Abelianization of the Second Non-Abelian Galois Cohomology

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Introduction. Let k be a field of characteristic 0, $\bar{k}$ an algebraic closure of k, $\bar{G}$ an algebraic group over $\bar{k}$. Let $L = (\bar{G}, \kappa)$ be a k-kernel (other terms: k-band, k-lien); see [Sp1] or 1.2 below for definition. In [Sp1] (see also 1.3 below) the second non-abelian Galois cohomology set $H^2(k, L)$ (or $H^2(k, \bar{G}, \kappa)$) was defined. (In a more general setting $H^2(k, L)$ was defined in [Gi].) The set $H^2(k, L)$ has a distinguished subset of *neutral* elements. Obstructions to some constructions over k lie in $H^2(k, L)$. A construction is possible if and only if the obstruction is trivial.

The set of neutral elements in $H^2(k, L)$ can be large. In particular, if k is a non-archimedian local field or a totally imaginary number field, and the group $\bar{G}$ is connected semisimple, then, as Douai [Do2] has proved, all the elements of $H^2(k, L)$ are neutral, though the set $H^2(k, L)$ may contain more than one element. It therefore would be convenient to define a map from $H^2(k, L)$ to some abelian group, such that the image of an element $\eta \in H^2(k, L)$ is zero if and only if η is neutral. This is just what we do here when k is a local field or a number field. We use this map to prove a Hasse principle for $H^2(k, L)$ and a Hasse principle for homogeneous spaces.

In [Bo1], for a connected group G over k we defined abelian groups $H^i_{ab}(k, G)$ for $i \geq -1$, and abelianization maps

$$\mathrm{ab}^i \colon H^i(k, G) \to H^i_{ab}(k, G)$$

for $i = 0, 1$. We proved that if k is a local field or a number field, then the map ab^1 is surjective.

In the present paper we define ab^2. Let $L = (\bar{G}, \kappa)$ be a connected k-kernel (i.e. $\bar{G}$ is connected). After some preparations in Sections 1–4, we define in Section 5 the abelian Galois cohomology group $H^2_{ab}(k, L)$ (which is an abelian group), and the abelianization map

$$\mathrm{ab}^2 \colon H^2(k, L) \to H^2_{ab}(k, L)$$

which takes the neutral elements to zero. Our main result is

THEOREM 0.1 (Theorem 5.6). *Let k be a local field or a number field, and L a connected k-kernel. A cohomology class $\eta \in H^2(k, L)$ is neutral if and only if* $\mathrm{ab}^2(\eta) = 0$.

In Section 6 we use Theorem 0.1 to show that in some cases the following Hasse principle holds for $H^2(k, L)$: an element $\eta \in H^2(k, L)$ is neutral if and only if its localizations $\mathrm{loc}_v(\eta) \in H^2(k_v, L)$ are neutral for all the places v of k. A particular case of our results is

THEOREM 0.2 (Consequence of Theorems 6.3 and 6.8). *Let $L = (\bar{G}, \kappa)$ be a connected semisimple k-kernel (i.e. $\bar{G}$ is connected semisimple).*

(i) ([Do2]) *If k is a non-archimedean local field, then any element $\eta \in H^2(k, L)$ is neutral.*

(ii) *If k is a number field, then an element $\eta \in H^2(k, L)$ is neutral if and only if $\mathrm{loc}_v(\eta)$ is neutral for any archimedean place v of k.*

In Section 7 we use the Hasse principle for $H^2(k, L)$ to give a new proof of most of the results of [Bo2] on the Hasse principle for homogeneous spaces. In particular, we give new proofs of Harder's result ([Ha2], 3.3) on the Hasse principle for projective homogeneous spaces, and of Rapinchuk's result ([Ra]) on the Hasse principle for symmetric homogeneous spaces.

In a sense, the map ab^2 is defined in Section 5 indirectly. In the Appendix we give an explicit formula (in terms of cocycles) for the map ab^2, and also explicit cocyclic formulas for the maps ab^0 and ab^1.

This paper emerged as a result of my correspondence with Lawrence Breen, in the course of which Breen defined the abelianization map $\mathrm{ab}^2: H^2(k, G) \to H^2_{\mathrm{ab}}(k, G)$ (using the cohomology theory of crossed modules of *gr*-categories, developped in [Br]). I am deeply grateful to him. It is a pleasure to thank Pierre Deligne for a series of valuable discussions on the Hasse principle for homogeneous spaces. It should be mentioned that when writing this paper, I was inspired by the paper [Ra] of Rapinchuk, who practically proved the Hasse principle for $H^2(k, G)$ when G is a semisimple group.

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Notation.

k is a field of characteristic 0, $\bar{k}$ is a fixed algebraic closure of k, $\Gamma = \mathrm{Gal}(\bar{k}/k)$.

$\bar{G}$ is a $\bar{k}$-group, G is a k-group, sometimes G is a k-form of $\bar{G}$.

Let G be a k-group. Then

G° is the connected component of G;

G^{u} is the unipotent radical of G°;

$G^{\mathrm{red}} = G/G^{\mathrm{u}}$ (this group is reductive);

G^{ss} is the derived group of G^{red} (it is semisimple);

G^{sc} is the universal covering of G^{ss} (it is simply connected);

$G^{\mathrm{tor}} = G^{\mathrm{red}}/G^{\mathrm{ss}}$ when G is connected (then G^{tor} is a k-torus).

Following Deligne we define the composition

$$\rho: G^{\mathrm{sc}} \to G^{\mathrm{ss}} \to G^{\mathrm{red}}.$$

When G is reductive we write G^{ad} for G/Z, where Z is the center or G.

Let $\bar{G}$ be a $\bar{k}$-group. We define $\bar{k}$-groups $\bar{G}^{\mathrm{red}}$, $\bar{G}^{\mathrm{ss}}$, $\bar{G}^{\mathrm{sc}}$ and, when $\bar{G}$ is connected, a $\bar{k}$-group $\bar{G}^{\mathrm{tor}}$, as above. We also define $\bar{\rho}: \bar{G}^{\mathrm{sc}} \to \bar{G}^{\mathrm{red}}$.

Let $L = (\bar{G}, \kappa)$ be a k-kernel (see 1.2 for the definition). We say that the kernel L is connected (reductive, semisimple, and so on) if $\bar{G}$ is so.

Let ψ be a cocycle. We write $\mathrm{Cl}(\psi)$ for the cohomology class of ψ.

1. Kernels and H^2. In this section we recall the definition of the second non-abelian Galois cohomology (cf. [Sp1]).

1.1. Let k be a field of characteristic 0, $\bar{k}$ an algebraic closure of k, and $\bar{G}$ an algebraic group over $\bar{k}$.

Consider the canonical morphisms

$$\bar{G} \xrightarrow{w} \operatorname{Spec} \bar{k} \longrightarrow \operatorname{Spec} k,$$

where w is the structure morphism. Let $\sigma \in \Gamma = \mathrm{Gal}(\bar{k}/k)$. A *$\sigma$-semialgebraic automorphism* of $\bar{G}$ over k is an automorphism s of $\bar{G}$ as a group scheme over k, such that s is compatible with σ, i.e. $w \circ s = \beta_\sigma \circ w$ where β_σ is the automorphism of $\operatorname{Spec} \bar{k}$ induced by σ. A *semialgebraic automorphism* of $\bar{G}$ over k is a σ-semialgebraic automorphism for some $\sigma \in \Gamma$; then such σ is unique.

Let $\mathrm{SAut}_k \bar{G}$ (or just $\mathrm{SAut}\, \bar{G}$) denote the group of semialgebraic automorphisms of $\bar{G}$ over k. As usual, $\mathrm{Aut}\, \bar{G}$ denotes the group of algebraic automorphisms of $\bar{G}$ over $\bar{k}$. We have an exact sequence

$$1 \longrightarrow \mathrm{Aut}\, \bar{G} \longrightarrow \mathrm{SAut}\, \bar{G} \longrightarrow \Gamma. \tag{1.1.1}$$

A k-form G of $\bar{G}$ defines a continuous homomorphism

$$\sigma \mapsto \sigma_* \colon \Gamma \longrightarrow \mathrm{SAut}\ \bar{G}. \tag{1.1.2}$$

which is a splitting of (1.1.1).

Let $\bar{Z}$ be the center of $\bar{G}$. Then $\mathrm{Int}\, \bar{G} = \bar{G}(\bar{k})/\bar{Z}(\bar{k})$. The subgroup $\mathrm{Int}\, \bar{G}$ is normal in $\mathrm{SAut}\, \bar{G}$. Set

$$\mathrm{Out}\, \bar{G} = \mathrm{Aut}\, \bar{G}/\mathrm{Int}\, \bar{G}$$
$$\mathrm{SOut}\, \bar{G} = \mathrm{SAut}\, \bar{G}/\mathrm{Int}\, \bar{G}$$

We have an exact sequence

$$1 \longrightarrow \mathrm{Out}\, \bar{G} \longrightarrow \mathrm{SOut}\, \bar{G} \xrightarrow{q} \Gamma \tag{1.1.3}$$

1.2. A *k-kernel* (k-band, k-lien) is a pair $L = (\bar{G}, \kappa)$, where $\bar{G}$ is a $\bar{k}$-group and κ is a splitting of (1.1.3), i.e. a continuous homomorphism $\kappa \colon \Gamma \to \mathrm{SOut}\, \bar{G}$ such that $q \circ \kappa$ is the identity map of Γ.

Let G be a k-group. Then G defines a homomorphism

$$\kappa_G \colon \Gamma \longrightarrow \operatorname{SAut} G_{\bar{k}} \longrightarrow \operatorname{SOut} G_{\bar{k}},$$

and thus a k-kernel $L_G = (G_{\bar{k}}, \kappa_G)$.

1.3. For a k-kernel $L = (\bar{G}, \kappa)$ we define the second Galois cohomology set $H^2(k, L) = H^2(k, \bar{G}, \kappa)$ in terms of cocycles. For a definitions in terms of extensions see [Sp1].

A 2-cocycle is a pair (f, u) of continuous maps

$$f : \Gamma \to \operatorname{SAut} \bar{G}, \quad u : \Gamma \times \Gamma \to \bar{G}(\bar{k})$$

such that for any $\sigma, \tau, \upsilon \in \Gamma$ the following holds:

$$\operatorname{int}(u_{\sigma,\tau}) \circ f_\sigma \circ f_\tau = f_{\sigma\tau} \tag{1.3.1}$$

$$u_{\sigma,\tau\upsilon} \cdot f_\sigma(u_{\tau,\upsilon}) = u_{\sigma\tau,\upsilon} \cdot u_{\sigma,\tau} \tag{1.3.2}$$

$$f_\sigma \bmod \operatorname{Int} \bar{G} = \kappa(\sigma) \tag{1.3.3}$$

Let $Z^2(k, L)$ denote the set of 2-cocycles with coefficients in L. The group $C(k, \bar{G})$ of continuous maps $c \colon \Gamma \to \bar{G}(\bar{k})$ acts on $Z^2(k, L)$ on the left by

$$c \cdot (f, u) = (f', u')$$

where

$$f'_\sigma = \operatorname{int}(c_\sigma) \circ f_\sigma \tag{1.3.4}$$

$$u'_{\sigma,\tau} = c_{\sigma\tau} \cdot u_{\sigma,\tau} \cdot f_\sigma(c_\tau)^{-1} \cdot c_\sigma^{-1} \tag{1.3.5}$$

The quotient set $H^2(k, L) = C(k, \bar{G}) \backslash Z^2(k, L)$ is called *the second cohomology set of k with coefficients in L*. If $(f, u) \in Z^2(k, L)$, we write $\operatorname{Cl}(f, u)$ for the cohomology class of (f, u) in $H^2(k, L)$.

Remark 1.3.1. Our notation slightly differs from that of [Sp1], who writes 2-cocycles in the form (f, g) where $g_{\sigma,\tau} = u_{\sigma,\tau}^{-1}$.

1.4. *A neutral 2-cocycle* is a cocycle of the form $(f, 1)$. *A neutral cohomology class* in $H^2(k, L)$ is the class of a neutral cocycle.

The set $H^2(k, L)$ does not necessarily contain a neutral element (for example, it may be empty). On the other hand, $H^2(k, L)$ may contain more than one neutral element.

Let G be a k-group. We write $H^2(k, G)$ for $H^2(k, L_G)$. The set $H^2(k, G)$ contains a canonical neutral element $\operatorname{Cl}(\sigma \mapsto \sigma_*, 1)$, where σ_* is as in (1.1.2).

1.5. Let $L = (\bar{G}, \kappa)$ be a k-kernel, and let $\bar{N} \subset \bar{G}$ be a normal $\bar{k}$-subgroup. Assume that $\bar{N}$ is invariant under all the semialgebraic automorphisms of $\bar{G}$. Set $\bar{G}' = \bar{G}/\bar{N}$.

Then the canonical homomorphism $\operatorname{SAut}\bar{G} \to \operatorname{SAut}\bar{G}'$ defines a homomorphism $\kappa' : \Gamma \to \operatorname{SOut}\bar{G}'$, and $L' = (\bar{G}', \kappa')$ is a k-kernel. We have then a canonical map $H^2(k, L) \to H^2(k, L')$ which takes neutral elements to neutral elements.

1.6. Let $L = (\bar{G}, \kappa)$ be a k-kernel, and let $\bar{Z}$ be the center of $\bar{G}$. Then κ defines a k-form Z of $\bar{Z}$. We say that Z is *the center of* L. The abelian group $H^2(k, Z)$ acts on the set $H^2(k, L)$ as follows:

$$\mathrm{Cl}(\varphi) \cdot \mathrm{Cl}(f, u) = \mathrm{Cl}(f, \varphi u)$$

where $\varphi \in Z^2(k, Z)$. This action is defined correctly.

LEMMA 1.7. *Let Z be the center of a k-kernel L. If $H^2(k, L) \neq \emptyset$, then the action of $H^2(k, Z)$ on $H^2(k, L)$ is simply transitive.*

In other words, the set $H^2(k, L)$ is either empty, or a principal homogeneous space of the abelian group $H^2(k, Z)$.

Proof. See [Sp1], 1.17 or [Mc], IV-8.8.

1.8. Let $\eta_1, \eta_2 \in H^2(k, L)$. By Lemma 1.7 there exists a unique element $\eta_Z \in H^2(k, Z)$ such that $\eta_2 = \eta_Z + \eta_1$. We write

$$\eta_Z = \eta_2 - \eta_1 .$$

2. Neutral cohomology classes. In this section we investigate the set of neutral cohomology classes in $H^2(k, \bar{G}, \kappa)$.

2.1. Let $\bar{G}$ be a $\bar{k}$-group. A k-form G of $\bar{G}$ defines a homomorphism

$$f : \Gamma \to \operatorname{SAut}\bar{G}, \qquad f_\sigma = \sigma_* ,$$

and thus a neutral 2-cocycle

$$(f, 1) \in Z^2(k, \bar{G}, \kappa_G), \text{ where } \kappa_G(\sigma) = f_\sigma \bmod \operatorname{Int}\bar{G} .$$

Set

$$n(G) = \mathrm{Cl}(f, 1) \in H^2(k, \bar{G}, \kappa_G).$$

We call $n(G)$ *the neutral cohomology class defined by* G.

Let

$$\psi : \Gamma \to (G/Z)(\bar{k}) = \operatorname{Int}\bar{G}$$

be a cocycle, where Z is the center of G. The twisted group $G' = {}_\psi G$ defines a homomorphism $f' : \Gamma \to \operatorname{SAut}\bar{G}$, and we have $f'_\sigma = \psi_\sigma f_\sigma$. We see that

$$f'_\sigma \bmod \operatorname{Int}\bar{G} \;=\; f_\sigma \bmod \operatorname{Int}\bar{G} \;=\; \kappa_G(\sigma),$$

and therefore the neutral cohomology class $n(G') = \mathrm{Cl}(f', 1)$, defined by $G' = {}_\psi G$, lies in $H^2(k, \bar{G}, \kappa_G) = H^2(k, G)$.

Now let $\varphi \in Z^1(k, G)$ be a cocycle with values in $G(\bar{k})$. One can easily check that $n({}_\varphi G) = n(G)$.

LEMMA 2.2. *Let $L = (\bar{G}, \kappa)$ be a k-kernel, and let $\eta \in H^2(k, L)$ be a neutral class. Then we have $\eta = n(G)$ for some k-form G of $\bar{G}$. This k-form G is defined uniquely modulo twisting by a cocycle $\varphi \in Z^1(k, G)$.*

Proof. Write $\eta = \mathrm{Cl}(f, 1)$. Then $f: \Gamma \to \mathrm{SAut}\, \bar{G}$ is a homomorphism, and it defines a k-form G of $\bar{G}$. We have then $\eta = n(G)$. We leave the rest to the reader.

The main result of this section is the following characterization of the neutral classes in $H^2(k, G)$.

PROPOSITION 2.3. *Let G be a k-group, Z its center. An element $\eta \in H^2(k, G)$ is neutral if and only if*

$$\eta - n(G) \in \mathrm{im}\, [\delta: H^1(k, G/Z) \to H^2(k, Z)],$$

where δ is the connecting (coboundary) map.

To prove the proposition we need two lemmas.

LEMMA 2.4. *An element $\eta \in H^2(k, G)$ is neutral if and only if $\eta = n({}_\psi G)$ for some $\psi \in Z^1(k, G/Z)$.*

Proof. If $\psi \in Z^1(k, G/Z)$, then ${}_\psi G$ defines a neutral class $n({}_\psi G) \in H^2(k, G)$, see 2.1. Conversely, let $\eta \in H^2(k, G)$ be a neutral element. Write

$$n(G) = \mathrm{Cl}(f, 1), \qquad \eta = \mathrm{Cl}(f', 1), \qquad f'_\sigma = \psi_\sigma f_\sigma\,,$$

where $f_\sigma = \sigma_*$. By (1.3.3) $\psi_\sigma \in \mathrm{Int}\, G_{\bar{k}} = (G/Z)(\bar{k})$. It follows from (1.3.1) that

$$\psi_{\sigma\tau} = \psi_\sigma f_\sigma \psi_\tau f_\sigma^{-1} = \psi_\sigma \cdot {}^\sigma\psi_\tau\,.$$

Hence $\psi: \Gamma \to (G/Z)(\bar{k})$ is a cocycle. We have $\eta = \mathrm{Cl}(\psi f, 1) = n({}_\psi G)$. The lemma is proved.

LEMMA 2.5. *Let G and Z be as in Proposition 2.3, and $\psi \in Z^1(k, G/Z)$. Then*

$$n({}_\psi G) - n(G) = \delta(\mathrm{Cl}(\psi))$$

where δ is the connecting map.

Proof. Let $f: \Gamma \to \mathrm{SAut}\, G_{\bar{k}}$ be the homomorphism $\sigma \mapsto \sigma_*$ defined by G. Let $f': \Gamma \to \mathrm{SAut}\, G_{\bar{k}}$ be the homomorphism defined by the inner form ${}_\psi G$ of G. Then $f' = \psi f$.

Let $\tilde{\psi}: \Gamma \to G(\bar{k})$ be a continuous map lifting ψ. By (1.3.4) and (1.3.5)

$$(\psi f, 1) = \tilde{\psi} \cdot (f, \lambda) \text{ where } \lambda_{\sigma,\tau} = \tilde{\psi}_{\sigma\tau}^{-1} \cdot \tilde{\psi}_\sigma \cdot f_\sigma(\tilde{\psi}_\tau).$$

Since ψ is a cocycle, $\lambda_{\sigma,\tau} \in Z(\bar{k})$), and we may write

$$\lambda_{\sigma,\tau} = \tilde{\psi}_\sigma \cdot f_\sigma(\tilde{\psi}_\tau) \cdot \tilde{\psi}_{\sigma\tau}^{-1}\,.$$

By definition $\mathrm{Cl}(\lambda) = \delta(\mathrm{Cl}(\psi))$ (see [Se], I-5.6 for the definition of δ). Thus we have

$$\mathrm{Cl}(\psi f, 1) = \mathrm{Cl}(f, \lambda) = \mathrm{Cl}(\lambda) + \mathrm{Cl}(f, 1) = \delta(\mathrm{Cl}(\psi)) + \mathrm{Cl}(f, 1),$$

whence

$$n({}_{\psi}G) - n(G) = \mathrm{Cl}(\psi f, 1) - \mathrm{Cl}(f, 1) = \delta(\mathrm{Cl}(\psi)),$$

which was to be proved.

2.6. *Proof of Proposition* 2.3. Let $\eta \in H^2(k, G)$ be a neutral element. By Lemma 2.4 $\eta = n({}_{\psi}G)$ for some $\psi \in Z^1(k, G/Z)$. By Lemma 2.5 then $\eta - n(G) = \delta(\mathrm{Cl}(\psi))$.

Conversely, suppose that $\eta - n(G) \in \mathrm{im}\ \delta$, i.e

$$\eta = \delta(\mathrm{Cl}(\psi)) + n(G)$$

for some $\psi \in Z^1(k, G/Z)$. By Lemma 2.5 $\eta = n({}_{\psi}G)$, hence η is neutral. The proposition is proved.

3. Connected reductive kernels. In this section we prove

PROPOSITION 3.1 ([Dol]). *Let $L = (\bar{G}, \kappa)$ be a connected reductive k-kernel. Then $H^2(k, L)$ contains a neutral element.*

To prove Proposition 3.1 we need the notion of based root datum.

3.2. Let G_0 be a *split* connected reductive group over k. Let $T \subset G_0$ be a split maximal torus, and B a Borel subgroup containing T. To the triple (G_0, B, T) we associate the based root datum $\Psi = \Psi(G_0, B, T)$ (cf. [Sp2], 2.3). By definition $\Psi = (X, X^\vee, \Phi, \Phi^\vee, \Pi, \Pi^\vee)$. Here X is the character group of T, and $X^\vee$ is the cocharacter group; Φ is the the root system of (G, T), and $\Phi^\vee$ is the corooot system; Π is the basis of Φ defined by B, and $\Pi^\vee$ is the dual basis of $\Phi^\vee$. We have an exact sequence

$$1 \longrightarrow G_0^{\mathrm{ad}}(\bar{k}) \longrightarrow \mathrm{Aut}\, G_{0\bar{k}} \longrightarrow \mathrm{Aut}\, \Psi \longrightarrow 1 \qquad (3.2.1)$$

(cf. [Sp2], 2.14).

For a root $\alpha \in \Phi$ let U_α be the corresponding one-parameter unipotent subgroup of B. For any $\alpha \in \Pi$ choose an element $u_\alpha \in U_\alpha(k)$. Such a choice defines a splitting

$$s\colon \mathrm{Aut}\, \Psi \longrightarrow \mathrm{Aut}_k G_0 \qquad (3.2.2)$$

of the exact sequence (3.2.1) (cf. [Sp2], 2.13), where $\mathrm{Aut}_k G_0 \subset \mathrm{Aut}\, G_{0\bar{k}}$ is the group of k-automorphisms of G_0.

The Galois group $\Gamma = \mathrm{Gal}(\bar{k}/k)$ acts on the terms of the exact sequence (3.2.1). Since it acts on Ψ trivially, the splitting s mentioned above is Γ-equivariant.

3.3. Let $\bar{G}$ be a connected reductive $\bar{k}$-group. It follows from Chevalley's theorem that $\bar{G}$ admits a split k-form G_0. Choose T and B as above and construct $\Psi = \Psi(G_0, B, T)$.

LEMMA 3.3.1 *There exists a canonical bijection between the set of k-kernels* $L = (\bar{G}, \kappa)$ *with given* $\bar{G}$ *and the set of continuous homomorphisms* $\mu: \Gamma \to \operatorname{Aut}\Psi$.

Proof. The split k-form G_0 of $\bar{G}$ defines a splitting of the exact sequence (1.1.3). Thus $\operatorname{SOut}\bar{G}$ becomes a semi-direct product of $\operatorname{Out}\bar{G}$ and Γ. The exact sequence (3.2.1) defines a Γ-equivariant isomorphism $\operatorname{Out}\bar{G} \to \operatorname{Aut}\Psi$. Since Γ acts on Ψ trivially, we obtain an isomorphism

$$\operatorname{Aut}\Psi \times \Gamma \xrightarrow{\sim} \operatorname{SOut}\bar{G}, \tag{3.3.2}$$

and the lemma follows.

3.4. *Proof of Proposition* 3.1. Let $L = (\bar{G}, \kappa)$ be a k-kernel. Let G_0 be a split form of $\bar{G}$, and let T, B, Ψ and $(u_\alpha)_{\alpha\in\Pi}$ be as in 3.2. Then by Lemma 3.3.1 κ defines a homomorphism $\mu: \Gamma \to \operatorname{Aut}\Psi$. Set $\psi = s \circ \mu$, where s is the splitting (3.2.2) of (3.2.1). Then ψ is a homomorphism, and $\psi_\sigma \in (\operatorname{Aut} G_0)(k)$ for any $\sigma \in \Gamma$. We see that ψ is a cocycle, $\psi \in Z^1(k, \operatorname{Aut} G_0)$. Set $G = {}_\psi G_0$. Then $n(G)$ is a neutral element of $H^2(k, L)$. The proposition is proved.

4. Non-reductive kernels. Let $L = (\bar{G}, \kappa)$ be an arbitrary k-kernel (we do not assume $\bar{G}$ to be connected). The normal subgroup $\bar{G}^{\mathrm{u}}$ of $\bar{G}$ (see Notation) is invariant under all the semialgebraic automorphisms of $\bar{G}$. Set $\bar{G}^{\mathrm{red}} = \bar{G}/\bar{G}^{\mathrm{u}}$. By 1.5 there exists a k-kernel $L^{\mathrm{red}} = (\bar{G}^{\mathrm{red}}, \kappa^{\mathrm{red}})$ and a canonical map $r: H^2(k, L) \to H^2(k, L^{\mathrm{red}})$. In this section we prove

PROPOSITION 4.1. *Let* $L = (\bar{G}, \kappa)$ *be a k-kernel. An element* $\eta \in H^2(k, L)$ *is neutral if and only if* $r(\eta)$ *is neutral.*

COROLLARY 4.2 ([Dol]). *Let* $(\bar{U}, \kappa)$ *be a unipotent k-kernel. Then any element* $\eta \in H^2(k, \bar{U}, \kappa)$ *is neutral.*

To prove Proposition 4.1 we need

LEMMA 4.3. *Let A be a commutative unipotent k-group. Then* $H^2(k, A) = 0$.

Proof. Since $\operatorname{char}(k) = 0$, A is isomorphic to a direct product of a number of copies of the additive group $\mathbb{G}_a$. We have $H^2(k, \mathbb{G}_a) = 0$, hence $H^2(k, A) = 0$, which was to be proved.

4.4. *Proof of Proposition* 4.1. We proceed by induction. We assume that $\bar{G}^{\mathrm{u}} \neq 1$. Let $\bar{A}$ be the center of $\bar{G}^{\mathrm{u}}$. Since $\bar{G}^{\mathrm{u}}$ is unipotent, we have $\dim \bar{A} > 0$ (cf. e.g. [Hu], 17.4 and 17.5). The subgroup $\bar{A}$ is normal in $\bar{G}$ and invariant under all the semialgebraic automorphisms of $\bar{G}$. Set $\bar{G}' = \bar{G}/\bar{A}$, then by 1.5 we get a k-kernel $L' = (\bar{G}', \kappa')$ and a canonical map $\nu: H^2(k, L) \to H^2(k, L')$. We have $\dim(\bar{G}')^{\mathrm{u}} < \dim \bar{G}^{\mathrm{u}}$. We therefore may and will assume that Proposition 4.1 holds for L'.

Let $\eta \in H^2(k, L)$ be a cohomology class, and suppose that $r(\eta)$ is neutral. We must prove that η is neutral. Since Proposition 4.1 holds for L', the image $\eta' = \nu(\eta)$ of η in

$H^2(k,L')$ is neutral. Write $\eta = \mathrm{Cl}(f,u)$, $\eta' = \mathrm{Cl}(f',u')$, where f' and u' are the maps defined by f and u, respectively. Since η' is neutral, we may choose the cocycle (f,u) in such a way that the cocycle (f',u') is neutral, i.e $u'=1$ and f' is a homomorphism. Then $\sigma \mapsto f_\sigma|_{\bar{A}}$ is a homomorphism; it defines a k-form A of $\bar{A}$. We can regard u as a map $\Gamma \to \bar{A}(\bar{k})$, and one can check that $u \in Z^2(k,A)$. By Lemma 4.3 $\mathrm{Cl}(u)=0$ in $H^2(k,A)$, i.e there exists a continuous map $c\colon \Gamma \to A(\bar{k})$ such that

$$c_{\sigma\tau} \cdot u_{\sigma,\tau} \cdot f_\sigma(c_\tau)^{-1} \cdot c_\sigma^{-1} = 1\,.$$

Then $c \cdot (f,u)$ is a neutral cocycle, and thus $\eta = \mathrm{Cl}(f,u)$ is neutral. The proposition is proved.

5. Abelianization. In this section for a connected k-kernel $L = (\bar{G},\kappa)$ we define an abelian group $H^2_{\mathrm{ab}}(k,L)$ and an abelianization map $\mathrm{ab}^2\colon H^2(k,L) \to H^2_{\mathrm{ab}}(k,L)$ which takes the neutral cohomology classes to zero. We prove that when k is a local field or a number field, an element $\eta \in H^2(k,L)$ is neutral if and only if $\mathrm{ab}^2(\eta)=0$.

5.1. First we assume that $L = (\bar{G},\kappa)$ is a *reductive* k-kernel. Let $\bar{G}^{\mathrm{ss}}$, $\bar{G}^{\mathrm{sc}}$ and $\bar{\rho}\colon \bar{G}^{\mathrm{sc}} \to \bar{G}$ be as in Notation. Let $\bar{Z}$ be the center of $\bar{G}$, $\bar{Z}^{(\mathrm{ss})}$ the center of $\bar{G}^{\mathrm{ss}}$, and $\bar{Z}^{(\mathrm{sc})}$ the center of $\bar{G}^{\mathrm{sc}}$. Note that κ defines k-forms Z of $\bar{Z}$, $Z^{(\mathrm{ss})}$ of $\bar{Z}^{(\mathrm{ss})}$, and $Z^{(\mathrm{sc})}$ of $\bar{Z}^{(\mathrm{sc})}$. The homomorphism

$$\bar{\rho}\colon Z^{(\mathrm{sc})} \to Z \tag{5.1.1}$$

is defined over k.

We regard the homomorphism (5.1.1) as a short complex of abelian k-groups

$$1 \longrightarrow Z^{(\mathrm{sc})} \xrightarrow{\rho} Z \longrightarrow 1 \tag{5.1.2}$$

where Z is in degree 0 and $Z^{(\mathrm{sc})}$ is in the degree -1. For $i \geq -1$ we set

$$H^i_{\mathrm{ab}}(k,L) = \mathbb{H}^i(k,\ Z^{(\mathrm{sc})} \to Z)$$

(the Galois hypercohomology group of k with coefficients in the complex (5.1.2)). In this paper we are interested in $H^2_{\mathrm{ab}}(k,L)$.

5.2. With the assumptions and notation of 5.1 consider the short exact sequence of complexes

$$1 \longrightarrow (1 \to Z) \longrightarrow (Z^{(\mathrm{sc})} \to Z) \longrightarrow (Z^{(\mathrm{sc})} \to 1) \longrightarrow 1$$

and the associated hypercohomology exact sequence

$$\ldots \longrightarrow H^2(k,Z^{(\mathrm{sc})}) \xrightarrow{\rho_*} H^2(k,Z) \longrightarrow H^2_{\mathrm{ab}}(k,L) \longrightarrow \ldots \tag{5.2.1}$$

Set

$$H^2_{\mathrm{q}}(k,L) = H^2(k,Z)/\rho_* H^2(k,Z^{(\mathrm{sc})})\,.$$

We call $H^2_{\mathrm{q}}(k,L)$ *the quotient cohomology group.* The exact sequence (5.2.1) defines an embedding $H^2_{\mathrm{q}}(k,L) \to H^2_{\mathrm{ab}}(k,L)$.

To define the abelianization map we need

LEMMA 5.3. *Let $L = (\bar{G}, \kappa)$ be a connected reductive k-kernel. Let $\eta, \eta' \in H^2(k, L)$ be two neutral elements. Then with the notation of* 5.1

$$\eta' - \eta \in \mathrm{im}\,[\rho_*: H^2(k, Z^{(sc)}) \to H^2(k, Z)].$$

Proof. By Lemma 2.2 $\eta = n(G)$ for some form G of $\bar{G}$. By Proposition 2.3

$$\eta' - \eta \in \mathrm{im}\,[\delta: H^1(k, G^{ad}) \to H^2(k, Z)].$$

From the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z^{(sc)} & \longrightarrow & G^{sc} & \longrightarrow & G^{ad} & \longrightarrow & 1 \\ & & \downarrow \rho & & \downarrow \rho & & \| & & \\ 1 & \longrightarrow & Z & \longrightarrow & G & \longrightarrow & G^{ad} & \longrightarrow & 1 \end{array} \tag{5.3.1}$$

we obtain the commutative diagram

$$\begin{array}{ccc} H^1(k, G^{ad}) & \xrightarrow{\delta'} & H^2(k, Z^{(sc)}) \\ \| & & \downarrow \rho_* \\ H^1(k, G^{ad}) & \xrightarrow{\delta} & H^2(k, Z) \end{array} \tag{5.3.2}$$

We see that $\mathrm{im}\,\delta = \mathrm{im}\,(\rho_* \circ \delta')$. Thus $\eta' - \eta \in \mathrm{im}\,\rho_*$, which was to be proved.

5.4. *The abelianization map.* Let $L = (\bar{G}, \kappa)$ be a connected reductive k-kernel. We define the abelianization map

$$H^2(k, L) \to H^2_{\mathrm{q}}(k, L) \to H^2_{\mathrm{ab}}(k, L)$$

as follows.

Let $\eta \in H^2(k, L)$. By Proposition 3.1 there exists a neutral element $\eta' \in H^2(k, L)$. We set

$$\mathrm{ab}^2(\eta) = (\eta - \eta') \bmod \rho_* H^2(k, Z^{(sc)}) \in H^2_{\mathrm{q}}(k, L) \subset H^2_{\mathrm{ab}}(k, L).$$

If $\eta'' \in H^2(k, L)$ is another neutral element, then by Lemma 5.3 $\eta' - \eta'' \in \rho_*(H^2(k, Z^{(sc)}))$, and we see that the image of η in $H^2_{\mathrm{ab}}(k, L)$ does not depend on the choice of η'. Thus the map ab^2 is defined correctly.

The map ab^2 takes the neutral cohomology classes to 0. The image of ab^2 is all the set $H^2_{\mathrm{q}}(k, L)$. Indeed, for any $\eta_Z \in H^2(k, Z)$ there exists $\eta \in H^2(k, L)$ such that $\eta - \eta' = \eta_Z$.

5.5. *The abelianization map (cont.).* Let $L = (\bar{G}, \kappa)$ be any connected k-kernel, not necessarily reductive. We set

$$H^2_{\mathrm{ab}}(k, L) = H^2_{\mathrm{ab}}(k, L^{\mathrm{red}})$$

and define the abelianization map as the composition

$$\mathrm{ab}^2\colon H^2(k, L) \longrightarrow H^2(k, L^{\mathrm{red}}) \xrightarrow{\mathrm{ab}^2} H^2_{\mathrm{ab}}(k, L^{\mathrm{red}}) = H^2_{\mathrm{ab}}(k, L)$$

It is clear that ab^2 takes the neutral elements to 0.

The main result of the present paper is

THEOREM 5.6. *Let k is a local field (archimedian or not) or a number field. Let $L = (\bar{G}, \kappa)$ be a connected k-kernel. Then an element $\eta \in H^2(k, L)$ is neutral if and only if* $\mathrm{ab}^2(\eta) = 0$.

To prove Theorem 5.6 we need

LEMMA 5.7. *Let G be a semisimple simply connected group over a field k of characteristic 0, and let Z be the center of G. If k is either a local field or a number field, then the connecting map $\delta\colon H^1(k, G^{\mathrm{ad}}) \to H^2(k, Z)$ is surjective.*

Proof. In the case of a non-archimedian local field see Kneser [Kn1] (see also [PR], §6.5, Theorem 21). In the real case the assertion follows from the existence of a maximal torus $T \subset G$ such that $H^2(k, T) = 0$, which was proved by Harder [Ha1], Lemma 4.2.3 (see also [PR], §6.5, Lemma 18). In the case of a number field see [Kn2], Ch. 5, Theorem 1.7, p. 77 (see also [Sa], 4.5, and [PR], §6.5, Theorem 20).

5.8. *Proof of Theorem* 5.6. If $\eta \in H^2(k, L)$ is neutral, then $\mathrm{ab}^2(\eta) = 0$. We must prove that if $\mathrm{ab}^2(\eta) = 0$, then η is neutral. By Proposition 4.1 it suffices to prove the assertion for L^{red}. We therefore may and will assume that L is reductive.

Let $\eta \in H^2(k, L)$ be an element such that $\mathrm{ab}^2(\eta) = 0$. By Proposition 3.1 and Lemma 2.2 there exists a neutral class $n(G)$ in $H^2(k, L)$, where G is a k-form of $\bar{G}$. Then

$$\eta - n(G) = \rho_*(\chi)$$

for some $\chi \in H^2(k, Z^{(\mathrm{sc})})$. By Lemma 5.7 there exists $\xi \in H^2(k, G^{\mathrm{ad}})$ such that $\chi = \delta'(\xi)$ with the notation of the commutative diagram (5.3.2). Then $\eta - n(G) = \delta(\xi)$. We see that $\eta - n(G) \in \operatorname{im} \delta$, thus by Proposition 2.3 η is neutral (namely, $\eta = n({}_{\psi}G)$, where $\psi \in Z^1(k, G^{\mathrm{ad}})$ is a cocycle representing ξ). The theorem is proved.

6. A Hasse principle for H^2.

In this section L is a connected k-kernel, where k is a non-archimedian local field or a number field. We apply Theorem 5.6 to prove a Hasse principle for non-abelian H^2.

6.1. Let $L = (\bar{G}, \kappa)$ be a connected k-kernel. We set $\bar{G}^{\mathrm{tor}} = \bar{G}^{\mathrm{red}}/\bar{G}^{\mathrm{ss}}$. The $\bar{k}$-group $\bar{G}^{\mathrm{tor}}$ is a torus, and the homomorphism κ defines a k-form G^{tor} of $\bar{G}^{\mathrm{tor}}$. We have a canonical map $t\colon H^2(k, L) \to H^2(k, G^{\mathrm{tor}})$.

Let Z, $Z^{(\mathrm{ss})}$ and $Z^{(\mathrm{sc})}$ be as in 5.1. From the short exact sequence of complexes

$$1 \longrightarrow (Z^{(\mathrm{sc})} \to Z^{(\mathrm{ss})}) \longrightarrow (Z^{(\mathrm{sc})} \to Z) \longrightarrow (1 \to G^{\mathrm{tor}}) \longrightarrow 1$$

we obtain the hypercohomology exact sequence

$$H^3(k, \ker \rho) \longrightarrow H^2_{\mathrm{ab}}(k, L) \xrightarrow{t_{\mathrm{ab}}} H^2(k, G^{\mathrm{tor}}), \tag{6.1.1}$$

because

$$\mathbb{H}^2(k, Z^{(\mathrm{sc})} \to Z^{(\mathrm{ss})}) = H^3(k, \ker \rho)$$
$$\mathbb{H}^2(k, Z^{(\mathrm{sc})} \to Z) = H^2_{\mathrm{ab}}(k, L).$$

One can check that the composition map

$$H^2(k,L)\xrightarrow{\mathrm{ab}^2}H^2_{\mathrm{ab}}(k,L)\xrightarrow{t_{\mathrm{ab}}}H^2(k,G^{\mathrm{tor}}) \tag{6.1.2}$$

is the canonical map $t\colon H^2(k,L)\to H^2(k,G^{\mathrm{tor}})$.

We consider the case of a non-archimedian local field k.

PROPOSITION 6.2. *Let $L=(\bar{G},\kappa)$ be a connected k-kernel, where k is a non-archimedian local field. Then an element $\eta\in H^2(k,L)$ is neutral if and only if $t(\eta)=0$, where $t\colon H^2(k,L)\to H^2(k,G^{\mathrm{tor}})$ is the canonical map.*

Proof. The group $\ker\rho$ is finite. Since k is a non-archimedian local field, we have $H^3(k,\ker\rho)=0$ (cf. [Se], II-5.3, Prop. 15). From (6.1.1) we see that the homomorphism t_{ab} is injective.

Let $\eta\in H^2(k,L)$. If η is neutral, then $t(\eta)=0$. Conversely, suppose that $t(\eta)=0$. Since $t=t_{\mathrm{ab}}\circ\mathrm{ab}^2$ and t_{ab} is injective, we conclude that $\mathrm{ab}^2(\eta)=0$. By Theorem 5.6 η is neutral. The proposition is proved.

THEOREM 6.3. *Let $L=(\bar{G},\kappa)$ be a connected k-kernel, where k is a non-archimedian local field. Assume that at least one of the following holds:*

(i) *G^{tor} is k-anisotropic;*

(ii) *$\bar{G}^{\mathrm{tor}}=1$;*

(iii) *$\bar{G}$ is semisimple.*

Then any element $\eta\in H^2(k,L)$ is neutral.

Proof. If (i) holds, then by the Tate–Nakayama duality (cf. [Se], II-5.8, or [Mi], I-2.4) $H^2(k,G^{\mathrm{tor}})=0$, and the assertion follows from Proposition 6.2. It is clear that (iii)⇒(ii)⇒(i). The theorem is proved.

Remark 6.3.1. Theorem 6.3 in the case (iii) was proved by Douai [Do2].

COROLLARY 6.4. *Let k be a non-archimedian local field, and let*

$$1\longrightarrow G_1\longrightarrow G_2\longrightarrow G_3\longrightarrow 1$$

be a short exact sequence of k-groups. If G_1 is connected and $G_1^{\mathrm{tor}}=1$, then the map $H^1(k,G_2)\to H^1(k,G_3)$ is surjective.

Proof. Let $\xi\in H^1(k,G_3)$, $\xi=\mathrm{Cl}(\psi)$. To the cocycle $\psi\in Z^1(k,G_3)$ Springer ([Spl], 1.20) associates a k-kernel $(G_{1\bar{k}},\kappa_\psi)$ and a cohomology class $\delta(\psi)\in H^2(k,G_{1\bar{k}},\kappa_\psi)$ which is the obstruction to lifting ξ to to $H^1(k,G_2)$. Since $G_1^{\mathrm{tor}}=1$, by Theorem 6.3 $\delta(\psi)$ is neutral, hence ξ comes from $H^1(k,G_2)$. The corollary is proved.

We now pass to the case of a number field k.

PROPOSITION 6.5. *Let k be a number field, and let $L = (\bar{G}, \kappa)$ be a connected k-kernel. Let $\eta \in H^2(k, L)$ be an element, which is locally neutral at the infinite places, i.e. such that the localization $\mathrm{loc}_v(\eta) \in H^2(k_v, L)$ is neutral for any infinite place v of k. Then η is neutral if and only if $t(\eta) = 0$.*

Proof. By [Bo1], Prop. 4.9, the group $H^2_{\mathrm{ab}}(k, L)$ is the fiber product of $H^2(k, G^{\mathrm{tor}})$ and $\prod_\infty H^2_{\mathrm{ab}}(k_v, L)$ over $\prod_\infty H^2(k_v, G^{\mathrm{tor}})$, where $\prod_\infty$ denotes product over the set of infinite places of k. If η is neutral, then $t(\eta) = 0$. Conversely, suppose that $t(\eta) = 0$. Then the image of $\mathrm{ab}^2(\eta)$ in in $H^2(k, G^{\mathrm{tor}})$ is zero (because $t = t_{\mathrm{ab}} \circ \mathrm{ab}^2$), and its image in $\prod_\infty H^2_{\mathrm{ab}}(k_v, L)$ is also zero (because η is locally neutral at the infinite places). We conclude that $\mathrm{ab}^2(\eta) = 0$. By Theorem 5.6 η is neutral. The proposition is proved.

6.6. Let T be a k-torus. The second Shafarevich-Tate group Ш^2 is defined by

$$Ш^2(k, T) = \ker[\mathrm{loc}\colon H^2(k, T) \to \prod_v H^2(k_v, T)]$$

where v runs over the set of all places of k. *A quasi-trivial torus* is a torus T such that its character group $\mathrm{X}(T_{\bar{k}})$ admits a Γ-stable basis. A torus T is quasi-trivial if and only if it is a product of tori of the form $\mathrm{R}_{K/k}\mathbb{G}_{\mathrm{m}}$, where K/k is a finite extension.

LEMMA 6.7. *Let T be a k-torus. Assume that at least one of the following holds:*
(i) *T is a quasi-trivial k-torus;*
(ii) *T_{k_v} is k_v-anisotropic for some place v of k;*
(iii) *T splits over a cyclic extension K/k;*
(iv) *T is one-dimensional.*
Then $Ш^2(k, T) = 0$.

Proof. For the cases (i) and (ii) see [Sa], 1.9. For the case (iii) see [Bo2], 3.4.1. The assumption (iv) implies (iii).

THEOREM 6.8 (A Hasse principle). *Let k be a number field and $L = (\bar{G}, \kappa)$ a connected k-kernel. Assume that at least one of the following holds:*
(i) $Ш^2(k, G^{\mathrm{tor}}) = 0$;
(ii) *The k-torus G^{tor} is as in Lemma 6.7;*
(iii) $\bar{G}^{\mathrm{tor}} = 1$;
(iv) *$\bar{G}$ is semisimple.*
Then an element $\eta \in H^2(k, L)$ is neutral if and only if its localizations $\mathrm{loc}_v\,\eta \in H^2(k_v, L)$ are neutral for all the places v of k.

Proof. If η is neutral, then $\mathrm{loc}_v\,\eta$ is neutral for any v. Conversely, suppose that $\mathrm{loc}_v\,\eta$ is neutral for any v. Then $\mathrm{loc}_v\, t(\eta) = 0$ for any v, hence $t(\eta) \in Ш^2(k, G^{\mathrm{tor}})$. Under any of the assumptions (i–iv) we have $Ш^2(k, G^{\mathrm{tor}}) = 0$. Thus $t(\eta) = 0$. By Proposition 6.5 η is neutral. The theorem is proved.

COROLLARY 6.9. *Let*

$$1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 1$$

be an exact sequence of algebraic groups over a number field k. Assume that G_1 is connected and that $\dim G_1^{\mathrm{tor}} \leq 1$. Let $\xi \in H^1(k, G_3)$ be a cohomology class. If for any place v of k, the localization $\mathrm{loc}_v\, \xi \in H^1(k_v, G_3)$ comes from $H^1(k_v, G_2)$, then ξ comes from $H^1(k, G_2)$.

Proof. Similar to that of Corollary 6.4.

7. Rational points in homogeneous spaces. In this section we apply Theorem 6.8 to prove a Hasse principle for homogeneous spaces. Our proof uses the Hasse principle for H^1 of a simply connected group.

7.1. Let H be a simply connected semisimple k-group. Let X be a right homogeneous space of H. This means that we are given a right algebraic action (defined over k)

$$X \times H \to X, \qquad (x, h) \mapsto x \cdot h$$

such that the action of $H(\bar{k})$ on $X(\bar{k})$ is transitive.

We are interested whether X has a rational point. Let $x \in X(\bar{k})$ be a $\bar{k}$-point. Let $\bar{G}$ be the stabiliser of x in $H_{\bar{k}}$ (we write $\bar{G} = \mathrm{Stab}(x)$). The subgroup $\bar{G}$ of $H_{\bar{k}}$ is not in general defined over k.

The homogeneous space X defines a k-form G^{tor} of $\bar{G}^{\mathrm{tor}}$. Moreover, X defines a k-kernel $L = (\bar{G}, \kappa)$. We construct L as follows.

For $\sigma \in \Gamma = \mathrm{Gal}(\bar{k}/k)$ write

$$ {}^{\sigma}x = x \cdot h_\sigma \tag{7.1.1}$$

where $h_\sigma \in H(\bar{k})$. Such an element h_σ is not unique; it is unique modulo left multiplication by an element of $\bar{G}(\bar{k})$. We can choose elements h_σ in such a way that the map $\sigma \mapsto h_\sigma$ is continuous. The σ-semialgebraic automorphism of $H_{\bar{k}}$

$$f_\sigma = \mathrm{int}(h_\sigma) \circ \sigma_*$$

takes $\bar{G}$ to itself. We will regard f_σ as a σ-semialgebraic automorphism of $\bar{G}$. Then the map $f\colon \Gamma \to \mathrm{SAut}\,\bar{G}$ is continuous, the composition

$$\kappa\colon \Gamma \xrightarrow{f} \mathrm{SAut}\,\bar{G} \longrightarrow \mathrm{SOut}\,\bar{G}$$

is a continuous homomorphism, and $L = (\bar{G}, \kappa)$ is a k-kernel.

THEOREM 7.2. *Let k be a non-archimedean local field. Let H be a simply connected semisimple k-group, and let X be a right homogeneous space of H. Assume that the stabilizer $\bar{G}$ of a point $x \in X(\bar{k})$ is connected, and that at least one of the following holds:*

(i) *The k-torus G^{tor} is k-anisotropic;*

(ii) $\bar{G}^{\text{tor}} = 1$*;*

(iii) $\bar{G}$ *is semisimple.*

Then X has a k-point.

THEOREM 7.3. *Let k be a number field. Let H be a simply connected semisimple k-group, and let X be a right homogeneous space of H. Assume that the stabilizer $\bar{G}$ of a point $x \in X(\bar{k})$ is connected, and that at least one of the following holds:*

(i) $Ш^2(k, G^{\text{tor}}) = 0$*;*

(ii) *The torus G^{tor} quasi-trivial;*

(iii) G^{tor} *is k_v-anisotropic for some place v of k;*

(iv) G^{tor} *splits over a cyclic extension of k;*

(v) $\bar{G}^{\text{tor}}$ *is one-dimensional;*

(vi) $\bar{G}^{\text{tor}} = 1$*;*

(vii) $\bar{G}$ *is semisimple.*

Then the Hasse principle holds for X, i.e if $X(k_v) \neq \emptyset$ for any place v of k, then $X(k) \neq \emptyset$.

Remark 7.3.1. Theorem 7.3 was proved in [Bo2] under slightly more general hypotheses on H and $\bar{G}$ (e.g. H may be adjoint) but with the additional non-necessary assumption that the pair $(H_{\bar{k}}, \bar{G})$ admits a k-form (H_0, G_0).

COROLLARY 7.4 ([Ha2], 3.3). *If X is a projective variety, then the Hasse principle holds for X.*

Proof. Since X is projective, $\bar{G}$ is a parabolic subgroup of $H_{\bar{k}}$. Harder [Ha2] shows that then G^{tor} is a quasi-trivial torus (cf. also [Bo2], 3.7). Now the corollary follows from Theorem 7.3, case (ii).

COROLLARY 7.5 ([Ra]). *Suppose that X is a symmetric homogeneous space of an absolutely simple k-group H, i.e. $\bar{G}$ is the group of invariants of an involution of $H_{\bar{k}}$. Then the Hasse principle holds for X.*

Proof. In this case $\dim \bar{G}^{\text{tor}} \leq 1$ (cf. [Ra] or [Bo2]), and the corollary follows from Theorem 7.3, cases (v) and (vi).

7.6. To prove Theorems 7.2 and 7.3 we construct an element $\eta(X) \in H^2(k, \bar{G}, \kappa)$, which is the obstruction to the existence of a principal homogeneous space over X (cf. also [Sp1], 1.27).

A principal homogeneous space of H over X is a pair (P, α), where P is a right principal homogeneous space of H and $\alpha: P \to X$ is an H-equivariant map (P and α are defined over k).

If X has a k-point x_0, then there exists a principal homogeneous space (P, α) over X. Indeed, we take $P = H$, $\alpha(h) = x_0 \cdot h$. Conversely, if there exists a principal homogeneous space (P, α) over X, and P has a k-point p_0, then X has a K-point $x_0 = \alpha(p_0)$.

We construct the cohomology class $\eta(X) \in H^2(k, L)$ mentioned above. Let $x \in X(\bar{k})$ be a $\bar{k}$-point. With the notation of 7.1 we set

$$\bar{P} = H_{\bar{k}}, \quad \bar{\alpha}(h) = x \cdot h \text{ for } h \in H_{\bar{k}} = \bar{P}.$$

Then $(\bar{P}, \bar{\alpha})$ is a principal homogeneous space over $X_{\bar{k}}$. We try to define $(\bar{P}, \bar{\alpha})$ over k.

For $\sigma \in \Gamma$ set

$$\nu_\sigma(h) = h_\sigma \cdot {}^\sigma h \qquad (h \in \bar{P} = H_{\bar{k}})$$

where h_σ is as in (7.1.1). Then ν_σ is a σ-semialgebraic $H_{\bar{k}}$-equivariant automorphism of $\bar{P}$, compatible with the σ-semialgebraic automorphism σ_* of $X_{\bar{k}}$. Set

$$\lambda_{\sigma,\tau} = \nu_{\sigma\tau} \circ \nu_\tau^{-1} \circ \nu_\sigma^{-1} \in \operatorname{Aut} \bar{P},$$

then $\lambda_{\sigma,\tau}(h) = u_{\sigma,\tau} \cdot h$, where

$$u_{\sigma,\tau} = h_{\sigma\tau} \cdot {}^\sigma h_\tau^{-1} \cdot h_\sigma^{-1} \in \bar{G}(\bar{k}).$$

Let f be the map defined in 7.1, then one can check that the pair (f, u) is a 2-cocycle, $(f, u) \in Z^2(k, L)$, where $L = (\bar{G}, \kappa)$. We set $\eta(X) = \operatorname{Cl}(f, u) \in H^2(k, L)$.

We show that the cohomology class $\eta(X)$ is neutral if and only if the pair $(\bar{P}, \bar{\alpha})$ can be defined over k. Indeed, let (P, α) be a k-form of $(\bar{P}, \bar{\alpha})$. We can take $\nu_\sigma(p) = {}^\sigma p$. Then the map $\nu: \Gamma \to \operatorname{SAut} \bar{P}$ is a homomorphism, hence $\lambda_{\sigma,\tau}(h) = h$ and $u_{\sigma,\tau} = 1$ for any $\sigma, \tau \in \Gamma$. Thus the class $\eta(X) = \operatorname{Cl}(f, u)$ is neutral. Conversely, if $\eta(X)$ is neutral, then we can define the elements $(h_\sigma)_{\sigma \in \Gamma}$ in such a way that $u_{\sigma,\tau} = 1$ for any $\sigma, \tau \in \Gamma$. Then ν is a homomorphism, and this homomorphism defines a k-form (P, α) of $(\bar{P}, \bar{\alpha})$.

Remark 7.6.1 In the language of gerbs [Gi] (see also [DM], Appendix), the fibered category $\mathcal{G}_X$ of such (P, α) is a gerb, and $\eta(X)$ is the class of $\mathcal{G}_X$ in $H^2(k, L)$. The gerb $\mathcal{G}_X$ is neutral if and only if there exists a pair (P, α) defined over k.

Now we can prove theorems 7.2 and 7.3. To prove Theorem 7.2 we need

LEMMA 7.7. *Let k, H and X be as in Theorem 7.2. If there exists a principal homogeneous space (P, α) over X, then X has a k-point.*

Proof. By Kneser's theorem ([Kn1]), the principal homogeneous space P of a simply connected group H over a non-archimedian local field k has a k-point p_0. Then $x_0 = \alpha(p_0)$ is a k-point of X. The lemma is proved.

7.8. *Proof of Theorem* 7.2. Let $\eta(X) \in H^2(k, L)$ be the cohomology class defined in 7.6. By Theorem 6.3 any element of $H^2(k, L)$ is neutral, thus $\eta(X)$ is neutral. It follows that there exists a principal homogeneous space (P, α) over X (see 7.6). By Lemma 7.7 $X(k) \neq \emptyset$. The theorem is proved.

To prove Theorem 7.3 we need

LEMMA 7.9. *Let k, H, X and $\bar{G}$ be as in Theorem 7.3. Suppose that $X(k_v) \neq \emptyset$ for any infinite place v of k, and suppose that there exists a principal homogeneous space (P, α) over X. Then $X(k) \neq \emptyset$.*

Proof. Set $G = \mathrm{Aut}_{X,H} P$. One can easily see that G is an algebraic group defined over k, and that $G_{\bar{k}}$ is isomorphic to $\bar{G}$. Thus G is connected.

We write $\mathcal{V}_\infty$ for the set of the infinite places of k. For any $v \in \mathcal{V}_\infty$ the homogeneous space X has a k_v-point. It follows that there exists a principal homogeneous space (P_v, α_v) of H_{k_v} over X_{k_v} (defined over k_v), such that P_v is trivial (i.e. has a k_v-point).

For any extension K/k, the isomorphism classes of K-forms of (P, α) correspond to elements of $H^1(K, G)$. In particular, for $v \in \mathcal{V}_\infty$ the k_v-form (P_v, α_v) of (P, α) defines a cohomology class $\xi_v \in H^1(k_v, G)$.

Consider the map

$$\mathrm{loc}_\infty : H^1(k, G) \to \prod_\infty H^1(k_v, G).$$

Since G is connected, the map loc_∞ is surjective ([Hal], 5.5.1; see also [PR], §6.5, Prop. 17). Hence there exists an element $\xi \in H^1(k, G)$ such that $\mathrm{loc}_v \xi = \xi_v$ for any $v \in \mathcal{V}_\infty$. In other words, there exists a form (P_*, α_*) of (P, α) such that $(P_*, \alpha_*)_{k_v} \simeq (P_v, \alpha_v)$. It is clear that P_* has a k_v-point for any $v \in \mathcal{V}_\infty$.

By the Hasse principle for H^1 of simply connected groups (Kneser–Harder–Chernousov, cf. [Hal] and [PR], Ch. 6), P_* has a k-point p_0. We set $x_0 = \alpha_*(p_0)$, then $x_0 \in X(k)$. The lemma is proved.

7.10. *Proof of Theorem* 7.3. Let $\eta(X) \in H^2(k, L)$ be the cohomology class defined in 7.6. For any place v of k there exists a k_v-point $x_v \in X(k_v)$, and therefore a principal homogeneous space (P_v, α_v) of H_{k_v} over X_{k_v}. It follows that the cohomology class $\mathrm{loc}_v\, \eta(X) \in H^2(k_v, L)$ is neutral for any v. By Theorem 6.8 $\eta(X)$ is neutral. This means that there exists a principal homogeneous space (P, α) of H over X. By Lemma 7.9 X has a k-point. The theorem is proved.

Appendix. Explicit formulas. Here we write down explicit formulas in terms of cocycles for the abelianization maps $\mathrm{ab}^i : H^i \to H^i_{\mathrm{ab}}$ $(i = 0, 1, 2)$. The maps

$$\mathrm{ab}^0 : H^0(k, G) \to H^0_{\mathrm{ab}}(k, G)$$
$$\mathrm{ab}^1 : H^1(k, G) \to H^1_{\mathrm{ab}}(k, G)$$

were defined (indirectly) in [Bo1]. The map

$$\mathrm{ab}^2 : H^2(k, L) \to H^2_{\mathrm{ab}}(k, L)$$

was defined (also indirectly) in Section 5.

Let G be a k-group. The k-group G^{sc} and k-homomorphism $\rho\colon G^{\mathrm{sc}} \to G$ are defined in Notation. Consider the complex $Z^{(\mathrm{sc})} \xrightarrow{\rho} Z$ of abelian k-groups, where $Z^{(\mathrm{sc})}$ and Z are the centers of G^{sc} and G, respectively. We set

$$H^i_{\mathrm{ab}}(k, G) = \mathbb{H}^i(k, Z^{(\mathrm{sc})} \to Z).$$

We define ab^0. We have

$$G(\bar{k}) = \rho(G^{\mathrm{sc}}(\bar{k})) \cdot Z(\bar{k}).$$

Let $g \in G(k) = H^0(k, G)$. We may write $g = \rho(g') \cdot z$, where $g' \in G^{\mathrm{sc}}(\bar{k}),\ z \in Z(\bar{k})$. Set

$$\varphi_\sigma = (g')^{-1} \cdot {}^\sigma g' \quad \text{for } \sigma \in \Gamma.$$

Then $\varphi_\sigma \in Z^{(\mathrm{sc})}(\bar{k})$, and the pair (φ, z) is a 0-cocycle, $(\varphi, z) \in Z^0(k, Z^{(\mathrm{sc})} \to Z)$. We set

$$\mathrm{ab}^0(g) = \mathrm{Cl}(\varphi, z) \in \mathbb{H}^0(k, Z^{(\mathrm{sc})} \to Z) = H^0_{\mathrm{ab}}(k, G).$$

We define ab^1. Let $\xi \in H^1(k, G),\ \xi = \mathrm{Cl}(\psi),\ \psi \in Z^1(k, G)$. Write

$$\psi_\sigma = \rho(\psi'_\sigma) \cdot z_\sigma$$

where the maps $\psi'\colon \Gamma \to G^{\mathrm{sc}}(\bar{k})$ and $z\colon \Gamma \to Z(\bar{k})$ are continuous. Set

$$\lambda_{\sigma,\tau} = \psi'_\sigma \cdot {}^\sigma\psi'_\tau \cdot (\psi'_{\sigma\tau})^{-1} \text{ for } \sigma, \tau \in \Gamma.$$

Then $\lambda_{\sigma,\tau} \in Z^{(\mathrm{sc})}(\bar{k})$, and the pair (λ, z) is a 1-cocycle, $(\lambda, z) \in Z^1(k, Z^{(\mathrm{sc})} \to Z)$. We set

$$\mathrm{ab}^1(\xi) = \mathrm{Cl}(\lambda, z) \in \mathbb{H}^1(k, Z^{(\mathrm{sc})} \to Z) = H^1_{\mathrm{ab}}(k, G).$$

Now let $L = (\bar{G}, \kappa)$ be a k-kernel. See Notation and 5.1 for the definitions of $\bar{G}$, $\bar{\rho}$, and the complex of k-groups $(Z^{(\mathrm{sc})} \to Z)$.

We define ab^2. Let $\eta \in H^2(k, L),\ \eta = \mathrm{Cl}(f, u)$. Write

$$u_{\sigma,\tau} = \bar{\rho}(u'_{\sigma,\tau}) \cdot z_{\sigma,\tau}$$

where the maps $u\colon \Gamma \times \Gamma \to \bar{G}^{\mathrm{sc}}(\bar{k})$ and $z\colon \Gamma \times \Gamma \to Z(\bar{k})$ are continuous. Set

$$\chi_{\sigma,\tau,\upsilon} = (u'_{\sigma,\tau})^{-1} \cdot (u'_{\sigma\tau,\upsilon})^{-1} \cdot u'_{\sigma,\tau\upsilon} \cdot f_\sigma({}^\sigma u'_{\tau,\upsilon}).$$

Then $\chi_{\sigma,\tau,\upsilon} \in Z^{(\mathrm{sc})}(\bar{k})$, and the pair (χ, z) is a 2-cocycle, $(\chi, z) \in Z^2(k, Z^{(\mathrm{sc})} \to Z)$. We set

$$\mathrm{ab}^2(\eta) = \mathrm{Cl}(\chi, z) \in \mathbb{H}^2(k, Z^{(\mathrm{sc})} \to Z) = H^2_{\mathrm{ab}}(k, L).$$

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