)$ where

$$f_m^1 = \Big(\frac{b_{1k}}{\sqrt{b_{1k}^2 + 2b_{1k}b_{2k}}}\Big)_{k=-m}^m, \quad g_m^1 = \Big(\frac{b_{2k}}{\sqrt{b_{1k}^2 + 2b_{1k}b_{2k}}}\Big)_{k=-m}^m, \tag{4.78}$$

□

The proof of Lemma 4.3 is similar. We get $(A^{-1}b, b) = \Delta(f_m^2, g_m^2)$ where

$$f_m^2 = \Big(\frac{b_{2k}}{\sqrt{b_{2k}^2 + 2b_{1k}b_{2k}}}\Big)_{k=-m}^m, \quad g_m^2 = \Big(\frac{b_{1k}}{\sqrt{b_{2k}^2 + 2b_{1k}b_{2k}}}\Big)_{k=-m}^m. \tag{4.79}$$

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[1] S. Albeverio and A. Kosyak, Quasiregular representations of the infinite-dimensional Borel group, J. Funct. Anal. 218/2 (2005) 445–474.

[2] S. Albeverio and A. Kosyak, Quasiregular representations of the infinite-dimensional nilpotent group, J. Funct. Anal. 236 (2006) 634–681.

[3] E.F. Beckenbach and R. Bellmann, Inequalities, Springer, Berlin, Göttingen, Heidelberg, 1961.

[4] Yu. M. Berezanskii, Selfadjoint operators in spaces of functions of infinitely many variables. Translated from the Russian by H. H. McFaden. Translations of Mathematical Monographs, 63. AMS, Providence, RI, 1986.

[5] J. Dixmier, Les C^*algèbres et leur représentations, Gautier-Villars, Paris, 1969.

[6] J. Dixmier, Les algèbres d'opérateurs dans l'espace hilbertien, 2nd Edition, Gauthier-Villars, Paris, 1969.

[7] R.F. Gantmacher, Matrizenrechnung. Teil 1. Veb Deutscher Verlag der Wissenschaften, Berlin, 1958.

[8] S. Helgason, Groups and geometric analysis. Integral geometry, invariant differential operators and spherical functions. Mathematical Surveys and Monographs, v 83, Academic Press, Orlando, 1984.

[9] R.A. Horn, C.R. Jonson, Matrix analysis, Cambridge University Press, Cambridge, 1989.

[10] R.A. Horn, C.R. Jonson, Topics in Matrix analysis, Cambridge University Press, Cambridge, 1991.

[11] S. Kakutani, On equivalence of infinite product measures, Ann. Math. 4 (1948) 214–224.

[12] A.W. Knapp, Representation theory of semisimple groups, An Overview Based on Examples, Princeton University Press, Princeton, New Jersey, 1986,

[13] B.O. Koopman, Hamiltonian systems and transformation in Hilbert space, Proc. Natl. Acad. Sci. U. S. A. 17 (1931), no. 5, 233–238.

[14] A.V. Kosyak, Irreducibility criterion for regular Gaussian representations of groups of finite upper triangular matrices, Funct. Anal. Appl. 24 (1990) no. 3, 243–245.

[15] A.V. Kosyak, Criteria for irreducibility and equivalence of regular Gaussian representations of group of finite upper triangular matrices of infinite order, Selecta Math. Soviet. 11 (1992) 241–291.

[16] A.V. Kosyak, Irreducible regular Gaussian representations of the group of the interval and the circle diffeomorphisms, J. Funct. Anal. 125 (1994) 493–547.

[17] A.V. Kosyak, The generalized Ismagilov conjecture for the group $B_0^{\mathbb{N}}$. I, Methods Funct. Anal. Topology. 8, No 2 (2002) 33–49.

[18] A.V. Kosyak, The generalized Ismagilov conjecture for the group $B_0^{\mathbb{N}}$. II, Methods Funct. Anal. Topology. 8, No 3 (2002) 27–45.

[19] A.V. Kosyak, Irreducibility criterion for quasiregular representations of the group of finite upper-trianguler matrices. in Funct. Anal. Appl. 37 (2003) no. 1, 65–68.

[20] A.V. Kosyak, Quasi-invariant measures and irreducible representations of the inductive limit of the special linear groups, Funct. Anal. Appl. 38 (2004) no.1, 67–68.

[21] A.V. Kosyak, Regular, quasiregular and induced representations of infinite-dimensional groups, Doctoral Dissertation, Kiev Institute of mathematics, June, 2010 (in Ukrainian).

[22] A.V. Kosyak, Induced representations of infinite-dimensional groups, I, J. Funct. Anal, 266 (2014) 3395–3434.

[23] H.H. Kuo, Gaussian measures in Banach spaces, in: Lecture Notes Mathematics, Vol. 463, Springer, Berlin, 1975.

[24] I. Penkov and K. Styrkas, Tensor representations of classical locally finite Lie algebras, in: Developments and Trends in Infinite-Dimensional Lie Theory, Progress in math, 288, K.-H. Neeb and Pianzola (eds.) Springer Basel, 2011.

[25] I. Schur, Über eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lassen. Dissertation. Berlin. 76 S (1901) JMF 32.0165.04

[26] I. Schur, Über die rationalen Darstellungen der allgemeinen linearen Gruppe. Sitzungsberichte Akad. Berlin 1927, 58–75 (1927) JMF 53.0108.05

[27] Tsilevich, N.V. and Vershik, A.M. Infinite-dimensional Schur-Weyl duality and the Coxeter–Laplace operator, Commun. Math. Phys., 327, Issue 3, (2014) 873–885.

[28] G.E. Shilov and Fan Dik Tun', Integral, Measure, and Derivative on Linear Spaces (Russian), Nauka, Moscow, 1967.

[29] A.V. Skorokhod. Integration in Hilbert space, Springer, Berlin, 1974.

[30] A. Weil, L'intégration dans les groupes topologique et ses application 2^e ed., Hermann, Paris, 1953.

[31] H. Weyl, The Classical Groups. Their Invariants and Representations. Princeton University Press, Princeton, N.J., 1939. xii+302 pp.

[32] Q. Yuan, Four flavors of Schur-Weyl duality, https://qchu.wordpress.com/2012/11/13/four-flavors-of-schur-weyl-duality/