On some classes of extensions of local fields

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ON SOME CLASSES OF EXTENSIONS OF LOCAL FIELDS

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In this note the theory of the Hasse–Herbrand function developed in section 3 Chap. III of [FV] is applied for study of several important classes of extensions of local fields.

The first section contains discussions of connections between the property of a totally ramified p-extension of a local field to be abelian and the property of its Galois group to possess integer jumps with respect to the upper numbering (Hasse–Arf property). It is shown that such an extension L/F is abelian if and only if for any totally ramified abelian extension E/F the extension LE/F satisfies the Hasse–Arf property. I formulate also an additional property to the Hasse–Arf property in terms of principal units which makes the extension abelian.

The second section deals with deeply ramified extensions introduced recently by J. Coates and R. Greenberg [CR]. Most of results (properties of deeply ramified extensions (2)–(5)) are due to them. The reason why they are included in the paper is a hope that the proofs of them are more elementary than in [CR]. A connection of these extensions with arithmetically profinite ones (playing a fundamental role in the theory of fields of norms of J.-M. Fontaine and J.-P. Wintenberger) is discussed as well.

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1. Hasse–Arf Property and Abelian Extensions

Let F be a complete (or Henselian) discrete valuation field with a perfect residue field of characteristic $p > 0$. For a finite Galois extension L/F let $h_{L/F}$ denote the Hasse-Herbrand function (it coincides with the inverse function to the function $\varphi_{L/F}$ in the ramification theory), see section 3 in Chap. III of [FV] or (3.2) of [F1].

The extension L/F is said *to satisfy Hasse–Arf property (HAP)*, if

$$\{v_L(\pi_L^{-1}\sigma\pi_L - 1) : \sigma \in \mathrm{Gal}(L/F)\} \subset h_{L/F}(\mathrm{N})$$

where π_L is a prime element in L and v_L is the discrete valuation on L, $v_L(\pi_L) = 1$.

Let $\overline{F}$ be the residue field of F and $\kappa = \dim_{\mathbf{F}_p} \overline{F}/\wp(\overline{F})$ where $\wp(X)$ is the polynomial $X^p - X$. Further we will assume that $\kappa \neq 0$ and apply local p-class field theory developed in [F2], the case $\kappa = 0$ when the field $\overline{F}$ is algebraically p-closed may be treated using the Serre geometric class field theory [Sr].

Let U_F be the group of units of the ring of integers of F and $U_{i,F}$ be the groups of higher principal units. The following assertion for totally ramified p-extensions is very well-known. We show how it easily follows from class field theory.

Theorem (Hasse–Arf). *Let L/F be a totally ramified abelian p-extension. Then L/F satisfies HAP.*

Proof. Let $\mathrm{Gal}(L/F)^*$ be the group of $\mathbf{Z}_p$-continuous homomorphisms from the Galois group of the maximal unramified abelian p-extension $\widetilde{F}/F$ to the discrete $\mathbf{Z}_p$-module $\mathrm{Gal}(L/F)$. Put as usually

$$\mathrm{Gal}(L/F)_i = \{\sigma \in \mathrm{Gal}(L/F) : \pi_L^{-1}\sigma\pi_L \in U_{i,L}\}.$$

The construction of the reciprocity map

$$\Psi_{L/F}\colon U_{1,F}/N_{L/F}U_{1,L} \to \mathrm{Gal}(L/F)^*$$

and the inverse isomorphism

$$\Upsilon_{L/F}\colon \mathrm{Gal}(L/F)^* \to U_{1,F}/N_{L/F}U_{1,L}$$

in section 1 of [F2] shows that $\Psi_{L/F}$ transforms $U_{i,L}N_{L/F}U_{1,L} - U_{i+1,L}N_{L/F}U_{1,L}$ onto $(\mathrm{Gal}(L/F)_{h_{L/F}(i)})^* - (\mathrm{Gal}(L/F)_{h_{L/F}(i)+1})^*$. Thus, any non-trival automorphism $\sigma \in \mathrm{Gal}(L/F)$ belongs exactly to $\mathrm{Gal}(L/F)_{h_{L/F}(i)}$ for some integer i. □

One can construct examples of non-abelian extensions (even totally ramified of degree a power of p) which satisfy HAP. Moreover, for any totally ramified non-abelian p-extension L/F (of degree a power of p) there exists a totally ramified p-extension E/F linearly disjoint with L/F and such that LE/E satisfies HAP, see Maus [M, Satz (3.7)]. Nevertheless, the following theorem provides a characterization of abelian totally ramified p-extensions in terms of HAP (the general case of totally ramified extensions see below in Remark 1).

Theorem. *Let L/F be a finite totally ramified Galois p-extension. Let M/F be the maximal abelian subextension in L/F. The following conditions are equivalent:*

(1) *L/F is abelian;*
(2) *for any totally ramified abelian p-extension E/F the extension LE/F satisfies HAP;*
(3) *for any totally ramified abelian p-extension E/F of M/F the extension LE/F satisfies HAP.*

Before starting the proof of the theorem we need to establish several auxiliary assertions. Every so often, we apply the description of the norm map on higher principal units (see (1.3) of [F2] or (3.1) of [F1]) and the properties of the Hasse-Herbrand function (see (3.2) of [F1] or section 3 in Chap. III of [FV]).

Lemma 1. *Let M/F be a totally ramified Galois p-extension. Let π_M be a prime element of M. Let an element $\alpha \in M^*$ be such that $v_M(\alpha - 1) = r = h_{M/F}(r_0)$, $r_0 \in \mathbb{N}$ and $N_{M/F}(\alpha) \in U_{r_0+1,F}$. Then there is $\tau \in \mathrm{Gal}(M/F)$ such that $\alpha\pi_M\tau(\pi_M^{-1})$ belongs to $U_{r+1,M}$.*

Proof. This is Theorem (4.2) of section 4 in [FV]. For the sake of completeness we indicate the arguements.

One may proceed by induction on the degree of M/F.

If M/F is of degree p, then the conditions of the lemma imply first of all that $\alpha = \gamma^{-1}\sigma(\gamma)$ for some $\gamma \in M^*$ and a generator σ of $\mathrm{Gal}(M/F)$. Then the commutative diagrams of section (3.1) of [F1] show that $r = v_M(\pi_M^{-1}\sigma\pi_M - 1)$ and $\alpha\pi_M\sigma^i(\pi_M^{-1}) \in U_{r+1,M}$ for a suitable $0 < i < p$.

Let M_1/F be a Galois subextension in M/F such that M/M_1 is of degree p. If $\beta = N_{M/M_1}\alpha$ belongs to $U_{r_1+1,F}$ for $r_1 = h_{M/M_1}^{-1}(r)$, then $r_1 = r$ and α can be written in the required form. If $\beta \notin U_{r_1+1,F}$, then β satisfies the conditions of the lemma for the extension M_1/F, therefore $\beta\pi_{M_1}\tau_1(\pi_{M_1}^{-1})$ belongs to U_{r_1+1,M_1} for $\pi_{M_1} = N_{M/M_1}\pi_M$ and a suitable $\tau_1 \in \mathrm{Gal}(M_1/F)$. According to the Herbrand Theorem there is an automorphism $\tau \in \mathrm{Gal}(M/F)$ such that $\tau|_{M_1} = \tau_1$ and $\pi_M\tau(\pi_M^{-1}) \in U_{r,M}$. Then $N_{M/M_1}(\alpha\pi_M\tau(\pi_M^{-1})) \in U_{r_1+1,M_1}$ and $\alpha\pi_M\tau(\pi_M^{-1})$ can be written either as an element of $U_{r+1,M}$ or as $(\pi_M^{-1}\sigma\pi_M)\varepsilon$ with $\varepsilon \in U_{r+1,M}$ for a suitable $\sigma \in \mathrm{Gal}(M/M_1)$ and in this case $\alpha\pi_M\sigma\tau(\pi_M^{-1})$ belongs to $U_{r+1,M}$. □

Lemma 2. *Let L/F be a totally ramified Galois p-extension, and $M = L \cap F^{\mathrm{ab}} \neq L$. Then there exists $\alpha \in U_{1,M}$ such that $N_{M/F}\alpha = 1$ and $\alpha \notin N_{L/M}U_{1,L}$.*

Proof. According to p-class field theory $N_{L/F}U_{1,L} = N_{M/F}U_{1,M}$ and $N_{L/M}U_{1,L} \neq U_{1,M}$. Let $\beta \in U_{1,M}, \beta \notin N_{L/M}U_{1,L}$. Then $N_{M/F}\beta = N_{L/F}\gamma$ for some $\gamma \in U_{1,L}$ and $\alpha = \beta N_{L/M}\gamma^{-1}$ is the required element. □

Proof of Theorem. The Hasse–Arf Theorem means that (1) implies (2) and (3). We will verify that (3) implies (1). Assume that L/F is non-abelian and (3) holds. Our aim is to obtain a contradiction.

Put $M = L \cap F^{\mathrm{ab}}$. It is sufficient to verify the required assertion for the case L/M is of degree p. Indeed, let M_1/F be a Galois subextension in L/F such that M_1/M is of degree p. If there is a $\tau \in \mathrm{Gal}(M_1E/F)$ such that $v_{M_1E}(\pi_{M_1E}^{-1}\tau\pi_{M_1E} - 1) \notin h_{M_1E/F}(\mathbb{N})$, then by the Herbrand Theorem there is a $\sigma \in \mathrm{Gal}(ME/F)$ such that $v_{ME}(\pi_{ME}^{-1}\sigma\pi_{ME} - 1) \notin h_{ME/F}(\mathbb{N})$.

Thus, we may assume that L/M is of degree p. Assume that $U_{s,M} \not\subset N_{L/M}U_{1,L}$, $U_{s+1,M} \subset N_{L/M}U_{1,L}$. Let π_L be a prime element in L. For arriving at a contradiction, it suffices to find a normic subgroup $\mathcal{N}$ in $U_{1,L}$ (see section 3 of [F2], for simplicity one can treat the case of a finite residue field, then the word "normic" can be replaced by "open") with the following properties: $U_{1,L}/\mathcal{N} \simeq \oplus_\kappa G$ for a finite abelian p-group G, $\ker N_{L/F} \subset \mathcal{N}$, $U_{t,L} \not\subset \mathcal{N}U_{t+1,L}$ for some $t < s$ such that $t \notin h_{L/F}(\mathrm{N})$. Indeed, let, according to the Existence Theorem in local p-class field theory, $\mathcal{N} = N_{T/L}U_{1,T}$, $\pi_L \in N_{T/L}T^*$ for a totally ramified abelian p-extension T/L. Then the sequence

$$1 \longrightarrow U_{1,L}/N_{T/L}U_{1,T} \xrightarrow{N^*_{L/F}} U_{1,F}/N_{T/F}U_{1,T} \longrightarrow U_{1,F}/N_{L/F}U_{1,L} \longrightarrow 1$$

is exact, where $N^*_{L/F}$ is induced by the norm map $N_{L/F}$. As $\alpha^{\tau-1} \in \mathcal{N}$ for any $\alpha \in L^*$, $\tau \in \mathrm{Gal}(L/F)$, the same theorem shows that T/F is a Galois extension. Now $U_{1,F}/N_{T/F}U_{1,T} \simeq \oplus_\kappa G'$ for an abelian p-group G' of order equal to $|T:F|p^{-1}$. This means that $|T:E| = p$ for the maximal abelian subextension E/F in T/F. The conditions on $\mathcal{N}$ imply that there exists a $\tau \in \mathrm{Gal}(T/L)$ such that $v_T(\pi_T^{-1}\tau\pi_T - 1) = h_{T/L}(t)$ for a prime element π_T in T. Then LE/F doesn't satisfy HAP.

Now we construct the desired group $\mathcal{N}$. By Lemma 2 there exists

$$t = \max(v_M(\alpha - 1) : N_{M/F}\alpha = 1, \alpha \notin N_{L/M}U_{1,L}).$$

Since $\pi_M^{-1}\tau\pi_M \in N_{L/M}U_{1,L}$ for a prime element π_M in M and $\tau \in \mathrm{Gal}(M/F)$, Lemma 1 implies $t \notin h_{L/F}(\mathrm{N}), t < s$. If it were $U_{t,L} \subset U_{t+1,L} \ker N_{L/F}$, then we would get $U_{t,M} \subset U_{t+1,M}N_{L/M}(\ker N_{L/F})$ that contradicts the choice of t. Therefore, there is a natural c such that $U_{t,L} \subset U_{t+1,L} \ker N_{L/F}U_{1,L}^{p^{c-1}}$, $U_{t,L} \not\subset U_{t+1,L} \ker N_{L/F}U_{1,L}^{p^c}$. Now one can take for the desired $\mathcal{N}$ any normic subgroup $\mathcal{N}$ in $U_{1,L}$ such that $\ker N_{L/F}U_{1,L}^{p^c} \subset \mathcal{N}, U_{t,L} \not\subset \mathcal{N}U_{t+1,L}$. □

Remark 1. *Let L/F be a finite totally ramified Galois extension, and M/F its maximal tamely ramified subextension. The extension M/F is a cyclic extension of degree prime to p. One can verify that L/F is abelian if and only if L/M is abelian and L/F satisfies HAP. Indeed, if L/F satisfies HAP and L/M is abelian, then all breaks in the upper numbering of the ramification subgroups of $\mathrm{Gal}(L/M)$ are divisible by $|M:F|$ and any $\alpha \in U_{1,M}$ can be written as $\prod(1+\theta_i\pi_F^{s_i}) \mod N_{L/M}U_{1,L}$ with $\theta_i \in U_F$ and a prime element π_F of F. Hence $\alpha^{\tau-1} \in N_{L/M}U_{1,L}$ for a $\tau \in \mathrm{Gal}(M/F)$. Furthermore, the extension L/F is abelian by the second commutative diagram in Proposition (1.8) of* [F2]. *Now it follows from the above-listed proof of the theorem that its assertions remain true if the words "p-extension" are replaced by "extension". Note that in the general case of a finite Galois extension with non-trivial unramified part there is no any similar characterization of abelian extensions in terms of HAP.*

Remark 2. *Let notations be the same as in the proof of the theorem, and* $|M : F| = p^n$. *Elementary calculations demonstrate that* $h_{L/F}^{-1}(s) \leqslant (n+p/(p-1))e_F$ *if* $\mathrm{char}(F) = 0$, *where* e_F *is the absolute ramification index of* F. *Then in terms of the proof*

$$c \leqslant 1 + n + \max_{1\leqslant m \leqslant p/(p-1)e_F} v_{\mathbb{Q}_p}(m).$$

Thus, in the case of $\mathrm{char}(F) = 0$ *for any extension* E/M *as in (3) of the theorem of sufficiently large degree the extension* LE/F *doesn't satisfy HAP.*

On the other hand, using the proof of the theorem one can construct examples of totally ramified non-abelian p*-extensions satisfying HAP. Let* M/F *be a totally ramified non-cyclic extension of degree* p^2 *such that*

$$s_2 = \max(v_M(\pi_M^{-1}\sigma\pi_M - 1) : \sigma \in \mathrm{Gal}(M/F), \sigma \neq 1) = v_M(\pi_M^{-1}\sigma\pi_M - 1),$$
$$s_1 = \max(v_M(\pi_M^{-1}\tau\pi_M - 1) < s_2 : \tau \in \mathrm{Gal}(M/F)) = v_M(\pi_M^{-1}\tau\pi_M - 1).$$

Let M_1 *be the fixed field of* σ. *Assume that* $s_2 < s \leqslant s_2 + p(s_1+1), s \in h_{M/F}(\mathrm{N})$. *Put* $r = s_2 + p^{-1}(s - s_2)$, *then* $r - s_1 - 1 \leqslant s_2$ *and there exists an alement* $\varepsilon \in U_{r-s_1-1,M_1}$ *with the properties:* $\varepsilon \notin N_{M/M_1}U_{1,M}, \varepsilon^{\tau-1} = N_{M/M_1}\beta, \beta \in U_{s-p,M}$. *If there are no non-trivial pth roots of unity in* F, *there exists a normic subgroup* $\mathcal{N}$ *in* $U_{1,M}$ *such that* $U_{s+1,M} \subset \mathcal{N}$, $U_{s,M} \not\subset \mathcal{N}, \beta \notin \mathcal{N}$ *and elements of the form* $\gamma^{\sigma-1}\delta^{\tau-1}$ *belong to* $\mathcal{N}$. *Hence there is a totally ramified extension* L/M *of degree* p *such that* L/F *is a non-abelian Galois extension. One can verify that* $t = s - p$ *in terms of the proof of the theorem. Now, if* $s = h_{M/F}(q)$ *and* $U_{q,F} \subset U_{q+1,F}U_{1,F}^{p^a}$, *then there exists an element* $\gamma \in U_{s,M}, \gamma \notin N_{L/M}U_{1,L}$ *such that* $N_{M/F}\gamma = N_{L/F}\delta^{p^a}$ *for some* $\delta \in U_{1,L}$. *Provided the residue field of* F *is of order* p *this implies that* $U_{j,L} \subset U_{j+1,L} \ker N_{L/F}$ *for* $j \notin h_{L/F}(\mathrm{N}), j < s, j \neq t$ *and* $U_{t,L} \subset U_{t+1,L} \ker N_{L/F}U_{1,L}^{p^a}$. *Therefore, in this case for any totally ramified abelian* p*-extension* E/F *with* $M \subset E, |E : M| \leqslant p^a$, *the extension* LE/F *satisfies HAP.*

Now let L/F be a finite totally ramified Galois p-extension. Put $\widetilde{L} = L\widetilde{F}$ and let $V(L|F)$ be the subgroup in $U_{1,\widetilde{L}}$ generated by $\varepsilon^{\sigma-1}$ where $\varepsilon \in U_{1,\widetilde{L}}, \sigma \in \mathrm{Gal}(\widetilde{L}/\widetilde{F})$. There is a homomorphism $i: \mathrm{Gal}(L/F) \to U_{1,\widetilde{L}}/V(L|F)$ defined by the formula $i(\sigma) = \pi^{-1}\sigma\pi \mod V(L|F)$ where π is a fixed prime element in $\widetilde{L}$ (i doesn't depend on the choice of π). The kernel of i coincides with the commutator subgroup of $\mathrm{Gal}(L/F)$, see, for instance, (1.4) of [F2]. A connection of the Hasse–Arf property and the extension to be abelian is contained in the following assertion.

Proposition. *The following two conditions are equivalent:*

(1) L/F *is abelian;*

(2) L/F *satisfies HAP, and if* $\varepsilon \in V(L|F)$, *then* $v_{\widetilde{L}}(\varepsilon - 1) \notin h_{L/F}(\mathrm{N})$.

Proof. If HAP holds, then for $\sigma \neq 1$ we get $i(\sigma) \in (U_{r,\widetilde{L}} - U_{r+1,\widetilde{L}})V(L|F)$ for some $r \in h_{L/F}(\mathrm{N})$ and the second condition of (2) means L/F is abelian.

In order to show that the first condition implies the second one, we may proceed by induction on the degree of L/F. If L/F is of degree p, then this follows immediately. In the general case let M/F be a subextension in L/F such that L/M is of degree p. Let an integer s be determined by the conditions $U_{s,M} \not\subset N_{L/M}U_{1,L}$, $U_{s+1,M} \subset N_{L/M}U_{1,L}$. Let α be an element of $V(L|F)$ and $v_{\widetilde{L}}(\alpha - 1) = r = h_{L/F}(q)$, for some $q \in \mathbb{N}$. Then by the induction assumption $N_{\widetilde{L}/\widetilde{M}}\alpha \in U_{h_{M/F}(q)+1,\widetilde{M}}$, since $N_{\widetilde{L}/\widetilde{M}}V(L|F) = V(M|F)$. In this case Lemma 1 implies $r = s$ and $\alpha = \pi_L^{\tau-1}\varepsilon$ for a prime element π_L in L, a generator τ of $\mathrm{Gal}(L/M)$, and some $\varepsilon \in U_{s+1,\widetilde{M}}$. We will show that this is impossible and thus complete the proof.

Let $\alpha_1 \in V(L|F)$, $\varepsilon_1 \in U_{s+1,\widetilde{L}}$ be such that $\alpha = \alpha_1^{\varphi-1}$, $\varepsilon = \varepsilon_1^{\varphi-1}$ for an extension φ of an automorphism $\psi \neq 1$ in $\mathrm{Gal}(\widetilde{F}/F)$ (α_1 and ε_1 exist by Lemma in (1.4) of [F2]). Then $N_{\widetilde{L}/\widetilde{F}}(\alpha_1\varepsilon_1^{-1}) \in U_{q+1,F}$. One may assume without loss of generality that $s \geqslant \max\{v_L(\pi_L^{-1}\tau\pi_L - 1) : 1 \neq \tau \in \mathrm{Gal}(L/F)\}$. Then it follows from the description of the norm map in (3.1) of [F2] that $U_{q+1,F} \subset N_{L/F}U_{1,L}$ and $\beta = N_{\widetilde{L}/\widetilde{F}}(\alpha_1\varepsilon_1^{-1}) \in N_{L/F}U_{1,L}$. Now the construction of the reciprocity map $\Psi_{L/F}$ in section 1 of [F2] implies that $\tau^{-1} = \Psi_{L/F}(\beta)(\varphi) = 1$, a contradiction. □

Remark 3. *One can verify proceeding by induction on the degree of the extension L/F that $V(L|F)U_{r+1,\widetilde{L}} \cap U_{r,\widetilde{L}} = U_{r,\widetilde{L}}$ for any $r \notin h_{L/F}(\mathbb{N})$. In addition, $V(L|F)U_{r+1,\widetilde{L}} \cap U_{r,\widetilde{L}} = U_{r+1,\widetilde{L}}$ for all $r \in h_{L/F}(\mathbb{N})$ if the extension L/F is abelian.*

Examples of non-abelian extensions satisfying HAP show that there exist totally ramified non-abelian p-extensions with $V(L|F)U_{r+1,\widetilde{L}} \cap U_{r,\widetilde{L}} \neq U_{r,\widetilde{L}}$ for $r \in h_{L/F}(\mathbb{N})$.

2. Deeply Ramified Extensions

Let F be a complete (or Henselian) discrete valuation field with a residue field of characteristic $p > 0$. We will assume in this section that extensions of fields are separable with separable residue field extensions. Let $\mathcal{F}/F$ be an extension (possibly infinite). Let $\mathfrak{M}_{\mathcal{F}}$ denote the maximal ideal of $\mathcal{F}$ with respect to the extension of the discrete valuation from F to $\mathcal{F}$. For a finite extension E/F we denote by $e(E|F)$ the ramification index of E/F and by $h_{E/F}$ the Hasse–Herbrand function of E/F, see section 3 Chap. III of [FV]. Finally, for a cyclic ramified extension L/F of a prime degree put $s(L|F) = v_L(\pi_L^{-1}\sigma\pi_L - 1)$ for a prime element π_L of L and a generator σ of $\mathrm{Gal}(L/F)$, then s is well defined and $s = 0, > 0$ for tamely totally ramified and wildly ramified extensions resp. (see for instance section 1 Chap. III of [FV]).

Theorem. *The following properties of an extension $\mathcal{F}/F$ are equivalent:*

(1) *for any $m \geqslant -1$ and any $\varepsilon > 0$ there exists a finite subextension E/F in $\mathcal{F}/F$ such that $h_{E/F}(m)/e(E|F) < \varepsilon$;*
(2) *$e(\mathcal{F}|F) = +\infty$ and for any cyclic ramified extension $\mathcal{F}'/\mathcal{F}$ of prime degree and any $\varepsilon > 0$ there exists a finite subextension E/F in $\mathcal{F}/F$ such that $\mathcal{F}'/\mathcal{F}$ is*

defined over E (i.e. $\mathcal{F}' = \mathcal{F}E'$ for a cyclic extension E'/E of the same degree) and $s(E'|E)/e(E|F) < \varepsilon$;

(3) *$e(\mathcal{F}|F) = +\infty$ and $H^1(\mathrm{Gal}(\mathcal{F}'/\mathcal{F}), \mathfrak{M}_{\mathcal{F}'}) = 0$ for any cyclic extension $\mathcal{F}'/\mathcal{F}$ of prime degree;*

(4) *$H^1(\mathrm{Gal}(\mathcal{F}'/\mathcal{F}), \mathfrak{M}_{\mathcal{F}'}) = 0$ for any finite extension $\mathcal{F}'/\mathcal{F}$;*

(5) *$\mathrm{Tr}_{\mathcal{F}'/\mathcal{F}} \mathfrak{M}_{\mathcal{F}'} = \mathfrak{M}_{\mathcal{F}}$ for any finite extension $\mathcal{F}'/\mathcal{F}$.*

Remark 1. *It follows immediately from the properties of the Hasse–Herbrand function ($h_{L/F} = h_{L/E} \circ h_{E/F}$, $h_{L/F}(x) \leqslant e(L|F)x$) that $h_{L/F} \leqslant e(L|E)h_{E/F}$ for a finite extension L/E. This implies that if $h_{E/F}(m)/e(E|F) < \varepsilon$, then $h_{L/F}(m)/e(L|F) < \varepsilon$ for a finite extension L/E. Note also that for a finite extension M/F and $m' = h_{M/F}(m)$*

$$h_{ME/M}(m')/e(ME|M) = e(M|F)h_{ME/F}(m)/e(ME|F).$$

We conclude that if the assertion (1) holds for $\mathcal{F}/F$, then it holds for $\mathcal{F}'/F$ and $\mathcal{F}/M$ for any extension $\mathcal{F}'/\mathcal{F}$ and finite subextension M/F in $\mathcal{F}/F$. In addition, if (1) holds for $\mathcal{F}/F$ and $\mathcal{F}/\mathcal{F}_0$ is finite, then (1) holds for $\mathcal{F}_0/F$.

Remark 2. *Let $s(E'|E)/e(E|F) < \varepsilon$ and $E'L \neq L$, then $s(E'L|L)/e(L|F) < \varepsilon$ for a finite extension L/E. This follows from*

$$s(E'L|L) = h_{E'L/E'}(s(E'|E)) = h_{L/E}(s(E'|E)) \leqslant e(L|E)s(E'|E).$$

Thus if the assertion (2) holds for E/F and L/E is a finite extension, then (2) holds for L/F (even with the same ε).

Proof of Theorem.

(1)⇒(2): (1) implies $e(\mathcal{F}|F) = +\infty$. Let $\mathcal{F}'/\mathcal{F}$ be defined over E_0 and let for $m = s(E_0'|E_0)$ the extension E/E_0 according to Remark 1 be such that the inequality $h_{E/E_0}(m)/e(E|E_0) < \varepsilon$ holds. Then for $E' = EE_0'$

$$s(E'|E)/e(E|F) = h_{E/E_0}(s(E_0'|E_0))/e(E|F) < \varepsilon/e(E_0|F).$$

(2)⇒(3): For a tamely ramified extension $\mathcal{F}'/\mathcal{F}$ (3) holds trivially. Assume that $\mathcal{F}'/\mathcal{F}$ is wildly ramified.

Let $\mathrm{Tr}_{\mathcal{F}'/\mathcal{F}}\, \alpha = 0$ for $\alpha \in \mathfrak{M}_{\mathcal{F}'}$. Take E/F for $\varepsilon = v_F(\alpha)$ as in (2). Then $v_{E'}(\alpha) > s(E'|E)$, $\mathrm{Tr}_{E'/E}\, \alpha = 0$, therefore $\alpha \in (\sigma - 1)\mathfrak{M}_{E'}$ for a generator σ of $\mathrm{Gal}(E'/E)$ according to standart properties of $s(E'|E)$, see for instance (1.4) Chap. III of [FV]. Thence $\alpha \in (\sigma - 1)\mathfrak{M}_{\mathcal{F}'}$.

(3)⇒(2): Assume that (2) doesn't hold. Then there exists $\varepsilon > 0$ such that for any finite extension E/F, $E \subset \mathcal{F}$ with $\mathcal{F}'/\mathcal{F}$ being defined over E the inequality

$s(E'|E)/e(E|F) \geqslant \varepsilon$ holds. Let E_0/F be a finite subextension in $\mathcal{F}/F$ such that $\varepsilon e(E_0|F) > 2$ and $\mathcal{F}'/\mathcal{F}$ is defined over E_0.

Let σ be a generator of $\mathrm{Gal}(E_0'/E_0)$, $s = s(E_0'|E_0) > 1$ and π a prime element of E_0'. Then $\alpha = (\sigma - 1)\pi^{i-s} \in \mathfrak{M}_{E_0'}$ for $i - s$ prime to p, $1 \leqslant i \leqslant 2$ and $\mathrm{Tr}_{E_0'/E_0}\alpha = 0$, $v_{E_0'}(\alpha) = i$. We claim that $\alpha \notin (\sigma - 1)\mathfrak{M}_{\mathcal{F}'}$. Indeed, otherwise there would exist a finite extension E/E_0 in $\mathcal{F}/F$ such that $\alpha \in (\sigma - 1)\mathfrak{M}_{E'}$. This is equivalent to $v_{E'}(\alpha) = ie(E|E_0) > s(E'|E)$. Then $s(E'|E)/e(E|F) < \varepsilon$, contradiction.

(2)⇒(1): Assume that (1) doesn't hold. Then there exist m and $\varepsilon > 0$ such that $h_{E/F}(m)/e(E|F) \geqslant \varepsilon$ for any finite subextension E/F in $\mathcal{F}/F$. Let M/F be a finite extension in $\mathcal{F}/F$ such that $h'_{E/M}(x) = e(E|M)$ for $x \geqslant h_{M/F}(m)$ and any E/M, $E \subset \mathcal{F}$.

Let $L_n - L_{n-1} - \cdots - L_0 = M$, $n \geqslant 1$, be an extension for which L_i/L_{i-1} is cyclic ramified of degree p with $s(L_n|L_{n-1}) > h_{L_{n-1}/F}(m)$. Such an extension exists: in characteristic p one can take L_1/L_0 as a suitable Artin–Schreier extension, in characteristic 0 one can take L_i/L_{i-1} as a suitable Artin–Schreier extension with $s(L_i|L_{i-1}) \approx pe(L_{i-1}|\mathbb{Q}_p)/(p-1)$ (see section 2 Chap. III of [FV]). Now, if $h_{L_{i-1}/F}(m) \geqslant s(L_i|L_{i-1})$, then $h_{L_i/F}(m)/e(L_i|F) \approx h_{M/F}(m)/e(M|F) - i$, and $pe(L_{n-1}|F)/(p-1) > s(L_n|L_{n-1}) > h_{L_{n-1}/F}(m)$ for sufficiently large n.

For any finite extension E/M in $\mathcal{F}/F$ with $EL_n \neq EL_{n-1}$ we get

$$s(EL_n|EL_{n-1}) > h_{EL_{n-1}/L_{n-1}}(h_{L_{n-1}/F}(m)) = h_{EL_{n-1}/F}(m),$$

and

$$\geqslant h_{E/F}(m) \geqslant \varepsilon_1 e(EL_{n-1}|F)$$

with $\varepsilon_1 = \varepsilon/e(EL_{n-1}|E)$. If E_1/E is a cyclic ramified extension of degree p with $E_1 \subset \mathcal{F}$ and $E_1 \not\subset EL_{n-1}$, then the choice of M/F implies $s(E_1|E) < h_{E/F}(m)$. Therefore, $s(E_1L_{n-1}|EL_{n-1}) = h_{EL_{n-1}/E}(s(E_1|E)) < h_{EL_{n-1}/F}(m)$, and hence $E_1L_{n-1} \neq EL_n$ and $E_1L_{n-1} \neq E_1L_n$.

Thus, $\mathcal{L}' = \mathcal{F}L_n$ is a ramified extension of $\mathcal{L} = \mathcal{F}L_{n-1}$ and assertion (2) doesn't hold for $\mathcal{L}/F$ by Remark 2. The same remark shows that (2) doesn't hold for $\mathcal{F}/F$.

(1)+(2)+(3)⇒(5)+(4); (5)⇒(2); (4)⇒(3): Assertion (5) for a cyclic ramified extension $\mathcal{F}'/\mathcal{F}$ of prime degree is equivalent to (2) as it follows from the well-known description of the trace $\mathrm{Tr}_{E'/E}\mathfrak{M}^i_{E'} = \mathfrak{M}_E^{s(E'|E)+1+[(i-1-s(E'|E))/p]}$ for $|E' : E| = p$, see for instance Proposition (1.4) Chap. III of [FV]. By Remark 1 we deduce now that property (5) holds for arbitrary finite extension. Then (5) and (3) imply (4).

It remains to show that assertion (4) implies $e(\mathcal{F}|F) = +\infty$. Indeed, if $e(\mathcal{F}) > 1$ then one can find a cyclic totally ramified extension $\mathcal{F}'/\mathcal{F}$ of degree p such that $1 < s(\mathcal{F}'|\mathcal{F}) < pe(\mathcal{F})/(p-1)$. Then in the same way as in the proof of (3)⇒(2) one obtains that (4) doesn't hold for $\mathcal{F}'/\mathcal{F}$. If $e(\mathcal{F}) = 1$ then (4) doesn't hold for $\mathcal{F}(\zeta)/\mathcal{F}$ where ζ is a primitive p^2th root of unity. □

An extension $\mathcal{F}/F$ satisfying one of the equivalent assertion of the theorem is called *deeply ramified* (in particular $e(\mathcal{F}|F) = +\infty$). According to Remark 1, if $\mathcal{F}/F$ is a deeply ramified extension, then $\mathcal{F}'/F$, $\mathcal{F}/M$, $\mathcal{F}_0/F$ are deeply ramified for an extension $\mathcal{F}'/\mathcal{F}$ and finite extensions M/F, $\mathcal{F}/\mathcal{F}_0$. In the case of characteristic 0 and perfect residue field the field $\mathcal{F}$ is called deeply ramified if it is a deeply ramified extension of its absolute inertia subfield (isomorphic to the quotient field of the Witt ring of the residue field).

Denote by $\delta_{E/F}$ the different of a finite extension E/F. An extension $\mathcal{F}/F$ is said to have infinite conductor if for any $m \geqslant 0$ there exists a finite subextension E/F in $\mathcal{F}/F$ such that $h'_{E/F}(m) \neq h'_{E/F}(m+1)$ (an equivalent condition is that G_F^m acts nontivially on $\mathcal{F}$ for any $m \geqslant 0$ where G_F^m is the mth ramification subgroup of the absolute group G_F of F with respect to upper numbering).

Corollary. *The following assertions of an extension $\mathcal{F}/F$ are equivalent to (1) – (5):*

(6) *the extension $\mathcal{F}/F$ has an infinite conductor*

(7) *for any finite extension $\mathcal{F}'/\mathcal{F}$ and any $\varepsilon > 0$ there exists a finite subextension E/F in $\mathcal{F}/F$ such that $\mathcal{F}'/\mathcal{F}$ is defined over E and $v_F(\delta_{E'/E}) < \varepsilon$*

(8) *for any $\varepsilon > 0$ there exists a finite subextension E/F in $\mathcal{F}/F$ such that $v_F(\delta_{E/F}) > \varepsilon$*

Proof. One can easily verify that (1) is equivalent to (6) (see also Remark 4 below). Using connections between the different and the s for a cyclic extension of prime degree and multiplicativity of the different one can easily show applying Remark 2 that (2) is equivalent to (7). Finally, similar observations imply equivalence of (6) and (8). □

Remark 3. *Remark 1 and Proposition imply that if $H^1(\mathrm{Gal}(\mathcal{F}^{\mathrm{sep}}/\mathcal{F}), \mathfrak{M}_{\mathcal{F}^{\mathrm{sep}}}) = 1$ then $H^1(\mathrm{Gal}(\mathcal{L}^{\mathrm{sep}}/\mathcal{L}), \mathfrak{M}_{\mathcal{L}^{\mathrm{sep}}}) = 1$ for an extension $\mathcal{L}/\mathcal{F}$ or a finite extension $\mathcal{F}/\mathcal{L}$.*

Remark 4. *Assume that the residue field of F is perfect. Any infinite arithmetically profinite extension (APF) (see* [FW], [W2], *section 5 Chap. III of* [FV]*) is deeply ramified: for an infinite extension $\mathcal{F}/F$ one can find an increasing sequence of real numbers (a_n) such that the property $G_F^{a+\varepsilon}G_{\mathcal{F}} \neq G_F^a G_{\mathcal{F}}$ for any $\varepsilon > 0$ holds only for $a \in (a_n)$ where G_F^i is the i-th ramification group (with respect to upper numbering) of the absolute Galois group of F. Then $\mathcal{F}/F$ is deeply ramified if and only if $a_n \to +\infty$, and $\mathcal{F}/F$ is APF if and only if $a_n \to +\infty$ and $|\mathcal{F}^{G_F^{a_n}} : F| < +\infty$. If there exists a constant c such that $|\mathcal{F}^{G_F^{a_{n+1}}} : \mathcal{F}^{G_F^{a_n}}| < c$ for all n then $\mathcal{F}/F$ is called strictly APF (see section 1 of* [W2]*). From Sen's and Wintenberger's theorems (*[Sn], [W1]*) it follows that any Galois extension of a local field with Galois group being a p-adic Lie group and with finite separable residue field extension is strictly APF.*

We note that there exist deeply ramified extensions in which any infinite subextension isn't APF (and in particular the Galois group of any infinite Galois subextension isn't a p-adic Lie group). Here is an example: let $\mathcal{F}$ be the union of fields

F_n where $F_0 = \mathbf{Q}_p$, F_{2n+1}/F_{2n} be a Galois extension such that F_{2n+1} is the compositum of all conjugates of L_{2n} over F_0 where L_{2n}/F_{2n} is a Galois totally ramified extension of degree p with $s(L_{2n}|F_{2n}) = 1$ (one can take suitable Artin–Schreier extensions for this purpose). Let F_{2n}/F_{2n-1} be a Galois totally ramified extension of degree p with $s(F_{2n}|F_{2n-1}) = h_{F_{2n-1}/F_0}(n+1)$. More concretely, let $\mathcal{N}$ be a subgroup of $N_{F_{2n-1}/F_0}F_{2n-1}^*$ such that $N_{F_{2n-1}/F_0}F_{2n-1}^{*p} \subset \mathcal{N}$ and $(\mathcal{N} \cap U_{i,F_0})U_{i+1,F_0} = (N_{F_{2n-1}/F_0}F_{2n-1}^* \cap U_{i,F_0})U_{i+1,F_0}$ for $i \leqslant n$, $(N_{F_{2n-1}/F_0}F_{2n-1}^* \cap U_{n+1,F_0})U_{n+2,F_0}/(\mathcal{N} \cap U_{n+1,F_0})U_{n+2,F_0}$ is of order p and generated by the element $\alpha_{n+1} = \alpha_n^p$ (pick an arbitrary $\alpha_1 \in U_{1,F_0} \setminus U_{2,F_0}$). The extension F_{2n}/F_{2n-1} can be chosen in such a way that $N_{F_{2n}/F_{2n-1}}F_{2n}^* = N_{F_{2n-1}/F_0}^{-1}(\mathcal{N})$. Then any nontrivial automorphism of $\mathrm{Gal}(F_{2n+1}/F_{2n})$ belongs exactly to $\mathrm{Gal}(F_0^{\mathrm{sep}}/F_0)^1$. From class field theory it follows that $U_{n+1,F_0} \subset N_{F_{2n-1}/F_0}F_{2n-1}^*$, $U_{n,F_0} \not\subset N_{F_{2n-1}/F_0}F_{2n-1}^*$, and the extension F_{2n}/F_0 is Galois. Thus, the extension $\mathcal{F}/F_0$ is Galois deeply ramified, and any its infinite subextension isn't arithmetically profinite, since it contains infinitely many automorphisms belonging exactly to $\mathrm{Gal}(F_0^{\mathrm{sep}}/F_0)^1$. The same is true for $F\mathcal{F}/FF_0$ where $F/\mathbf{Q}_p$ is a finite extension.

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