GORENSTEIN MODELS OF DEL PEZZO SURFACES OF DEGREE 1 OVER DEDEKIND SCHEMES.

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ABSTRACT. Let R be a Dedekind scheme, η its generic point, X and V del Pezzo surfaces of degree 1 over R that are Gorenstein Mori fiber spaces (as 3-folds germs over the ground field). We study birational maps $\varphi : X \dashrightarrow V$ over R which are isomorphisms over the generic point of R. We put down normal forms of such transformations (in suitable coordinates) and give some properties of X and V. In particular, we prove the uniqueness of a smooth model (corollary 3.2).

1. PRELIMINARY.

The aim of this note is the following. Running the Sarkisov program in order to factorize a birational map between two Mori fibrations by elementary links (see, for instance, [1]), we have to consider birational maps that induce an isomorphism of fibers over generic points. This is especially important in the birational rigidity problem. It is amazing but during a long time we did not know nearly anything about such transformations in cases of 3-fold fibrations on del Pezzo surfaces of degree 1,2 or 3, except, probably, the Corti paper [3]. Here, arguing in "coordinates" almost by the same way as in [3], we try to clear this subject in the practically important case of Gorenstein fibrations on del Pezzo surfaces of degree 1. Our argumentation is rather elementary and allows us to construct various interesting examples.

Everywhere in the sequel the following conditions hold. The characteristic of the ground field k is equal to 0. Let $\mathcal{O}$ be a DVR, $R = \mathbf{Spec}\ \mathcal{O}$, K the field of functions of $\mathcal{O}$, X and V del Pezzo surfaces of degree 1 over R. We assume X and V to be Gorenstein Mori fibrations over R. In particular, the fibers X_K and V_K over the generic point of R have the Picard number 1. We denote X_0 and V_0 the central fibers,

$$\varphi : X \dashrightarrow V$$

a birational map over R inducing an isomorphism $\varphi_K : X_K \simeq V_K$.

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Remark 1.1. Recently in Park's preprint [5] it was proved that for degree of fibers up to 4, φ is an isomorphism always if the central fibers are smooth.

It is well known that $|-K_X|$ embeds X into the weighted projective space $\mathbb{P}_{\mathcal{O}}(1,1,2,3)$ over $\mathcal{O}$, so that X_0 avoids the singular points of $\mathbb{P}_k(1,1,2,3)$ because of the Gorenstein property. As to X_0 itself, there are two possibilities ([4], [6]):

Proposition 1.2. *If X_0 is normal, then it has either a unique singular point which is minimally elliptic, or at most du Val singularities.*

If X is not normal, then its normalization is $\mathbb{P}^2$, and X is obtained by gluing $\mathbb{P}^2$ along a quadric (possibly, reducible or non-reduced).

Note that in both the cases there exists a degree 2 morphism from X onto a quadratic cone in $\mathbb{P}^3$ branched along a cubic section that does not pass through the cone vertex. In fact, this morphism is the restriction of a projection $\mathbb{P}(1,1,2,3) \to \mathbb{P}(1,1,2)$ to X (the last weighted space is nothing but a quadratic cone).

In order to separate terminal cases, we will need the following statement ([7]):

Proposition 1.3. *Let U be a germ of a 3-fold terminal singularity, $S \in |-K_U|$ a general element. Then S has at most du Val singularities.*

This statement will be used together with the so-called "recognition principle" (see [2]), which allows us to discern du Val singularities by an equation not in normal form.

The remaining part of the paper is organized as follows. First, we give a suitable system of coordinates in $\mathbb{P}(1,1,2,3)$, that allows us to define equations for φ, X, and V in a convenient form (section 2). In section 3, we produce the main division into cases and deal with each of them. Then, we give some examples in section 4. Finally, section 5 contains some remarks that are closely related to the subject.

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2. Suitable coordinates.

We denote $\mathfrak{m}$ the maximal ideal of $\mathcal{O}$. We may suppose that it is generated by t, i.e., $\mathfrak{m} = (t)\mathcal{O}$.

Then, we fix a copy of a weighted projective space $P_{1_{\mathcal{O}}} \cong \mathbb{P}_{\mathcal{O}}(1,1,2,3)$ to which X is embedded. By P_1 and P_{1_K} we denote the specializations of $P_{1_{\mathcal{O}}}$ over the central point $t = 0$ and the generic point respectively. Let $(x : y : z : w)$ be the coordinates in $P_{1_{\mathcal{O}}}$ of weights (1,1,2,3) (we will use the same denotations of coordinates for the specializations). $P_{2_{\mathcal{O}}}$, P_2, P_{2_K}, and $(p : q : r : s)$ are used for V in the same sense.

Further, X is defined by a homogeneous polynomial of degree 6 in $P_{1_{\mathcal{O}}}$:

$$aw^2 + bz^3 + cwzf_1(x, y) + z^2 f_2(x, y) + zf_4(x, y) + f_6(x, y) = 0,$$

where $a, b, c \in \mathcal{O}$; f_i denotes a homogeneous polynomial of degree i. But X_0 does not pass through points $(0 : 0 : 1 : 0)$ and $(0 : 0 : 0 : 1)$, so a and b are invertible in $\mathcal{O}$, and we suppose $a = 1$. As to b, we may also assume $b = 1$ since it is not important in the sequel, or, if it is more convenient for the reader, we may extend $\mathcal{O}$ by the cubic root of b and put $z := z\sqrt[3]{b}$. Note that such a substitution can not increase the Picard number of X_K, since the new K will be an extension of degree 3, i.e., not even.

Now it is clear that substituting the coordinates w and z by a suitable manner, we obtain the following equation for X:

$$w^2 + z^3 + zf_4(x, y) + f_6(x, y) = 0 \tag{2.1}$$

(f_i's may be different from the previous ones, of course).

Further, since for $1 \leq i \leq 3$ $H^1(P_{1_K}, \mathcal{I}_{X_K} \times \mathcal{O}(i)) = 0$, we see that

$$H^0(X_K, -iK_{X_K}) \simeq H^0(P_{1_K}, \mathcal{O}(i)).$$

So the isomorphism $\varphi_K^{-1} : V_K \simeq X_K$ induces isomorphisms

$$(\varphi_K^{-1})^* : H^0(P_{1_K}, \mathcal{O}(i)) \simeq H^0(P_{2_K}, \mathcal{O}(i)),$$

which yields an isomorphism

$$\varphi_K^{-1} : P_{2_K} \simeq P_{1_K}.$$

Thus, for the coordinates in P_{2_K} we may put

$$\begin{cases} p = (\varphi_K^{-1})^*(x) \\ q = (\varphi_K^{-1})^*(y) \\ r = (\varphi_K^{-1})^*(z) \\ s = (\varphi_K^{-1})^*(w) \end{cases}$$

It remains to take into account a projection

$$P_{2_K} = P_{2_{\mathcal{O}}} \underset{\mathcal{O}}{\otimes} \mathbf{Spec}\ K \to P_{2_{\mathcal{O}}},$$

and then, multiplying by a suitable element of $\mathcal{O}$, for the coordinates in $P_{2_{\mathcal{O}}}$ we get $p = Ax$, $q = By$, $r = Cz$, and $s = Dw$, where

$A, B, C, D \in \mathcal{O}$. Now it is clear that we may suppose (possibly, stretching the coordinates by invertible elements)

$$\varphi = \left\{ \begin{array}{ccc} p & = & t^a x \\ q & = & t^b y \\ r & = & t^c z \\ s & = & t^d w \end{array} \right\} \tag{2.2}$$

where the set (a, b, c, d) contains at least one 0. It is obvious that

$$\varphi^{-1} = \left\{ \begin{array}{ccc} x & = & t^\alpha p \\ y & = & t^\beta q \\ z & = & t^\gamma r \\ w & = & t^\delta s \end{array} \right\} \tag{2.3}$$

with the same condition on the set $(\alpha, \beta, \gamma, \delta)$. Moreover, taking into account the weights, we get

$$6(a + \alpha) = 6(b + \beta) = 3(c + \gamma) = 2(d + \delta).$$

Then, substituting (2.3) for the coordinates in (2.1) and using the condition that V_0 does not pass through points (0:0:1:0) and (0:0:0:1), we obtain an equation of V

$$s^2 + r^3 + rg_2(p, q) + g_6(p, q) = 0 \tag{2.4}$$

and a condition $3\gamma = 2\delta$, where g_i are some homogeneous polynomials of degree i.

Finally, we see that there exists a positive integer m such that the following conditions hold:

$$\left\{ \begin{array}{rcl} a + \alpha & = & m \\ b + \beta & = & m \\ c + \gamma & = & 2m \\ d + \delta & = & 3m \\ 2d & = & 3c \\ 2\delta & = & 3\gamma \end{array} \right. \tag{2.5}$$

3. Main division into cases.

Using the results of section 2 and the symmetry of the situation, we can produice the following division into four cases:

Case	(a,b,c,d)	$(\alpha,\beta,\gamma,\delta)$	Remarks
A	$(a,m,0,0)$	$(\alpha,m,2m,3m)$	$a+\alpha=m$ $a,\alpha>0$
B	$(0,m,0,0)$	$(m,0,2m,3m)$	
C	$(m,m,0,0)$	$(0,0,2m,3m)$	
D	$(0,m,2k,3k)$	$(m,0,2l,3l)$	$k+l=m$ $m\geq 2$ $k,l>0$

We will show that

The cases A, B, and C can not occur.

Here our main tools will be proposition 1.3, the "recognition principle", and a condition that the central fibers are Gorenstein.

First, assuming X and V to be defined by equations (2.1) and (2.4) while φ and φ^{-1} are in form (2.2) and (2.3), we obtain the following conditions:

$$\begin{cases} f_4(x,y) &=& t^{c-2d}g_4(t^a x, t^b y) \\ f_6(x,y) &=& t^{-2d}g_6(t^a x, t^b y) \end{cases}$$
$$\begin{cases} g_4(p,q) &=& t^{\gamma-2\delta}f_4(t^\alpha x, t^\beta y) \\ g_6(p,q) &=& t^{-2\delta}f_6(t^\alpha x, t^\beta y) \end{cases} \tag{3.1}$$

Then, we will suppose that g_4 and g_6 are defined by

$$g_4(p,q) \;=\; \textstyle\sum_{i=0}^4 a_i p^{4-i}q^i; \qquad g_6(p,q) \;=\; \sum_{i=0}^6 b_i p^{6-i}q^i. \tag{3.2}$$

Case A. Here we have

$$\begin{array}{rcccl} f_4(x,y) &=& \sum_{i=0}^4 a_i t^{4a+(m-a)i}x^{4-i}y^i &=& t^{4a}g_4(x,t^{m-a}y), \\ f_6(x,y) &=& \sum_{i=0}^6 b_i t^{6a+(m-a)i}x^{6-i}y^i &=& t^{6a}g_6(x,t^{m-a}y). \end{array}$$

So X is defined by

$$w^2+z^3+t^{4a}zg_4(x,t^{m-a}y)+t^{6a}g_6(x,t^{m-a}y)=0.$$

Since $a>0$, the curve $\{t=w=z=0\}$ lies on X, and X is singular along it. So it can not be a terminal case.

Case C. The same reason as above: simply substitute m for a.

Case B. We have

$$\begin{array}{rcl} f_4(x,y) &=& \sum_{i=0}^4 a_i t^{mi}x^{4-i}y^i, \\ f_6(x,y) &=& \sum_{i=0}^6 b_i t^{mi}x^{6-i}y^i, \end{array}$$

and X is

$$w^2+z^3+z(\sum_{i=0}^{4}a_it^{mi}x^{4-i}y^i)+\sum_{i=0}^{6}b_it^{mi}x^{6-i}y^i=0.$$

Note that at least one of the coefficients is not identically 0. Choose an affine piece $\{y\neq 0\}$. Using the same denotations for the coordinates in this affine piece, we see that a point $A=\{t=x=z=w=0\}$ is singular on X.

Then, a general element of $|-K_X|$ through A has the form $\{x=th\}$ in the affine part, where $h\in\mathcal{O}$. Thus, we obtain that such an element is defined by

$$w^2+z^3+uzt^4+vt^6=0,$$

where $u,v\in\mathcal{O}$ and at least one of them is not 0. Due the "recognition principle", it can not be an equation of a du Val singularity. By proposition 1.3, X can not be terminal.

So we proved that the cases A,B, and C can not occur. Now we shall deal with the case D.

The unique possibility: case D. In the sequel we will always assume that

$$\varphi=\left\{\begin{array}{ccc}p&=&x\\q&=&t^my\\r&=&t^{2k}z\\s&=&t^{3k}w\end{array}\right\},\quad \varphi^{-1}=\left\{\begin{array}{ccc}x&=&t^mp\\y&=&q\\z&=&t^{2l}r\\w&=&t^{3l}s\end{array}\right\},$$

where $k+l=m$, and $0<k\leq l$, so $k\leq\frac{m}{2}$.

Using (3.1), we see that

$$f_4(x,y)=t^{-4k}g_4(x,t^my),\qquad f_6(x,y)=t^{-6k}g_6(x,t^my).$$

Thus, there exist $\alpha_0,\alpha_1,\beta_0,\beta_1,\beta_2\in\mathcal{O}$ such that

$$\left\{\begin{array}{ccc}a_0&=&\alpha_0t^{4k}\\a_1&=&\alpha_1t^{4k-m}\\b_0&=&\beta_0t^{6k}\\b_1&=&\beta_1t^{6k-m}\\b_2&=&\beta_2t^{6k-2m}\end{array}\right. \tag{3.3}$$

For convenience, let us put down all equations that we will need in the sequel:

$$
\begin{aligned}
f_4(x,y) &= \alpha_0 x^4+\alpha_1 x^3y+a_2t^{2m-4k}x^2y^2+a_3t^{3m-4k}xy^3+ \\
&\quad +a_4t^{4m-4k}y^4,\\
f_6(x,y) &= \beta_0x^6+\beta_1x^5y+\beta_2x^4y^2+b_3t^{3m-6k}x^3y^3+ \\
&\quad +b_4t^{4m-6k}x^2y^4+b_5t^{5m-6k}xy^5+b_6t^{6m-6k}y^6,\\
g_4(p,q) &= \alpha_0t^{4k}p^4+\alpha_1t^{4k-m}p^3q+a_2p^2q^2+a_3pq^3+a_4q^4,\\
g_6(p,q) &= \beta_0t^{6k}p^6+\beta_1t^{6k-m}p^5q+\beta_2t^{6k-2m}p^4q^2+b_3p^3q^3+ \\
&\quad b_4p^2q^4+b_5pq^5+b_6q^6,
\end{aligned}
\tag{3.4}
$$

and

$$
X:\ \begin{cases} w^2+z^3+z\left(\alpha_0x^4+\alpha_1x^3y+t^{2m-4k}\sum_{i=2}^4 a_it^{m(i-2)}x^{4-i}y^i\right)+ \\ +\beta_0x^6+\beta_1x^5y+\beta_2x^4y^2+t^{3m-6k}\sum_{i=3}^6 b_it^{m(i-3)}x^{6-i}y^i=0, \end{cases}
$$

$$
V:\ \begin{cases} s^2+r^3+r\left(\alpha_0t^{4k}p^4+\alpha_1t^{4k-m}p^3q+\sum_{i=2}^4 a_ip^{4-i}q^i\right)+ \\ +\beta_0t^{6k}p^6+\beta_1t^{6k-m}p^5q+\beta_2t^{6k-2m}p^4q^2+\sum_{i=3}^6 b_ip^{6-i}q^i=0. \end{cases}
\tag{3.5}
$$

Note that φ is not defined along the curve $\{t=x=0\}$, and φ^{-1} along $\{t=q=0\}$. Then, V_0 is contracted to the point $A=\{t=0,(0:1:0:0)\}\in X$, and X_0 to the point $B=\{t=0,(1:0:0:0)\}\in V$.

Claim 3.1. *X is always singular.*

Proof. Since $2m-4k\geq 0$, it is easy to check that the point A is singular.

In particular, we proved the following important result:

Corollary 3.2. *Let U/T be a Mori fibration on del Pezzo surfaces of degree 1 over a curve T. Denote*

$$\mathcal{MF}_{base}=\{\textit{Mori fibrations that are birational over } T \textit{ to } U\}.$$

Then $\mathcal{MF}_{base}$ contains at most one non-singular element.

Lemma 3.3. *Always $m\leq 6k$.*

Proof. Assume the converse, i.e., $m>6k$. Then $\alpha_1,\beta_1\in\mathfrak{m}$ and $\beta_2\in\mathfrak{m}^6$ because of (3.3). Let $S=\{x=th\}$, $h\in\mathcal{O}$, be a general element of $|-K_X-A|$ (in the affine piece $y\neq 0$). Then S has the form

$$w^2+z^3+uzt^4+vt^6=0$$

for some $u, v \in \mathcal{O}$. So S is not canonical, which contradicts to proposition 1.3.

Now we consider two possible cases: $k = l$, i.e., $m = 2k$, and $k < l$.
Case $\mathbf{k = l}$. In this case, let us consider the affine part $\{p \neq 0\}$ of V. Then V is defined by

$$s^2+r^3+r(\alpha_0 t^{4k}+q(\ldots\text{terms}\ldots))+\beta_0 t^{6k}+\beta_1 t^{4k}q+q^2(\ldots\text{terms}\ldots) = 0,$$

and we see that the point $B \in V$ is singular. Thus, for $k = l$, such a fiber-wise transformations only exist between singular varieties.
Case $\mathbf{k < l}$. Here, it would be interesting to know under which conditions V is non-singular. Since X_0 is contracted to the point B, we should first check V at this point. Take an affine piece $\{p \neq 0\}$. Then from equations (3.5) it follows that $\beta_1 t^{6k-m} \notin \mathfrak{m}$, i.e., $6k - m \geq 0$. Thus, $m = 6k$ because of lemma 3.3, and β_1 is invertible in $\mathcal{O}$.

So, if V is smooth, then $(a, b, c, d) = (0, 6k, 2k, 3k)$ and $(\alpha, \beta, \gamma, \delta) = (6k, 0, 10k, 15k)$. Let us note that $\alpha_1 \in \mathfrak{m}^{2k}$ and $\beta_2 \in \mathfrak{m}^{6k}$ in this case. It easy to check also that X_0 only has a E_8-singularity at A, so X has cE_8 there.

4. Examples.

Here we give examples with some interesting properties.
Example 1. Smooth case. Suppose $(a, b, c, d) = (0, 6, 2, 3)$ and $(\alpha, \beta, \gamma, \delta) = (6, 0, 10, 15)$. X and V is defined by

$$X: \qquad w^2 + z^3 + x^5 y + t^{24} x y^5 = 0,$$

$$V: \qquad s^2 + r^3 + p^5 q + p q^5 = 0.$$

It is easy to see that V is non-singular, X has a cE_8-singularity at the point $\{t = 0, (0 : 1 : 0 : 0)\}$.

Example 2. Birational automorphism. Let $(a, b, c, d) = (2, 0, 2, 3)$ and $(\alpha, \beta, \gamma, \delta) = (0, 2, 2, 3)$. Equations:

$$X: \qquad w^2 + z^3 + t^4 x^5 y + x y^5 = 0,$$

$$V: \qquad s^2 + r^3 + p^5 q + t^4 p q^5 = 0.$$

Note that X and V are isomorphic to each other: simply take $w = s$, $z = r$, $x = q$, and $y = p$. So we may assume that $\varphi \in \mathrm{Bir}(X)$ is defined by a transformation of coordinates in P_{1_K}:

$$\begin{array}{rcl} x & \longmapsto & t^{-1}y; \\ y & \longmapsto & tx; \\ z & \longmapsto & z; \\ w & \longmapsto & w. \end{array}$$

X has a cE_8-singularity in the central fiber.

Example 3. Non-normal fibers. We give two examples with the same weights $(a, b, c, d) = (2, 0, 2, 3)$ and $(\alpha, \beta, \gamma, \delta) = (0, 2, 2, 3)$ such that X_0 and V_0 are not normal. Both the cases are also birational automorphisms. The first one is

$$X: \quad w^2 + z^3 + tzx^2y^2 + tx^6 + t^7y^6 = 0,$$

$$V: \quad s^2 + r^3 + trp^2q^2 + t^7p^6 + tq^6 = 0.$$

X_0 is non-normal with an equation $w^2 + z^3 = 0$ (the "cusp" case, see [6], 1.4.). X has a cE_7-singularity in the central fiber.

The second is

$$X: \quad w^2 + z^3 - 3zx^2y^2 + tx^6 + 2x^3y^3 + t^7y^6 = 0,$$

$$V: \quad s^2 + r^3 - 3rp^2q^2 + t^7p^6 + 2p^3q^3 + tq^6 = 0.$$

X_0 is defined by $w^2 + (z - xy)^2(z + 2xy) = 0$ (the "node" case), and X is cD_4.

5. Remarks.

As it was shown, both φ and φ^{-1} contract central fibers to points. It gives us some special conditions on central fibers and elements of $|-K|$.

Central fibers. Since X and V are assumed to be terminal, i.e., having only rational singularities, their central fibers X_0 and V_0 must be rational. Consider X_0. If it is not normal, then its normalization is $\mathbb{P}^2$, so X_0 is rational (example 3 of the previous section). If X_0 is normal, then it is rational if and only if it has at most du Val singularities. So the remaining case of minimally elliptic singularities (defined by $w^2 + z(z^2 - x^4) = 0$) is not possible.

Anticanonical divisors. Let $S \in |-K_X|$ be a general element, $S' \in |-K_V|$ its strict transform. Then S is a smooth elliptic surface. We will show that the elliptic surface S' has an elliptic (non-minimal in general case) singularity.

Denote s and s' sections of S and S' that are the base locus of anticanonical divisors, $\rho : T \to S'$ the minimal resolution of singularities of S'. Since S is relatively minimal over R, it yields a birational morphism $\mu : T \to S$, which is factorized by contractions of **-1**-curves. Denote C and C' the central fibers of S and S', $\tilde{C}$ and $\tilde{C}'$ their strict transforms on T, and s_T the pre-image of s (or s') on T. Obviously, C is either an elliptic curve, or a rational curve with a double point ("cusp" or "node"). We also know that ρ contracts $\tilde{C}$ to a point on S', and μ contracts $\tilde{C}'$. Then, since C' is irreducible, $\tilde{C}'$ is a unique **-1**-curve on T. Moreover, $s' \cap C'$ is a non-singular point on S', thus s_T does not intersect the exeptional curves. Further, the point $s \cap C$ is a non-singular point of C, so $\tilde{C}$ has the arithmetic genus 1. The following picture clarifies the geometric situation.

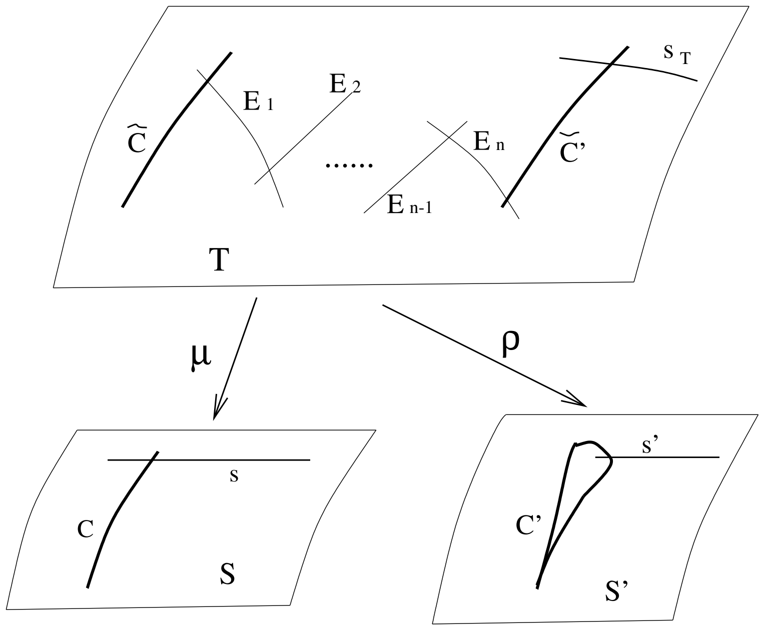

FIGURE 1

Here $E_1, \ldots, E_n$, $n \geq 0$, are **-2**-curves. Let us note also that C' can not have a node point, so it is a cusp.

Thus, we have the following intersection numbers: $\tilde{C}^2 = \tilde{C}'^2 = -1$, $E_i^2 = -2$, $E_i \circ E_{i+1} = 1$.

Then, comparing K_T for morphisms ρ and μ, we see that

$$K_T \sim -(n+1)\tilde{C} - nE_1 - (n-1)E_2 - \ldots - E_n$$

and

$$K_T \sim E_1 + 2E_2 + \ldots nE_n + (n+1)\tilde{C}',$$

whence

$$(n+1)(\tilde{C} + E_1 + \ldots + E_n + \tilde{C}') \sim 0$$

(in fact, it is obvious, since $\tilde{C}+E_1+\ldots+E_n+\tilde{C}'$ is a fiber of T). This formula can be explained as follows. Suppose for instance that S and S' are projective elliptic surfaces, f and f' are their classes of a fiber. There exist integers m and m' such that $K_S \sim mf$ and $K_{S'} \sim m'f'$. Then $m' = m+(n+1)$.

Now it is clear that S' has an elliptic singularity with $\tilde{C}$ as the fundamental cycle, so it is minimally elliptic if and only if $n=0$.

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