A Note On Intrinsic Regularization Method

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ABSTRACT

There exist certain intrinsic relations between the ultraviolet divergent graphs and the convergent ones at the same loop order in renormalizable quantum field theories. Whereupon we may establish a new method, the intrinsic regularization method, to regulate those divergent graphs. In this note, we present a proposal, the inserter proposal, to the method. The ϕ^4 theory and QED at the one loop order are dealt with in some detail. Inserters in the standard model are given. Some applications to SUSY-models are also made at the one loop order.

Introduction

As is well known, various regularization schemes have been developed in the quantum field theory. However, the topic is still one of the important and fundamental issues under investigation. One of the most challenging problems is perhaps how to preserve all symmetries and topological properties manifestly and consistently.

It has been found that there exist certain intrinsic relations between the ultraviolet divergent graphs and the convergent ones at the same loop order in renormalizable QFT

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[1-5]. Whereupon we should be able to establish a new method, the intrinsic regularization method, for regularization of those divergent graphs. In this note, we present a proposal to this method, the inserter proposal. We deal with the ϕ^4 theory and QED at the one loop order. We also make some applications to SUSY-models at the one loop order by means of the SUSY version of the inserter proposal and explain how to apply it to other cases. This proposal may shed light on that challenge.

The key point of the new method is, in fact, based upon the following simple observation: For a given ultraviolet divergent function at certain loop order in a renormalizable QFT, there always exists a set of convergent functions at the same loop order such that their Feynman graphs share the same loop skeleton and the main difference is that the convergent ones have additional vertices of certain kind and the original one is the case without these vertices. This is, in fact, a certain intrinsic relation between the original ultraviolet divergent graph and the convergent ones in the QFT. It is this relation that indicates it is possible to introduce the regulated function for the divergent function with the help of those convergent ones so that the potentially divergent integral of the graph can be rendered finite while for the limiting case of the number of the additional vertices $q \to 0$ the divergence again becomes manifest in pole(s) of q.

It is very simple why there always exists such kind of intrinsic relations in renormalizable QFTs. Let us consider some Feynman graphs at the L loop order with I internal lines of any kind and V vertices of any kind. The topological formula

$$L - I + V = 1$$

shows that for fixed L I increases the same as V does so that the superficial degree of divergence decreases. Therefore, for a given divergent Feynman graph at certain loop order, the topological formula insures that one can always reach a set of convergent graphs in a suitable perturpation expansion series in the order of some coupling constant , which always appears with a vertex of certain kind, as long as the original divergent graph is included in the series. In fact, this topological formula is a cornerstone of the intrinsic regularization method. In general, however, the procedure may be very involved. The aim of the inserter proposal to be presented in this note is just to simplify the procedure. In fact, by means of this proposal the fore-mentioned limiting procedure allows us to unambiguously calculate the various Feynman graphs.

To be concrete, let us consider a 1PI graph with I internal lines at one loop order in the ϕ^4 theory. Its superficial degree of divergences in the momentum space is

$$\delta = 4 - 2I.$$

When $I = 1$ or 2, the graph is divergent. Obviously, there exists such kind of graphs that they have additional q four-ϕ-vertices in the internal lines. Then the number of internal lines in these graphs is $I + q$ so that the divergent degree of the new 1PI graphs become

$$\delta' = 4 - 2(I + q).$$

If q is large enough, the new ones are convergent and the original divergent one is the case of $q = 0$. Thus, a certain intrinsic relation has been reached between the original divergent 1PI graph and the new convergent ones at the same loop order.

2

In the inserter proposal, we take all external lines in the additional vertices with zero momenta and call such a vertex an *inserter*. Thus those convergent graphs can simply be regarded as the ones given by suitably inserting q-inserters in all internal lines in the given divergent graph and the powers of the propagators are simply raised so as to those integrals for the new graphs become convergent. It is clear that these new graphs share the same loop skeleton with the original divergent one and the main differences between those graphs are the number of inserters as well as the dimension in mass and the order in the coupling constant due to the insertions of the inserter. Thus it is possible to regulate the divergent graph based upon the intrinsic relations of this type as long as we may get rid of those differences and deal with those convergent functions on an equal footing. In fact, to this end we introduce a well-defined convergent 1PI function, the regulated function, by taking the arithmetical average of those convergent 1PI functions and changing their dimension in mass, their order in λ etc. to the ones in the original divergent function. Thus this new function renders the divergent integral in the original function finite. Evaluating it and continuing q analytically from the integer to the complex number, the divergent function of the original 1PI graph is recovered as the $q \to 0$ limiting case of such a regulated function.

It is not hard to see that in any given QFT as long as a suitable kind of inserters are constructed with the help of the Feynman rules of the theory and some intrinsic relations between the divergent functions and convergent ones at the same loop order are found by inserting the inserters, this inserter proposal should work in principle. Of course, special attention should be devoted to each specified case. In QED, for example, since the electron-photon vertex carries a γ-matrix and is a Lorentz vector, therefore, how to construct a suitable inserter is the first problem to be solved in addition to the above procedure in the ϕ^4 theory. Otherwise, simply inserting the vertex would increase the rank of the functions as Lorentz tensors and the problem could become quite complicated. We will be back to this point at the end of this note. In order to avoid this complication, we borrow an inserter of the Yukawa type for the massive fermions from the standard model (see the appendix) and employ it for the purpose in QED. By inserting this inserter to the internal fermion lines in the graph of a given 1PI n-point divergent function, a set of new convergent functions can be obtained if the number of inserters, q, is large enough. Then we introduce a new convergent function, the regulated function, it not only should have the same dimension in mass, and the same order in the coupling constant and so on as the original divergent function but also should preserve the gauge invariance. Thus the potential infinity in the original 1PI n-point function may be recovered as the $q \to 0$ limiting case of that function.

In what follows, we concentrate on how to regulate the divergent graphs at one loop order in the ϕ^4 theory and QED. We present the main steps and the results of the inserter regularization procedure for each of them. We also describe briefly the SUSY version of the inserter proposal and make some applications to SUSY-models at the one loop order. Finally, we end with some discussion and remarks. In the appendix, different kinds of inserters in the standard model are given.

Intrinsic Regularization In ϕ^4 Theory

The action of the ϕ^4 theory is

$$S[\phi] = \int d^4x \, \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right) \tag{1}$$

and its Feynman rules are well known.

The main steps of the inserter proposal for the ϕ^4 theory may more concretely be stated as follows. First, we should construct the inserter in the ϕ^4 theory. As mentioned above, it is a four-ϕ-vertex with two zero momentum external lines whose Feynman rule is the same as the vertex

$$I^\phi(p) = -i\lambda. \tag{2}$$

For a given 1PI n-point divergent function at the one loop order $\Gamma^{(n)}(p_1, \cdots, p_n)$, we consider all $n + 2q$-point functions $\Gamma^{(n+2q)}(p_1, \cdots, p_n; q)$ which are the amplitudes of the graphs corresponding to all possible q insertions of the inserter on the internal lines of the given n-point graph. If q is large enough, $\Gamma^{(n+2q)}(p_1, \cdots, p_n; q)$ become convergent. And for $q = 0$ it is the case of original n-point function. This is a relation between the given divergent function and those convergent ones. With the help of this relation, we introduce a new function by taking the arithmetical average of these convergent functions, i.e. the summation of these functions divided by N_q, the total number of such inserted functions, and let it have the same dimension in mass and the same order in λ as the original 1PI n-point function:

$$\Gamma^{(n)}(p_1, \cdots, p_n; q; \mu) = (-i\mu^2)^q(-i\lambda)^{-q}\frac{1}{N_q}\sum \Gamma^{(n+2q)}(p_1, \cdots p_n; q) \tag{3}$$

where μ is an arbitrary reference mass parameter. Note that this function is no longer an $n + 2q$-point function rather a regulated n-point function since it is at the same order in the coupling constant λ as the original function. Now we evaluate it and analytically continue q from the integer to the complex number. Then the original potentially divergent 1PI n-point function is recovered by

$$\Gamma^{(n)}(p_1, \cdots, p_n; \mu) = \lim_{q\to 0}\Gamma^{(n)}(p_1, \cdots, p_n; q; \mu), \tag{4}$$

and the original infinity arises manifestly as pole in q. Obviously, this procedure should in principle work for the cases at higher loop orders.

At the one loop order there are only two divergent graphs in the ϕ^4 theory, the tadpole (t) and the fish (f). In the momentum space, the amplitude of (t) and (f) are

$$
\begin{aligned}
(t) &= \tfrac{1}{2}\int \tfrac{d^4l}{(2\pi)^4} \ \tfrac{\lambda}{l^2-m^2}, \\
(f) &= \tfrac{1}{2}\int \tfrac{d^4l}{(2\pi)^4} \ \tfrac{\lambda^2}{(l^2-m^2)((p_1+p_2+l)^2-m^2)} + (p_2 \to p_3) + (p_2 \to p_4).
\end{aligned}
\tag{5}
$$

They are quadratically and logarithmically divergent respectively and needed to be regulated.

In order to regulate the tadpole, we attach q inserters to the internal line of the graph. Then (t) becomes a $2 + 2q$-point function (t_q). For q large enough, (t_q) is convergent. We now introduce a new function (t_q') which has the same dimension in mass and the same order in λ with (t) and when $q = 0$, $(t_q')|_{q=0} = (t)$.

The amplitude of (t_q') can be expressed as [5]

$$(t_q') = (-i\mu^2)^q(-i\lambda)^{-q}(t_q) = \frac{1}{2}\mu^{2q} \int \frac{d^4l}{(2\pi)^4} \frac{\lambda}{(l^2 - m^2)^{q+1}}. \tag{6}$$

It can be easily integrated and expressed in terms of the *gamma* functions of q:

$$(t_q') = \mu^{2q}\frac{i}{2}\frac{\lambda}{(4\pi)^2}\frac{\Gamma(q-1)}{\Gamma(q+1)(-m^2)^{q-1}}. \tag{7}$$

Now we analytically continue q from the integer to the complex number. The original tadpole function (t) is then recovered as the $q \to 0$ limiting case of (t_q'):

$$(t) = \lim_{q \to 0}(t_q') = \frac{i}{2}\frac{\lambda}{(4\pi)^2}m^2[\frac{1}{q} + 1 + \ln(-\frac{\mu^2}{m^2}) + o(q)]. \tag{8}$$

In order to regulate the fish, we attach to its internal lines q inserters and it turns to a set of the graphs $(f_{q,i})$ with i inserters inserted on one internal line while $q - i$ on the other. For q large enough, all $(f_{q,i})$ are convergent. Let us introduce their arithmetical average $(f_q) = \frac{1}{N_q}\sum_{i=0}^{q}(f_{q,i})$, $N_q = q + 1$ and a new function (f_q') which has the same dimension in mass and the same order in λ with that of (f) and when $q = 0$, $(f_q')|_{q=0} = (f)$.

The amplitude of (f_q') can be expressed as

$$\begin{aligned}(f_q') &= (-i\mu^2)^q(-i\lambda)^{-q}(f_q) \\ &= \frac{\mu^{2q}}{2(q+1)}\sum_{i=0}^{q}\int \frac{d^4l}{(2\pi)^4}\frac{\lambda^2}{(l^2-m^2)^{i+1}((p_1+p_2+l)^2-m^2)^{q-i+1}} \\ &\quad +(p_2 \to p_3) + (p_2 \to p_4).\end{aligned} \tag{9}$$

And we can get

$$(f_q') = \frac{i}{2(4\pi)^2}\lambda^2\mu^{2q}\frac{1}{q(q+1)}\int_0^1 d\alpha\frac{1}{[\alpha(1-\alpha)(p_1+p_2)^2-m^2]^q} + (p_2 \to p_3) + (p_2 \to p_4). \tag{10}$$

We now analytically continue q from the integer to the complex number. Then the fish function (f) is reached by the $q \to 0$ limiting case of (f_q'),

$$\begin{aligned}(f) &= \lim_{q \to 0}(f_q') \\ &= i\frac{\lambda^2}{(4\pi)^2}[\frac{3}{2q} + \frac{3}{2} + \frac{3}{2}\ln(-\frac{\mu^2}{m^2}) + A(p_1,\cdots,p_4) + o(q)],\end{aligned} \tag{11}$$

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[5]In this note, the order of the inserted inserters in each inserted graph is always fixed so that the relevant combinatory factor is simply fixed to be one as well.

where

$$A(p_1, \cdots, p_4) = -\frac{1}{2}\sqrt{1 - \frac{4m^2}{(p_1 + p_2)^2}} \ln \frac{\sqrt{1 - \frac{4m^2}{(p_1+p_2)^2}} + 1}{\sqrt{1 - \frac{4m^2}{(p_1+p_2)^2}} - 1} + (p_2 \to p_3) + (p_2 \to p_4).$$

Thus we complete the regularization of the ϕ^4 theory at the one loop order by means of the inserter proposal.

Intrinsic Regularization In QED

The action of the QED is

$$S[\mathbf{A}, \psi] = \int d^4x \{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial \cdot \mathbf{A})^2 + \bar{\psi}(i\not{\partial} - m)\psi - e\bar{\psi}\not{A}\psi \}. \qquad (12)$$

and the Feynman rules are well known.

As was mentioned above, we first employ an *inserter* borrowed from the standard model. It is an $ff\phi$-vertex of the Yukawa type with a zero momentum Higgs external line. The Feynman rule of such an inserter is

$$I^{\{f\}}(p) = -i\lambda_f, \qquad (13)$$

where λ_f takes value $\frac{g}{2}m_f/M_W$ in the standard model, but here its value is irrelevant for our purpose. Then for a given divergent 1PI amplitude $\Gamma_*^{(n_f, n_g)}(p_1, \cdots, p_{n_f}; k_1, \cdots, k_{n_g})$ of rank $*$ Lorentz tensor at the one loop order with n_f external fermion lines and n_g external photon lines, we consider a set of 1PI amplitudes $\Gamma_*^{(n_f, n_g, 2q)}(p_1, \cdots, p_{n_f}; k_1, \cdots, k_{n_g}; q)$ which correspond to the graphs with all possible $2q$ insertions of the inserter in the internal fermion lines in the original graph. Because each insertion decreases the divergent degree by 1, the divergent degree becomes:

$$\delta = 4 - (I_f + 2q) - 2I_g.$$

If q is large enough, $\Gamma_*^{(n_f, n_g, 2q)}(p_1, \cdots, p_{n_f}; k_1, \cdots, k_{n_g}; q)$ are convergent and the original divergent function is the case of $q = 0$. Thus we reach a relation between the given divergent 1PI function and a set of convergent 1PI functions at the one loop order. In fact, the function of inserting such an inserter to an internal fermion line is simply to raise the power of the propagator of the line and to decrease the degree of divergence of given graph.

In order to regulate the given divergent function with the help of this relation, we need to deal with those convergent functions on an equal footing and pay attention to their differences due to the insertions. To this end, we introduce a new function:

$$\Gamma_*^{(n_f, n_g)}(p_1, \cdots, p_{n_f}; k_1, \cdots, k_{n_g}; q; \mu)$$

$$= (-i\mu)^{2q}(-i\lambda_f)^{-2q}\frac{1}{N_q}\sum \Gamma_*^{(n_f, n_g, 2q)}(p_1, \cdots, p_{n_f}; k_1, \cdots, k_{n_g}; q) \qquad (14)$$

6

where μ is an arbitrary reference mass parameter as in the last section, the factor $(-i\lambda_f)^{-2q}$ introduced here is to cancel the one coming from $2q$-inserters and the summation is taken over the entire set of such N_q inserted functions. It is clear that this function is the arithmetical average of those convergent functions and has the same dimension in mass, the same order in e with the original divergent 1PI function. Now we evaluate it and continue analytically q from the integer to the complex number. Finally, the original 1PI function should be recovered as its $q \to 0$ limiting case:

$$\Gamma_*^{(n_f, n_g)}(p_1, \cdots, p_{n_f}; k_1, \cdots, k_{n_g}) = \lim_{q \to 0} \Gamma_*^{(n_f, n_g)}(p_1, \cdots, p_{n_f}; k_1, \cdots, k_{n_g}; q; \mu), \qquad (15)$$

and the original infinity would appear as pole in q. Similarly, this procedure should work for the cases at the higher loop orders in principle.

The divergent 1PI graphs at the one loop order in QED are those contribute to the vacuum polarization $\Pi_{\mu\nu}(k)$, the electron self-energy $\Sigma(p)$, the vertex function $\Lambda_\mu(p', p)$ and the photon-photon scattering function $\Gamma_{\mu\nu\rho\sigma}(p_1, \cdots, p_4)$. Their integral expressions in the momentum space are as follows (For simplicity, we take the Feynman gauge $\xi = 1$.):

$$\Pi_{\mu\nu}(k) = -e^2 \int \frac{d^4p}{(2\pi)^4} Tr[\gamma_\mu \frac{1}{\not{p} - \not{k} - m} \gamma_\nu \frac{1}{\not{p} - m}],$$

$$\Sigma(p)_{\beta\alpha} = -e^2 \int \frac{d^4k}{(2\pi)^4} (\gamma^\mu \frac{1}{\not{k} - m} \gamma_\mu)_{\beta\alpha} \frac{1}{(p - k)^2},$$

$$\Lambda_\mu(p', p)_{\beta\alpha} = -e^3 \int \frac{d^4l}{(2\pi)^4} (\gamma^\rho \frac{1}{\not{l} - \not{k} - m} \gamma_\mu \frac{1}{\not{l} - m} \gamma_\rho)_{\beta\alpha} \frac{1}{(p - l)^2}, \qquad (16)$$

$$\Gamma_{\mu\nu\rho\sigma}(p_1, \cdots, p_4) = -e^4 \int \frac{d^4k}{(2\pi)^4} Tr\{\gamma_\mu \frac{1}{\not{k} - m} \gamma_\nu \frac{1}{\not{k} + \not{p_2} - m} \gamma_\rho \frac{1}{\not{k} + \not{p_2} + \not{p_3} - m} \gamma_\sigma \frac{1}{\not{k} - \not{p_1} - m}\}$$

$$+ (\mu \leftrightarrow \nu, p_1 \leftrightarrow p_2) + (\mu \leftrightarrow \rho, p_1 \leftrightarrow p_3)$$

$$+ (\mu \leftrightarrow \sigma, p_1 \leftrightarrow p_4) + (\nu \leftrightarrow \rho, p_2 \leftrightarrow p_3)$$

$$+ (\rho \leftrightarrow \sigma, p_3 \leftrightarrow p_4)$$

which are superficially quadratically, linearly and logarithmically ultraviolet divergent respectively. Let us now render them finite by means of the inserter procedure.

To regulate the divergent vacuum polarization function $\Pi_{\mu\nu}(k)$, we attach to one internal fermion line with i inserters and to the other with $2q - i$ ones. Then we get a set of $2 + 2q$-point functions $\Pi_{\mu\nu}^{\{q, i\}}(k; q)$. If q is large enough, all these $2 + 2q$-point functions are convergent. Then we introduce a new function

$$\Pi_{\mu\nu}(k; q; \mu) = (-i\mu)^{2q} (-i\lambda_f)^{-2q} \frac{1}{N_q} \sum_{i=0}^{2q} \Pi_{\mu\nu}^{\{q, i\}}(k; q), \qquad (17)$$

which has the same dimension in mass, the same order in e with the original function $\Pi_{\mu\nu}(k)$. It is not hard to prove that this function can be expressed as

$$\Pi_{\mu\nu}(k; q; \mu) = -\mu^{2q} e^2 \frac{1}{N_q} \sum_{i=0}^{2q} \int \frac{d^4p}{(2\pi)^4} Tr[\gamma_\mu \left(\frac{1}{\not{p} - \not{k} - m}\right)^{i+1} \gamma_\nu \left(\frac{1}{\not{p} - m}\right)^{2q-i+1}]$$

7

and satisfies the gauge invariant condition:

$$k^\mu \Pi_{\mu\nu}(k; q; \mu) = 0. \tag{18}$$

Continuing q to the complex number, thus the original amplitude $\Pi_{\mu\nu}$ is recovered as

$$\Pi_{\mu\nu}(k) = \lim_{q \to 0} \Pi_{\mu\nu}(k; q; \mu). \tag{19}$$

If we denote

$$\Pi_{\mu\nu}(k^2) \equiv (k_\mu k_\nu - k^2 g_{\mu\nu})\Pi(k^2), \quad \Pi(k^2) = \Pi(0) + \Pi^f(k^2), \tag{20}$$

by some calculation, we get

$$\Pi(0) = e^2 \frac{4i}{(4\pi)^2}[\frac{1}{3q} + C + \frac{1}{3}\ln(-\frac{\mu^2}{m^2}) + o(q)],$$

$$\Pi^f(k^2) = -\frac{ie^2}{2\pi^2} \int_0^1 d\alpha \alpha(1 - \alpha)\ln[1 - \frac{\alpha(1-\alpha)k^2}{m^2}]. \tag{21}$$

where C is some constant. The finite part $\Pi^f(k^2)$ is the same as that derived in other regularization procedures.

To regulate the electron self-energy function $\Sigma(p)$, we attach to the internal fermion line with $2q$ inserters and the graph $\Sigma(p)$ is turned to a $2 + 2q$-point convergent function $\Sigma(p; q)$ if q is large enough. Then we introduce a new function:

$$\Sigma(p; q; \mu) = (-i\mu)^{2q}(-i\lambda_f)^{-2q}\Sigma(p; q), \tag{22}$$

which can be expressed as

$$\Sigma(p; q; \mu)_{\beta\alpha} = -\mu^{2q}e^2 \int \frac{d^4k}{(2\pi)^4}\Big(\gamma^\mu \frac{(\not k + m)^{1+2q}}{(k^2 - m^2)^{1+2q}}\gamma_\mu\Big)_{\beta\alpha}\frac{1}{(p - k)^2}.$$

Continuing q to the complex number, the original function $\Sigma(p)$ is reached by

$$\Sigma(p)_{\beta\alpha} = \lim_{q \to 0} \Sigma(p; q; \mu)_{\beta\alpha}. \tag{23}$$

Denoting

$$\Sigma(p) = mA(p^2) + iB(p^2) \not p, \tag{24}$$

we may finally get

$$A(p^2) = -\frac{ie^2}{4\pi^2}\{\frac{1}{q} + 3 - \ln\frac{p^2-m^2}{\mu^2} + \frac{m^2}{p^2}\ln(1 - \frac{p^2}{m^2}) + A^f(p^2)\},$$

$$B(p^2) = \frac{e^2}{(4\pi)^2}\{\frac{1}{q} + \frac{1}{2} + \frac{m^2}{p^2} - \ln\frac{p^2-m^2}{\mu^2} - (\frac{m^2}{p^2})^2\ln(1 - \frac{p^2}{m^2})\} + B^f(p^2),$$

both $A^f(p^2)$ and $B^f(p^2)$ are finite function.

Similarly, to regulate the vertex function, $\Lambda_\mu(p', p)$, we attach to the internal fermion lines with $2q$ inserters to get a set of $(3 + 2q)$-point functions $\Lambda_\mu^{\{q,i\}}(p', p; q)$ with i inserters

on one internal fermion line and $q - i$ inserters on the other. Then we introduce a new function

$$\Lambda_\mu(p', p; q; \mu) = (-i\mu)^{2q}(-i\lambda_f)^{-2q}\frac{1}{N_q}\sum_{i=0}^{2q}\Lambda_\mu^{\{q,i\}}(p', p; q), \tag{25}$$

which is convergent if q is large enough and has the same dimension in mass, the same order in e with the original vertex function $\Lambda_\mu(p', p)$. It can be expressed as

$$\Lambda_\mu(p', p; q; \mu) = -\frac{\mu^{2q}e^3}{2q+1}\sum_{i=0}^{2q}\int\frac{d^4l}{(2\pi)^4}\left(\gamma^\rho\frac{(\not{l}-\not{k}+m)^{1+i}}{[(l-k)^2-m^2]^{i+1}}\gamma_\mu\frac{(\not{l}+m)^{1+2q-i}}{(l^2-m^2)^{1+2q-i}}\gamma_\rho\right)\frac{1}{(p-l)^2}.$$

Continuing q to the complex number, the original vertex function is then recovered as:

$$\Lambda_\mu(p', p) = \lim_{q\to 0}\Lambda_\mu(p', p; q; \mu). \tag{26}$$

Finally, we find that

$$\Lambda_\mu(p', p) = \frac{-ie^3\mu^{2q}}{(4\pi)^2}\gamma_\mu$$

$$[\frac{1}{q} - \frac{3}{2} - \int_0^1 d\alpha\int_0^{1-\alpha}d\beta\ln\{\beta(1-\beta)k^2 + \alpha(1-\alpha)p^2 - 2\alpha\beta k\cdot p - (1-\alpha)m^2\}] \tag{27}$$

$$- \frac{ie^3}{2(4\pi)^2}\int_0^1 d\alpha\int_0^{1-\alpha}d\beta\frac{\gamma^\rho[(\beta-1)\not{k}+\alpha\not{p}+m]\gamma_\mu[\beta\not{k}+\alpha\not{p}+m]\gamma_\rho}{\beta(1-\beta)k^2+\alpha(1-\alpha)p^2-2\alpha\beta\cdot p-(1-\alpha)m^2} + o(q).$$

The observable part Λ_μ^f of the vertex function is defined by

$$\Lambda_\mu = K\gamma_\mu + \Lambda_\mu^f,$$

where K contains the pole in q when $q \to 0$ and Λ_μ^f is finite from which the anomalous magnetic moment of the electron can be derived. The result is the same as in other approaches.

Similar procedure may also be applied to the photon-photon scattering $\Gamma_{\mu\nu\rho\sigma}(p_1\cdots p_4)$. We check its gauge invariance by the inserter proposal. Attaching $2q$ inserters to internal fermion lines in all possible ways we get a set of convergent functions if q is large enough. Then we introduce a new function

$$\Gamma_{\mu\nu\rho\sigma}(p_1\cdots p_4; q; \mu) = (-i\mu)^{2q}(-i\lambda_f)^{-2q}\frac{1}{N_q}\sum_{i=0}^{2q}\sum_{j=0}^{2q-i}\sum_{l=0}^{2q-i-j}\Gamma_{\mu\nu\rho\sigma}^{\{q,i,j,l\}}(p_1\cdots p_4), \tag{28}$$

which has required properties with respect to the original function. It can be proved that this function satisfies the gauge invariant condition. Continuing q to the complex number, the original function is then recovered by

$$\Gamma_{\mu\nu\rho\sigma}(p_1\cdots p_4) = \lim_{q\to 0}\Gamma_{\mu\nu\rho\sigma}(p_1\cdots p_4; q; \mu). \tag{29}$$

By some straightforward calculation, we may explicitly show that

$$\Gamma_{\mu\nu\rho\sigma}(p_1\cdots p_4)|_{p_1=\cdots=p_4=0} = 0. \tag{30}$$

This also coincides with the gauge invariance.

Thus we complete the regularization of QED at the one loop order by means of the inserter proposal.

Some Applications to SUSY-Models

We now apply the inserter proposal for the intrinsic regularization to some SUSY-models at one-loop order. We will not present any detailed calculation here. The aim is to show that the SUSY version of the inserter proposal should work and preserve supersymmetry manifestly and consistently by reexaming some well-known and simple examples at the one loop order.

Let us first consider an example in the massive Wess-Zumino model. It is well known that at the one loop level, the self-energy graph of antichiral-chiral superfield propagator $\bar{\phi}\phi$ is divergent. After some D-algebraic manipulation, it is left a divergent integral

$$\int d^4\theta \ \phi(-p,\theta)\bar{\phi}(p,\theta) \ A(p,m),$$ (31)

where

$$A(p,m) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+m^2)((p+k)^2+m^2)}.$$ (32)

To regulate this integral by means of the inserter proposal, we need first to construct an antichiral-chiral superfield inserter. For such an inserter, we take a pair of vertices linked by a $\bar{\phi}\phi$-internal line with a pair of chiral and antichiral external legs carrying zero momenta. Its Feynman rule can easily be written down. Now we may utilize this inserter to insert q-times the internal lines in the divergent graph. Then we get a set of convergent graphs with i-inserters on one internal line and $q-i$ on the other. Similarly, after some D-algebraic manipulation, the corresponding convergent function $I^{\{q,i\}}(p)$ is proportional to

$$\int d^4\theta \ \phi(-p,\theta)\bar{\phi}(p,\theta) \ A^{\{q,i\}}(p,m),$$ (33)

where

$$A^{\{q,i\}}(p,m) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+m^2)^{2i}((p+k)^2+m^2)^{2q-2i+1}}.$$ (34)

It is very similar to the ones in the case of inserted fish functions, accept the sign of the mass due to the convention. Now we may almost repeat the procedure in the ϕ^4 theory to define the regulated function and so on. The original divergent function is recovered in the limiting case of $q \to 0$ and the divergence manifestly appears as a pole of q. It is so analogous to the case of the fish in the ϕ^4 theory that we do not need to repeat it here.

It is easy to see that the SUSY version of the inserter proposal may also be applied to the massless Wess-Zumino model as well as other models. Let us now consider a most general $N = 1$ supersymmetric renormalizable model invariant under a gauge group G contains

chiral superfields ϕ^a in a representation R of G and $N = 1$ Yang-Mills field contained in the general superfield V.

The one-loop correction to the $\phi^a\bar{\phi}_a$ propagator are given by two divergent graphs. One is the same as in the Wess-Zumino model while the other is the one with an internal antichiral-chiral line replaced by a V-line. They lead to the expression:

$$g^2 \int d^4\theta \ \phi^b(-p,\theta)(S_b^a - C_2(R)\delta_b^a)\bar{\phi}_a(p,\theta) \ A(p), \tag{35}$$

where $A(p) = A(p, m = 0)$ and

$$2g^2 S_b^a = d^{bce}d_{ace},$$

d^{bce} are the couplings of the ϕ^3-term. the cancellation condition $S_b^a = C_2(R)\delta_b^a$ should hold much safer if the divergent integrals $A(p)$ in two graphs can be regulated in a way of preserving supersymmetry manifestly and consistently. This can be done by means of the SUSY-version of the inserter proposal. To this end, in addition to the inserter for the chiral superfield constructed above (the internal representation indices should be paired here), we need an inserter for the V-internal line as well. In fact, it may be constructed in such a way that two $V\phi\phi$-vertices linked by an internal antichiral-chiral line with two external ϕ^a, $\bar{\phi}_b$ legs carrying zero momenta and paired representation indices. Then it is easy to see that by inserting these two inserters to the internal V line and the internal antichiral-chiral line respectively, we can always get the same regulated functions for the both graphs. Therefore, the cancellation can be insured at the regulated function level as well.

The one-loop correction to the vector superfield propagator is given by three graphs with V-loop, ϕ-loop and the ghost loop respectively. The SUSY version of the inserter proposal also ensures the corresponding cancellation condition holds at the regulated function level as long as we employ the ghost inserter as a pair of the ghost-V vertices linked by an internal ghost antichiral-chiral line with two external V legs carrying zero momenta in addition to the fore-mentioned inserters for the internal V line and the internal antichiral-chiral line.

The SUSY version of the inserter proposal may also be combined with the background approach. For example, in the background field approach the above one-loop contribution to the V self-energy from a massive chiral superfield leads to one divergent integral only:

$$\frac{1}{4}C_2(R) \ tr \int d^4\theta \ W^\alpha(p,\theta)\Gamma_\alpha(-p,\theta) \ A(p,m), \tag{36}$$

where $W^\alpha(p,\theta)$ is the superfield strength and $\Gamma_\alpha(-p,\theta)$ the background field connection. Again, we may utilize the an antichiral-chiral superfield inserter in the background field to get a set of convergent integrals and to regulate this divergent one in the way of preserving supersymmetry manifestly and consistently.

It should be noticed that all construction for the superfield inserters and regularization for the divergent graphs are made with the help of the super-Feynman rules. It is natural to expect that the SUSY version of the inserter proposal does preserve supersymmetry manifestly and consistently not only for the one-loop cases but also for the high-loop cases. We will explore this issue in detail elsewhere.

11

Further Remarks

We have shown the main steps and results for the regularization of the divergent 1PI functions at the one loop order in both ϕ^4 theory and QED by means of the inserter proposal for the intrinsic regularization method. Some applications to SUSY-models are also made at the one loop order by means of the SUSY version of the inserter proposal. The results are satisfactory. It is naturally to expect that this proposal should be available to the cases at higher loop orders in principle.

The crucial point of this approach, in fact, is very simple but fundamental. That is, the entire procedure is intrinsic in the QFT. There is nothing changed, the action, the Feynman rules, the spacetime dimensions etc. are all the same as that in the given QFT. Although for QED, the inserter we have employed is borrowed from the standard model, QED is in fact unified with the weak interaction in the standard model. Therefore, it is still intrinsic in the standard model. Consequently, in applying to other cases all symmetries and topological properties there should be preserved in principle. This is a very important property which should shed light on that challenging problem. It is reasonable to expect that this proposal should be able to apply consistently to those cases where the symmetries and topological properties are sensitive to the spacetime dimensions, the number of fermionic degrees of freedom, such as chiral symmetry, anomalies, SUSY theories etc.. As was shown in the last section, it is the case for some SUSY-models at the one loop order. Of course, for each case some special care should be taken. For the non-Abelian gauge theories, like QCD and the standard model, for instance, special attention should be devoted to the Lorentz indices and those indices of the internal gauge symmetries in constructing the inserters. The Lorentz indices can be handled by contracting pairly by the spacetime metric. Similarly, the internal gauge symmetry indices may also be dumbed by the Killing-Cartan metrics in the corresponding representations. In the appendix, we construct the inserters in the standard model. It is straightforward to apply them at the one loop order. For higher loop orders and other theories, we will study them in detail elsewhere.

The renormalization of the QFT under consideration in this scheme should be the same as in usual approaches. Namely, we may subtract the divergent part of the n-point functions at each loop order by adding the relevant counterterms to the action. The renormalized n-point functions are then evaluated from the renormalized action. In the limiting case, we get the finite results for all correlation functions.

In our proposal the inserters play an important role. The zero-momentum-line(s) in the inserters do not of course correspond to realistic particles. But, it may have some physical explanation. Namely, for each inserted internal line, the virtual particle always emits and/or absorbs via the inserters other far-infrared "particles" that carry zero momenta from the vacuum. In other wards, the vacuum is full of such far-infrared "particles" that they always have or pair together with the vacuum quantum numbers, i.e. zero momenta, singlet(s) in all internal symmetries (including gauge symmetries) and scalar(s) in the spacetime symmetries. The ill-definess of those divergent graphs can be handled by taking into account the role played by these far-infrared "particles". This is just what has been done in the intrinsic inserter proposal. An analogous explanation may also be made for the

SUSY version of the inserter proposal in terms of the superspace and superfields.

We have not devoted any attention in this note to the infrared divergences at all. It is in fact another most challenging problem to the regularization schemes. In the course of application to QED, the internal photon lines are the same as the original ones. It is intriguing to see, however, as far as the vacuum picture is concerned, certain kind of inserters should be constructed and some intrinsic relation between the divergent function and convergent ones may also be established in the infrared region. Then the intrinsic inserter proposal may work in this region as well. We will also investigate this issue elsewhere.

On the other hand, however, as was fore-mentioned, although the inserter proposal works it is still a simplified procedure from the point of view of the intrinsic regularization method. In fact, what have been taken into account is a proper set of all convergent 1PI functions which share the same loop skeleton with the given divergent 1PI functions. Not all of them. It is obvious that the simplification of the inserter proposal certainly leads to a question what role should be played by other convergent functions which can not be given by inserting the inserters. As a matter of fact, we may propose an alternative approach to the method. For the ϕ^4 theory and for some SUSY-models, for example, the (massive) Wess-Zomino model, it is the same as the inserter proposal or its SUSY-version. But for QED, it is different: In spite of the complication mentioned before, we first simply attach $2q$ fermion-photon vertices with zero momentum photon lines to the internal fermion line(s) in the graph of given divergent 1PI n-point function with rank r as a Lorentz tensor. By doing so, if q is large enough, we can get a set of convergent 1PI $n+2q$-point functions of rank $r+2q$ Lorentz tensors. And the original one is the case of $q = 0$. Thus we also reach an intrinsic relation between the original divergent function and those convergent ones. Although not only they are at different order in the coupling constant e but also they have different ranks as Lorentz tensors, it is still possible to define a regulated function with the help of this relation. To this end, we may take all possible ways of contracting those additional Lorentz indices in convergent functions by the spacetime metric to reduce the rank of Lorentz tensor to the one in the original Lorentz tensor function. Then we may employ the same procedure as that in the inserter proposal to introduce the regulated function. It is easy to see that the total number of the convergent graphs in this approach could be different from and larger than that in the inserter proposal. While the total number of the convergent functions after contracting the additional Lorentz indices is even much bigger than the one in the inserter proposal. Of course, this approach is much more complicated than the inserter proposal and the calculation is also more tedious. But, as far as the topological formula and those intrinsic relations are concerned, this alternative approach may also be available. Thus, it is of course interesting to see whether there are some essential differences between these two approaches. We would leave this classification problem for further investigation.

Finally, it should be mentioned that some idea of the inserter proposal was first presented for the ϕ^4 theory in [1] by ZHW and HYG as what is called the intrinsic vertex regularization. Later, the intrinsic loop regularization method has been studied in [2-5]. Most results for the ϕ^4 theory and QED at the one loop order in [2,3,5] are very similar to what have been given in this note. The mass shifting in that approach, however, is not really intrinsic and do not completely work for the theories with self-interacting massless particles, like QCD,

the standard model, SUSY-models etc.. The approach presented here should be able to get rid of all those problems.

Appendix: Inserters In The Standard Model

In order to apply the intrinsic inserter proposal to SM, different kinds of inserters are needed for inserting the internal lines of quarks, leptons, gauge bosons, Higgs and ghosts in the Feynman graphs with divergent amplitudes to be regulated. To construct appropriate inserters we choose suitablely a vertex or a pair of vertices linked by an internal line and make merely use of the Feynman rules in SM. In all inserters, the external leg or legs are all being managed in such a way that they always carry the vacuum quantum numbers, i.e. zero momentum, singlet in internal and gauge symmetries, and scalar in the spacetime symmetry. For some specified internal line(s), different inserter may be employed for different purpose.

1. The fermion-inserters:

There are two types of inserters. The Yukawa-inserters for massive fermions and the one for the neutrinos. For the Yukawa-inserters, we take them as corresponding $ff\phi$-vertices with zero-momentum Higgs lines. The Feynman rule is:

$$I^{\{f\}}(p) = -i\frac{g}{2}\frac{m_f}{M_W}.$$

For each neutrino-inserter, we take a pair of $\nu_l\nu_l Z$-vertices linked by an internal neutrino line such that two γ-matrices are contracted and Z-external lines carry zero momenta. The Feynman rule is then:

$$I^{\{\nu_l\}}(p) = \frac{g^2}{4cos^2\theta_W}\frac{i}{\not{p} + i\epsilon}(1 - \gamma_5).$$

2. The gauge-boson-inserters:

For the gauge bosons such as gluons, $W^\pm$ and Z, there are some options. We may take an inserter as a 4-gauge-boson vertex with two zero-momentum lines whose indices are dumbed by the spacetime metric and the Killing-Cartan metric of the gauge algebra respectively. Their Feynman rules are easily be given. For example, for the gluon-inserter:

$$I^{\{g\}ab}_{\mu\nu}(p) = -6ig_c^2 C_2(8)g_{\mu\nu}\delta^{ab},$$

where $C_2(8)$ is the second Casimir operator valued in the adjoint representation of $SU_c(3)$ algebra. On the other hand, we may also take a pair of 3-gauge-boson vertices linked by an internal gauge-boson line with two zero-momentum lines and dumbed pairly indices.

3. The Higgs-inserters:

Similar to the one in the ϕ^4 theory, each Higgs-inserter may be taken as a suitable 4-ϕ-vertex with two zero momentum lines. Their Feynman rules are easily be given as well. For example, the inserter for $\phi^i, i = 1, 2$, is

$$I^{\{\phi^i\}}(p) = -3ig^2\frac{\mu^2}{2M_W^2}.$$

On the other hand, a pair of 3-ϕ-vertices may also make a Higgs-inserter.

4. The ghost-inserters:

For the ghost-inserter in QCD, for example, we take a tree graph with two ghost-gluon vertices linked by a ghost line with two gluon lines carrying zero momenta whose Lorentz indices and color indices are contracted by the spacetime metric and the Killing-Cartan metric respectively. Its Feynman rule is given by

$$I^{\{gh\}}_{a_1 a_2}(p) = -ig_c^2 C_2(\mathbf{8})\delta_{a_1 a_2}.$$

For other ghost-inserters, it is easy to construct in a similar way.

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