THE EFFECTIVE CONE OF $\overline{\mathcal{M}}_g$ AND THE SLOPE CONJECTURE

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ABSTRACT: This is an shortened written version of my talk at the Mathematische Arbeitstagung 2003, reporting on joint work with M. Popa and mostly contained in the papers [FP1] and [FP2].

We denote by $\overline{\mathcal{M}}_g$ the Deligne-Mumford moduli space of stable curves of genus g. A fundamental problem which goes back to Mumford is to describe the cone $\mathrm{Eff}(\overline{\mathcal{M}}_g)$ of effective divisors on $\overline{\mathcal{M}}_g$. Knowledge of $\mathrm{Eff}(\overline{\mathcal{M}}_g)$ will have wide ranging applications to the birational geometry of $\overline{\mathcal{M}}_g$, in particular giving a description of all rational maps from $\overline{\mathcal{M}}_g$ to other projective varieties.

We start by recalling a few facts about $\overline{\mathcal{M}}_g$. The boundary $\overline{\mathcal{M}}_g - \mathcal{M}_g$ corresponding to singular stable curves, breaks up into a union of irreducible divisors $\Delta_0 \cup \ldots \cup \Delta_{[g/2]}$. We denote by $\delta_i := [\Delta_i] \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ the associated class in the Picard group of the moduli stack, and by λ the Hodge class. It is known that for $g \geq 3$, the group $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ is freely generated by the classes $\lambda, \delta_0, \ldots, \delta_{[g/2]}$.

If $D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ is such that Δ_i is not contained in $\mathrm{supp}(D)$ for $i = 0, \ldots, [g/2]$, it is easy to show that we can write

$$D \equiv a\lambda - b_0\delta_0 - \cdots - b_{[g/2]}\delta_{[g/2]}, \text{ with } a, b_i \geq 0.$$

For such a divisor D one defines the *slope* by the formula

$$s(D) := \frac{a}{\min_{i=0}^{[g/2]} b_i} \geq 0.$$

Since the class λ is big and nef on $\overline{\mathcal{M}}_g$, it is very easy to write down effective divisors on $\overline{\mathcal{M}}_g$ of large slope. One is particularly interested in effective divisors of small slope and in particular the number $\inf_{D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)} s(D)$ is a very subtle invariant of $\overline{\mathcal{M}}_g$ encoding a lot of informations on the nature of $\overline{\mathcal{M}}_g$ as an algebraic variety. We have the following conjecture due to J. Harris and I. Morrison (cf. [HMo]):

Conjecture 0.1. For every effective divisor D on $\overline{\mathcal{M}}_g$ we have the inequality $s(D) \geq 6 + 12/(g+1)$, with equality if and only if $g+1$ is composite and D is essentially a combination of Brill-Noether divisors.

We recall that the Brill-Noether divisors on $\overline{\mathcal{M}}_g$ for $g+1$ composite are defined as loci $\mathcal{M}^r_{g,d} := \{[C] \in \mathcal{M}_g : C \text{ has a } \mathfrak{g}^r_d\}$, for all positive integers r and d

such that

$$\rho(g,r,d) = g - (r+1)(g-d+r) = -1.$$

Here $\mathfrak{g}^r_d$, as usual, is a linear system of degree d and projective dimension r.

In such a case $\mathcal{M}^r_{g,d}$ is an irreducible divisor, and the class of its compactification has been computed by Eisenbud and Harris (cf. [EH]):

$$\overline{\mathcal{M}}^r_{g,d} = c((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i\geq 1} i(g-i)\delta_i).$$

Thus the Slope Conjecture would single out the Brill-Noether divisors as those minimizing the slope function. The Slope Conjecture holds for $g \leq 9$ and $g = 11$. The main evidence for it comes from the large number of divisor class calculations on $\overline{\mathcal{M}}_g$ where in each case it turns out that the slope exceeds $6 + 12/(g+1)$ which is the slope of the Brill-Noether divisors.

If true, the Slope Conjecture would imply that the Kodaira dimension of $\mathcal{M}_g$ is $-\infty$ for $g \leq 22$. This is known at the moment only for $g \leq 16$. Recall that for $g \geq 24$ we have the celebrated result of Harris, Mumford and Eisenbud stating that $\mathcal{M}_g$ is of general type (cf. [HM]), while for the intermediate case $g = 23$ we know that $\kappa(\mathcal{M}_{23}) \geq 2$ (cf. [Fa]). The Slope Conjecture would also provide myriads of new Schottky relations coming from Siegel modular forms of small slope.

It turns out that the Slope Conjecture is intimately related to curves sitting on $K3$ surfaces. If we denote by $\mathcal{K}_g := \{[C] \in \mathcal{M}_g : C \text{ sits on a } K3 \text{ surface}\}$ we have the following:

Proposition 0.2. *For every $D \in Eff(\overline{\mathcal{M}}_g)$ with $s(D) < 6 + 12/(g+1)$, we must have $\mathcal{K}_g \subset D$.*

Due to work of Mukai it is known that $\dim(\mathcal{K}_g) = 19 + g$ for $g \geq 13$. The naive dimension count breaks down for $g = 10$, where because of the existence of a homogeneous variety associated to the exceptional Lie group G_2 we obtain that $\mathcal{K} = \mathcal{K}_{10}$ is a divisor on $\mathcal{M}_{10}$.

We prove the following result:

Theorem 0.3. *The class of the divisor $\overline{\mathcal{K}}$ in $\mathrm{Pic}(\overline{\mathcal{M}}_{10})$ is given by*

$$\overline{\mathcal{K}} \equiv 7\lambda - \delta_0 - 5\delta_1 - 9\delta_2 - 12\delta_3 - 14\delta_4 - 15\delta_5,$$

where λ is the class of the Hodge bundle and $\delta_0, \ldots, \delta_5$ are the boundary classes corresponding to singular curves.

We compute the class of $\overline{\mathcal{K}}$ by reinterpreting $K3$ sections in a way that makes no reference to $K3$ surfaces:

Theorem 0.4. *The divisor $\mathcal{K}$ has four incarnations as a geometric subvariety of $\mathcal{M}_{10}$:*

(1) *(By definition) The locus of curves sitting on a* $K3$ *surface.*
(2) *The locus of curves* C *with a non-surjective Wahl map* $\psi_K : \wedge^2 H^0(K_C) \to H^0(3K_C)$.
(3) *The locus of curves* C *carrying a stable rank two vector bundle* E *with* $\wedge^2(E) = K_C$ *and* $h^0(E) \geq 7$.
(4) *The locus of genus 10 curves sitting on a quadric in an embedding* $C \subset \boldsymbol{P}^4$ *with* $deg(C) = 12$.

The equivalence of descriptions (1) and (2) was established before by Cukierman and Ulmer (cf. [CU]). Description (3) reveals the true nature of the divisor $\mathcal{K}$ as a higher rank Brill-Noether divisor. Moreover, characterizations (3) and (4) are generalizable to other genera to provide new counterexamples to the Slope Conjecture. For instance we can show that on $\mathcal{M}_{13}$ the locus D consisting of genus 13 curves C which lie on a quadric in an embedding in $\mathbf{P}^5$ with $\deg(C) = 16$, is a divisor which also violates the Slope Conjecture.

There is a striking similarity between the class of $\overline{\mathcal{K}}$ and the class of the Brill-Noether divisors on $\overline{\mathcal{M}}_{11}$. On this space there are two distinct Brill-Noether divisors, namely $\overline{\mathcal{M}}^1_{11,6}$ and $\overline{\mathcal{M}}^2_{11,9}$, which both have classes proportional to the $\mathbb{Q}$-class

$$BN = 7\lambda - \delta_0 - 5\delta_1 - 9\delta_2 - 12\delta_3 - 14\delta_4 - 15\delta_5.$$

This similarity, especially surprising since $\mathcal{K}$ behaves very differently from all Brill-Noether divisors, might suggest the existence of a correspondence between genus 10 and genus 11 curves, that would associate to a $K3$-section of genus 10 a genus 11 curve with a $\mathfrak{g}^1_6$.

We also studied divisors on $\overline{\mathcal{M}}_g$ having minimal slope $6+12/(g+1)$ (for those g for which the original Slope Conjecture holds on $\mathcal{M}_g$. We have the following:

Theorem 0.5. *The Iitaka dimension of the Brill-Noether linear system on* $\overline{\mathcal{M}}_{11}$ *is equal to* 19.

This seems to contradict the hypothesis formulated in [HMo] that the Brill-Noether divisors are essentially the only effective divisors on $\overline{\mathcal{M}}_g$ having slope $6 + 12/(g + 1)$.

Although the Slope Conjecture fails, a weakened version appears to be true:

Conjecture 0.6. There exists a constant $\epsilon > 0$ such that that $s(D) \geq \epsilon$, for every $D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ and for every g.

If true this would highlight a fundamental difference between $\mathcal{M}_g$ and the moduli space of abelian varieties $\mathcal{A}_g$ on which there are Siegel modular forms (leading to effective divisors on $\mathcal{A}_g$) and having arbitrarily small slope. We also believe that the following should hold:

Conjecture 0.7. For every g there exists $\epsilon_g > 0$ such that for all $D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)$ with $a/b_0 \leq \epsilon_g$, we have that $b_i \geq b_0$, hence $s(D) = a/b_0$.

We proved that this holds on $\overline{\mathcal{M}}_g$ for all $g \leq 23$ (cf. [FP1]).

References

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