Einstein Metrics on
Rational Homology 7-Spheres

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Abstract: In this note we demonstrate the existence of Sasakian-Einstein structures on certain 2-connected rational homology 7-spheres. Aside from the regular homogeneous Sasakian-Einstein structure on spherical space forms, the real Stiefel manifold of 2-frames in $\mathbb{R}^{4n+1}$, and $G_2/Sp(1)$ these appear to be the first non-regular examples of Sasakian-Einstein metrics on rational homology spheres.

Introduction

Dimension seven appears to be rather special when it comes to examples of compact Einstein manifolds. It is perhaps the prominent rôle such manifolds have played in physics ever since the early days of Kaluza-Klein supergravity that made both theoretical physicists and mathematicians alike particularly interested in them. Discoveries of many different constructions followed as a result of this interest.

Arguably, today a special place among all compact Einstein 7-manifolds is reserved for the so-called Sasakian-Einstein spaces. They are defined to be Riemannian manifolds with the property that the metric cone on them is a Calabi-Yau 4-fold and, in particular, they are always of positive scalar curvature. All regular Sasakian-Einstein manifolds are circle bundles of Fano 3-folds which admit Kähler-Einstein metrics. Non-regular ones fiber over compact Kähler-Einstein Fano 3-folds with orbifold singularities. An interesting subfamily of the family of Sasakian-Einstein 7-manifolds consists of the so-called 3-Sasakian spaces. They are characterized by fact that their metric cone is not just Calabi-Yau but also hyperkähler and are all orbifold fibrations over compact Kähler-Einstein Fano 3-folds which admit a complex contact structure.

Regular and non-regular examples of both Sasakian-Einstein and 3-Sasakian manifolds are now plentiful and they were extensively studied by the first two authors [BG1, BG2]. There is an example of a regular Sasakian-Einstein (4n+3)-manifold which is worthy of some further discussion. It is the homogeneous Stiefel manifold of 2-frames in (2n+1)-dimensional Euclidean space, $V_2(\mathbb{R}^{2n+1}) = SO(2n+1)/SO(2n-1)$ which is a circle bundle over the oriented Grassmannian, $\widetilde{G}_2(\mathbb{R}^{2n+1})$. From the point of view of an algebraic geometer it is a classical fact that $\widetilde{G}_2(\mathbb{R}^{2n+1})$ is diffeomorphic to the complex quadric Q_{2n-1} in $\mathbb{CP}^{2n}$ which is well-known to be Fano and to admit a Kähler-Einstein metric. It is perhaps less well-known that the quadric Q_{2n-1} has the same cohomology groups as $\mathbb{CP}^{2n-1}$, but differs in the ring structure. Hence, $V_2(\mathbb{R}^{2n+1})$ is a rational homology sphere with $H_{n-1}(V_2(\mathbb{R}^{2n+1}), \mathbb{Z}) \approx \mathbb{Z}_2$. Now it has been know for quite some time

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that $V_2(\mathbb{R}^{2n+1})$ carries a Sasakian-Einstein structure [BGFK,BG1]. Up to date, apart from $S^{2n+1}, V_2(\mathbb{R}^{2n+1})$, and the 3-Sasakian homogeneous 11-manifold $G_2/Sp(1)$ we are not aware of any other examples of simply connected rational homology spheres which are also known to admit Sasakian-Einstein structures. In this paper we shall demonstrate that for $n=2$, quite to the contrary, there are many examples of such structures, 184 to be precise. These examples are obtained as hypersurfaces in certain weighted projective 4-spaces, but we certainly expect the phenomena to occur in arbitrary dimension.

The key to this construction is a recent paper of Johnson and Kollár [JK2]. There they give a list of 4442 quasi-smooth Fano 3-folds $\mathcal{Z}$ anticanonically embedded in weighted projective 4-spaces $\mathbb{P}(\mathbf{w})$. Moreover, they show that 1936 of these 3-folds admit Kähler-Einstein metrics. According to our general theory [BG1] such Fano 3-folds give rise to Sasakian-Einstein metrics on smooth 7-manifolds M^7. Moreover, these 7-manifolds arise as links of isolated hypersurface singularities associated to certain weighted homogeneous polynomials in $\mathbb{C}^5$. As in [JK1] Johnson and Kollár [JK2] only consider the case when the orbifold Fano index is one, and as the authors showed in [BGN1] for log del Pezzo surfaces, there should be many more interesting examples of quasi-smooth Fano 3-folds with higher orbifold Fano index. This is currently under study.

In this note we prove

Theorem A: *There are1936 distinct Sasakian-Einstein structures on certain 2-connected 7-manifolds $M^7_{\mathbf{w},d}$ realized as links of weighted homogeneous polynomials in $\mathbb{C}^5$ with weight vector $\mathbf{w}=(w_0,w_1,w_2,w_3,w_4)$ and degree d. In particular, there are 184 2-connected rational homology spheres which are listed in the Table below. In addition to the weight vector $\mathbf{w}$ and degree d the Table lists the Milnor number μ of the link and the order of $H_3(M^7_{\mathbf{w},d},\mathbb{Z})$.*

We have not answered the question as to whether two distinct or non-conjugate Sasakian-Einstein structures on the same link M^7 could belong to the same underlying Riemannian metric g. Indeed, this can happen, but if g is not the standard round metric on S^7 then by a Theorem of Tachibani and Yu [TY], the two Sasakian-Einstein structures must belong to a 3-Sasakian structure. But then by a Theorem of Galicki and Salamon [GS], we must have $b_3=0$, so M^7 must be a rational homology sphere.

2. The Sasakian Geometry of Links of Weighted Homogeneous Polynomials

In this section we briefly review the Sasakian geometry of links of isolated hypersurface singularities defined by weighted homogeneous polynomials. Consider the affine space $\mathbb{C}^{n+1}$ together with a weighted $\mathbb{C}^*$-action given by $(z_0,\ldots,z_n)\mapsto(\lambda^{w_0}z_0,\ldots,\lambda^{w_n}z_n)$, where the *weights* w_j are positive integers. It is convenient to view the weights as the components of a vector $\mathbf{w}\in(\mathbb{Z}^+)^{n+1}$, and we shall assume that they are ordered $w_0\leq w_1\leq\cdots\leq w_n$ and that $\gcd(w_0,\ldots,w_n)=1$. Let f be a quasi-homogeneous polynomial, that is $f\in\mathbb{C}[z_0,\ldots,z_n]$ and satisfies

2.1 $$f(\lambda^{w_0}z_0,\ldots,\lambda^{w_n}z_n)=\lambda^d f(z_0,\ldots,z_n),$$

where $d\in\mathbb{Z}^+$ is the degree of f. We are interested in the *weighted affine cone* C_f defined by the equation $f(z_0,\ldots,z_n)=0$. We shall assume that the origin in $\mathbb{C}^{n+1}$ is an isolated singularity, in fact the only singularity, of f. Then the link L_f defined by

2.2 $$L_f=C_f\cap S^{2n+1},$$

where

$$S^{2n+1} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} | \sum_{j=0}^{n} |z_j|^2 = 1\}$$

is the unit sphere in $\mathbb{C}^{n+1}$, is a smooth manifold of dimension $2n-1$. Furthermore, it is well-known [Mil] that the link L_f is $(n-2)$-connected.

On S^{2n+1} there is a well-known [YK] "weighted" Sasakian structure $(\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ which in the standard coordinates $\{z_j = x_j + iy_j\}_{j=0}^{n}$ on $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ is determined by

$$\eta_{\mathbf{w}} = \frac{\sum_{i=0}^{n}(x_i dy_i - y_i dx_i)}{\sum_{i=0}^{n} w_i(x_i^2 + y_i^2)}, \qquad \xi_{\mathbf{w}} = \sum_{i=0}^{n} w_i(x_i\partial_{y_i} - y_i\partial_{x_i}),$$

and the standard Sasakian structure (ξ, η, Φ, g) on S^{2n+1}. The embedding $L_f \hookrightarrow S^{2n+1}$ induces a Sasakian structure on L_f [BG3].

Given a sequence $\mathbf{w} = (w_0, \ldots, w_n)$ of ordered positive integers one can form the graded polynomial ring $S(\mathbf{w}) = \mathbb{C}[z_0, \ldots, z_n]$, where z_i has grading or *weight* w_i. The weighted projective space [Dol, Fle] $\mathbb{P}(\mathbf{w}) = \mathbb{P}(w_0, \ldots, w_n)$ is defined to be the scheme $\mathrm{Proj}(S(\mathbf{w}))$. It is the quotient space $(\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*(\mathbf{w})$, where $\mathbb{C}^*(\mathbf{w})$ is the weighted action defined in 2.1, or equivalently, $\mathbb{P}(\mathbf{w})$ is the quotient of the weighted Sasakian sphere $S^{2n+1}_{\mathbf{w}} = (S^{2n+1}, \xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ by the weighted circle action $S^1(\mathbf{w})$ generated by $\xi_{\mathbf{w}}$. As such $\mathbb{P}(\mathbf{w})$ is also a compact complex orbifold with an induced Kähler structure. We have from [BG3]

THEOREM 2.3: *The quadruple $(\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ gives L_f a quasi-regular Sasakian structure such that there is a commutative diagram*

$$\begin{array}{ccc} L_f & \longrightarrow & S^{2n+1}_{\mathbf{w}} \\ \downarrow \pi & & \downarrow \\ \mathcal{Z}_f & \longrightarrow & \mathbb{P}(\mathbf{w}), \end{array}$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are principal S^1 V-bundles and orbifold Riemannian submersions. Moreover, if $\mathcal{Z}_f$ is Fano, L_f is the total space of the principal S^1 V-bundle over the orbifold $\mathcal{Z}_f$ whose first Chern class in $H^2_{orb}(\mathcal{Z}_f, \mathbb{Z})$ is $c_1(\mathcal{Z}_f)/I$, where I is the index.

We should also mention that $c_1(\mathcal{Z}_f)$ pulls back to the basic first Chern class $c_1^B \in H^2_B(\mathcal{F}_{\xi_{\mathbf{w}}})$ and $\eta_{\mathbf{w}}$ is the connection in this V-bundle whose curvature is $d\eta = \frac{2ni}{I}\pi^*\omega_{\mathbf{w}}$, where $\omega_{\mathbf{w}}$ is the Kähler form on $\mathcal{Z}_f$.

Now conditions on the weights that guarantee that the hypersurface $C_f \subset \mathbb{C}^{n+1}$ have only an isolated singularity at the origin are well-known [Fle,JK1]. These conditions become more complicated as the dimension increases [Fle,JK2]; however, in this paper we are only interested in the $n = 4$ case of hypersurfaces in a weighted complex projective 4-space. These conditions, known as *quasi-smoothness* conditions guarantee that $\mathcal{Z}_f$ is smooth in the orbifold sense, that is, at a vertex $P_i \in \mathbb{P}(\mathbf{w})$ the preimage of $\mathcal{Z}_f$ in the orbifold chart of $\mathbb{P}(\mathbf{w})$ is smooth. It is easy to see that one can formulate all these conditions as follows [Fle,JK2]:

Quasi-Smoothness Conditions 2.4:

I. For each $i = 0, \cdots, 4$ there is a j and a monomial $z_i^{m_i} z_j \in \mathcal{O}(d)$.
Here $j = i$ is possible.

II. For all distinct i, j either there is a monomial $z_i^{b_i} z_j^{b_j} \in \mathcal{O}(d)$.
or there exist monomials $z_i^{n_1} z_j^{m_1} z_k, z_i^{n_2} z_j^{m_2} z_l \in \mathcal{O}(d)$ with $\{k, l\} \neq \{i, j\}$ and $k \neq l$.

III. For every i, j there exists a monomial of degree d that does not involve either z_i or z_j.

There is another condition apart from quasi-smoothness that assures us that the adjunction theory behaves correctly, and that $\mathbb{P}(\mathbf{w})$ does not have any orbifold singularities of codimension 1. It is [Dol,Fle]

Well-formedness Condition 2.5

IV. For each i we have $\gcd(w_0, \cdots, \hat{w}_i, \cdots, w_4) = 1$.
Here the ˆ means skip that element.

Condition IV guarantees that the canonical V-bundle $K_{\mathcal{Z}}$ is determined in terms of the degree and index by

2.6 $$K_{\mathcal{Z}} \simeq \mathcal{O}(-I) = \mathcal{O}(d - |\mathbf{w}|),$$

where $|\mathbf{w}| = \sum_i w_i$.

In this note we shall only consider the anticanonically embedded Fano 3-folds of [JK2], that is, we shall assume hereafter that $I = |\mathbf{w}| - d = 1$. The examples we consider are from the list sporadic.txt of Johnson and Kollár [JK2] which is found at:

http://www.math.princeton.edu/˜ jmjohnso.

3. The Topology of the Link $M^7_{\mathbf{w},d}$

The topology of a link L_f of an isolated hypersurface singularity is encoded in the characteristic polynomial $\Delta(t)$ of the monodromy map. $\Delta(t)$ is an important link invariant that generalizes the Alexander polynomial of a knot, and is often called the "Alexander polynomial" of the link [HZ]. Let us recall the well-known construction of Milnor [Mil] concerning isolated hypersurface singularities: There is a fibration of $(S^{2n+1} - L_f) \longrightarrow S^1$ whose fiber F is an open manifold that is homotopy equivalent to a bouquet of n-spheres $S^n \vee S^n \cdots \vee S^n$. The *Milnor number* μ of L_f is the number of S^n's in the bouquet. It is an invariant of the link which can be calculated explicitly in terms of the degree d and weights $(w_0, \ldots, w_n)$ by the formula [MO]

3.1 $$\mu = \mu(L_f) = \prod_{i=0}^{n} \left(\frac{d}{w_i} - 1\right).$$

The closure $\bar{F}$ of F has the same homotopy type as F and is a compact manifold whose boundary is precisely the link L_f. So the reduced homology of F and $\bar{F}$ is only non-zero in dimension n and $H_n(F,\mathbb{Z}) \approx \mathbb{Z}^\mu$. Using the Wang sequence of the Milnor fibration together with Alexander-Poincare duality gives the exact sequence [Mil]

$$0 \longrightarrow H_n(L_f,\mathbb{Z}) \longrightarrow H_n(F,\mathbb{Z}) \xrightarrow{\mathbb{I}-h_*} H_n(F,\mathbb{Z}) \longrightarrow H_{n-1}(L_f,\mathbb{Z}) \longrightarrow 0, \tag{3.2}$$

where h_* is the *monodromy* map (or characteristic map) induced by the $S^1_{\mathbf{w}}$ action. From this we see that $H_n(L_f,\mathbb{Z}) = \ker(\mathbb{I}-h_*)$ is a free Abelian group, and $H_{n-1}(L_f,\mathbb{Z}) = \mathrm{coker}(\mathbb{I}-h_*)$ which in general has torsion, but whose free part equals $\ker(\mathbb{I}-h_*)$. There is a well-known algorithm due to Milnor and Orlik [MO] for computing the free part of $H_{n-1}(L_f,\mathbb{Z})$ in terms of the characteristic polynomial $\Delta(t) = \det(t\mathbb{I}-h_*)$ of the monodromy map. The Betti number $b_n(L_f) = b_{n-1}(L_f)$ equals the number of factors of $(t-1)$ in $\Delta(t)$. Generally, finding the torsion is much more difficult. However, in the case of rational homology spheres, $b_n(L_f) = b_{n-1}(L_f) = 0$, the group $H_{n-1}(M,\mathbb{Z})$ is a torsion group of order $\Delta(1)$.

It is not our purpose in this note to give a systematic study of the Johnson-Kollár list. This requires a computer program for computing the Betti numbers which is currently under study. Here we are content with giving algorithm for finding special cases when rational homology spheres occur. We have written a MAPLE program which allows us to search the JK list, sporatic.txt and pick out certain rational homology spheres. We emphasize that this procedure does not necessarily find all rational homology spheres, but only all of those that satisfy certain additional conditions. Actually there are two distinct types of conditions on the weights that allow us to find rational homology spheres and they are described in the lemmas below. The first and simplest is that the weights are all relatively prime to the degree.

Lemma 3.4: *Let $\mathbf{w} = (w_0,w_1,w_2,w_3,w_4)$ be the weights of a quasi-smooth Fano 3-fold, $\mathcal{Z}_f$ of degree d and index 1. Suppose further that $\gcd(w_i,d) = 1$ for all $i = 0,\cdots,4$. Then there exists an integer $N(\mathbf{w})$ such that the Alexander polynomial $\Delta(t)$ of the link $M^7_{\mathbf{w},d}$ has the form*

$$\Delta(t) = \frac{(t^d-1)^{N(\mathbf{w})}}{t-1}$$

Hence, the Betti number b_3 of the link $M^7_{\mathbf{w},d}$ is given by $b_3(M^7_{\mathbf{w},d}) = N(\mathbf{w})-1$.

Proof: The Milnor and Orlik [MO] algorithm for computing the characteristic polynomial of the monodromy operator for weighted homogeneous polynomials is as follows: First associate to any monic polynomial F with roots $\alpha_1,\ldots,\alpha_k \in \mathbb{C}^*$ its divisor

$$\mathrm{div}\ F = <\alpha_1> + \cdots + <\alpha_k>$$

as an element of the integral ring $\mathbb{Z}[\mathbb{C}^*]$ and let $\Lambda_n = \mathrm{div}\ (t^n-1)$. The rational weights w'_i used in [MO] are related to our integer weights w_i by $w'_i = \frac{d}{w_i}$, and we write the $w'_i = \frac{u_i}{v_i}$ in irreducible form. The divisor div Δ is given by

$$\mathrm{div}\ \Delta = \Big(\frac{\Lambda_{u_0}}{v_0}-1\Big)\cdots\Big(\frac{\Lambda_{u_4}}{v_4}-1\Big) \tag{3.6}$$

which can be reduced to the form

$$\mathrm{div}\ \Delta(t) = \sum_j a_j\Lambda_j - 1 \tag{3.7}$$

for some integers a_j upon using the relations $\Lambda_a\Lambda_b = \gcd(a,b)\Lambda_{lcm(a,b)}$. The characteristic polynomial $\Delta(t)$ is then determined from its divisor by

3.8 $$\Delta(t) = \frac{\prod(t^j-1)^{a_j}}{t-1},$$

and the third Betti number is given by

$$b_3(M^7_{\mathbf{w},d}) = \sum_j a_j - 1.$$

In our case we have $\gcd(w_i,d)=1$ so equation 3.7 must take the form

3.9 $$\mathrm{div}\ \Delta(t) = N(\mathbf{w})\Lambda_d - 1$$

where $N(\mathbf{w})$ is an integer. ∎

The integer $N(\mathbf{w})$ can be computed explicitly from the above procedure in terms of the weights and index. We find

3.10 $$N(\mathbf{w}) = \frac{d(dr_{01}r_{23}+r_{01}+r_{23})}{w_4} + \frac{1}{w_4} - (dr_{01}r_{23}+r_{01}+r_{23})$$

where

$$d = |\mathbf{w}|-1, \qquad r_{ij} = \frac{d}{w_iw_j} - \frac{1}{w_i} - \frac{1}{w_j}.$$

We should remark here that although it is far from manifest in equation 3.10, under the hypothesis of Lemma 3.4 the function $N(\mathbf{w})$ is invariant under a permutation of the weights, i.e. if Σ_5 denotes the permutation group on 5 letters, then $N(\sigma(\mathbf{w})) = N(\mathbf{w})$ for any $\sigma \in \Sigma_5$. We have an immediate

COROLLARY 3.11: *Let $M^7_{\mathbf{w},d}$ be the link of an isolated hypersurface singularity defined by a weighted homogeneous polynomial f with well-formed weights $\mathbf{w} = (w_0,w_1,w_2,w_3,w_4)$ and degree d which satisfy $\gcd(w_i,d)=1$ for all $i=0,\cdots,4$. Then $M^7_{\mathbf{w},d}$ is a rational homology sphere if and only if $N(\mathbf{w})=1$. Furthermore, in this case the Milnor number $\mu = d-1$, and the order of $H_3(M^7_{\mathbf{w},d},\mathbb{Z})$ equals the degree d.*

PROOF: The only part that we need to compute is the order of H_3. Since for a 2-connected rational homology sphere $H_4(M^7_{\mathbf{w},d},\mathbb{Z}) = 0$, the exact sequence 3.2 shows [Mil] that the order of $H_3(M^7_{\mathbf{w},d},\mathbb{Z})$ equals $\Delta(1)$. But from 3.8 and 3.9 we see that in our special case the characteristic polynomial takes the form

$$\Delta(t) = \frac{t^d-1}{t-1} = t^{d-1}+\cdots+t+1$$

from which the result follows. ∎

We now describe the second type of condition.

Lemma 3.12: *Let $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ be the weights of a quasi-smooth Fano 3-fold, $\mathcal{Z}_f$ of degree d and index 1. Suppose further that the degree can be written as $d = m_3 m_2$, where m_2 and m_3 are relatively prime, and that the "rational weights" $\frac{d}{w_i}$ take the form $\frac{m_3}{v_i}$ for 3 values of i and $\frac{m_2}{v_i}$ for the other 2 values of i. Then there exist positive integers $l = l(\mathbf{w})$, and $n = n(\mathbf{w})$, depending on the weights $\mathbf{w}$, such that the Alexander polynomial $\Delta(t)$ of the link $M^7_{\mathbf{w},d}$ takes the form*

$$\Delta(t) = \frac{(t^d-1)^{ln}(t^{m_3}-1)^l}{(t-1)(t^{m_2}-1)^n}.$$

Hence,

$$b_3(M^7_{\mathbf{w},d}) = (n(\mathbf{w})+1)(l(\mathbf{w})-1).$$

Proof: Computing as in the proof of Lemma 3.4, we see that from the Milnor-Orlik procedure [MO] that the divisor of the Alexander polynomial must take the form

$$\text{div}\ \Delta(t) = l(\mathbf{w})n(\mathbf{w})\Lambda_d + l(\mathbf{w})\Lambda_{m_3} - n(\mathbf{w})\Lambda_{m_2} - 1 \tag{3.13}$$

for some positive integers $l(\mathbf{w})$ and $n(\mathbf{w})$ depending on the weights. The above form of the Alexander polynomial then follows from equations 3.7 and 3.8. The explicit form of the functions $l(\mathbf{w})$ and $n(\mathbf{w})$ are also easily calculated. Let i_1, i_2, i_3 denote the 3 indices whose rational weights take the form $\frac{m_3}{v_i}$ and similarly let j_1, j_2 denote the indices corresponding to the rational weights $\frac{m_2}{v_j}$. Then one finds

$$l(\mathbf{w}) = \frac{m_3^2}{v_{i_1}v_{i_2}v_{i_3}} - m_3\Big(\frac{1}{v_{i_1}v_{i_2}} + \frac{1}{v_{i_1}v_{i_3}} + \frac{1}{v_{i_2}v_{i_3}}\Big) + \frac{1}{v_{i_1}} + \frac{1}{v_{i_2}} + \frac{1}{v_{i_3}} \tag{3.14}$$

$$n(\mathbf{w}) = \frac{m_2}{v_{j_1}v_{j_2}} - \frac{1}{v_{j_1}} - \frac{1}{v_{j_2}} \tag{3.15}$$

The expression for b_3 follows directly from the expression for $\Delta(t)$. ∎

Corollary 3.16: *Let $\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$ be the weights of a quasi-smooth Fano 3-fold, $\mathcal{Z}_f$ of degree d and index 1. Suppose further that the hypothesis of Lemma 3.12 is satisfied, then $M^7_{\mathbf{w},d}$ is a rational homology sphere if and only if $l(\mathbf{w}) = 1$. Furthermore, in this case the Milnor number $\mu = (m_3-1)(n(\mathbf{w})m_2+1)$, and the order of $H_3(M^7_{\mathbf{w},d}, \mathbb{Z})$ is $m_3^{n(\mathbf{w})+1}$.*

Proof: Cancel the $t-1$ factors in $\Delta(t)$ and evaluate at $t = 1$.

Remark 3.17: One can also write $d = m_4 m_1$ or $d = m_{2,1}m_{2,2}m_1$ where in each case the m_s are pairwise relatively prime positive integers. Also in the first case $d/w_i = m_4/v_i$ for 4 values of i and $d/w_j = m_1/v_j$ for the remaining index. In the second case $d/w_i = m_{2,1}/v_i$ and $d/w_j = m_{2,2}/v_j$ for two pairs of index and $d/w_k = m_1/v_k$ for the remaining index. In both cases one finds rational homology spheres without any further conditions; however, one can also easily show that the weights are not well-formed in either case.

Table: $\mathbb{Q}$-Homology 7-Spheres $M^7_{\mathbf{w},d}$ admitting S-E Structures			
$\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$	d	μ	Order of $H_3(M^7_{\mathbf{w},d}, \mathbb{Z})$
$(17, 34, 75, 125, 175)$	425	4416	$582622237229761 = 17^{12}$
$(17, 238, 381, 635, 889)$	2159	16272	$118587876497 = 17^9$
$(19, 57, 100, 125, 175)$	475	3168	$16983563041 = 19^8$
$(49, 334, 525, 668, 763)$	2338	4995	$37259704 = 2^3 \cdot 167^3$
$(49, 573, 1862, 2483, 4393)$	9359	36720	$282475249 = 7^{10}$
$(50, 65, 73, 73, 105)$	365	1152	$28398241 = 73^4$
$(52, 127, 381, 533, 559)$	1651	5040	$260144641 = 127^4$
$(55, 160, 373, 373, 905)$	1865	5932	$19356878641 = 373^4$
$(87, 558, 687, 1331, 1331)$	3993	4320	$1771561 = 11^6$
$(93, 459, 780, 1331, 1331)$	3993	4320	$1771561 = 11^6$
$(93, 1011, 2298, 3401, 3401)$	10203	13600	$11566801 = 19^2 \cdot 179^2$
$(97, 1531, 2201, 2775, 3253)$	9856	9855	$9856 = 2^7 \cdot 7 \cdot 11$
$(99, 318, 465, 881, 881)$	2343	3520	$776161 = 881^2$
$(101, 439, 559, 579, 619)$	2296	2295	$2296 = 2^3 \cdot 7 \cdot 41$
$(101, 1597, 1996, 2695, 3693)$	10081	10080	$10081 = 17 \cdot 593$
$(101, 1597, 2096, 2495, 3793)$	10081	10080	$10081 = 17 \cdot 593$
$(101, 1697, 2296, 2695, 4093)$	10881	10880	$10881 = 3^3 \cdot 13 \cdot 31$
$(103, 1321, 2337, 2845, 3251)$	9856	9855	$9856 = 2^7 \cdot 7 \cdot 11$
$(108, 267, 507, 881, 881)$	2643	3520	$776161 = 881^2$
$(109, 1616, 2047, 2693, 4417)$	10881	10880	$10881 = 3^3 \cdot 13 \cdot 31$
$(111, 329, 407, 423, 470)$	1739	1728	$1369 = 37^2$
$(111, 658, 2303, 3071, 6031)$	12173	24600	$35611289 = 7^3 \cdot 47^3$
$(111, 768, 2523, 3401, 3401)$	10203	13600	$11566801 = 19^2 \cdot 179^2$
$(113, 1115, 6021, 8362, 9589)$	25199	50064	$1142897 = 113^3$
$(113, 1561, 3345, 8362, 11819)$	25199	50064	$1142897 = 113^3$
$(115, 341, 523, 591, 727)$	2296	2295	$2296 = 2^3 \cdot 7 \cdot 41$
$(115, 797, 949, 987, 2050)$	4897	4896	$4897 = 59 \cdot 83$
$(125, 1732, 4577, 5567, 6433)$	18433	18432	18433
$(125, 2599, 4208, 5569, 9901)$	22401	22400	$22401 = 3^2 \cdot 19 \cdot 131$
$(127, 1888, 2643, 4657, 6671)$	15985	15984	$15985 = 5 \cdot 23 \cdot 139$
$(127, 2266, 3651, 6043, 8435)$	20521	20520	20521
$(127, 2392, 3399, 6043, 8561)$	20521	20520	20521
$(127, 2770, 4407, 7429, 10325)$	25057	25056	25057
$(129, 511, 1192, 1235, 1831)$	4897	4896	$4897 = 59 \cdot 83$
$(133, 346, 379, 527, 857)$	2241	2240	$2241 = 3^3 \cdot 83$
$(136, 337, 421, 455, 893)$	2241	2240	$2241 = 3^3 \cdot 83$
$(136, 2023, 3237, 5395, 7553)$	18343	17280	$289 = 17^2$
$(137, 1223, 1427, 2786, 4349)$	9921	9920	$9921 = 3 \cdot 3307$
$(137, 1495, 1699, 3466, 5301)$	12097	12096	12097
$(138, 171, 393, 701, 701)$	2103	2800	$149401 = 701^2$
$(139, 2343, 3721, 6202, 10061)$	22465	22464	$22465 = 5 \cdot 4493$

$\mathbb{Q}$-Homology 7-Spheres $M^7_{\mathbf{w},d}$ admitting S-E Structures (cont.)			
$\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$	d	μ	Order of $H_3(M^7_{\mathbf{w},d}, \mathbb{Z})$
$(139, 3171, 5101, 8548, 13787)$	30745	30744	$30745 = 5 \cdot 11 \cdot 13 \cdot 43$
$(141, 652, 1909, 2375, 2701)$	7777	7776	$7777 = 7 \cdot 11 \cdot 101$
$(141, 1259, 1492, 3031, 4663)$	10585	10584	$10585 = 5 \cdot 29 \cdot 73$
$(141, 2224, 3475, 4031, 9729)$	19599	19600	$19881 = 3^2 \cdot 47^2$
$(141, 2363, 3058, 4309, 9729)$	19599	19600	$19881 = 3^2 \cdot 47^2$
$(143, 194, 209, 231, 291)$	1067	1152	$9409 = 97^2$
$(143, 1135, 2057, 3476, 5675)$	12485	13608	$1288225 = 5^2 \cdot 227^2$
$(143, 2981, 6530, 9795, 16467)$	35915	39168	$10660225 = 5^2 \cdot 363^2$
$(145, 2157, 3451, 5752, 9347)$	20851	20160	$841 = 29^2$
$(146, 869, 1955, 2969, 3983)$	9921	9920	$9921 = 3 \cdot 3307$
$(147, 207, 230, 245, 299)$	1127	1152	$2401 = 7^4$
$(147, 255, 1056, 1457, 1457)$	4371	5824	$2122849 = 31^2 \cdot 47^2$
$(154, 535, 739, 1427, 2115)$	4969	4968	4969
$(154, 763, 1297, 2975, 3891)$	9079	10368	$1682209 = 1297^2$
$(155, 921, 1612, 2149, 4681)$	9517	9792	$94249 = 307^2$
$(155, 1075, 3532, 5835, 7064)$	17660	21186	$12475024 = 3532^2$
$(155, 2309, 7543, 10006, 12469)$	32481	32480	$32481 = 3^4 \cdot 401$
$(155, 2617, 8467, 11392, 14163)$	36793	36792	36793
$(157, 269, 637, 665, 1090)$	2817	2816	$2817 = 3^2 \cdot 313$
$(157, 436, 1401, 1775, 1993)$	5761	5760	$5761 = 7 \cdot 823$
$(157, 545, 1051, 1401, 2608)$	5761	5760	$5761 = 7 \cdot 823$
$(157, 883, 1558, 2597, 4311)$	9505	9504	$9505 = 5 \cdot 1901$
$(157, 1195, 2182, 3689, 6027)$	13249	13248	13249
$(157, 2339, 5146, 7641, 12943)$	28225	28224	$28225 = 5^2 \cdot 1129$
$(157, 2651, 5770, 8733, 14659)$	31969	31968	$31969 = 7 \cdot 4567$
$(159, 2365, 3784, 6307, 12455)$	25069	25488	$223729 = 11^2 \cdot 43^2$
$(163, 1939, 2747, 6786, 8887)$	20521	20520	20521
$(163, 1939, 3070, 5171, 10179)$	20521	20520	20521
$(163, 2101, 3394, 5657, 11151)$	22465	22464	$22465 = 5 \cdot 4493$
$(166, 1237, 4371, 5773, 7175)$	18721	18720	$18721 = 97 \cdot 193$
$(166, 1399, 2551, 5513, 7077)$	16705	16704	$16705 = 5 \cdot 13 \cdot 257$
$(167, 717, 1324, 1765, 3805)$	7777	7776	$7777 = 7 \cdot 11 \cdot 101$
$(169, 2015, 6549, 8732, 10915)$	28379	26208	$169 = 13^2$
$(170, 1267, 3041, 4477, 7687)$	16641	16640	$16641 = 3^2 \cdot 43^2$
$(171, 247, 556, 695, 973)$	2641	2520	$361 = 19^2$
$(175, 271, 299, 306, 751)$	1801	1800	1801
$(175, 289, 925, 2312, 3525)$	7225	14688	$180124137569 = 17^6$
$(175, 2434, 10605, 13387, 15995)$	42595	43776	$1481089 = 1217^2$
$(176, 295, 317, 481, 973)$	2241	2240	$2241 = 3^3 \cdot 83$
$(176, 1135, 1397, 4103, 5675)$	12485	13608	$1288225 = 5^2 \cdot 227^2$
$(177, 997, 3695, 5044, 6217)$	16129	16128	$16129 = 3^6$

$\mathbb{Q}$-Homology 7-Spheres $M^7_{\mathbf{w},d}$ admitting S-E Structures (cont.)			
$\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$	d	μ	Order of $H_3(M^7_{\mathbf{w},d}, \mathbb{Z})$
$(177, 2275, 5950, 7175, 15399)$	30975	30976	$31329 = 3^2 \cdot 59^2$
$(178, 883, 2031, 3973, 5033)$	12097	12096	12097
$(181, 1613, 4301, 5197, 11110)$	22401	22400	$22401 = 3^2 \cdot 19 \cdot 131$
$(181, 1618, 7011, 8809, 10607)$	28225	28224	$28225 = 5^2 \cdot 1129$
$(181, 2338, 10251, 12949, 15467)$	41185	41184	$41185 = 5 \cdot 8237 \cdot 131$
$(183, 787, 1453, 2422, 4661)$	9505	9504	$9505 = 5 \cdot 1901$
$(183, 1031, 2608, 4003, 6793)$	14617	14616	$14617 = 47 \cdot 311$
$(185, 477, 991, 1321, 2788)$	5761	5760	$5761 = 7 \cdot 823$
$(187, 409, 539, 1320, 2045)$	4499	4896	$167281 = 409^2$
$(187, 631, 1299, 2746, 3563)$	8425	8424	$8425 = 5^2 \cdot 337$
$(187, 781, 2306, 3459, 5951)$	12683	13824	$1329409 = 1153^2$
$(187, 1853, 2594, 6485, 10931)$	22049	23328	$1682209 = 1297^2$
$(187, 2416, 8177, 10965, 19328)$	41072	43470	$5837056 = 2^8 \cdot 151^2$
$(191, 235, 433, 509, 904)$	2241	2240	$2241 = 3^3 \cdot 83$
$(191, 2467, 7990, 10627, 18787)$	40041	40040	$40041 = 3^3 \cdot 1483$
$(191, 3607, 11770, 15757, 27717)$	59041	59040	$59041 = 17 \cdot 23 \cdot 151$
$(193, 3247, 4202, 10877, 18335)$	36863	36864	$37249 = 193^2$
$(194, 539, 1155, 2425, 3157)$	7469	7488	$9409 = 97^2$
$(194, 693, 847, 2425, 3311)$	7469	7488	$9409 = 97^2$
$(195, 484, 613, 1291, 2387)$	4969	4968	4969
$(196, 681, 827, 2383, 3259)$	7345	7344	$7345 = 5 \cdot 13 \cdot 113$
$(196, 1119, 1411, 4135, 5741)$	12601	12600	12601
$(196, 2337, 7595, 10127, 17917)$	38171	37440	$2401 = 7^4$
$(197, 881, 4111, 5188, 6265)$	16641	16640	$16641 = 3^2 \cdot 43^2$
$(197, 1273, 6071, 7736, 9205)$	24481	24480	24481
$(199, 305, 377, 628, 1309)$	2817	2816	$2817 = 3^2 \cdot 313$
$(199, 376, 831, 1603, 2177)$	5185	5184	$5185 = 5 \cdot 17 \cdot 61$
$(199, 673, 2811, 3880, 4751)$	12313	12312	$12313 = 7 \cdot 1759$
$(199, 1973, 3157, 7300, 12429)$	25057	25056	25057
$(203, 1409, 1912, 4931, 8251)$	16705	16704	$16705 = 5 \cdot 13 \cdot 257$
$(205, 389, 1093, 1389, 2686)$	5761	5760	$5761 = 7 \cdot 823$
$(205, 1629, 7127, 10588, 12421)$	31969	31968	$31969 = 7 \cdot 4567$
$(206, 247, 259, 319, 771)$	1801	1800	1801
$(206, 1331, 4607, 6143, 10955)$	23241	23240	$23241 = 3 \cdot 61 \cdot 127$
$(207, 2468, 8021, 10695, 21183)$	42573	43120	$380689 = 617^2$
$(208, 439, 1007, 2091, 2737)$	6481	6480	6481
$(209, 707, 2038, 3161, 5407)$	11521	11520	$11521 = 41 \cdot 281$
$(209, 1351, 5092, 6859, 12159)$	25669	27000	$1825201 = 7^2 \cdot 193^2$
$(211, 339, 436, 985, 1631)$	3601	3600	$3601 = 13 \cdot 277$
$(211, 1886, 6077, 8173, 16135)$	32481	32480	$32481 = 3^4 \cdot 401$
$(211, 2306, 7547, 10063, 19915)$	40041	40040	$40041 = 3^3 \cdot 1483$

$\mathbb{Q}$-Homology 7-Spheres $M^7_{\mathbf{w},d}$ admitting S-E Structures (cont.)			
$\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$	d	μ	Order of $H_3(M^7_{\mathbf{w},d}, \mathbb{Z})$
$(214, 489, 659, 1849, 2551)$	5761	5760	$5761 = 7 \cdot 823$
$(217, 731, 1075, 2752, 4557)$	9331	9504	$47089 = 7^2 \cdot 31^2$
$(217, 817, 946, 2795, 4557)$	9331	9504	$47089 = 7^2 \cdot 31^2$
$(217, 2795, 7310, 13115, 23219)$	46655	46656	$47089 = 7^2 \cdot 31^2$
$(217, 3440, 5375, 14405, 23219)$	46655	46656	$47089 = 7^2 \cdot 31^2$
$(220, 237, 1021, 1367, 1477)$	4321	4320	$4321 = 29 \cdot 149$
$(221, 317, 805, 1073, 2194)$	4609	4608	$4609 = 11 \cdot 419$
$(221, 2416, 5491, 13617, 19328)$	41072	43470	$5837056 = 2^8 \cdot 151^2$
$(223, 256, 563, 1041, 1519)$	3601	3600	$3601 = 13 \cdot 277$
$(223, 2212, 3539, 9511, 15261)$	30745	30744	$30745 = 5 \cdot 11 \cdot 13 \cdot 43$
$(223, 2437, 13071, 18166, 20825)$	54721	54720	54721
$(223, 4211, 9087, 22606, 31915)$	68041	68040	68041
$(226, 2245, 12123, 16837, 19307)$	50737	50400	$12769 = 113^2$
$(226, 3143, 6735, 16837, 23797)$	50737	50400	$12769 = 113^2$
$(227, 721, 856, 2523, 4099)$	8425	8424	$8425 = 5^2 \cdot 337$
$(227, 901, 1051, 3228, 5179)$	10585	10584	$10585 = 5 \cdot 29 \cdot 73$
$(227, 1015, 3496, 4737, 9247)$	18721	18720	$18721 = 97 \cdot 193$
$(227, 1241, 4400, 5867, 11507)$	23241	23240	$23241 = 3 \cdot 61 \cdot 127$
$(229, 2503, 5461, 13652, 19341)$	41185	41184	$41185 = 5 \cdot 8237$
$(235, 323, 334, 891, 1459)$	3241	3240	$3241 = 7 \cdot 463$
$(237, 275, 766, 1021, 2023)$	4321	4320	$4321 = 29 \cdot 149$
$(238, 1301, 7451, 10289, 11827)$	31105	31104	$31105 = 5 \cdot 6221$
$(241, 639, 3275, 4792, 5671)$	14617	14616	$14617 = 47 \cdot 311$
$(242, 385, 409, 1419, 2045)$	4499	4896	$167281 = 409^2$
$(243, 457, 2339, 3280, 3979)$	10297	10296	$10297 = 7 \cdot 1471$
$(245, 2434, 5355, 13387, 21175)$	42595	46776	$1481089 = 1217^2$
$(247, 292, 799, 1583, 2121)$	5041	5040	$5041 = 71^2$
$(247, 1351, 3439, 8474, 12159)$	25669	27000	$1825201 = 1361^2$
$(247, 3190, 6871, 17177, 27237)$	54721	54720	54721
$(250, 275, 393, 917, 1441)$	3275	3168	$625 = 5^4$
$(250, 1367, 3231, 8077, 11557)$	24481	24480	24481
$(251, 561, 592, 1963, 3115)$	6481	6480	6481
$(253, 311, 1096, 1377, 2725)$	5761	5760	$5761 = 7 \cdot 823$
$(253, 1507, 6530, 9795, 17831)$	35915	39168	$10660225 = 5^2 \cdot 653^2$
$(255, 844, 1519, 4135, 6497)$	13249	13248	13249
$(259, 285, 407, 950, 1615)$	3515	3456	$1369 = 37^2$
$(259, 643, 1993, 3536, 6171)$	12601	12600	12601
$(261, 287, 410, 957, 1653)$	3567	3520	$1681 = 41^2$
$(261, 491, 1762, 2773, 4795)$	10081	10080	$10081 = 17 \cdot 593$
$(262, 291, 331, 883, 1475)$	3241	3240	$3241 = 7 \cdot 463$
$(262, 443, 469, 1641, 2371)$	5185	5184	$5185 = 5 \cdot 17 \cdot 61$

$\mathbb{Q}$-Homology 7-Spheres $M^7_{\mathbf{w},d}$ admitting S-E Structures (cont.)			
$\mathbf{w} = (w_0, w_1, w_2, w_3, w_4)$	d	μ	Order of $H_3(M^7_{\mathbf{w},d}, \mathbb{Z})$
$(263, 1699, 3921, 9802, 15421)$	31105	31104	$31105 = 5 \cdot 6221$
$(268, 511, 911, 2599, 3777)$	8065	8064	$8065 = 5 \cdot 1613$
$(271, 377, 673, 1696, 2745)$	5761	5760	$5761 = 7 \cdot 823$
$(275, 729, 2734, 4465, 7927)$	16129	16128	$16129 = 127^2$
$(277, 441, 2591, 3748, 4465)$	11521	11520	$11521 = 41 \cdot 281$
$(299, 325, 1869, 2492, 3115)$	8099	7488	$169 = 13^2$
$(301, 363, 1294, 2257, 3851)$	8065	8064	$8065 = 5 \cdot 1613$
$(301, 3289, 17342, 24219, 44849)$	89999	90000	$90601 = 7^2 \cdot 43^2$
$(301, 3887, 12259, 28704, 44849)$	89999	90000	$90601 = 7^2 \cdot 43^2$
$(307, 2441, 13120, 18307, 33867)$	68041	68040	68041
$(309, 349, 2279, 3244, 3901)$	10081	10080	$10081 = 17 \cdot 593$
$(311, 495, 2104, 3403, 6001)$	12313	12312	$12313 = 7 \cdot 1759$
$(311, 2473, 8037, 18856, 29365)$	59041	59040	$59041 = 17 \cdot 23 \cdot 151$
$(316, 1727, 9577, 13345, 24648)$	49612	49770	$49856 = 2^4 \cdot 79^2$
$(316, 2041, 6751, 15857, 24648)$	49612	49770	$49856 = 2^4 \cdot 79^2$
$(328, 347, 449, 1571, 2347)$	5041	5040	$5041 = 71^2$
$(339, 383, 1780, 2839, 4957)$	10297	10296	$10297 = 7 \cdot 1471$
$(341, 407, 2306, 3459, 6171)$	12683	13824	$1329409 = 1153^2$
$(356, 387, 2225, 2967, 5547)$	11481	11440	$7921 = 89^2$
$(357, 388, 2231, 2975, 5593)$	11543	11520	$9409 = 97^2$

4. Discussion of the Table

In this section we give a discussion about the representatives listed in the Table. It is easy to notice the existence of *twins* in the Table. These are rational homology 7-spheres with the same degree d, Milnor number μ and order of H_3. Twins often occur as adjacent listings with the same w_0, but this is not always the case as with twins $d = |H_3| = 10881$, $\mu = 10880$ and $w_0 = 101$ and 109, and the twins $d = |H_3| = 7777$ with $w_0 = 141$ and $w_0 = 167$. Twins may also be members of a larger set, such as the *septuplets* with $d = |H_3| = 5761$ and $\mu = 5760$. These have $w_0 = 157, 157, 185, 205, 214, 253, 271$, respectively. Since twins have the same Milnor number, it is tempting to conjecture that twins correspond to homeomorphic or even diffeomorphic links, but we have no proof as of yet. In fact, except for cases where the order of H_3 contains no primes of order larger than one in its prime decomposition, we don't even know that twins have isomorphic H_3's. Notice that the order of H_3 tends to be quite large varying from $169 = 13^2$ to 17^{12} a number over 500 trillion.

Another interesting fact is that of the 184 rational homology 7-spheres listed in the Table, only 10 have even degree, while the remaining 174 have odd degree, and the degree is even if and only if the order of H_3 is even. But even more intriguing is the fact that for all 174 rational homology 7-spheres with odd degree, the order $|H_3| \equiv 1 \bmod(8)$. In [BGN4] we construct positive Sasakian structures on homotopy 9-spheres using the rational homology 7-spheres listed in the Table. There we show that the exotic Kervaire sphere can only occur when the degree of the rational homology sphere is odd.

Also of interest are invariants of the underlying contact, and almost contact structures. The underlying almost contact structures are classified [Sa] by homotopy classes of maps $[M^7, SO(8)/U(4)]$, and Morita [Mo] shows that for Brieskorn spheres this is a function of the Milnor number μ. It seems reasonable that a similar result holds true in our case. There are candidates for this in the table. For example the rational homology 7-spheres with weights $\mathbf{w} = (196, 2337, 7595, 10127, 17917)$ and degree $d = 38171$, and with weights $\mathbf{w} = (147, 207, 230, 245, 299)$ and degree $d = 1127$ both have $|H_3| = 7^4$, so they could be diffeomorphic. But they have very different Milnor numbers, namely, 37440 and 1152, respectively, so they could belong to distinct almost contact structures. Similarly there are 4 rational homology 7-spheres with $|H_3| = 97^2$, two are twins having the same Milnor number, but the other two have different Milnor numbers both different than the Milnor number of the twins. Moreover, twins probably belong to the same underlying almost contact structures, but could possibly belong to distinct contact structures. It appears that nothing is known beyond the standard spheres [Us] about distinct contact structures within the same underlying almost contact structures.

5. Some Comments on Regular Rational Homology Spheres

In this section we discuss some rational homology spheres that are regular, in particular the homogeneous ones. The following result follows easily from previous work [BG1] together with the well-known classification of del Pezzo surfaces:

PROPOSITION 5.1: *Let $\mathcal{S} = (g, \xi, \eta, \Phi)$ be a regular positive Sasakian structure on a smooth compact 5-manifold M^5. Then M^5 is a rational homology sphere if and only if M^5 is covered by S^5 and $\mathcal{S}$ is homologous to the standard Sasakian structure with the round metric g_0.*

It is well-known that the standard Sasakian structure is a homogeneous Sasakian-Einstein structure. Dimension seven is a bit more interesting:

THEOREM 5.2: *Let $\mathcal{S} = (g, \xi, \eta, \Phi)$ be a regular positive Sasakian structure on a smooth compact 7-manifold M^7. Then M^7 is a rational homology sphere if and only if it is one of the following:*

1. *$M^7 = S^7$ and $\mathcal{S}$ is homologous to the standard Sasakian structure with the round metric.*
2. *$M^7 = V_2(\mathbb{R}^5)$ the Stiefel manifold of 2-frames in $\mathbb{R}^5$ and $\mathcal{S}$ is homologous to the standard homogeneous Sasakian-Einstein structure on $V_2(\mathbb{R}^5)$ [BG1,BG2].*
3. *M^7 is a circle bundle over a smooth variety V_5 of degree 5 in $\mathbb{CP}^6$ with a compatible Sasakian structure $\mathcal{S}$.*
4. *M^7 is a circle bundle over a smooth variety V_{22} of degree 22 in $\mathbb{CP}^{13}$ with a compatible Sasakian structure $\mathcal{S}$.*

Furthermore, M^7 admits a homogeneous Sasakian-Einstein structure if and only if $M^7 = S^7$ or $V_2(\mathbb{R}^5)$.

PROOF: By [BG1] M^7 is a regular rational homology sphere with a Sasakian structure $\mathcal{S}$ if and only if it is the total space of an S^1 bundle over a smooth projective 3-fold $\mathcal{Z}$ with the same rational homology groups as projective space $\mathbb{CP}^3$. Furthermore, $\mathcal{S}$ is positive [BGN3] if and only if $\mathcal{Z}$ is Fano. Thus, $\mathcal{Z}$ must occur on Iskovskikh's list [Isk] of smooth Fano 3-folds of the first kind, and there are precisely four which have the same rational cohomology groups as $\mathbb{CP}^3$. This gives the four cases above. The last statement follows from Corollary 4.1.3 of [BG2]. ∎

We do not know whether cases 3 and 4 in Theorem 5.2 admit Sasakian-Einstein structures.

There is a straightforward procedure for finding all rational homology spheres M^{2n+1} that admit a homogeneous Sasakian-Einstein structure. By Theorem 3.2 of [BG1] M^{2n+1} must fibre over a generalized flag manifold G/P, where G is a complex semi-simple Lie group, and P is a parabolic subgroup. In order that M^{2n+1} be a rational homology sphere, it is necessary that G/P have the rational homology of a projective space. Hence, we may restrict ourselves to the case where G is simple and P is maximal parabolic. The procedure for computing the cohomology ring of G/P is outlined in Baston and Eastwood [BE]. All G/P's with G simple are realized by crossing out nodes in each Dynkin diagram of G. When P is maximal parabolic only one node is crossed out. The rank of the cohomology groups is determined by the Hasse diagram $W^{\mathfrak{p}}$ which is the coset space $W_{\mathfrak{g}}/W_{\mathfrak{p}}$ where $W_{\mathfrak{g}}$ is the Weyl group of the Lie algebra $\mathfrak{g}$ of G, and $W_{\mathfrak{p}}$ is the Weyl group of the Levi factor of the Lie algebra $\mathfrak{p}$ of P. Then the cohomology groups of G/P will have the same rank as $\mathbb{CP}^n$ if and only if $W^{\mathfrak{p}}$ has precisely one element of length l for each $l = 1, \cdots, n$. One then needs to check all maximal parabolics for all Dynkin diagrams, and compute the Hasse diagram for each case. There are many cases and repetitions can and do occur. Here we mention the Stiefel manifolds $V_2(\mathbb{R}^{2n+1})$ which are circle bundles over the odd quadrics Q_{2n-1} and the homogeneous 3-Sasakian rational homology sphere $G_2/Sp(1)_+$ (cf. [BGP] and Remark 5.6(2) below). A Gysin sequence or spectral sequence argument shows that

5.3
$$H^p(V_2(\mathbb{R}^{2n+1}), \mathbb{Z}) \approx \begin{cases} \mathbb{Z} & \text{if } p = 0, 4n-1; \\ \mathbb{Z}_2 & \text{if } p = 2n; \\ 0 & \text{otherwise.} \end{cases}$$

5.4
$$H^p(G_2/Sp(1)_+, \mathbb{Z}) \approx \begin{cases} \mathbb{Z} & \text{if } p = 0, 11; \\ \mathbb{Z}_3 & \text{if } p = 4, 8; \\ 0 & \text{otherwise.} \end{cases}$$

We have

PROPOSITION 5.5: *The Stiefel manifold $V_2(\mathbb{R}^{2n+1})$ and $G_2/Sp(1)_+$ are simply connected rational homology spheres which admit homogeneous Sasakian-Einstein structures.*

REMARKS 5.6: (1) Since $V_2(\mathbb{R}^{2n+1})$ can be represented as the link of the quadric hypersurface singularity, 5.3 can be derived from the Milnor-Orlik algorithm described in section 3. (2) There are two non-conjugate $Sp(1)$ subgroups of the exceptional Lie group G_2, denoted in [BGP] as $Sp(1)_\pm$. The quotient $G_2/Sp(1)_+$ has a homogeneous 3-Sasakian structure, whereas the quotient $G_2/Sp(1)_-$ does not. It does, however, have a homogeneous Sasakian-Einstein structure, and as homogeneous Sasakian-Einstein manifolds $G_2/Sp(1)_- \approx V_2(\mathbb{R}^7)$.

There is an obvious corollary of Theorem 4.2.6 and Proposition 5.4.4 of [BG2], viz.

COROLLARY 5.7: *Let M^{4n+3} be a rational homology sphere that admits a 3-Sasakian homogeneous structure. Then M^{4n+3} is either $S^{4n+3}, \mathbb{RP}^{4n+3}$, or $G_2/Sp(1)_+$.*

Appendix

In this appendix we give a proof of the condition on the degree and weights of a weighted homogeneous hypersurface that guarentees the existence of a Kähler-Einstein metric. The condition is stated in [JK2] without proof. Here we give a generalization to the case of arbitrary orbifold Fano index I. We refer to our previous work [BGN1] and the book [KM] for background material.

Suppose $X \subset \mathbb{P}(\mathbf{w}) = \mathbb{P}(w_0, \ldots, w_n)$ is an irreducible hypersurface in weighted projective space. We assume that X has at worst quotient singularities (at singular points of $\mathbb{P}(\mathbf{w})$) and that X is Fano. We would like to study under what conditions (X, D) is klt whenever $D \equiv -\alpha K_X$ where $\alpha > \frac{n-1}{n}$ and D is effective. Under these circumstances X admits a Kähler-Einstein metric of positive scalar curvature [DM,JK1]. We shall prove

THEOREM A.1: *Suppose $X \subset \mathbb{P}(\mathbf{w}) = \mathbb{P}(w_0, \ldots, w_n)$ is a normal Fano variety of index I satisfying*

$$\deg(X) < \frac{nw_0w_1}{(n-1)I}.$$

Then there exists $\epsilon > 0$ such that (X, D) is klt for every effective $D \equiv -\frac{n-1+\epsilon}{n} K_X$.

PROOF: By induction on dimension, which reduces to the surface case already handled in [JK1] and [BGN1]. Suppose for a moment that X is a surface with at worst quotient singularities. Johnson and Kollar [JK1] give sufficient conditions for (X, D) to be klt:

1. If $D = \sum_{i=1}^r \alpha_i C_i$ then $\alpha_i < 1$ for all i,
2. for all smooth points $x \in X$, $\mathrm{mult}_x(D) \leq 1$,
3. if $P \subset X$ is a singular point and $\pi : Y \to X$ a local finite cover resolving the singularity at P then $\pi^* D$ has multiplicity at most one at $Q = \pi^{-1}(P)$.

Condition 1 is also necessary, while neither 2 nor 3 is necessary. We will now formulate analogues of 1,2, and 3 above, for an arbitrary (X, D), designed to guarantee that 1,2, and 3 will hold when we cut X with the appropriate number of hypersurfaces, allowing us to apply the inversion of adjunction formula to conclude that (X, D) is klt.

(i) If $D = \sum_{i=1}^r \alpha_i D_i$ then $\alpha_i < 1$ for all i, and

(ii) for all smooth points $x \in X$, $\mathrm{mult}_x(D) \leq 1$, and

(iii) suppose $x \in X$ is a singular point of X with local group G_x and that $H_1, H_2, \ldots, H_n$ are general hypersurfaces through x; then we ask that the intersection number at x, $i(x, X \cdot H_1 \cdot \ldots \cdot H_n; \mathbb{P}(\mathbf{w}))$ satisfies

$$i(x, X \cdot H_1 \cdot \ldots \cdot H_n; \mathbb{P}(\mathbf{w})) \leq \frac{1}{|G_x|}.$$

Suppose that conditions i, ii, and iii are satisfied by a divisor D on X and suppose $x \in X$. Choose general hypersurfaces $H_1, H_2, \ldots, H_n$ through x and let $V = X \cap H_1 \cap \ldots \cap H_{n-2}$ with $D_V = D \cap V$. Consider the pair (V, D_V). We see that V is a surface and we will show that (V, D_V) satisfies 1,2, and 3 above and hence (V, D_V) is klt at x. Using the inversion of adjunction, (X, D) is klt at x as well.

Write $D_V = \sum b_j C_j$. Since the hypersurfaces H_i are general, we can use Remark 8.2 of [Ful] to see that $b_j < 1$ for all j as each b_j is equal to one of the w_i. Similarly, the multiplicity of D at x will be preserved under intersection by general hyperplanes and so $\mathrm{mult}_x(D_V) \leq 1$ (see [Ful] Corollary 12.4). Finally, under hypothesis iii above, if $\pi : Y \to V$ is a local cover of the quotient singularity at x, we see, using [Ful] 8.3.12, that 3 is satisfied. More precisely, letting $y = \pi^{-1}(x)$,

$$\begin{aligned}\mathrm{mult}_y(\pi^* D_V) &\le \pi^* D_V \cdot \pi^* H_{n-1} \cdot \pi^* H_n \\ &\le |G_x| i(x, D_V \cdot H_{n-1} \cdot H_n; \mathbb{P}(\mathbf{w})) \\ &\le 1,\end{aligned}$$

the last inequality coming from iii. Thus it is sufficient, in order to prove Theorem 0.1, to verify conditions i, ii, and iii above where $D \equiv -\frac{n-1+\epsilon}{n} K_X$ is an effective divisor.

Suppose then that we write

$$D = \sum_{i=1}^{r} \alpha_i D_i.$$

We will show that $\alpha_1 < 1$, the other cases being identical. Suppose $x \in D_1$ is a smooth point of D_1 and X; here we use the fact that X is normal and hence smooth along D_1. To simplify notation in what follows, we let $E = \alpha_1 D_1$. We will assume for simplicity, rearranging the coordinates if necessary, that

$$x = (z_0, \ldots, z_k, 0, \ldots, 0).$$

Consider the hyperplane H_1 given by $z_n = 0$. If $E \subset H_1$ then replace H_1 with H_2, given by $z_{n-1} = 0$ and start over. Assuming that H_1 meets E properly, write

$$H_1 \cdot E = \sum \beta_j V_j.$$

We then repeat the above procedure, replacing E with $H_1 \cdot E$ and intersecting with H_2. We continue up to and including H_{n-k} given by $z_{k+1} = 0$. Next consider the hypersurface D_i for $1 \le i \le k$ defined by

$$c_{i-1} z_{i-1}^{w_i} - c_i z_i^{w_{i-1}}$$

where the c_i are chosen so that $x \in D_i$. The divisors $H_1, \ldots, H_{n-k}, D_1, \ldots, D_k$ cut out the point x set theoretically. Thus we can continue the intersection process above, using the $\mathbf{Q}$-divisors $D_i / \min(w_{i-1}, w_i)$, until we obtain a cycle Z of dimension zero. Bounding the degree of Z by the degree of the total intersection class we have

$$\begin{aligned}\deg(Z) &\le w_n \ldots w_3 \deg(E) \\ &\le w_n \ldots w_3 \deg(D),\end{aligned}$$

since the worst case scenario for the degree of Z occurs when all possible intersections are proper and where no reordering of variables has been necessary; note that the degrees in this formula are relative to $\mathcal{O}_{\mathbb{P}}(\mathbf{w})(1)$. On the other hand since $\mathrm{mult}_x(H_i) \ge 1$ for all i and

$$\frac{\mathrm{mult}_x(D_i)}{\min(w_{i-1}, w_i)} \ge 1,$$

it follows by [Ful] Corollary 12.4 that

$$\deg(Z) \ge \mathrm{mult}_x(E).$$

Hence

$$\mathrm{mult}_x(E) \le w_n \ldots w_3 \deg(D).$$

But

$$\deg(D) = \frac{n-1+\epsilon}{n\prod_{i=0}^{n} w_i} I \deg(X),$$

where $\deg(X)$ denotes the degree of the polynomial defining X. So if we choose ϵ so that

$$\frac{n-1+\epsilon}{n} I \deg(X) < w_0 w_1$$

we see that

$$\alpha_1 \mathrm{mult}_x(D_1) = \mathrm{mult}_x(E) < 1/w_2 \leq 1.$$

Since we have assumed that X and D_1 are smooth at x it follows that $\mathrm{mult}_x(D_1) = 1$ and hence $\alpha_1 < 1$ as desired.

Next we address property ii. The argument is essentially identical to the verification of property i. In particular, let

$$x = (z_0, z_1, \ldots, z_n);$$

again we will assume that $x_0, \ldots, x_k$ are non–zero and that the rest of the coordinates are zero, possibly after reordering the coordinates. Intersecting with the divisors H_i and $\frac{D_j}{w_j}$ as above will show that

$$\mathrm{mult}_x(D) < \frac{1}{w_2} \leq 1.$$

Again, this result holds with the original w_2 since the initial ordering of the w_i's is increasing.

Finally we establish property iii. We begin with a specific example so that the coordinates are simple and the argument transparent. Suppose $x = (0, \ldots, 0, 1) \in X$. Thus $|G_x| \leq w_n$. To see that

$$i(x, X \cdot H_1 \cdot \ldots \cdot H_n; \mathbb{P}(\mathbf{w})) \leq \frac{1}{w_n}$$

we consider the intersection theoretic argument above. In this case, x is cut out by $H_i = \{z_i = 0\}$ for $0 \leq i \leq n-1$. Thus we never need to intersect with $\{z_n = 0\}$ and consequently instead of

$$\deg(Z) \leq w_n \ldots w_3 \deg(D),$$

we will find

$$\deg(Z) \leq w_{n-1} \ldots w_2 \deg(D).$$

Writing $Z = m[x] + Z'$ where Z' is a zero cycle not supported at x, we find, plugging in the value of $\deg(D)$, that

$$m \leq \frac{1}{w_n}.$$

But $i(x, H_1 \cdot \ldots \cdot H_n; \mathbb{P}(\mathbf{w}))$ is the minimal intersection number supported on x which one can obtain but cutting out x and thus

$$i(x, H_1 \cdot \ldots \cdot H_n; \mathbb{P}(\mathbf{w})) \leq \frac{1}{w_n}$$

as desired.

More generally, if P is some other singular point of X, with coordinates $(z_0, \cdots, z_n)$ then some of the coordinates $\{z_i\}$ must be zero. If j is the smallest index such that $z_j \neq 0$ then $|G_x| \leq w_j$. The argument then procedes as in case ii, the w_j in the denominator coming from taking the worst case scenario in computation of intersection product.

Note that the most difficult of the three conditions to satisfy is definitely the third. This is the same as in the surface case but the higher dimensional case will be more difficult to deal with as, in the surface case, condition iii was dealt with using the inversion of the adjunction formula. In higher dimension, this will be more complicated.

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