# A WEIL-BARSOTTI FORMULA FOR DRINFELD MODULES

MATTHEW A. PAPANIKOLAS AND NIRANJAN RAMACHANDRAN

Abstract. We study the group of extensions in the category of Drinfeld modules and Anderson's $t$-modules, and we show in certain cases that this group can itself be given the structure of a $t$-module. Our main result is a Drinfeld module analogue of the Weil-Barsotti formula for abelian varieties. Extensions of general $t$-modules are also considered, in particular extensions of tensor powers of the Carlitz module. We motivate these results from various directions and compare to the situation of elliptic curves.

## 1. Introduction and statement of results

In this paper, we investigate extensions of Drinfeld modules using the well-known analogy between abelian varieties and Drinfeld modules. Associated with an abelian variety $A$ are two other group varieties: (i) the dual abelian variety $A^\vee$, and (ii) the universal vectorial (or additive) extension $A^\natural$.

The Weil-Barsotti formula identifies $\mathrm{Ext}^1(A, \mathbb{G}_m)$ in the category of group schemes with $A^\vee$ [14]. Likewise, $\mathrm{Ext}^1(A^\vee, \mathbb{G}_m) \cong A$. Theorem 1.1 below provides a Drinfeld module analogue of this result.

There is also an important relationship between the de Rham cohomology (in characteristic zero) of $A$ and the variety $A^\natural$ [13]. We study generalizations of a de Rham theory for Drinfeld modules based on additive extensions, which has been developed by G. Anderson, P. Deligne, E.-U. Gekeler, and J. Yu [11].

Moreover, Anderson has remarked that Deligne's 1-motives [9, §10.1] provide a fruitful analogue for Drinfeld modules. Drinfeld modules (or $t$-modules) include analogues of $\mathbb{G}_m$, abelian varieties, and semi-abelian varieties, all of which are examples of 1-motives. In [9], Deligne defines duality and de Rham realizations of 1-motives using extensions by $\mathbb{G}_m$ and $\mathbb{G}_a$, thereby generalizing the already existing theory for abelian varieties. We pursue these ideas in the context of general extensions of Drinfeld modules.

*Notation.* Let $K$ be a perfect field of characteristic $p > 0$, and let $\mathbb{F}_q[t]$ be the polynomial ring in one variable over the finite field $\mathbb{F}_q$ where $q = p^m$. Fix an $\mathbb{F}_q$-linear homomorphism $\iota \colon \mathbb{F}_q[t] \to K$ with $\theta := \iota(t)$. Throughout, all Drinfeld modules and $t$-modules are defined with respect to the map $\iota$, and in particular all Drinfeld modules are $\mathbb{F}_q[t]$-modules.

Let $C$ denote the Carlitz module, $C \colon \mathbb{F}_q[t] \to K\{\tau\}$, defined by $C(t) = \theta + \tau$, where $K\{\tau\}$ is the ring of twisted polynomials in $K$ such that for $x \in K$, $\tau x = x^q \tau$.

We take $\mathrm{Ext}^1(\cdot,\cdot)$ to be the bifunctor $\mathrm{Ext}^1$ from the additive category of $t$-modules to the category of abelian groups. In §2 we see that, for two $t$-modules $E$ and $F$,

*Date*: July 20, 2001.

1991 *Mathematics Subject Classification.* 11G09.

*Key words and phrases.* Drinfeld modules, $t$-modules, extensions.

those extensions which induce trivial $t$-module extensions of their respective tangent spaces comprise a canonical subgroup $\mathrm{Ext}_0^1(E,F) \subset \mathrm{Ext}^1(E,F)$. For a $t$-module $E$, we let $E^\vee := \mathrm{Ext}_0^1(E,C)$.

**Theorem 1.1.** *Let $E$ be a Drinfeld module of rank $r \geq 2$.*

(a) *The group $\mathrm{Ext}^1(E,C)$ is naturally a $t$-module of dimension $r$ and sits in an exact sequence of $t$-modules*

$$0 \to E^\vee \to \mathrm{Ext}^1(E,C) \to \mathbb{G}_a \to 0.$$

*Furthermore, $E^\vee$ is the Cartier-Taguchi dual $t$-module associated to $E$* [19]*, and in particular, $E^\vee$ is isomorphic to the $(r-1)$-st exterior power $E^{\wedge(r-1)}$ of $E$.*

(b) *The group $\mathrm{Ext}^1(E^\vee,C)$ is also naturally a $t$-module of dimension $r$ and sits in an exact sequence*

$$0 \to E \to \mathrm{Ext}^1(E^\vee,C) \to \mathbb{G}_a^{r-1} \to 0.$$

*Moreover, $(E^\vee)^\vee \cong E$.*

Because of some interesting similarities with the classical result, we are tempted to call this result a Weil-Barsotti formula for Drinfeld modules. For Drinfeld modules of rank 2, this result has been proved by S. S. Woo [20]. Furthermore, Y. Taguchi shows that one has a Weil pairing on the torsion points of $E$ and $E^\vee$ [19].

Theorem 1.1 requires us to work outside the category of Drinfeld modules, and one may ask for general $t$-modules $E$ and $F$ over $K$ whether $\mathrm{Ext}^1(E,F)$ has the structure of a $t$-module. In this vein we have the following result. Let $C^{\otimes n}$ denote the $n$-th tensor power of the Carlitz module [2].

**Theorem 1.2.** *If $n > m$, then $\mathrm{Ext}^1(C^{\otimes m}, C^{\otimes n})$ has the structure of a $t$-module, and there is an exact sequence of $t$-modules*

$$0 \to C^{\otimes(n-m)} \to \mathrm{Ext}^1(C^{\otimes m}, C^{\otimes n}) \to L \to 0,$$

*where $L$ is an $m$-dimensional iterated extension of $\mathbb{G}_a$. Moreover, $\mathrm{Ext}_0^1(C^{\otimes m}, C^{\otimes n}) \cong C^{\otimes(n-m)}$.*

Since the (tractable) period $\widetilde{\pi}^{n-m}$ of $C^{\otimes(n-m)}$ is a power of the period $\widetilde{\pi}$ of the Carlitz module (see D. Goss [12, Ch. 3]), one should compare Theorem 1.2 with the isomorphism

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(m), \mathbb{Z}(n)) \cong \mathbb{C}/(2\pi i)^{n-m}\mathbb{Z}, \quad n > m,$$

from the theory of mixed Hodge structures [8].

The question of whether an analogue of the Weil-Barsotti formula holds for general $t$-modules is also interesting. Experimental evidence suggests that the general situation is subtle and that such formulas are not always valid for pure $t$-modules, e.g. in the form of Theorem 1.1, a Weil-Barsotti formula does not hold for $C^{\otimes n}$.

The outline of this paper is as follows. In §2, we present definitions and fundamental results on extensions of $t$-modules. We prove Theorems 1.1 and 1.2 in §§3–4. In §5, we consider extensions of $t$-modules from an analytic viewpoint, so as to motivate the expectation that $\mathrm{Ext}^1(E,F)$ can be represented by a $t$-module for certain $t$-modules $E$ and $F$. We consider the situation of elliptic curves in §6 and compare our results to a beautiful unpublished theorem of S. Lichtenbaum about extensions of elliptic curves over $\mathbb{C}$. We conclude in §7 with some remarks about extensions of $t$-motives.

## 2. Extensions of $t$-modules and biderivations

In this section we establish definitions and results about extensions of $t$-modules. For general definitions of $t$-modules we follow the terminology in Goss [12, Ch. 5]. We point out that the results in this section remain valid in the case that $K$ is not perfect.

Let $E$ and $F$ be $t$-modules over $K$. An extension of $E$ by $F$ is a $t$-module $X$ fitting into an exact sequence of $t$-modules

$$0 \to F \to X \to E \to 0. \tag{1}$$

Then $\mathrm{Ext}^1(E, F)$ is defined to be the group (under Baer sum) of of $t$-module extensions of $E$ by $F$ up to Yoneda equivalence.

The main tool which enables us to compute this group is that of biderivations. The following definitions run parallel to those of W. D. Brownawell and the first author [6] and Gekeler [11], where extensions of $t$-modules by $\mathbb{G}_a$ were investigated.

Let $\Phi\colon \mathbb{F}_q[t] \to \mathrm{Mat}_d(K)\{\tau\}$ and $\Psi\colon \mathbb{F}_q[t] \to \mathrm{Mat}_e(K)\{\tau\}$ be choices of coordinates for $E$ and $F$ respectively, where $\mathrm{Mat}_d(K)\{\tau\}$ is the ring of twisted polynomials with matrix coefficients. A $(\Phi, \Psi)$-*biderivation* is an $\mathbb{F}_q$-linear map

$$\delta : \mathbb{F}_q[t] \to \mathrm{Mat}_{e\times d}(K)\{\tau\},$$

which satisfies

$$\delta(ab) = \Psi(a)\delta(b) + \delta(a)\Phi(b), \quad \forall a, b \in \mathbb{F}_q[t]. \tag{2}$$

The $\mathbb{F}_q$-vector space of all $(\Phi, \Psi)$-biderivations is denoted $\mathrm{Der}(\Phi, \Psi)$. It is straightforward to check that a biderivation $\delta$ is uniquely determined by the single value $\delta(t)$, and so if $V \in \mathrm{Mat}_{e\times d}(K)\{\tau\}$, we define $\delta_V \in \mathrm{Der}(\Phi, \Psi)$ to be that biderivation such that $\delta_V(t) = V$. In this way, we have an isomorphism

$$V \mapsto \delta_V : \mathrm{Mat}_{e\times d}(K)\{\tau\} \xrightarrow{\sim} \mathrm{Der}(\Phi, \Psi),$$

of $\mathbb{F}_q$-vector spaces. A biderivation $\delta$ is called *inner* if for some $U \in \mathrm{Mat}_{e\times d}(K\{\tau\})$ we have

$$\delta(a) = \delta^{(U)}(a) := U\Phi(a) - \Psi(a)U, \quad \forall a \in \mathbb{F}_q[t]. \tag{3}$$

The subspace of $\mathrm{Der}(\Phi, \Psi)$ of inner biderivations is denoted $\mathrm{Der}_{in}(\Phi, \Psi)$.

Every $(\Phi, \Psi)$-biderivation $\delta$ gives rise to an extension $X = (\mathbb{G}_a^{d+e}, \Upsilon)$ of $E$ by $F$ by defining

$$\Upsilon(a) := \begin{pmatrix} \Phi(a) & 0 \\ \delta(a) & \Psi(a) \end{pmatrix}, \quad \forall a \in \mathbb{F}_q[t].$$

Again it is straightforward, using (2), to check that $\Upsilon$ is well-defined. Moreover, every extension of $E$ by $F$ defines a unique biderivation.

We note that if $\delta^{(U)}$ is an inner biderivation then in fact $X$ is split. In this case the matrix $\Theta := \left(\begin{smallmatrix} I_d & 0 \\ U & I_e \end{smallmatrix}\right)$ provides the splitting, where $I_d$, $I_e$ are identity matrices:

$$\Theta^{-1}\Upsilon(a)\Theta = \begin{pmatrix} \Phi(a) & 0 \\ 0 & \Psi(a) \end{pmatrix}, \quad \forall a \in \mathbb{F}_q[t].$$

Furthermore, it follows from the above discussion that every split extension arises in this way.

Suppose we are given two extensions of $E$ by $F$ which are Yoneda equivalent. It follows easily from the definition of Yoneda equivalence that the corresponding biderivations differ by an inner biderivation. It is straightforward to check that the (Baer) sum on $\mathrm{Ext}^1(E,F)$ corresponds to usual addition on the level of biderivations.

Now the endomorphisms of $E$ and $F$ induce (identical) $\mathbb{F}_q[t]$-module structures on $\mathrm{Ext}^1(E,F)$. That is, if $X$ represents a class in $\mathrm{Ext}^1(E,F)$ and $b \in \mathbb{F}_q[t]$, we can define two $t$-modules $X \cdot b$ and $b \cdot X$, which ultimately represent the same class in $\mathrm{Ext}^1(E,F)$. Explicitly, suppose $\delta$ is the $(\Phi,\Psi)$-biderivation corresponding to $X$ and $\pi\colon X \to E$ is the natural map in (1). Let

$$X \cdot b := \ker\left((e,x) \mapsto \Phi(b)(e) - \pi(x) : E \oplus X \to E\right).$$

Then $X \cdot b$ is itself a $t$-module extension of $E$ by $F$, and the operation of $\mathbb{F}_q[t]$ on it is given by

$$(\Upsilon \cdot b)(a) = \begin{pmatrix} \Phi(a) & 0 \\ \delta(a)\Phi(b) & \Psi(a) \end{pmatrix}, \quad \forall a \in \mathbb{F}_q[t].$$

On the other hand, we can similarly use endomorphisms of $F$ to define an extension $b \cdot X$ whose $t$-module structure is given by

$$(b \cdot \Upsilon)(a) = \begin{pmatrix} \Phi(a) & 0 \\ \Psi(b)\delta(a) & \Psi(a) \end{pmatrix}, \quad \forall a \in \mathbb{F}_q[t].$$

To see that $X \cdot b$ and $b \cdot X$ are equivalent extensions, we note that $\varepsilon\colon \mathbb{F}_q[t] \to \mathrm{Mat}_{e\times d}(K)\{\tau\}$ defined by

$$\varepsilon(a) := \delta(a)\Phi(b) - \Psi(b)\delta(a), \quad \forall a \in \mathbb{F}_q[t],$$

is in fact the inner biderivation $\delta^{(U)}$, with $U = \delta(t)$ in (3). That is,

$$b\colon \delta(\cdot) \mapsto \delta(\cdot)\Phi(b) \quad \text{and} \quad b\colon \delta(\cdot) \mapsto \Psi(b)\delta(\cdot) \tag{4}$$

define $\mathbb{F}_q[t]$-module structures on $\mathrm{Der}(\Phi,\Psi)$ which are the same modulo $\mathrm{Der}_{in}(\Phi,\Psi)$. We record the results from the preceding paragraphs in the following lemma.

**Lemma 2.1.** *Let $E = (\mathbb{G}_a^d, \Phi)$ and $F = (\mathbb{G}_a^e, \Psi)$ be $t$-modules. Then*

$$\mathrm{Ext}^1(E,F) \cong \mathrm{Der}(\Phi,\Psi)/\,\mathrm{Der}_{in}(\Phi,\Psi)$$

*as $\mathbb{F}_q[t]$-modules.*

For $U \in \mathrm{Mat}_{d_1\times d_2}(K)\{\tau\}$, we let $dU \in \mathrm{Mat}_{d_1\times d_2}(K)$ be the constant term of $U$ as a polynomial in $\tau$, and we define the following subspaces of $\mathrm{Der}(\Phi,\Psi)$:

$$\mathrm{Der}_0(\Phi,\Psi) := \{\delta \in \mathrm{Der}(\Phi,\Psi) : d\delta(t) = 0\},$$
$$\mathrm{Der}_{si}(\Phi,\Psi) := \{\delta^{(U)} \in \mathrm{Der}_{in}(\Phi,\Psi) : dU = 0\}.$$

Note that $\mathrm{Der}_0$ differs here from definitions in [6] and [11]. The utility of $\mathrm{Der}_0(\Phi,\Psi)$ is derived from the following lemma, whose immediate corollary follows from Lemma 2.1. Biderivations in $\mathrm{Der}_{si}(\Phi,\Psi)$ are called *strictly inner*, and clearly $\mathrm{Der}_{si}(\Phi,\Psi) \subset \mathrm{Der}_0(\Phi,\Psi)$. We will study $\mathrm{Der}_{si}(\Phi,\Psi)$ in more detail in §5.

The map $d\Phi\colon \mathbb{F}_q[t] \to \mathrm{Mat}_d(K)$ defines a $t$-module whose underlying space is the tangent space $\mathrm{Lie}(E) \cong K^d$. Furthermore, the map $\delta \mapsto d\delta\colon \mathrm{Der}(\Phi,\Psi) \to \mathrm{Der}(d\Phi,d\Psi)$ is $\mathbb{F}_q$-linear, and it is $\mathbb{F}_q[t]$-linear modulo inner biderivations.

**Lemma 2.2.** *Let $E = (\mathbb{G}_a^d, \Phi)$ and $F = (\mathbb{G}_a^e, \Psi)$ be $t$-modules. The following is an exact sequence of $\mathbb{F}_q[t]$-modules:*

$$0 \to \frac{\mathrm{Der}_0(\Phi,\Psi)}{\mathrm{Der}_0(\Phi,\Psi)\cap \mathrm{Der}_{in}(\Phi,\Psi)} \to \frac{\mathrm{Der}(\Phi,\Psi)}{\mathrm{Der}_{in}(\Phi,\Psi)} \to \frac{\mathrm{Der}(d\Phi,d\Psi)}{\mathrm{Der}_{in}(d\Phi,d\Psi)}. \tag{5}$$

*If $\theta \in K$ is transcendental over $\mathbb{F}_q$, then the final map in this sequence is surjective.*

We define $\mathrm{Ext}_0^1(E,F)$ to be the $\mathbb{F}_q[t]$-submodule of $\mathrm{Ext}^1(E,F)$ corresponding to $\mathrm{Der}_0(\Phi,\Psi)/(\mathrm{Der}_0(\Phi,\Psi)\cap \mathrm{Der}_{in}(\Phi,\Psi))$.

**Corollary 2.3.** *The sequence in Lemma 2.2 corresponds to an exact sequence of $\mathbb{F}_q[t]$-modules,*

$$0 \to \mathrm{Ext}_0^1(E,F) \to \mathrm{Ext}^1(E,F) \to \mathrm{Ext}^1(\mathrm{Lie}(E),\mathrm{Lie}(F)),$$

*where the final map is surjective if $\theta \in K$ is transcendental over $\mathbb{F}_q$.*

*Proof of Lemma 2.2.* Injectivity on the left of (5) is clear. To show exactness in the center, first any $\delta \in \mathrm{Der}_0(\Phi,\Psi)$ maps to 0 in $\mathrm{Der}(d\Phi,d\Psi)$. On the other hand, suppose $\delta \in \mathrm{Der}(\Phi,\Psi)$ and $d\delta \in \mathrm{Der}(d\Phi,d\Psi)$ is inner, say $d\delta(t) = Ud\Phi(t) - d\Psi(t)U$ with $U \in \mathrm{Mat}_{e\times d}(K)\{\tau\}$. Then $\delta - \delta^{(U)}_{\Phi,\Psi} \in \mathrm{Der}_0(\Phi,\Psi)$ represents the same class as $\delta$ in $\mathrm{Der}(\Phi,\Psi)/\mathrm{Der}_{in}(\Phi,\Psi)$.

In the case that $\theta \in K$ is transcendental over $\mathbb{F}_q$, we show surjectivity on the right. We suppose $d\Phi(t) = I_d\theta + M$ and $d\Psi(t) = I_e\theta + N$, where $M$ and $N$ are nilpotent. Without loss of generality we can assume $M$ and $N$ are both upper triangular. The spaces of biderivations $\mathrm{Der}(d\Phi,d\Psi)$ and $\mathrm{Der}_{in}(d\Phi,d\Psi)$ are both naturally $K$-linear, and the map defining elements of $\mathrm{Der}_{in}(d\Phi,d\Psi)$,

$$U \mapsto \delta^{(U)}_{d\Phi,d\Psi}(t) : \mathrm{Mat}_{e\times d}(K)\{\tau\} \to \mathrm{Mat}_{e\times d}(K)\{\tau\}, \tag{6}$$

is $K$-linear and respects the grading by degrees in $\tau$. Now

$$\delta^{(U)}_{d\Phi,d\Psi}(t) = U\theta - \theta U + UM - NU. \tag{7}$$

If $U_{ij}$ is the matrix in $\mathrm{Mat}_{e\times d}(K)$ with a 1 in the $ij$-th entry and zeros elsewhere, then $\{U_{m1}\tau^r, U_{m-1,1}\tau^r, \dots\} = \{U_{ij}\tau^r\}_{i=m,\dots,e}^{j=1,\dots,d}$ is an ordered $K$-basis for $\mathrm{Mat}_{e\times d}(K)\tau^r$. Using this basis, it follows from (7) and the fact that $M$ and $N$ are both upper triangular that the map in (6), restricted to $\mathrm{Mat}_{e\times d}(K)\tau^r$, is lower triangular with $\theta^{q^r} - \theta$ along the diagonal. Since $\theta^{q^r} - \theta$ is non-zero by our assumption on $\theta$,

$$U \mapsto \delta^{(U)}_{d\Phi,d\Psi}(t) : \mathrm{Mat}_{e\times d}(K)\{\tau\}\tau \xrightarrow{\sim} \mathrm{Mat}_{e\times d}(K)\{\tau\}\tau,$$

is an isomorphism of $K$-vector spaces. Therefore, for every $\delta \in \mathrm{Der}(d\Phi,d\Psi)$, if $d\delta(t) = 0$, then $\delta$ is inner, and so the right-hand side of (5) is surjective. □

## 3. A Weil-Barsotti formula

In this section, we prove Theorem 1.1. Let $E = (\mathbb{G}_a, \Phi)$ be a Drinfeld module of rank $r \geq 2$, where $\Phi(t) = \theta + a_1\tau + \cdots + a_r\tau^r$. For an extension $X$ of $E$ by $C$, we let $\delta_X \in \mathrm{Der}(\Phi, C)$ be its associated biderivation.

*Proof of Theorem 1.1(a).* An extension $X$ of $E$ by $C$ splits if and only if $\delta_X \in \mathrm{Der}_{in}(\Phi, C)$, i.e. if there exists $u \in K\{\tau\}$ such that $\delta_X(t) = u\Phi(t) - (\theta+\tau)u$. If $u = c\tau^m$, then we obtain an inner biderivation $\delta^{(u)}$ such that

$$\begin{aligned} \delta^{(u)}(t) &= u\Phi(t) - (\theta+\tau)u \\ &= c\theta^{q^m}\tau^m + c{a_1}^{q^m}\tau^{m+1} + \cdots + c{a_r}^{q^m}\tau^{m+r} - \theta c\tau^m - c^q\tau^{m+1}. \end{aligned} \tag{8}$$

If $n := \deg_\tau(\delta_X(t))$ is greater than $r-1$, we can repeatedly subtract (8) from $\delta_X(t)$, with $m = n-r,\ n-r-1, \dots, 0$, to reduce the $\tau$-degree of $\delta_X(t)$; eventually this degree will be $< r$. Namely, any extension $X$ is equivalent to an extension $X'$ with $\deg_\tau(\delta_{X'}(t)) \le r-1$, which we call the reduced representative of $X$.

According to (8), the non-zero inner biderivations of least degree have degree $r$, so two extensions $Z$ and $Y$ satisfying $\deg_\tau(\delta_Z(t)) \le r-1$, $\deg_\tau(\delta_Y(t)) \le r-1$, and $\delta_Z(t) \ne \delta_Y(t)$, are inequivalent. Therefore the map

$$X \mapsto \delta_{X'}(t) : \mathrm{Ext}^1(E,C) \xrightarrow{\sim} V := \{b_0 + b_1\tau + \cdots + b_{r-1}\tau^{r-1} : b_i \in K\}, \tag{9}$$

where $X'$ is the reduced representative of $X$, induces an isomorphism of $\mathbb{F}_q[t]$-modules.

We now turn to the $t$-module structure on $\mathrm{Ext}^1(E,C)$. Recall from (4) that multiplication-by-$t$ on $\mathrm{Ext}^1(E,C)$ is defined by $t\alpha = (\theta+\tau)\alpha = \alpha\Phi(t)$. Here we think of $\alpha$ as being an element of $V$. In order to see the action explicitly, it is enough to consider $\alpha = b_i\tau^i$:

$$\begin{aligned} t(b_0) &= (\theta+\tau)b_0 = \theta b_0 + b_0^q\tau, \\ t(b_1\tau) &= (\theta+\tau)b_1\tau = \theta b_1\tau + b_1^q\tau^2, \\ &\ \ \vdots \qquad\qquad \vdots \\ t(b_{r-1}\tau^{r-1}) &= (\theta+\tau)b_{r-1}\tau^{r-1} = \theta b_{r-1}\tau^{r-1} + b_{r-1}^q\tau^r. \end{aligned} \tag{10}$$

Using (8) with $m=0$, $c = b_{r-1}^q/a_r$, we can rewrite the last identity of (10) as

$$t(b_{r-1}\tau^{r-1}) = \Bigl(\frac{b_{r-1}^{q^2}}{a_r^q} - \frac{b_{r-1}^q a_1}{a_r}\Bigr)\tau - \frac{b_{r-1}^q}{a_r}(a_2\tau^2 + \cdots + a_{r-2}\tau^{r-2}) + \Bigl(\theta b_{r-1} - \frac{b_{r-1}^q a_{r-1}}{a_r}\Bigr)\tau^{r-1}.$$

Thus in terms of the elements $e_i = \tau^i$ $(i = 0, \dots, r-1)$ of $V$, the $t$-module structure on $\mathrm{Ext}^1(E,C)$ can be expressed by the map $\Pi\colon \mathbb{F}_q[t] \to \mathrm{Mat}_r(K)\{\tau\}$, defined by

$$\Pi(t) := \begin{pmatrix} \theta & 0 & \cdots & \cdots & 0 & 0 \\ \tau & \theta & 0 & & \vdots & -\frac{a_1}{a_r}\tau + \frac{1}{a_r^q}\tau^2 \\ 0 & \tau & \theta & 0 & \vdots & -\frac{a_2}{a_r}\tau \\ \vdots & 0 & \tau & \theta & 0 & \vdots \\ \vdots & & 0 & \tau & \theta & -\frac{a_{r-2}}{a_r}\tau \\ 0 & \cdots & \cdots & 0 & \tau & \theta - \frac{a_{r-1}}{a_r}\tau. \end{pmatrix}.$$

Comparing with Taguchi in [19, §5], it is clear that the t-module $\mathrm{Ext}^1(E,C)$ is an extension of $\mathbb{G}_a$ by the t-module denoted there $\check{E}$. By Corollary 2.3 and the characterization of $V$ in (9), it is clear $\mathrm{Ext}^1(E,C)$ is an extension of $\mathbb{G}_a$ by $E^\vee$. Thus $E^\vee$ is the same as Taguchi's $t$-module. Moreover, Taguchi shows [19, Thm. 5.1] that $E^\vee$ is a pure $t$-module isomorphic to $E^{\wedge(r-1)}$. ☐

We turn to part (b), and for simplicity we assume that $a_r = 1$; the general case follows similarly. The $t$-module structure on $E^\vee$ is then defined by

$$\Psi(t) := \begin{pmatrix} \theta & 0 & \cdots & 0 & -a_1\tau + \tau^2 \\ \tau & \theta & 0 & \vdots & -a_2\tau \\ 0 & \tau & \theta & 0 & \vdots \\ \vdots & 0 & \tau & \theta & -a_{r-2}\tau \\ 0 & \cdots & 0 & \tau & \theta - a_{r-1}\tau. \end{pmatrix}.$$

*Proof of Theorem 1.1(b).* Consider the biderivation $\delta_X : \mathbb{F}_q[t] \to \mathrm{Mat}_{1\times(r-1)}(K)\{\tau\}$ of any extension $X$ of $E^\vee$ by $C$. The inner biderivations are of the form $\delta^{(U)} = U\Psi - (\theta+\tau)U$ for $U = (u_i) \in \mathrm{Mat}_{1\times(r-1)}(K)\{\tau\}$. Explicitly, $v = (v_i) \in \mathrm{Mat}_{1\times(r-1)}(K)\{\tau\}$ defines an inner biderivation, $\delta^{(U)} = \delta_v$, if it has the form

$$\begin{aligned} v_i &= u_i\theta - \theta u_i + u_{i+1}\tau - \tau u_i \quad (1 \le i \le r-2), \\ v_{r-1} &= u_{r-1}\theta - \theta u_{r-1} + u_1\tau^2 - \tau u_{r-1} - \textstyle\sum_{j=1}^{j=r-1} u_j a_j \tau. \end{aligned}$$

An inner biderivation $\delta_v$ is said to be basic if $u_i = 0$ for all $i \neq s$ and $u_s = c\tau^m$ with $c \in K$; we write $v = v(s,c,m) = (v_1, \ldots, v_{r-1})$. Explicitly written, these are

$$\begin{aligned} v(1,c,m) &= (c\theta^{q^m}\tau^m - c\theta\tau^m - c^q\tau^{m+1}, 0, \ldots, 0, c\tau^{m+2} - ca_1^{q^m}\tau^{m+1}), \\ v(2,c,m) &= (c\tau^{m+1}, c\theta^{q^m}\tau^m - c\theta\tau^m - c^q\tau^{m+1}, 0, \ldots, 0, -ca_2^{q^m}\tau^{m+1}), \\ v(z,c,m) &= (0, \ldots, c\tau^{m+1}, c\theta^{q^m}\tau^m - c\theta\tau^m - c^q\tau^{m+1}, 0, \ldots, 0, -ca_z^{q^m}\tau^{m+1}), \\ v(r-1,c,m) &= (0, \ldots, 0, c\tau^{m+1}, c\theta^{q^m}\tau^m - c\theta\tau^m - c^q\tau^{m+1} - ca_{r-1}^{q^m}\tau^{m+1}), \end{aligned} \tag{11}$$

where for $2 \le z \le r-2$ the possible non-zero coordinates of $v(z,c,m)$ are $v_{z-1}$, $v_z$, and $v_{r-1}$. Every inner biderivation arises from an additive combination of basic $\delta_v$.

Consider $G \subset \mathrm{Mat}_{1\times(r-1)}(K)\{\tau\}$ consisting of elements $u := (u_i)$ with $\tau$-degrees of $u_1, \ldots, u_{r-2}$ zero and the $\tau$-degree of $u_{r-1}$ less than one. In other words, $u_1, \ldots, u_{r-2} \in K$ and $u_{r-1} = c + d\tau$ with $c, d \in K$. Elements of $G$ give rise to biderivations which we will call reduced.

**Lemma 3.1.** *The map $u \mapsto \delta_u : G \to \mathrm{Der}(\Psi, C)$ induces an isomorphism*

$$G \cong \frac{\mathrm{Der}(\Psi, C)}{\mathrm{Der}_{in}(\Psi, C)} \cong \mathrm{Ext}^1(E^\vee, C)$$

*of $\mathbb{F}_q$-vector spaces.*

*Proof of Lemma.* We need only to prove the first isomorphism by Lemma 2.1. Let $X$ be an extension. We want to subtract appropriate $v$'s in (11) from $\delta_X(t) = u = (u_1, u_2, \ldots, u_{r-1})$ so that the resulting biderivation is reduced. In this process, we need to keep track of the $\tau$-degrees of the $u_j$'s. We define the $\tau$-degree of $u$ to be the vector $d_u := (d_1(u), \ldots, d_{r-1}(u))$ with $d_j(u) = \tau$-degree of $u_j$. Given two vectors $d$ and $d'$ with integer coefficients, we shall say $d \le d'$ if $d_i \le d'_i$ for all $i$. So our claim is that $u$ can be reduced to a biderivation $\tilde{u}$ such that $d_{\tilde{u}} \le (0, \ldots, 0, 1)$.

Let $n$ be the maximum of the integers $d_j(u)$; one has $d_j(u) \le n$ for all $j$. We can modify $u$ by $v(2, c, d_1 - 1)$ for an appropriate $c \in K$ to obtain $u'$ such that $d_1(u') < d_1(u)$ and $d_g(u') \le n$ for $g \ge 2$. Subtracting an appropriate $v(3, c, d_2(u'))$, we obtain $u''$ such that $d_2(u'') < d_2(u')$ and $d_g(u'') \le n$ for $g \ge 3$. Repeating this for $z = 4, \dots, r-2$ using appropriate $v(z,c,m)$, we obtain a $w \in \mathrm{Mat}_{1\times(r-1)}(K)\{\tau\}$ whose degree vector is less than or equal to $(n-1, n-1, \dots, n-1, n)$.

If $n \leq 1$, we are done. If not ($n \geq 2$), we can subtract an appropriate $v(1,c,n-2)$ to obtain a vector $w$ whose degree vector is less than or equal to $(n-1, n-1, \dots, n-1, n-1)$. We repeat the procedure in the two paragraphs above until we arrive at a vector $w$ whose degree vector is less than or equal to $(0, \dots, 0, 1)$. □

We can now determine the $t$-module structure on $\mathrm{Ext}^1(E^\vee, C)$. By the previous lemma, it suffices to see this structure on $G$. Consider the elements $e_i \in G \subset \mathrm{Mat}_{1\times(r-1)}(K)\{\tau\}$, $1 \leq i \leq r$, defined as follows. For $1 \leq i \leq r-1$, we take $e_i$ to be the vector with 1 in the $i$-th coordinate and zeros elsewhere. We take $e_r$ to be the vector with $\tau$ in the last coordinate and zeros elsewhere. The structure of a $t$-module on $G$ is completely described by the action of $t$ on elements of the form $be_i$, $b \in K$, since additive combinations of such elements give all of $G$.

Consider $t(be_i) = (\theta+\tau)be_i = \theta e_i + b^q\tau e_i$. The last is no longer an element of $G$, and we need its equivalent vector in $G$. Using the reduction procedure of Lemma 3.1, it is easily computed that

$$\begin{aligned}
(\theta+\tau)(be_i) &= \theta b e_n + (\textstyle\sum_{f=0}^{r-i-1} b^{q^{r-i-f}} a_{r-f})e_r, \quad 1 \leq i \leq r-1,\\
(\theta+\tau)(be_{r-1}) &= \theta b e_{r-1} + b^q e_r\\
(\theta+\tau)(be_{r-2}) &= \theta b e_{r-2} + (b^{q^2} + b^q a_{r-1})e_r\\
(\theta+\tau)(be_r) &= (\theta + \textstyle\sum_{f=1}^{r} b^{q^f} a_f)e_r.
\end{aligned}$$

Thus the $t$-module structure $\Xi \colon \mathbb{F}_q[t] \to \mathrm{Mat}_r(K)\{\tau\}$ on $\mathrm{Ext}^1(E^\vee, C)$ is completely described by

$$\Xi(t) := \begin{pmatrix} \theta & 0 & \cdots & \cdots & 0 \\ 0 & \theta & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \theta & 0 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{r-1} & \alpha_r \end{pmatrix}.$$

Here $\alpha_r = \theta + a_1\tau + a_2\tau^2 + \cdots + a_{r-1}\tau^{r-1} + \tau^r$, and the others are given by $\alpha_n = \sum_{f=0}^{r-n-1} \tau^{r-n-f} a_{r-f}$, for $1 \leq n \leq r-1$. Moreover, $\mathrm{Ext}^1(E^\vee, C)$ is an extension of $\mathbb{G}_a^{r-1}$ by $E$. By Lemma 3.1, $(E^\vee)^\vee$ is 1-dimensional, which completes the proof. □

## 4. Extensions of tensor powers of the Carlitz module

In this section we prove Theorem 1.2. Recall that the $n$-th tensor power of the Carlitz module is the $n$-dimensional pure $t$-module $C^{\otimes n} \colon \mathbb{F}_q[t] \to \mathrm{Mat}_n(K)\{\tau\}$ defined by

$$C^{\otimes n}(t) := \begin{pmatrix} \theta & 1 & 0 \\ \vdots & \ddots & 1 \\ \tau & \cdots & \theta \end{pmatrix}.$$

That is, $C^{\otimes n}(t) = \theta I_n + N_n + E_n\tau$, where

$$N_n := \begin{pmatrix} 0 & 1 & 0 \\ \vdots & \ddots & 1 \\ 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad E_n := \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 1 & \cdots & 0 \end{pmatrix}.$$

See Anderson and D. S. Thakur [2] for more details.

Fix $m < n$. The following lemma determines representatives for elements of $\mathrm{Der}(C^{\otimes m}, C^{\otimes n})/\mathrm{Der}_{in}(C^{\otimes m}, C^{\otimes n})$.

**Lemma 4.1.** *Let*

$$G := \left\{ \begin{pmatrix} * & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ * & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{Mat}_{n\times m}(K) \right\}.$$

*Then* $V \mapsto \delta_V \colon G \xrightarrow{\sim} \mathrm{Der}(C^{\otimes m}, C^{\otimes n})/\mathrm{Der}_{in}(C^{\otimes m}, C^{\otimes n})$ *is an isomorphism of* $\mathbb{F}_q$*-vector spaces.*

*Proof.* Let $Q_{ij}$ be the $n \times m$ matrix with a 1 in the $ij$-th entry and zeros elsewhere. For $c \in K$ and $k \geq 0$, by taking $cQ_{ij}\tau^k$ for $U$ in $\delta^{(U)}(t)$, we define

$$\begin{aligned} \delta_{ijk}^{[c]} &:= (cQ_{ij}\tau^k)C^{\otimes m}(t) - C^{\otimes n}(t)(cQ_{ij}\tau^k) \\ &= c(\theta^{q^k} - \theta)Q_{ij}\tau^k + c(Q_{ij}N_m - N_nQ_{ij})\tau^k + (cQ_{ij}E_m - c^qE_nQ_{ij})\tau^{k+1}. \end{aligned} \tag{12}$$

Biderivations arising from (12) generate $\mathrm{Der}_{in}(C^{\otimes m}, C^{\otimes n})$ as an $\mathbb{F}_q$-vector space.

Suppose that $V = (v_{ij}) \in \mathrm{Mat}_{n\times m}(K\{\tau\})$ is arbitrary and that $\deg_\tau(v_{ij}) \leq r$ for some $r \geq 1$. We will show that by subtracting matrices in (12) from $V$ in various ways we can replace $V$ by a matrix $V'$ which has each entry of $\tau$-degree $\leq r-1$ and also $\delta_{V'}$ equivalent to $\delta_V$ modulo $\mathrm{Der}_{in}(C^{\otimes m}, C^{\otimes n})$.

We bootstrap our way through the entries of $V$ in the following way. Let

$$D_{ij} := \{\delta_{ijk}^{[c]} : k \geq 0, c \in K\}.$$

Define a function $F$ from the set of subsets of $I := \{(i,j) : 1 \leq i \leq n, 1 \leq j \leq m\}$ to itself. We set $F(S)$ to be those entries of $V$ whose degrees in $\tau$ can be decreased by subtracting an element of $\sum_{(i,j)\in S} D_{ij}$, without increasing the degrees of the other entries. Our claim then is that $F(I) = I$. The following containments can be easily checked:

$$\begin{array}{lll} F(\{2 \leq i \leq n; j = m\}) & \supset & \{2 \leq i \leq n; j = 1\}, \\ F(\{2 \leq i \leq n; j = m\} \cup \{i - j \geq 2\}) & \supset & \{i - j \geq 1\}, \\ F(\{j = m\} \cup \{i - j \geq 2\}) & \supset & \{i - j \geq 1\} \cup \{(1,1),(1,m)\}, \\ F(\{j = m\} \cup \{i - j \geq 1\}) & \supset & \{i - j \geq 0\} \cup \{(1,m)\}, \\ F(\{j = m\} \cup \{i - j \geq -\ell\}) & \supset & \{i - j \geq -\ell - 1\} \cup \{(1,m)\}, \end{array} \tag{13}$$

where the last containment holds for all $\ell = 0, \ldots, m-1$.

Therefore, we can assume that every entry of $V$ is a constant from $K$. Now that the $\tau$-degree of each entry of $V$ is 0, one checks the containments in (13) still hold with the exception that all sets on the right-hand side must have $\{j = 1\}$ removed. That is, we can adjust $V$ so that it can be replaced by a matrix in $G$, but elements of $G$ can be reduced no further. □

*Proof of Theorem 1.2.* Let $G$ be given as in Lemma 4.1 so that by Lemma 2.1

$$G \cong \mathrm{Ext}^1(C^{\otimes m}, C^{\otimes n}).$$

Recalling the definition of $Q_{ij}$ from the proof above, let

$$e_1 := Q_{11}, \ldots, e_n := Q_{n1},$$

be basis vectors for $G$ over $K$. Combining Lemmas 2.1 and 4.1, $G$ has a natural $\mathbb{F}_q[t]$-module structure which we now make explicit. For $c \in K$,

$$t \cdot (ce_1) = C^{\otimes n}(t)(ce_1) = \theta c e_1 + c^q \tau e_n, \tag{14}$$

and

$$t \cdot (ce_i) = C^{\otimes n}(t)(ce_i) = \theta c e_i + c e_{i-1}, \quad 2 \le i \le n. \tag{15}$$

We note that as defined in (12),

$$\sum_{i=0}^{m-1} \delta_{n-i,m-i,0}^{[c^q]} = c^q Q_{n1}\tau - c^q Q_{n-m,1} = c^q \tau e_n - c^q e_{n-m},$$

and since this sum defines an inner $(C^{\otimes m}, C^{\otimes n})$-biderivation, we subtract it from (14) and find that

$$t \cdot (ce_1) = \theta c e_1 + c^q e_{n-m}.$$

Therefore, combining this with (15), we see that the $\mathbb{F}_q[t]$-module structure on $G$ can be expressed as a $t$-module by the map $\Pi \colon \mathbb{F}_q[t] \to \mathrm{Mat}_n(K)\{\tau\}$ defined by

$$\Pi(t) := \left( \begin{array}{c|ccc} & 0 & \cdots & 0 \\ C^{\otimes(n-m)}(t) & \vdots & & \vdots \\ & 1 & \cdots & 0 \\ \hline 0 & & I_m\theta + N_m & \end{array} \right).$$

Thus there is an exact sequence of $t$-modules,

$$0 \to C^{\otimes(n-m)} \to G \to L \to 0,$$

where $L$ is an $m$-dimensional iterated extension of $\mathbb{G}_a$, and one checks that moreover $C^{\otimes(n-m)} = \mathrm{Ext}_0^1(C^{\otimes m}, C^{\otimes n})$. □

## 5. Periods of $t$-modules and extensions

In this section we would like to motivate the expectation that $\mathrm{Ext}^1(E, F)$, for $t$-modules $E$ and $F$, can be given the structure of a $t$-module. Our insights here come from the analytic theory of $t$-modules, and our main tool will be generalizations of quasi-periodic functions defined in [6] and [11] for extensions by $\mathbb{G}_a$. Moreover we study an analogue of the de Rham map of Gekeler [11]. For more details on the general analytic theory of $t$-modules see Goss [12, Ch. 5].

Let $\mathbb{K}$ be the completion of the algebraic closure of the Laurent series field $\mathbb{F}_q(\!(1/\theta)\!)$, where $\theta$ is an independent variable. Let $\iota \colon \mathbb{F}_q[t] \to \mathbb{K}$ be defined by $t \mapsto \theta$.

Let $E = (\mathbb{G}_a^d, \Phi)$ and $F = (\mathbb{G}_a^e, \Psi)$ be uniformizable $t$-modules over $\mathbb{K}$. We take $\mathrm{Exp}_E \colon \mathrm{Lie}(E)(\mathbb{K}) \to E(\mathbb{K})$ to be the exponential map of $E$. Because $E$ is uniformizable, there is an exact sequence of $\mathbb{F}_q[t]$-modules

$$0 \to \Lambda_\Phi \to \mathbb{K}^d \xrightarrow{\mathrm{Exp}_\Phi} \mathbb{K}^d \to 0,$$

where $\mathbb{F}_q[t]$ operates by $d\Phi$ on the central $\mathbb{K}^d$ and by $\Phi$ on the right one. Also, $\Lambda_\Phi$ is the period lattice of $\Phi$ and is a discrete $\mathbb{F}_q[t]$-submodule of $\mathbb{K}^d$. Similarly, we define $\mathrm{Exp}_\Psi$ and $\Lambda_\Psi$ for $F$.

**Lemma 5.1.** *For each $\delta \in \mathrm{Der}_0(\Phi,\Psi)$ there is a unique $\mathbb{F}_q$-linear entire function $F_\delta\colon \mathbb{K}^d \to \mathbb{K}^e$ such that for $z \in \mathbb{K}^d$,*

$$F_\delta(z) \equiv 0 \pmod{\deg q},$$
$$F_\delta(d\Phi(a)z) = \Psi(a)F_\delta(z) + \delta(a)\operatorname{Exp}_\Phi(z), \quad \forall a \in \mathbb{F}_q[t].$$

*Proof.* The proof here is essentially the same as the that of the existence of the exponential function, and in particular we can easily adapt the proof of Proposition 2.1.4 in Anderson [1] to this situation. □

**Lemma 5.2.** *Suppose $\delta^{(U)} \in \mathrm{Der}_0(\Phi,\Psi) \cap \mathrm{Der}_{in}(\Phi,\Psi)$. Then*

$$F_\delta(z) = U\operatorname{Exp}_\Phi(z) - \operatorname{Exp}_\Psi(dU \cdot z).$$

*Proof.* This follows directly from Lemma 5.1, using the fact that $\delta^{(U)} \in \mathrm{Der}_0(\Phi,\Psi)$ if and only if $dU\, d\Phi(a) = d\Psi(a)\, dU$ for all $a \in \mathbb{F}_q[t]$. □

If $X = (\mathbb{G}_a^{d+e}, \Upsilon)$ is an extension of $E$ by $F$ defined by a biderivation $\delta \in \mathrm{Der}_0(\Phi,\Psi)$, then Lemma 5.1 and the uniqueness of the exponential function imply that

$$\operatorname{Exp}_\Upsilon \begin{pmatrix} z \\ u \end{pmatrix} = \begin{pmatrix} \operatorname{Exp}_\Phi(z) \\ \operatorname{Exp}_\Psi(u) + F_\delta(z) \end{pmatrix}$$

is the exponential function for $X$. Also, since $E$ and $F$ are both uniformizable, so is $X$. Moreover, the period lattice of $X$ is

$$\Lambda_\Upsilon = \left\{ \begin{pmatrix} \lambda \\ \eta \end{pmatrix} : \lambda \in \Lambda_\Phi, \operatorname{Exp}_\Psi(\eta) + F_\delta(\lambda) = 0 \right\}.$$

For each $(\lambda,\eta) \in \Lambda_\Upsilon$, it follows that $(\lambda, \eta + d\Psi(a)\mu) \in \Lambda_\Upsilon$ for all $a \in \mathbb{F}_q[t]$ and $\mu \in \Lambda_\Psi$.

**Proposition 5.3.** *The following map is a homomorphism of $\mathbb{F}_q[t]$-modules:*

$$\begin{aligned} \mathrm{DR} := \mathrm{DR}_{(\Phi,\Psi)} :\ \mathrm{Der}_0(\Phi,\Psi)/\mathrm{Der}_{si}(\Phi,\Psi) &\to \mathrm{Hom}_{\mathbb{F}_q[t]}(\Lambda_\Phi, \Psi(\mathbb{K})), \\ \delta &\mapsto (\lambda \mapsto F_\delta(\lambda)). \end{aligned}$$

*Proof.* The map $\mathrm{DR}_0\colon \delta \mapsto (\lambda \mapsto F_\delta(\lambda))$ on $\mathrm{Der}_0(\Phi,\Psi) \to \mathrm{Hom}_{\mathbb{F}_q[t]}(\Lambda_\Phi, \Psi(\mathbb{K}))$ is clearly well-defined and $\mathbb{F}_q$-linear by Lemma 5.1. If $\delta^{(U)} \in \mathrm{Der}_{si}(\Phi,\Psi)$, then by definition $dU = 0$, and so by Lemma 5.2, $\mathrm{DR}_0(\delta^{(U)}) = 0$. Therefore DR is well-defined. Furthermore, from (4) and Lemma 5.1, it follows that

$$F_{a\cdot\delta}(z) = F_{\Psi(a)\delta(\cdot)}(z) = \Psi(a)F_\delta(z), \quad \forall a \in \mathbb{F}_q[t],$$

and so $\mathrm{DR}(a\cdot\delta)(\lambda) = \Psi(a)(\mathrm{DR}(\delta)(\lambda))$, for all $a \in \mathbb{F}_q[t]$. □

*Remark* 5.4. In the case of extensions of Drinfeld modules by $\mathbb{G}_a$, the map DR specializes to the de Rham homomorphism of Gekeler [11]. Gekeler shows in this case that the de Rham map is in fact an isomorphism. In general, determining the kernel and cokernel of DR is a delicate matter and will require further study. However, Theorems 1.1 and 1.2 can be used to imply that the de Rham maps in their respective situations are indeed isomorphisms. A straightforward modification of the proof of [11, Thm. 3.1] yields the following partial result.

**Proposition 5.5.** *Suppose $E = (\mathbb{G}_a, \Phi)$ and $F = (\mathbb{G}_a, \Psi)$ are Drinfeld modules of ranks $r$ and $s$ respectively. If $r > s$, then $\mathrm{DR}_{(\Phi,\Psi)}$ is injective.*

For the remainder of this section we will consider the implications of the assumption that DR is an isomorphism. In this way we attempt to motivate the idea that $\mathrm{Ext}^1_0(E,F)$ can be given the structure of a $t$-module.

Suppose $\mathbb{K}^d$ is an $\mathbb{F}_q[t]$-module defined by an $\mathbb{F}_q$-linear representation $\phi\colon \mathbb{F}_q[t] \to \mathrm{Mat}_d(\mathbb{K})$ such that $\phi(t) = \theta I_d + N$ with $N$ nilpotent. If $\Lambda$ is any finitely generated discrete $\mathbb{F}_q[t]$-submodule of $\mathbb{K}^d$ of rank $r$, then following the language of Anderson [1, §4.4], we call $\mathbb{K}^d/\Lambda$ a *$t$-torus* of dimension $d$ and rank $r$. Through its exponential function, every uniformizable $t$-module is isomorphic to a $t$-torus.

Let $\lambda_1, \dots, \lambda_r$ be an $\mathbb{F}_q[t]$-basis for $\Lambda_\Phi$. Choosing this basis amounts to fixing isomorphisms

$$\mathrm{Hom}_{\mathbb{F}_q[t]}(\Lambda_\Phi, \Psi(\mathbb{K})) \cong \Psi(\mathbb{K})^r \cong (\mathbb{K}^e)^r/(\Lambda_\Psi)^r. \tag{16}$$

We let

$$W := W_{(\Phi,\Psi)} := \left\{ (U\lambda_1, \dots, U\lambda_r) \in \mathrm{Mat}_{e\times r}(\mathbb{K}) : \begin{array}{c} U \in \mathrm{Mat}_{e\times d}(\mathbb{K}), \\ U d\Phi(t) = d\Psi(t) U \end{array} \right\}. \tag{17}$$

Suppose $\delta^{(U)} \in \mathrm{Der}_0(\Phi,\Psi) \cap \mathrm{Der}_{in}(\Phi,\Psi)$ for $U \in \mathrm{Mat}_{e\times d}(\mathbb{K})$. By Lemma 5.2 and (16),

$$\mathrm{DR}(\delta^{(U)}) = (\mathrm{Exp}_\Psi(U\lambda_1), \dots, \mathrm{Exp}_\Psi(U\lambda_r)) \in \mathrm{Mat}_{e\times r}(\mathbb{K}),$$

and so $\mathrm{DR}(\mathrm{Der}_0(\Phi,\Psi) \cap \mathrm{Der}_{in}(\Phi,\Psi)) = \mathrm{Exp}_\Psi^{\oplus r}(W)$. Thus by Corollary 2.3, $\mathrm{DR}_{(\Phi,\Psi)}$ induces a homomorphism of $\mathbb{F}_q[t]$-modules,

$$\mathrm{DR}_{(\Phi,\Psi)} : \mathrm{Ext}^1_0(E,F) \to \mathrm{Mat}_{e\times r}(\mathbb{K})/(W + \Lambda_\Psi^r), \tag{18}$$

which is an isomorphism if the original $\mathrm{DR}_{(\Phi,\Psi)}$ is one. Depending on $E$ and $F$, the right-hand side may or may not be isomorphic (rigid analytically) to a $t$-torus. We consider the following situations.

*Example* 5.6. Suppose $E = (\mathbb{G}_a, \Phi)$ and $F = (\mathbb{G}_a, \Psi)$ are Drinfeld modules of rank 2 and $s$ respectively. Let $\lambda_1$, $\lambda_2$ and $\mu_1, \dots, \mu_s$ be generators for their period lattices over $\mathbb{F}_q[t]$. The exact sequence

$$0 \to W \to \mathbb{K}^2 \xrightarrow{f} \mathbb{K} \to 0,$$

where $f(x,y) = \lambda_1 y - \lambda_2 x$, provides a choice of coordinates on the right-hand side of (18), and $f(\Lambda_\Psi \oplus \Lambda_\Psi)$ is the $\mathbb{F}_q[t]$-submodule generated by $\{\lambda_i\mu_j\}_{i=1,2}^{j=1,\dots,s}$.

If $s = 1$, then Proposition 5.5 implies that $\mathrm{Ext}^1_0(E,F) \hookrightarrow \mathbb{K}/(\mathbb{F}_q[t]\lambda_1\mu_1 + \mathbb{F}_q[t]\lambda_2\mu_1)$, and Theorem 1.1 (and also Woo [20, Prop. 7]) confirms that this is an isomorphism of $t$-tori. If $s \geq 2$, then it is easy to construct examples where $f(\Lambda_\Psi \oplus \Lambda_\Psi)$ is not discrete in $\mathbb{K}$, and so in such cases the right-hand side of (18) is not a $t$-torus (cf. Theorem 6.1).

**Proposition 5.7.** *Suppose $E = (\Phi, \mathbb{G}_a)$ and $F = (\Psi, \mathbb{G}_a)$ are Drinfeld modules of rank $r$ and $s$ respectively, with $r > s$. If $\mathrm{DR}_{(\Phi,\Psi)}$ is an isomorphism, then $\mathrm{Ext}^1_0(E,F)$ is isomorphic as an $\mathbb{F}_q[t]$-module to a $t$-torus of dimension $r-1$ and rank $rs$.*

*Proof.* Let $\lambda_1, \dots, \lambda_r$ and $\mu_1, \dots, \mu_s$ be $\mathbb{F}_q[t]$-bases for $\Lambda_\Phi$ and $\Lambda_\Psi$ respectively. Because $E$ and $F$ are both 1-dimensional, $\mathrm{Der}_{in}(\Phi,\Psi) \subset \mathrm{Der}_0(\Phi,\Psi)$. As in (17),

$$W = W_{(\Phi,\Psi)} = \{(U\lambda_1, \dots, U\lambda_r) \in \mathrm{Mat}_{1\times r}(\mathbb{K}) : U \in \mathbb{K}\}.$$

We claim that $W \cap (\Lambda_\Psi)^r = \{0\}$. Suppose $w \in W \cap (\Lambda_\Psi)^r$. Then

$$w = (U\lambda_1, \ldots, U\lambda_r) = (\nu_1, \ldots, \nu_r), \quad \nu_i \in \Lambda_\Psi.$$

Because $r > s$, there is a non-trivial $\mathbb{F}_q[t]$-linear dependency $\sum d\Psi(a_i)\nu_i = 0$, and thus $U \cdot \sum d\Phi(a_i)\lambda_i = 0$. Since $\lambda_1, \ldots, \lambda_r$ are linearly independent over $\mathbb{F}_q[t]$, it follows that $U = 0$.

By the following argument, the image of $\Lambda_\Psi^r$ is discrete in $\mathrm{Mat}_{1\times r}(\mathbb{K})/W$. We proceed by induction on $r$, for which the base case ($r = 2$) is trivial. Furthermore, it suffices to continue with $r = s + 1$, which we will now assume; the cases where $r > s + 1$ follow as straightforward consequences. The $\mathbb{K}$-linear map

$$\begin{array}{rccc} f: & \mathrm{Mat}_{1\times r}(\mathbb{K}) & \to & \mathrm{Mat}_{1\times(r-1)}(\mathbb{K}), \\ & (x_1, \ldots, x_r) & \mapsto & (\lambda_2 x_1 - \lambda_1 x_2, \ldots, \lambda_r x_{r-1} - \lambda_{r-1} x_r), \end{array}$$

has kernel $W$. We need to show that $V := \mathbb{F}_q((1/\theta)) \cdot f(\Lambda_\Psi) \subset \mathrm{Mat}_{1\times(r-1)}(\mathbb{K})$ is an $rs$-dimesional vector space over $\mathbb{F}_q((1/\theta))$. The typical element of $V$ has the form

$$\nu = (\mu_1, \ldots, \mu_s)\, A \begin{pmatrix} \lambda_2 & \cdots & 0 \\ -\lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \lambda_r \\ 0 & \cdots & -\lambda_{r-1} \end{pmatrix}, \tag{19}$$

where $A = (\alpha_{ij}) \in \mathrm{Mat}_{s\times r}(\mathbb{F}_q((1/\theta)))$. Let $B$ denote the $r \times (r-1)$ matrix in the above formula. By our assumption that $s = r - 1$, the matrix $AB$ is square, and its determinant is

$$\det(AB) = \lambda_2 \cdots \lambda_{r-1} \sum_{j=1}^{r} (-1)^{r-j} \lambda_j \det(A_j),$$

where $A_j$ is the $s \times s$ minor of $A$ with the $j$-th column removed. If $\nu = 0$, then this implies that $\det(AB) = 0$, and since $\lambda_1, \ldots, \lambda_r$ are $\mathbb{F}_q((1/\theta))$-linearly independent, $\det(A_j) = 0$ for each $j$. Thus the rank of the matrix $A$ is less than $s$, and we can rewrite the last row of $A$ as an $\mathbb{F}_q((1/\theta))$-linear combination of the other rows, say $\alpha_{sj} = \sum_{i=1}^{s-1} \beta_i \alpha_{ij}$ with $\beta_i \in \mathbb{F}_q((1/\theta))$. The formulation in (19) can be rewritten as

$$\nu = (\mu_1 + \beta_1\mu_s, \ldots, \mu_{s-1} + \beta_{s-1}\mu_s)\widetilde{A}B,$$

where $\widetilde{A}$ is the $(s-1)\times r$ matrix obtained by removing the last row of $A$. Again $\nu = 0$, but then our induction hypothesis with $r$ replaced by $r-1$ allows us to conclude that $\widetilde{A} = 0$. □

## 6. Elliptic curves

Let $E_1$ and $E_2$ be elliptic curves over $\mathbb{C}$. Let $G := \mathrm{Ext}^1_{\mathbb{C}}(E_1, E_2)$ be the extension group in the category of complex abelian varieties. By the Poincaré reducibility theorem, this is a torsion group. The set of complex points $E(\mathbb{C})$ of an elliptic curve $E$ over $\mathbb{C}$ can be viewed as a complex torus written $E^{an}$; so one may also consider $A := \mathrm{Ext}^1(E_1^{an}, E_2^{an})$, the extension group in the category of complex tori. There is a natural homomorphism $G \to A$; the image is the torsion subgroup of $A$ (see [4, Rmk. 6.2, p. 23]).

We present next a theorem of Lichtenbaum (1960's, unpublished); however, our proof is different in that it is based on periods. By comparison to Example 5.6 and

Proposition 5.7, the theorem below indicates that the situation for elliptic curves runs quite parallel to our own. Specifically, in light of in Example 5.6, it suggests that $\mathrm{Ext}^1(E,F)$ will rarely be representable as a $t$-module for nonisogenous Drinfeld modules $E$ and $F$ of rank 2. Moreover, it suggests that $\mathrm{Ext}^1(E,F)$, for general $t$-modules $E$ and $F$, will not always have the structure of a $t$-module.

**Theorem 6.1** (Lichtenbaum). *Let $E_1$ and $E_2$ be elliptic curves over $\mathbb{C}$, and let $G := \mathrm{Ext}^1(E_1,E_2)$ and $A := \mathrm{Ext}^1(E_1^{an},E_2^{an})$.*

(a) *If $E = E_1 = E_2$ has complex multiplication, then $A$ is naturally isomorphic to $E(\mathbb{C})$ (as abelian groups). Under this isomorphism, the group $G$ is identified with the torsion points of $E(\mathbb{C})$.*
(b) *If $E_1$ and $E_2$ are isogenous and admit complex multiplication, then $A$ is isogenous to $E_1(\mathbb{C})$ and $E_2(\mathbb{C})$.*
(c) *If at least one of $E_1$ and $E_2$ does not admit complex multiplication, then the natural topology on $G$ is non-Hausdorff. Therefore, $G$ is not the set of complex points of a complex algebraic variety (with the classical topology).*

*Proof.* Given $E_i$ $(i = 1,2)$ as the quotient of $\mathbb{C}$ by lattices $\mathbb{Z}+\mathbb{Z}\tau_i$, where each $\tau_i$ may be taken to be have positive imaginary part, we see from [4, Prop. 5.7, p. 21] that $A$ is naturally the quotient of $\mathbb{C}$ by the subgroup $\Lambda$ generated by $1,\tau_1,\tau_2$ and $\tau_1\tau_2$.

Let us suppose that an elliptic curve $X$ has complex multiplication; let us think of $X(\mathbb{C})$ as a quotient of $\mathbb{C}$ by $\mathbb{Z}+\mathbb{Z}\tau$. By the theory of complex multiplication [18], we have the following: (i) $\tau$ lies in an imaginary quadratic field; (ii) $\tau^2 = a\tau + b$ for some integers $a$ and $b$; and (iii) if $X'$ (with $X'(\mathbb{C}) = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau')$) is an elliptic curve isogenous to $X$, then $\tau'$ and $\tau$ lie in the same imaginary quadratic field and $X'$ also has complex multiplication with the same CM-field.

If $X(\mathbb{C}) = E = E_1 = E_2$, then we have $\tau = \tau_1 = \tau_2$. By (i) and (ii), we get that $\Lambda$ is the lattice $\mathbb{Z}+\mathbb{Z}\tau$. Thus we obtain that $A$ is naturally identified with $E(\mathbb{C})$. The last statement in (a) follows from [4, Rmk. 6.2, p. 23].

If $E_1$ and $E_2$ are isogenous curves with complex multiplication, then $\tau_1$ and $\tau_2$ lie in the same imaginary quadratic field $F$; in this case, $\Lambda$ is isomorphic to a fractional ideal of an order of $K$. This proves (b).

For (c), suppose at least one of $E_1$ and $E_2$ does not admit complex multiplication. By the fundamental theorem of complex multiplication, $\tau_1$ and $\tau_2$ are not both contained in an imaginary quadratic field. In other words, in this case $\Lambda$ is a subgroup of $\mathbb{Z}$-rank greater than two, and so it is not a discrete subgroup of $\mathbb{C}$. ☐

*Remark* 6.2. (C. Schoen) (a) If $A$ and $B$ are abelian varieties defined over $\overline{\mathbb{Q}}$, then the natural map $\mathrm{Ext}^1_{\overline{\mathbb{Q}}}(A,B) \to \mathrm{Ext}^1_{\mathbb{C}}(A,B)$ is an isomorphism.

(b) Any abelian surface corresponding to an element of $\mathrm{Ext}^1(E_1,E_2)$ is isogenous to the product $E_1 \times E_2$. Any complex abelian variety, which is isogenous to a product of CM-elliptic curves, is itself a product of CM-elliptic curves [17].

## 7. Extensions of $t$-motives

In this section we explore extensions from the standpoint of $t$-motives and examine avenues for further study. Given two $t$-modules $E$, $F$ over $K$, we can consider the associated Anderson $t$-motives $M(E)$ and $M(F)$ [1]. Since the functor $M$ which sends a $t$-module to its associated $t$-motive is contravariant, we obtain a map

$$M^* : \mathrm{Ext}^1(E,F) \to \mathrm{Ext}^1((M(F),M(E));$$

because $M$ gives an anti-equivalence of categories of $t$-modules and $t$-motives [1, Thm. 1], $M^*$ is an isomorphism. If one is interested in computing just the group of extensions of $t$-modules, then it is relatively easy to compute extensions in the category of $t$-motives. We formulate this precisely in the next lemma.

The evident functor $f$ from the category $\mathcal{T}$ of $t$-motives to the category $\mathcal{C}$ of left $K[t,\tau]$-modules is fully faithful [1, §1.2]. Here $K[t,\tau]$ is the noncommutative ring generated by $t$ and $\tau$ with the relations, $t\tau = \tau t, xt = tx, \tau x = x^q\tau$, for all $x \in K$.

**Lemma 7.1.** *For any $t$-motives $A$ and $B$ over $K$, the natural map*

$$f^* : \mathrm{Ext}^1_{\mathcal{T}}(A, B) \to \mathrm{Ext}^1_{\mathcal{C}}(f(A), f(B))$$

*is an isomorphism.*

*Proof.* The fact that $f$ is fully faithful implies that $f^*$ is injective in the following way. Suppose the images of $\alpha$ and $\beta$ under $f^*$ coincide. Pick representatives of $\alpha$ and $\beta$, i.e. extensions $X_1$ and $X_2$ of $A$ by $B$ which satisfy $f(X_1) \cong f(X_2)$. We obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f(Y) & \longrightarrow & f(X_1) & \longrightarrow & f(A) & \longrightarrow & 0 \\ & & \| & & \gamma\downarrow & & \| & & \\ 0 & \longrightarrow & f(Y) & \longrightarrow & f(X_2) & \longrightarrow & f(A) & \longrightarrow & 0, \end{array}$$

where $\gamma$ is an isomorphism of $K[t,\tau]$-modules. Since $f$ is fully faithful, $\gamma$ is an isomorphism of $t$-motives. More precisely, $\gamma$ is induced by an isomorphism $X_1 \xrightarrow{\sim} X_2$ of $t$-motives.

It remains to show the surjectivity of $f^*$. For this, we have to show the following: given any extension $X$ of $f(A)$ by $f(B)$ in $\mathcal{C}$, the left $K[t,\tau]$-module $X$ is a $t$-motive, i.e. (i) it is free and finitely generated as a $K[\tau]$-module and (ii) the associated primes of $X' := X/\tau X$, viewed as a module over the commutative ring $R := K[t]$, consist only of the principal ideal $I := (t-\theta)$. Geometrically, we want the coherent sheaf associated to $X'$ on $\mathbb{A}^1$ to be supported only at the point $t = \theta$.

Condition (i) is clearly satisfied by $X$ by general properties of modules over the ring $K[\tau]$ [12, Prop. 5.4.9]. For (ii), consider an extension $Q$ of $P$ by $N$ where $P$ and $N$ are finitely generated $R$-modules. Every associated prime of $Q$ is an associated prime of either $P$ or $N$. Also, if $H$ is a quotient module of $N$, then every associated prime of $H$ is an associated prime of $N$ (this is easy to see via the geometric interpretation).

Now, by assumption, the associated primes of the $R$-modules $A' := f(A)/\tau f(A)$ and $B' := f(B)/\tau f(B)$ consist of just the ideal $I$. The $R$-module $X'$ is an extension of $A'$ by $B''$ (= a quotient module of $B'$). So we may apply the comments in the previous paragraph to the extension $X'$ to deduce that $X'$ satisfies (ii). □

*Remark* 7.2. Lemma 7.1 shows that extensions may be computed via resolutions of $t$-motives by free $K[t,\tau]$-modules. Furthermore, it implies the injectivity of the map

$$f^* : \mathrm{Ext}^2_{\mathcal{T}}(A, B) \to \mathrm{Ext}^2_{\mathcal{C}}(f(A), f(B)).$$

For $M$ and $N$ as in Remark 7.4 (and $K = \mathbb{F}_q$), this gives $\mathrm{Ext}^2_{\mathcal{T}}(M, N) = 0$.

*Remark* 7.3 (Analogy with $\mathcal{D}$-modules). A. Rosenberg points out that $K[t,\tau]$ and, more generally, skew polynomial algebras are analogous to Weyl algebras in $\mathcal{D}$-module theory (the first Weyl algebra is $\mathbb{C}[x, \partial]$ with the relation $\partial x - x\partial = 1$) in that they are all special cases of *hyperbolic* algebras [16, Ch. II]. Since Ext's of certain (but not all) $\mathcal{D}$-modules possess a nontrivial structure of a $\mathcal{D}$-module, one expects the same to be true for $t$-modules.

The analogy with $\mathcal{D}$-modules is best viewed within the context of opers (see [3, §7.3.14]; a Drinfeld module is an example of a Frobenius oper) and noncommutative algebraic geometry (see [5, Rmk. 5.3.5, §6], [7, §0.6], [16]).

*Remark* 7.4. Geometrically interpreting the definition of a $t$-motive [1, §1.2], a $t$-motive $M$ is a special sheaf $\mathcal{F}_M$ over a noncommutative surface $S$ [16] given by the product of the noncommutative affine line $\mathbb{A}^1_{nc}$ by the commutative affine line $\mathbb{A}^1$. We think of $S$ as fibered over $\mathbb{A}^1$ (viewed horizontally) with (vertical) fibers $\mathbb{A}^1_{nc}$. The sheaf $\mathcal{F}_M$ is the ideal sheaf of a curve $X_M \subset S$ for which the following hold.

(a) $X_M$ is finite over the two axes, via the projections to the components; in other words, $X_M$ is transversal to the horizontal and vertical fibers.

(b) The intersection of $X_M$ with the horizontal axis (which corresponds to $\tau = 0$) is a nilpotent subscheme of $\mathbb{A}^1$ supported at the point $t = \theta$; (a) assures us that the intersection is a proper subscheme of $\mathbb{A}^1$.

If $M$ and $N$ are distinct $t$-motives, then the group $\mathrm{Ext}^1_{\mathcal{T}}(M, N)$ can be interpreted via the "intersection scheme" of the curves $X_M$ and $X_N$ in $S$. We can view the subgroup $\mathrm{Ext}^1_0(M, N)$ (cf. Corollary 2.3) of $\mathrm{Ext}^1_{\mathcal{T}}(M, N)$ as the nontrivial part of the intersection locus corresponding to points distinct from $t = \theta, \tau = 0$, as in (b).

The geometric situation is especially clear in the case $K = \mathbb{F}_q$: $S$ is the usual commutative affine plane $\mathbb{A}^2$; if $g(t, \tau)$ and $h(t, \tau)$ (assumed to have no common factors) are defining equations for the curves $X_M$ and $X_N$, then $\mathrm{Ext}^1_{\mathcal{T}}(M, N)$ is isomorphic to the quotient module $\mathbb{F}_q[t, \tau]/(g, h)$ of the commutative ring $\mathbb{F}_q[t, \tau]$.

*Remark* 7.5. The results and ideas in this paper are used in an ongoing project with Thakur, whose aim is to relate extension groups of $t$-modules to values of zeta functions in the spirit of [2], [10], [15].

## Acknowledgements

We wish to thank G. Anderson, W. D. Brownawell, D. Goss, S. Lichtenbaum, and D. Thakur for encouragement, useful discussions, and helpful advice. Part of the work on this paper was done when the second author was visiting MPIM (Bonn), which he thanks for its hospitality, support, and excellent working conditions.

## References

[1] G. W. Anderson, *t-motives*, Duke Math. J. **53** (1986), 457–502.

[2] G. W. Anderson and D. S. Thakur, *Tensor powers of the Carlitz module and zeta values,* Ann. of Math. (2) **132** (1990), 159–191.

[3] D. Ben-Zvi and E. Frenkel, *Spectral curves, opers and integrable systems*, Inst. Hautes Études Sci. Publ. Math. (to appear), math.AG/9902068.

[4] C. Birkenhake and H. Lange, *Complex tori*, Birkhäuser, Boston, 1999.

[5] A. Blum and U. Stuhler, *Drinfeld modules and elliptic sheaves*, Vector bundles on curves—new directions, Lecture Notes in Math., vol. 1649, Springer, Berlin, 1997, pp. 110–193.

[6] W. D. Brownawell and M. A. Papanikolas, *Linear independence of Gamma values in positive characteristic*, preprint, 2000, math.NT/0106054.

[7] H. Carayol, *Variétés de Drinfel'd compactes, d'après Laumon, Rapoport et Stuhler*, Astérisque (1992), no. 206, Exp. No. 756, 5, 369–409, Séminaire Bourbaki, Vol. 1991/92.

[8] J. A. Carlson, *Extensions of mixed Hodge structures*, Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, Sijthoff & Noordhoff, Alphen aan den Rijn, 1980, pp. 107–127.

[9] P. Deligne, *Théorie de Hodge III*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 44, 5–77.
[10] P. Deligne, *Le groupe fondamental de la droite projective moins trois points*, Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987) (Y. Ihara, K. Ribet, and J.-P. Serre, eds.), Springer, New York, 1989, pp. 79–297.
[11] E.-U. Gekeler, *On the de Rham isomorphism for Drinfeld modules*, J. Reine Angew. Math. **401** (1989), 188–208.
[12] D. Goss, *Basic structures of function field arithmetic*, Springer-Verlag, Berlin, 1996.
[13] W. Messing, *The universal extension of an abelian variety by a vector group*, Symposia Mathematica, Vol. XI (Convegno di Geometria, INDAM, Rome, 1972), Academic Press, London, 1973, pp. 359–372.
[14] J. S. Milne, *Abelian varieties*, Arithmetic Geometry (G. Cornell and J. H. Silverman, eds.), Springer-Verlag, New York, 1986, pp. 103–150.
[15] J. S. Milne and N. Ramachandran, *Integral motives and special values of zeta functions*, preprint, 2001.
[16] A. L. Rosenberg, *Noncommutative algebraic geometry and representations of quantized algebras*, Kluwer Academic Publishers, Dordrecht, 1995.
[17] C. Schoen, *Produkte Abelscher Varietäten und Moduln über Ordnungen*, J. Reine Angew. Math. **429** (1992), 115–123.
[18] J.-P. Serre, *Complex multiplication*, Algebraic Number Theory (J. W. S. Cassels and A. Fröhlich, eds.), Academic Press, San Diego, 1967, pp. 292–296.
[19] Y. Taguchi, *A duality for finite $t$-modules*, J. Math. Sci. Univ. Tokyo **2** (1995), 563–588.
[20] S. S. Woo, *Extensions of Drinfeld modules of rank 2 by the Carlitz module*, Bull. Korean Math. Soc. **32** (1995), 251–257.

Department of Mathematics, Brown University, Providence, RI 02912 USA
*E-mail address*: map@math.brown.edu

Department of Mathematics, University of Maryland, College Park, MD 20742 USA

Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany
*E-mail address*: atma@math.umd.edu