Canonical form of the 4th order polynomial in the normal equation of a real hypersurface in $\mathbb{C}^3$

Vladimir V. Ežov *
University of Adelaide
Department of Pure Mathematics
Adelaide, SA 5005, South Australia
vezhov@maths.adelaide.edu.au

Aleksandr V. Loboda †
Voronezh State Academy of Architecture and Construction
Voronezh, 20-let Oktyabrya 84, Russia, 394006
lob@vgasa.voronezh.su

Gerd Schmalz‡
Mathematisches Institut der Universität Bonn,
Beringstraße 1, D-53115 Bonn
schmalz@uni-bonn.de

June 25, 1998

Abstract

We classify the first non-quadratic term in the Chern-Moser normal form of a real hypersurface in $\mathbb{C}^3$. Our classification is based on the

*Research of the first author was supported by ARC and MPI Bonn.
†Research of the second author was supported by SFB 256 of DFG.
‡Research of the third author was supported by SFB 256 of DFG.

study of two real forms of the 5-dimensional irreducible representation of the group $\mathrm{SL}(2,\mathbb{C})$. This result can be applied to the problem of classification of homogeneous real hypersurfaces by means of the normal equation.

Let $M \subset \mathbb{C}^3$ be a smooth real hypersurface with non-degenerate Levi form. According to the results by Chern and Moser [CM74], there exist coordinates $z_1, z_2, w = u + iv$ such that the equation of M takes the form

$$v = \langle z, z\rangle + F_{22} + O((|z| + |u|)^5), \tag{1}$$

where $\langle z, z\rangle$ is a hermitian form and F_{22} is a polynomial of degree 2 in z, and of degree 2 in $\bar{z}$. Thus, up to order three, any hypersurface with non-degenerate Levi form is equivalent to the quadric

$$v = \langle z, z\rangle.$$

This means that F_{22} is the lowest order term that can be different in the normalised equations for two hypersurfaces with given Levi form. In dependence of the signature of the Levi form we distinguish two cases

$\langle z, z\rangle_+ = \lvert z_1\rvert^2 + \lvert z_2\rvert^2$	pseudo-convex case
$\langle z, z\rangle_- = 2\,\mathrm{Im}\, z_1\bar{z}_2$	indefinite case.

The polynomial F_{22} satisfies the “harmonicity”condition $\Delta F_{22} = 0$, where

$$\Delta_+ = \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2}{\partial z_2 \partial \bar{z}_2} \qquad \text{in the pseudo-convex case}$$

$$\Delta_- = \frac{\partial^2}{\partial z_1 \partial \bar{z}_2} - \frac{\partial^2}{\partial z_2 \partial \bar{z}_1} \qquad \text{in the indefinite case.}$$

Below we study the question to what form one can reduce the polynomial F_{22} by means of biholomorphic transformations that preserve the origin and respect the normal form of the equation of M.

We prove the following result.

Theorem 1 *In the case of an indefinite Levi form the polynomial $F_{22}(z, \bar{z})$ in the normalised equation (1) can be reduced to one of the following:*

1. $\sigma_1 + a\sigma_0 + \sigma_{-1}$ *with* $a \geq 0$,
2. $\sigma_2 + a\sigma_0 + \sigma_{-2}$ *with* $a \geq 2$,
3. $\sigma_2 + \sigma_0$,
4. σ_2,
5. σ_1,
6. σ_0

where

$$\sigma_{-2} = |z_2|^4, \quad \sigma_{-1} = \mathrm{Re}(z_1\bar{z}_2)|z_2|^2, \quad \sigma_0 = \frac{2}{3}|z_1|^2|z_2|^2 + \frac{1}{3}\,\mathrm{Re}\, z_1^2\bar{z}_2^2$$
$$\sigma_1 = \mathrm{Re}(z_1\bar{z}_2)|z_1|^2, \quad \sigma_2 = |z_1|^4.$$

In the pseudo-convex case the polynomial $F_{22}(z,\bar{z})$ *in the normalised equation (1) can be reduced to one of the following:*

1. $\pm(\eta_{-2} + ia\eta_{-1} + 2\sigma_0 - ia\eta_1 + \eta_2)$ *with* $0 \leq a \leq 1$,
2. $\pm(\eta_{-2} - 2\eta_{-1} + 2\sigma_0 - 2\eta_1 + \eta_2)$,
3. $\pm i(\eta_{-2} + 2\eta_{-1} + 2\eta_1 + \eta_2)$,
4. $\pm(\eta_{-2} + 4\eta_{-1} + 6\sigma_0 + 4\eta_1 + \eta_2)$,
5. $\pm(\eta_{-1} + a\eta_0 + 2\eta_1)$ *with* $a \geq 0$,
6. $\pm\eta_0$,

where

$$\eta_{-2} = \bar{\eta}_2 = z_1^2\bar{z}_2^2,\ \eta_{-1} = \bar{\eta}_1 = (z_1 z_2 \bar{z}_2^2 - z_1^2\bar{z}_1\bar{z}_2),\ \eta_0 = |z_1|^4 - 4|z_1|^2|z_2|^2 + |z_2|^4.$$

Proof. According to Chern Moser's theorem, any biholomorphic mapping Φ that preserves the origin and respects the normal form conditions equals, up to 4-th order terms, to a fractional-linear automorphism of the quadric

$$
\begin{aligned}
z^* &= \lambda\, \mathrm{e}^{i\theta}\, J^\varepsilon U \frac{z + aw}{1 - 2i\langle z, a\rangle - (r + i\langle a, a\rangle)w} \\
w^* &= (-1)^\varepsilon \lambda^2 \frac{w}{1 - 2i\langle z, a\rangle - (r + i\langle a, a\rangle)w},
\end{aligned}
$$

where $\lambda \in \mathbb{R}^+, \theta \in [0, 2\pi), a \in \mathbb{C}^2, r \in \mathbb{R}$. The matrix U is unitary, resp. pseudo-unitary, and unimodular,

$$
J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
$$

and the parameter ε is always 0 in the pseudo-convex case and is 0 or 1 in the indefinite case.

Only the linear part of the mapping Φ effects the polynomial

$$
F_{22}(z, \bar{z}) = (-1)^\varepsilon \lambda^2 F^*_{22}(Uz, \overline{Uz}).
$$

Thus, we have to study a linear action of the group of matrices $C = \lambda\, \mathrm{e}^{i\theta}\, U$ at the space of harmonic polynomials of bidegree $(2, 2)$. This decomposition is unique up to a multiplier -1 if one takes U unimodular. One sees at once that $\mathrm{e}^{i\theta}$ does not effect F_{22} at all and that λ acts by simple multiplication with λ^2.

Consider the indefinite case. Then U is from $\mathrm{SL}(2, \mathbb{R})$, i.e., we have to describe an $\mathrm{SL}(2, \mathbb{R})$ action at the 5-dimensional space $\mathcal{F}_{22}$ of harmonic polynomials of bidegree $(2, 2)$ spanned by σ_j $j = -2, \ldots, 2$. In order to do this, we pass to the isomorphic 5-dimensional standard representation of $\mathrm{SL}(2, \mathbb{R})$ at the space $\mathcal{P}_5 = \mathbb{R}^{(4)}[X : Y]$ of real homogeneous polynomials of degree 4 in X, Y. The isomorphism is established by the assignment

$$
\sigma_j \mapsto \hat{\sigma_j} = X^{2-j} Y^{2+j}.
$$

Since $SL(2, \mathbb{R})$ is connected it suffices to check that the mapping is equivariant with respect to the corresponding $\mathfrak{sl}\,(2, \mathbb{R})$ actions $\tau : \mathfrak{sl}\,(2, \mathbb{R}) \to \mathfrak{gl}\,(\mathcal{F}_{22})$, resp. $\hat{\tau} : \mathfrak{sl}\,(2, \mathbb{R}) \to \mathfrak{gl}\,(\mathcal{P}_5)$. We have then

$$\tau(T_-) = 2\,\mathrm{Re}\, z_2\frac{\partial}{\partial z_1}, \qquad \tau(T_+) = 2\,\mathrm{Re}\, z_1\frac{\partial}{\partial z_2}, \quad \tau(T_0) = 2\,\mathrm{Re}\left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}\right),$$

$$\hat\tau(T_-) = X\frac{\partial}{\partial Y}, \qquad \hat\tau(T_+) = Y\frac{\partial}{\partial X}, \qquad \hat\tau(T_0) = X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y},$$

$$\text{where} \quad T_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad T_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad T_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the standard generators in $\mathfrak{sl}(2,\mathbb{R})$. The bases

$$v_j = \frac{4!}{(2+j)!}\sigma_j, \quad \text{resp.,} \ \hat v_j = \frac{4!}{(2+j)!}\hat\sigma_j, \text{ for } j = -2, \dots, 2$$

are then the canonical ones, i.e.,

$$\tau(T_0)v_j = 2jv_j, \quad \tau(T_-)v_j = v_{j-1}, \quad \tau(T_+)v_j = (2-j)(4+j)v_{j+1},$$

and analogously for $\hat\tau$.

Now we consider the $\mathrm{SL}(2,\mathbb{R})$ orbits of $\mathcal{P}_5$. Therefore, let $P = \sum \alpha_j \hat\sigma_j$. To P we assign the polynomial $\tilde P = \sum_{j=0}^4 \alpha_{2-j}\xi^j$. The group $\mathrm{SL}(2,\mathbb{R})$ acts at the roots of the latter polynomial by Möbius transformations of the upper half-plane. In dependence of the distribution of the roots of $\tilde P$ we distinguish the following cases:

1. $\tilde P$ has at least two different real roots. By means of a Möbius transformation, we send the root of maximal multiplicity to infinity and another root to the origin. One can choose the latter root so that the images of the remaining roots have non-positive real part. Thus $\tilde P$ is in the orbit of a polynomial of degree ≤ 3 and without constant term. The remaining freedom is then a map $(X,Y) \mapsto (\mu X, \mu^{-1}Y)$ with $\mu \in \mathbb{R}^+$. One achieves that, up to a real multiplier, P takes one of the forms
   (a) $P = \hat\sigma_1 + a\hat\sigma_0 + \hat\sigma_{-1}$ with $a > 2$, if all roots are different and real
   (b) $P = \hat\sigma_1 + a\hat\sigma_0 + \hat\sigma_{-1}$ with $0 \leq a < 2$, if there is a pair of complex conjugate roots in addition to two real ones.
   (c) $P = \hat\sigma_1$, if there is a root of multiplicity 3.
   (d) $P = \hat\sigma_2 - \hat\sigma_0$, if there is a root of multiplicity 2 (to be sent to infinity) and two more different real roots (to be sent to ± 1).

(e) $P = \hat{\sigma}_0$ in the case of two roots of multiplicity 2.

2. $\tilde{P}$ has only one real root of multiplicity 4. One can send this root to infinity. Thus, this case is equivalent to $P = \hat{\sigma}_2$.

3. $\tilde{P}$ has a real root of multiplicity 2 and a pair of complex conjugate roots. It is then possible to send the real root to infinity and one complex root to the imaginary unit. This gives

$$P = \hat{\sigma}_2 + \hat{\sigma}_0.$$

4. $\tilde{P}$ has two pairs of complex conjugate roots. We can achieve that the roots from the upper half-plane shift to the imaginary unity and to a purely imaginary number in the upper half-plane. We obtain

$$P = \hat{\sigma}_2 + a\hat{\sigma}_0 + \hat{\sigma}_{-2}, \quad \text{with } a \geq 2.$$

   The parameter $a = 2$ corresponds to the case of a double complex root.

By means of the scaling $(z, w) \mapsto (\lambda z, \lambda^2 w)$ we make the said multiplier equal to ± 1. Since the remaining freedom in the change of coordinates $(z, w) \mapsto (Jz, -w)$ transforms σ_j to $-\sigma_{-j}$ we can always make the multiplier equal to 1.

We turn now to the pseudo-convex case. Any harmonic polynomial F_{22} can be written in the form

$$h_{22} = \alpha_2\eta_2 + \alpha_1\eta_1 + \alpha_0\eta_0 + \bar{\alpha}_1\eta_{-1} + \bar{\alpha}_2\eta_{-2},$$

with $\alpha_2, \alpha_1 \in \mathbb{C}$ and $c \in \mathbb{R}$. By $\mathcal{Q}_5 \subset \mathbb{C}^{(4)}[X : Y]$ we denote the subspace of the space of homogeneous polynomials with degree 4 in two variables $Q(X, Y)$ such that $Q(Y, X) = \overline{Q(\bar{X}, \bar{Y})}$ with the standard SU(2)-action. The latter action is isomorphic to the SU(2)-action on the space of harmonic polynomials via

$$\eta_2 \mapsto X^4, \quad \eta_1 \mapsto X^3Y, \quad \eta_0 \mapsto X^2Y^2, \quad \eta_{-1} \mapsto XY^3, \quad \eta_{-2} \mapsto Y^4.$$

We consider the SU(2) orbits of $\mathcal{Q}_5$. Let

$$Q = \alpha_2 X^4 + \alpha_1 X^3 Y + \alpha_0 X^2 Y^2 + \bar{\alpha}_1 X Y^3 + \bar{\alpha}_2 Y^4,$$
$$\tilde{Q} = \alpha_2 \xi^4 + \alpha_1 \xi^3 + \alpha_0 \xi^2 + \bar{\alpha}_1 \xi + \bar{\alpha}_2.$$

One easily checks that if a root a of $\tilde{Q}$ is not contained in the unit circle then $\bar{a}^{-1}$ is also a root. The group SU(2) acts at the roots of the polynomial $\tilde{Q}$ by Möbius transformations of the unit disk. The possible distributions of these roots lead to the following list of the representatives of orbits (up to a real multiplier):

1. $\tilde{Q}$ has at least two different roots and all are unimodular. We send the root of maximal multiplicity to 1 and another root to -1. One can choose the latter root so that the images of the remaining roots are in the upper half-plane. Moreover, if the remaining roots are different, one can achieve that the product of their images is -1 (they are symmetric with respect to the imaginary axis). We obtain the following options

   (a) $Q = \hat{\sigma}_{-2} + ia\hat{\sigma}_{-1} - 2\hat{\sigma}_0 - ia\hat{\sigma}_1 + \hat{\sigma}_2$ with $0 \leq a < 1$, if all roots are different and unimodular.

   (b) $Q = \hat{\sigma}_{-2} - 2\hat{\sigma}_{-1} + 2\hat{\sigma}_0 - 2\hat{\sigma}_1 + \hat{\sigma}_2$ if there are three unimodular roots, one of them of multiplicity 2. Then the double root will be sent to 1 and the other two to $\pm i$.

   (c) $Q = i\hat{\sigma}_{-2} + 2i\hat{\sigma}_{-1} - 2i\hat{\sigma}_1 - i\hat{\sigma}_2$, if there is a root of multiplicity 3. The triple root is sent to -1 and the other root to 1.

   (d) $Q = \hat{\sigma}_{-2} - 2\hat{\sigma}_0 + \hat{\sigma}_2$, if there are two double roots (to be sent to ± 1). This is (1.a) with $a = 1$.

2. $\tilde{Q}$ has only one unimodular root of multiplicity 4. One can send this root to -1. Thus, this case is equivalent to $Q = \hat{\sigma}_{-2} + 4\hat{\sigma}_{-1} + 6\hat{\sigma}_0 + 4\hat{\sigma}_1 + \hat{\sigma}_2$.

3. $\tilde{Q}$ has a two unimodular and a pair of non-unimodular roots. It is then possible to send a non-unimodular root to the origin and the unimodular roots to mutually complex conjugate elements with non-positive real part. This gives

$$Q = \hat{\sigma}_{-1} + a\hat{\sigma}_0 + \hat{\sigma}_1, \quad 0 \leq a \leq 2,$$

where $a = 2$ corresponds to the case of a unimodular double root.

4. $\tilde{Q}$ has two pairs of non-unimodular roots. We can achieve that one root goes to the origin and the other two are real non-positive. We obtain

$$Q = \hat{\sigma}_{-1} + a\hat{\sigma}_0 + \hat{\sigma}_1, \quad \text{with } a > 2.$$

5. $\tilde{Q}$ has a pair of non-unimodular double roots. Then

$$Q = \hat{\sigma}_0.$$

By the scaling $(z, w) \mapsto (\lambda z, \lambda^2 w)$ we make the multiplier equal to ± 1. $\square$

References

[CM74] S.S. Chern and J.Moser. Real hypersurfaces in complex manifolds. *Acta Math.*, 133(3-4):219–271, 1974.