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Abstract

Let $\mathbb{X} \subset \mathbb{P}(V)$ be a projective variety, which is not contained in a hyperplane. Then every vector v in V can be written as a sum of vectors from the affine cone X over $\mathbb{X}$. The minimal number of summands in such a sum is called the rank of v. The set of vectors of rank r is denoted by X_r and its projective image by $\mathbb{X}_r$. The r-th secant variety of X is defined as $\sigma_r(\mathbb{X}) := \overline{\sqcup_{s\leq r}\mathbb{X}_s}$; it is called *tame* if $\sigma_r(\mathbb{X}) = \sqcup_{s\leq r}\mathbb{X}_s$ and *wild* if the closure contains elements of higher rank. In this paper, we classify all equivariantly embedded homogeneous projective varieties $\mathbb{X} \subset \mathbb{P}(V)$ with tame secant varieties. Classical examples are: the variety of rank one matrices (Segre variety with two factors) and the variety of rank one quadratic forms (quadratic Veronese variety). In the general setting, $\mathbb{X}$ is the orbit in $\mathbb{P}(V)$ of a highest weight line in an irreducible representation V of a reductive algebraic group G. Thus, our result is a list of all irreducible representations of reductive groups, where the resulting $\mathbb{X}$ has tame secant varieties.

Contents

1 Introduction

Let V be a simple module of a semi-simple Lie group G. In this work we consider two different notions of rank on V and provide a list of modules (G, V) for which these notions coincide.

There is a well known notion of rank for matrices of size $m \times n$, $n \times n$ symmetric and antisymmetric matrices. These spaces are in a natural way simple modules of the reductive groups $\mathrm{GL}_m \times \mathrm{GL}_n$ and GL_n respectively. For these three modules (G, V) there is a common way to restore the rank function from the action of the group G.

Let $\mathbb{X}(G, V) \subset \mathbb{P}(V)$ be the projectivization of the set of highest weight vectors with respect to G. Then $\mathbb{X}(G, V)$ is a complete G-homogeneous space. Every point in V can be expressed as a linear combination of points of $\mathbb{X}(G, V)$; the rank of a point is the minimal number of summands in such a linear combination. For the matrix cases mentioned above the varieties $\mathbb{X}(G, V)$ are the Segre, Veronese and Grassmann varieties, respectively. Also, in these cases the notion of rank coincides with the classical one.

In general, one can define a notion of rank on V for any $\mathbb{X} \subset \mathbb{P}(V)$ such that $\mathbb{X}$ spans V. In the same way every point in V can be expressed as a linear combination of points of $\mathbb{X}$ and the rank of a point is the minimal number of summands in such a linear combination. We define the r-th secant variety of $\mathbb{X}$ as the closure of the projectivization of the union of vectors of rank r or less. The matrices of rank r or less form a closed subvariety which equals to the r-th secant variety. This closedness condition fails in general and this is a source of difficulties in the study of secant varieties and rank. The goal of this work is to determine all simple representations V of reductive groups G for which the closedness condition holds. In short: there are not too many such representations, there are some such representations, some of these representations are unexpected to us.

We now turn from a general discussion to a world of formal definitions. First we fix some notation. We work over an algebraically closed field $\mathbb{F}$ of characteristic zero. Let V denote a vector space of dimension $N \geq 2$. If $v \in V$ is a nonzero vector, we shall denote by $[v]$ its image in the projective space $\mathbb{P}(V)$. Let $\mathbb{X} \subset \mathbb{P}(V)$ be a locally closed, nondegenerate subvariety. For any $[v] \in \mathbb{P}(V)$ the number

$$\mathrm{rk}[v] = \mathrm{rk}_{\mathbb{X}}[v] = \min\{r \in \mathbb{N} : v = v_1 + ... + v_r \text{ with } [v_j] \in \mathbb{X}\} .$$

is called the *rank of* $[v]$. Set

$$\mathbb{X}_r = \{[v] \in \mathbb{P}(V) : \mathrm{rk}[v] = r\} \quad , \quad r = 1, 2, ...$$

There exists an r_g, called the *typical* (or generic or essential) rank, such that the set $\mathbb{X}_{r_g}$ is Zariski dense in $\mathbb{P}(V)$. For $1 \leq r \leq r_g$, the Zariski closure $\overline{\mathbb{X}_r}$ is called the r-th secant variety of $\mathbb{X}$ and denoted by $\sigma_r(\mathbb{X})$. For any $[v] \in V$ the smallest r such that $[v] \in \sigma_r(\mathbb{X})$ is called the *border rank* of $[v]$ and we denote it by $\underline{\mathrm{rk}}_{\mathbb{X}}[v]$. By definition, $\mathrm{rk}_{\mathbb{X}}[v] \geq \underline{\mathrm{rk}}_{\mathbb{X}}[v]$.

Definition 1.1. A point $[v] \in \mathbb{P}(V)$, for which $\mathrm{rk}_{\mathbb{X}}[v] \neq \underline{\mathrm{rk}}_{\mathbb{X}}[v]$, is called *exceptional.* We say that a subvariety $\mathbb{X} \subset \mathbb{P}(V)$ is *wild* if there exists an exceptional with respect to $\mathbb{X}$ point $[v] \in \mathbb{P}(V)$. Otherwise, we say that $\mathbb{X}$ is *tame* in $\mathbb{P}(V)$.

Clearly, if $[v]$ is exceptional, then $\underline{\mathrm{rk}}_{\mathbb{X}}[v] < \mathrm{rk}_{\mathbb{X}}[v]$, so exceptional points are points which can be approximated by points of lower rank.

Let $G \subset GL(V)$ be a connected reductive algebraic subgroup such that V is a simple G-module. Then the variety $\mathbb{P}(V)$ contains a unique closed G-orbit, which we denote by $\mathbb{X}(G, V)$.

Theorem 1.1. Assume that $\dim V \geq 2$. Let G' be the commutator subgroup of G. Then the variety $\mathbb{X}(G, V)$ is tame in $\mathbb{P}(V)$ if and only if the pair (G', V) appears in the following table.

Group G'	Representation V	Highest weight of V
Simple classical groups		
SL_n	$\mathbb{F}^n, (\mathbb{F}^n)^*, (\Lambda^2\mathbb{F}^n), (\Lambda^2\mathbb{F}^n)^*$, $\mathrm{S}^2\mathbb{F}^n$, $(\mathrm{S}^2\mathbb{F}^n)^*$, $\mathfrak{sl}_n$	$\pi_1,\ \pi_{n-1},\ \pi_2,\ \pi_{n-2}$, $2\pi_1,\ 2\pi_{n-1},\ \pi_1+\pi_{n-1}$
SO_n	$\mathbb{F}^n, RSpin_n(n \leq 10)$	$\pi_1,\ \pi_{\frac{n}{2}}(2 \mid n),\ \pi_{\frac{n}{2}} - 1(2 \mid n)$, $\pi_{\frac{n-1}{2}}(2 \nmid n)$
Sp_{2n}	$\mathbb{F}^{2n}, \Lambda_0^2\mathbb{F}^{2n}$, $\mathrm{S}^2\mathbb{F}^{2n} \cong \mathfrak{sp}_{2n}$	$\pi_1, \pi_2, 2\pi_1$
Simple exceptional groups		
E_6	$\mathbb{F}^{27}, (\mathbb{F}^{27})^*$	π_1, π_5
F_4	$\mathbb{F}^{26}$	π_1
G_2	$\mathbb{F}^7$	π_1
Non-simple groups		
$SL_m \times SL_n$	$\mathbb{F}^m \otimes \mathbb{F}^n$	$\pi_1 \oplus \pi_1$
$SL_m \times Sp_{2n}$	$\mathbb{F}^m \otimes \mathbb{F}^{2n}$	$\pi_1 \oplus \pi_1$
$Sp_{2m} \times Sp_{2n}$	$\mathbb{F}^{2m} \otimes \mathbb{F}^{2n}$	$\pi_1 \oplus \pi_1$

, (1)

where by $RSpin_n$ we denote (any) spinor representation of the simply connected cover of SO_n, by $\Lambda_0^2\mathbb{F}^{2n}$ we denote the second fundamental representation of Sp_{2n} (which is identified with a hyperplane in $\Lambda^2\mathbb{F}^{2n}$), by $\mathbb{F}^{27}$ we denote any one of two the smallest fundamental representations of E_6, by $\mathbb{F}^{26}$ we denote the smallest fundamental representations of F_4, by $\mathbb{F}^7$ we denote the smallest fundamental representations of G_2. In the third column we write the highest weight of V with respect to G' (we use here and throughout the paper the enumeration convention of [VO90]).

Moreover, if (G, V) is wild, then it contains an exceptional vector of border rank 2.

We give a proof of Theorem 1.1 in Section 3. As a corollary, we obtain the list of homogeneous projective varieties given below. The list of varieties is shorter, because in certain cases there are subgroups of the automorphism group of the variety acting transitively.

Corollary 1.2. The tame projective varieties $\mathbb{X} \subset \mathbb{P}(V)$ with transitive linear automorphism group are the following:

Notation for $\mathbb{X}$	Ambient $\mathbb{P}(V)$	Group G	Max $\mathrm{rk}_{\mathbb{X}}$
$\mathbb{P}(\mathbb{F}^n)$	$\mathbb{P}(\mathbb{F}^n)$	SL_n	1
$\mathrm{Ver}_2(\mathbb{P}(\mathbb{F}^n))$	$\mathbb{P}(\mathrm{S}^2\mathbb{F}^n)$	SL_n	n
$\mathrm{Gr}_2(\mathbb{F}^n)$	$\mathbb{P}(\Lambda^2\mathbb{F}^n)$	SL_n	$\lfloor \frac{n}{2} \rfloor$
$\mathrm{Fl}(1, n-1; \mathbb{F}^n)$	$\mathbb{P}(\mathfrak{sl}_n)$	SL_n	n
Q^{n-2}	$\mathbb{P}(\mathbb{F}^n)$	SO_n	2
S^{10}	$\mathbb{P}(\mathbb{F}^{16})$	$Spin_{10}$	2
$\mathrm{Gr}_\omega(2, \mathbb{F}^{2n})$	$\mathbb{P}(\Lambda_0^2\mathbb{F}^{2n})$	Sp_{2n}	n
E^{16}	$\mathbb{P}(\mathbb{F}^{27})$	E_6	3
F^{15}	$\mathbb{P}(\mathbb{F}^{26})$	F_4	3
$\mathrm{Segre}(\mathbb{P}(\mathbb{F}^m) \times \mathbb{P}(\mathbb{F}^n))$	$\mathbb{P}(\mathbb{F}^m \otimes \mathbb{F}^n)$	$SL_m \times SL_n$	$\min\{m, n\}$

In all cases $\mathbb{X}$ coincides with the projectivization of the set of highest weight vectors of the respective representation.

Let us say a few words about previous work on this and related topics, in order to embed our result into a variety of results of other people. Secant varieties of projective varieties have been an object of study for a long time. Many results and further references can be found in the recent monograph [L12]. Questions of interest are: What is the dimension of the r-th secant variety? What is the typical rank? What is the ideal of the r-th secant variety? What is the rank of a given element? Much of the activity is concentrated on some special cases of varieties of the form $\mathbb{X}(G, V)$, e.g. Segre, Veronese and Grassmann varieties, as well as spinor and adjoint varieties. First, there is the Segre variety

$$\mathrm{Segre}(\mathbb{P}^{n_1-1} \times ... \times \mathbb{P}^{n_k-1}) = \mathbb{X}(SL_{n_1} \times ... \times SL_{n_k}, \mathbb{F}^{n_1} \otimes ... \otimes \mathbb{F}^{n_k})$$

and the related notion of tensor rank, see e.g. [CGG02], [LM04]. The Segre variety is tame if and only if $k = 2$, i.e. in the classical case of matrices. Second, there is the Veronese variety

$$\mathrm{Ver}_k(\mathbb{P}^{n-1}) = \mathbb{X}(SL_n, \mathrm{S}^k\mathbb{F}^n).$$

Here the rank is called *Waring rank* and equals the number of k-th powers of linear forms necessary to express a given polynomial as a sum. The general solution to the problem of determining the Waring rank of a given polynomial is still unknown. However, there are recent results which give, among other things, the ranks of monomials and sums of coprime monomials, see [LT10], [CCG12]. The dimensions of the secant varieties of Veronese varieties have been computed in [AH95]. The Veronese variety $\mathrm{Ver}_k(\mathbb{P}^{n-1})$ is tame if and only if $k = 2$. Third, there is the Grassmann variety in its Plücker embedding

$$\mathrm{Gr}_k(\mathbb{F}^n) \cong \mathbb{X}(SL_n, \Lambda^k\mathbb{F}^n) \hookrightarrow \mathbb{P}(\Lambda^k\mathbb{F}^n).$$

The corresponding secant varieties and rank sets have been studied considerably less than those of the Segre and Veronese varieties, see e.g. [CGG05] (and some further references) where one finds results on dimension of the secant varieties. The Grassmann variety is tame if and only if $k \leq 2$ or $|n - k| \leq 2$. Another class of varieties $\mathbb{X}(G, V)$, which has been a subject of research is the class of *adjoint varieties* $\mathbb{X}(G, \mathfrak{g})$ for a simple group G, see [Kaji98], [KY00], [BD04]. In particular, in [KY00] it is shown that $\mathbb{X}(G, \mathfrak{g})$ is tame if and only if G is of type A_n or C_n. Thus, the tame cases are rather scarce and many of them have interesting descriptions as set of matrices, where the rank is actually a classical, well-known invariant.

Let us emphasize, that the second secant variety plays a prominent role both in this paper and in the literature. Sometimes the second secant variety is called just "the secant variety". This variety is much more accessible than the higher secant varieties: for example for a simple G-module the second secant variety has an open G-orbit [Zak93, Ch. III, Thm 1.4].

The starting point of this study is the paper [BL13], where it is shown that an interesting class of homogeneous projective varieties is tame: the subcominuscule varieties. Let us recall that a variety $\mathbb{X}$ is called *subcominuscule*, if $\mathbb{X} = \mathbb{X}(G, V)$, where $G \to GL(V)$ is the isotropy representation of a compact Hermitian symmetric space (such a representation is called *subminuscule*). In this case, the sets $\mathbb{X}_r$ are exactly the G-orbits in $\mathbb{P}(V)$, see [LM03], [BL13]. Further, the result on subminuscule representations allows us to deduce that a larger class of representations is tame, namely, the irreducible representations where G acts spherically on $\mathbb{P}(V)$. We call these representations spherical; the irreducible spherical representations are classified [Kac80] (see also [Knop98]). In fact, a quick examination of the lists shows that, for every irreducible spherical representation $G \to GL(V)$, there exists a subminuscule representations $\tilde{G} \to GL(V)$, such $\mathbb{X}(G, V) = \mathbb{X}(\tilde{G}, V)$. The group $\tilde{G}$ can be taken to be the simply connected cover of the automorphism group of $\mathbb{X}(G, V)$. Similarly, one can enlarge further the list of tame representations, but not of tame varieties, by taking a tame variety $\mathbb{X}(G, V)$ and a subgroup of $G_1 \subset G$ acting transitively on $\mathbb{X}$. Then V automatically remains irreducible over G_1 and, since the notion of rank is independent of the group acting, we still have a tame variety. For instance, the subgroup $Sp_{2n} \subset SL_{2n}$ acts transitively on the quadratic Veronese variety $\mathrm{Ver}_2(\mathbb{P}^{2n-1})$. It is, however, not true that the subcominuscule varieties give a full list of tame homogeneous projective varieties. From our classification we see that the exceptions are $\mathrm{Fl}(1, n-1; \mathbb{F}^n)$, $\mathrm{Gr}_\omega(2, \mathbb{F}^{2n})$ and F^{15}.

This leads us to the following observation. If $\mathbb{X} \subset \mathbb{P}(V)$ is a tame homogeneous variety, then $\mathbb{X}$ is either a subcominuscule variety or a hyperplane section of a subcominuscule variety. If $\mathbb{X}$ is subcominuscule, then the maximal rank in $\mathbb{P}(V)$ with respect to $\mathbb{X}$ is equal to the rank of $\mathbb{P}(V)$ as a spherical variety with respect to the automorphism group of $\mathbb{X}$. If $\mathbb{X}$ is a hyperplane section of a subcominuscule variety, then its rank function is obtained by restriction.

The paper is organized as follows. In Subsection 2.1, we recall the notions of secant varieties, rank and border rank, with their basic properties. In Subsection 2.2, we recall some basic notions about algebraic groups: Borel subgroup, Cartan subgroup, weight lattice, root system, Weyl chamber. We also introduce the notion of chopping (this is a simple combinatorial procedure) and provide some facts on $\mathbb{X}_2(G, V)$ and $\sigma_2(\mathbb{X}(G, V))$ playing a crucial role in this paper.

In Section 3, we present a plan of our proof of Theorem 1.1, the main theorem of our article. Essentially, this proof is a compilation of Propositions 4.6, 4.8 and Theorems 6.1 and 7.1. We prove Propositions 4.6, 4.8 in Section 4. We prove Theorems 6.1, 7.1 in Section 6 and 7 respectively.

In Section 5, we prove a strong necessary condition for tameness of a representation in terms of its choppings, formulated in Proposition 5.1.

The main statement of Section 6 is Theorem 6.1. In this theorem we find out which fundamental representations of classical groups are tame and which are wild. This is done in the following way: for fundamental modules

$$\mathbb{F}^n, \Lambda^2\mathbb{F}^n \text{ for } SL_n,\ \Lambda^3\mathbb{F}^6 \text{ for } SL_6,\ \Lambda_0^2\mathbb{F}^{2n} \text{ and } \Lambda_0^3\mathbb{F}^{2n} \text{ for } Sp_{2n},$$
$$\Lambda^2\mathbb{F}^n \text{ for } SO_n,\ RSpin_n \text{ for } Spin_n (n \leq 12)$$

we check wildness/tameness in a straightforward way. From these data we deduce wildness of all other modules using the notion of chopping.

The main statement of Section 7 is Theorem 7.1. In this theorem we find out which fundamental representations of exceptional groups are tame and which are wild. This is done via case by case checking of 27 fundamental representations of 5 exceptional Lie algebras. For any such a representation we find some arguments by which it is wild/tame. For most of the representations the arguments are quite short, but for three representations:

$$V(\pi_1), V(\pi_2) \text{ for } F_4 \text{ and } V(\pi_1) \text{ for } E_7$$

we are able to find only quite long arguments presented in the corresponding subsections.

2 Preliminaries

In this section, we recall some definitions and elementary facts about secant varieties and rank. The goal is to introduce notation and perhaps help the unexperienced reader to become more familiar with these notions. We also fix some standard notation for reductive algebraic groups, their Lie algebras and their representations.

Throughout the paper we use the following notation. The letter $\mathbb{X}$ is always used for a projective variety and X denote the cone of it. For any subset $S \subset V$ we denote by $\langle S \rangle \subset V$ the span of S. For any subset $\mathbb{S} \in \mathbb{P}(V)$ we denote by $\langle S \rangle \subset V$ the span of the cone S of $\mathbb{S}$ in V. For any non-zero vector $v \in V$ we denote by $[v]$ the class of it in $\mathbb{P}(V)$. If $v = 0$, we set $[v] := 0$.

2.1 Secant varieties and rank: general definitions

Let $\mathbb{X} \subset \mathbb{P}$ be an algebraic variety and $X \subset V$ denote the affine cone over $\mathbb{X}$. We denote by $\mathbb{P}\langle\mathbb{X}\rangle = \mathbb{P}(\langle X \rangle)$ the corresponding projective subspace of $\mathbb{P}$. We say that $\mathbb{X}$ spans $\mathbb{P}$ if $\mathbb{P}\langle\mathbb{X}\rangle = \mathbb{P}$; this is equivalent to the requirement that X contains a basis of V. Assume that this is the case. Then every point in V can be written as a linear combination of points in X. This allows us to define the notion of rank already given in the introduction: the rank of $[\psi] \in \mathbb{P}$ with respect to $\mathbb{X}$ is the minimal number of elements of X necessary to express ψ as a linear combination. Thus, the space $\mathbb{P}$ is partitioned into the rank subsets,

$$\mathbb{P} = \mathbb{X}_1 \sqcup \mathbb{X}_2 \sqcup ....$$

Since $\mathbb{X}$ spans $\mathbb{P}$, we have $\mathbb{X}_r = \emptyset$ for $r > N$.

The following properties of varieties $\mathbb{X}_r$ hold:

(i) $\mathbb{X}_1 = \mathbb{X}$.

(ii) There exists a maximal $r_m \in \{1, ..., N\}$, such that $\mathbb{X}_{r_m} \neq \emptyset$ and $\mathbb{X}_r = \emptyset$ for $r > r_m$.

(iii) If $r \in \{1, ..., r_m\}$, then $\mathbb{X}_r \neq \emptyset$.

(iv) The projective space $\mathbb{P}$ can be written as a disjoint union $\mathbb{P} = \mathbb{X}_1 \sqcup \cdots \sqcup \mathbb{X}_{r_m}$.

Let $r \in \{2, ..., r_m\}$. The subset $\mathbb{X}_r \subset \mathbb{P}$ is not closed, because we have $\mathbb{X} \subset \overline{\mathbb{X}_r}$ and $\mathbb{X} \not\subset \mathbb{X}_r$. (Here and in what follows we use $\overline{S}$ to denote the Zariski closure of a subset S of some algebraic variety.) The r-th secant variety of $\mathbb{X}$ is defined as

$$\sigma_r(\mathbb{X}) = \overline{\bigsqcup_{s \leq r} \mathbb{X}_s} \subset \mathbb{P} \ .$$

It can also be written as

$$\sigma_r(\mathbb{X}) = \overline{\bigcup_{x_1, ..., x_r \in \mathbb{X}} \mathbb{P}_{x_1 \dots x_r}} \ ,$$

where $\mathbb{P}_{x_1 \dots x_r}$ stands for the projective subspace of $\mathbb{P}$ spanned by the points $x_1, ..., x_r$.

The following properties of secant varieties $\sigma_r(\mathbb{X})$ hold:

(i) $\sigma_1(\mathbb{X}) = \mathbb{X}_1 = \mathbb{X}$.

(ii) $\sigma_r(\mathbb{X}) \subset \sigma_{r+1}(\mathbb{X})$.

(iii) If $\mathbb{X}$ is irreducible, then $\sigma_r(\mathbb{X})$ is also irreducible.

(iv) There exists a minimal number $r_g \in \{1, ..., r_m\}$ such that $\sigma_{r_g}(\mathbb{X}) = \mathbb{P}$ and $\sigma_{r_g - 1}(\mathbb{X}) \neq \mathbb{P}$.

(v) For $r \in \{1, ..., r_g\}$ the rank subset $\mathbb{X}_r$ is dense in $\sigma_r(\mathbb{X})$, i.e. we have $\sigma_r(\mathbb{X}) = \overline{\mathbb{X}_r}$.

Definition 2.1. The number r_g from part (iv) of the above proposition is called the *typical rank* of $\mathbb{P}$ with respect to $\mathbb{X}$.

Let $[\psi] \in \mathbb{P}$. The border rank of $[\psi]$ with respect to $\mathbb{X}$ is defined as

$$\underline{\mathrm{rk}}[\psi] := \underline{\mathrm{rk}}_{\mathbb{X}}[\psi] := \min\{r \in \mathbb{N} : [\psi] \in \overline{\mathbb{X}_r}\} \ .$$

Definition 2.2. Points $[\psi] \in \mathbb{P}$, for which $\mathrm{rk}[\psi] \neq \underline{\mathrm{rk}}[\psi]$, are called *exceptional*.

Clearly, $\mathrm{rk}[\psi] \geq \underline{\mathrm{rk}}[\psi]$ and $[\psi]$ is exceptional exactly when $\underline{\mathrm{rk}}[\psi] < \mathrm{rk}[\psi]$. So, exceptional points are points which can be approximated by points of lower rank. Also, we have

$$\underline{\mathrm{rk}}[\psi] = \min\{r \in \mathbb{N} : [\psi] \in \sigma_r(\mathbb{X})\}.$$

This leads us to the next definition.

Definition 2.3. The secant variety $\sigma_r(\mathbb{X})$ is called *wild* if it contains exceptional vectors and *tame* if it does not contain exceptional vectors. The embedding $\mathbb{X} \subset \mathbb{P}$ is called *wild/tame*, if some of the secant varieties $\sigma_r(\mathbb{X})$ is wild/tame. The embedding $\mathbb{X} \subset \mathbb{P}$ is called r-tame, if $\sigma_r(\mathbb{X})$ is tame. The embedding $\mathbb{X} \subset \mathbb{P}$ is called *r-wild*, if $\sigma_r(\mathbb{X})$ is wild and $\sigma_s(\mathbb{X})$ is not wild for $s < r$.

We record another list of simple statements, which are derived immediately from the above definitions.

1) The secant variety $\sigma_r(\mathbb{X})$ is wild if and only if $\sigma_r(\mathbb{X}) \neq \mathbb{X}_1 \sqcup \mathbb{X}_2 \sqcup ... \sqcup \mathbb{X}_r$.

2) If $\sigma_r(\mathbb{X})$ is wild, then, for $s \geq r$, $\sigma_s(\mathbb{X})$ is also wild.

3) The embedding $\mathbb{X} \subset \mathbb{P}$ is wild if and only if $\mathrm{rk}_{\mathbb{X}} \neq \underline{\mathrm{rk}}_{\mathbb{X}}$.

We denote by $X_r \subset V$ the cone over $\mathbb{X}_r$ without 0 and by $\sigma_r(X) \subset V$ the cone over $\sigma_r(\mathbb{X})$ with 0.

2.2 Irreducible representations of reductive groups

Here we fix some notation concerning semisimple or reductive algebraic groups and their representations. All notions from this theory used by us can be found in [GW09] (the notation, however, may differ).

Let G be a connected reductive algebraic group over $\mathbb{F}$ and $\mathfrak{g}$ be its Lie algebra. We assume that the semisimple part G' of G is simply connected. Let $B \subset G$ be a Borel subgroup and $H \subset B$ a Cartan subgroup. Let $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ denote the respective subalgebras of $\mathfrak{g}$. Let $\Lambda \subset \mathfrak{h}^*$ be the integral weight lattice and $\Delta \subset \Lambda$ the root system. Let $\Delta = \Delta^+ \sqcup \Delta^-$ be the partition of the root system into positive and negative roots corresponding to the Borel subgroup B. Let $\Pi = \{\alpha_1, ..., \alpha_\ell\}$ be the basis of simple roots in Δ^+.

The Cartan-Killing form of $\mathfrak{g}$ determines a scalar product $(\cdot,\cdot)$ on $\mathfrak{h}^*$. This scalar product $(\cdot,\cdot)$ defines a Dynkin diagram (graph) Dyn, whose vertices are labeled by the simple roots $\alpha_1, ..., \alpha_\ell \in \Pi$. Also Π and the scalar product define the dominant Weyl chamber Λ and the monoid of dominant weights $\Lambda^+ \subset \Lambda$, generated by the fundamental weights $\pi_1, ..., \pi_\ell$. Hence $\lambda \in \Lambda^+$ defines a function $f_\lambda : \Pi \to \mathbb{Z}_{\geq 0}$ such that $\lambda = f_\lambda(\alpha_1)\pi_1 + ... + f_\lambda(\alpha_\ell)\pi_\ell$. This defines a one-to-one correspondence between the set Λ^+ of dominant weights and functions from the set of vertices of Dyn to the non-negative integers. We denote

$$h(\lambda) := f_\lambda(\alpha_1) + ... + f_\lambda(\alpha_\ell).$$

The set Λ^+ is also in a one-to-one correspondence with the set of isomorphism classes of simple finite-dimensional G-modules. We denote by $V(\lambda)$ the irreducible representation of G corresponding to $\lambda \in \Lambda^+$ (λ is a highest weight of $V(\lambda)$ and $V(\lambda)$ contains a unique up to scaling vector v^λ of weight λ). Set $\mathbb{P}(\lambda) := \mathbb{P}(V(\lambda))$. We denote by $X(\lambda)$ and $\mathbb{X}(\lambda)$ the orbits of v^λ and $[v^\lambda]$ in $V(\lambda)$ and $\mathbb{P}(\lambda)$ respectively. Note that $\mathbb{X}(\lambda)$ is a unique closed G-orbit on $\mathbb{P}(\lambda)$.

Any subdiagram of a Dynkin diagram is again a Dynkin diagram. Thus, by chopping down some vertices (along with the adjacent edges) we obtain a new diagram $\underline{Dyn}$, which corresponds to a semisimple Levi subgroup $\underline{G} \subset G$. The restriction of f_λ to $\underline{Dyn}$ defines a simple representation $\underline{V}$ of the group $\underline{G}$.

Definition 2.4. We say that a $\underline{G}$-representation $\underline{V}$ is a *chopping* of a G-representation V, if $\underline{V}$ is obtained from V via the above construction.

Remark 2.1. Note that a chopping of a fundamental representation is either fundamental, or trivial one-dimensional.

The Dynkin diagram determines the Weyl group $\mathcal{W}$. We denote by w_0 the longest element of $\mathcal{W}$ with respect to the Bruhat order. The weights of the form $w\lambda$, with $w \in \mathcal{W}$, are called *extreme weights* of the module $V(\lambda)$ and the corresponding weight vectors are called extreme weight vectors. For any $w \in \mathcal{W}$ we denote by $v^{w\lambda}$ the unique up to scaling vector of weight $w\lambda$. The weight $w_0\lambda$ is called *the lowest weight* of $V(\lambda)$.

For $\lambda_1, \lambda_2 \in \Lambda^+$, G-module $V(\lambda_1) \otimes V(\lambda_2)$ contains a unique up to scaling vector of weight $\lambda_1 + \lambda_2$. This vector is contained in a simple G-submodule, which is isomorphic to $V(\lambda_1 + \lambda_2)$. We call this submodule the *Cartan component* of $V(\lambda_1) \otimes V(\lambda_2)$. It is well known that the Cartan component does not depend on a choice of Borel subgroup $B \subset G$.

Let us fix $\lambda \in \Lambda^+$ and put $V = V(\lambda)$ and $\mathbb{P} = \mathbb{P}(\lambda) := \mathbb{P}(V(\lambda))$. The group G acts on the projective space $\mathbb{P}$ and has a unique closed orbit therein, namely, the orbit through the highest weight line, to be denoted by $\mathbb{X} = \mathbb{X}(\lambda) := G[v^\lambda]$. We have $\mathbb{X} = G/P$, where P denotes the stabilizer of $[v^\lambda] \in \mathbb{P}$ in G. This P is a standard parabolic subgroup, i.e. a closed subgroup of G containing the fixed Borel subgroup B. The cosets of G by parabolic subgroups are called the flag varieties of G. Thus we have an equivariantly embedded flag variety $\mathbb{X} = G/P \subset \mathbb{P}$. In fact, all equivariantly embedded homogeneous projective varieties are obtained in this fashion. Note, that the variety $\mathbb{X}$ is the set of highest weight vectors with respect to all possible choices of Borel subgroups $B \subset G$.

The irreducibility of V implies that $\mathbb{X}$ spans $\mathbb{P}$. Hence, we have well defined rank and border rank functions on $\mathbb{P}$ with respect to $\mathbb{X}$, as well as secant varieties $\sigma_r(\mathbb{X}) \subset \mathbb{P}$. Since the group G acts on V by invertible linear transformations, it follows immediately that rank and border rank are G-invariant functions. Hence, the rank sets $\mathbb{X}_r$ and the secant varieties $\sigma_r(\mathbb{X})$ are preserved by G.

Definition 2.5. Let $V = V(\lambda)$, with $\lambda \in \Lambda^+$, be an irreducible representation of a reductive linear algebraic group G. Let $\mathbb{X} = \mathbb{X}(\lambda)$ be the unique closed G-orbit in $\mathbb{P} = \mathbb{P}(\lambda)$. The G-module V is called *tame* (resp. *wild*, r*-tame*, r*-wild*), if the the variety $\mathbb{X} \subset \mathbb{P}$ is tame (resp. wild, r-tame, r-wild).

Our goal is to classify all tame irreducible representations of semisimple algebraic groups.

Remark 2.2. In some of our constructions we consider reductive groups, rather than semisimple groups, just because this simplifies some steps. The actions of G and G' on $\mathbb{P}$ coincide and we are concerned with properties of the embedding $\mathbb{X} \subset \mathbb{P}$. Thus the classification of tame representations of reductive groups can be easily obtained from the one for semisimple groups.

Below we assume that $\lambda \neq 0$ (this corresponds to the inequality $\dim V \geq 2,\ V = V(\lambda)$, assumed in Theorem 1.1).

We denote by $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$ the simple simply connected algebraic groups with the corresponding Dynkin diagrams. We denote by $\mathfrak{a}_n, \mathfrak{b}_n, \mathfrak{c}_n, \mathfrak{d}_n, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2$ the corresponding Lie algebras.

3 Plan of proof of Theorem 1.1

Theorem 1.1 follows from a number of propositions and theorems proved throughout the article. Thus the proof explains the role of different parts of this text.

In Proposition 4.6, we prove that, if an irreducible representation $V(\lambda)$ with highest weight λ is tame, then $h(\lambda) < 3$. In Proposition 4.8, we classify all irreducible tame modules $V(\lambda)$ with $h(\lambda) = 2$.

The tame fundamental representations (i.e. irreducible G-modules $V(\lambda)$ with $h(\lambda) = 1$) are classified in Theorems 6.1 and 7.1. This is done in a case by case study. We consider separately representations of classical groups (Theorem 6.1), and representations of exceptional groups (Theorem 7.1). This completes the proof of Theorem 1.1.

Let us say a few words about the proofs of Theorems 6.1 and 7.1. The tame representations are considered individually and for each case we provide specific arguments. Most representations are wild and thus we have to check wildness of a huge amount of cases. The number of cases to be considered is greatly reduced by Proposition 5.1, where we show that, if $\underline{G}$-representation $\underline{V}$ is a chopping of G-representation V and $\underline{V}$ is wild, V is also wild. Thus, it suffices to check wildness directly for only few basic cases, then we are able to deduce wildness for most of the fundamental representations.

4 Non-fundamental tame modules

The goal of this section is to classify all non-fundamental tame modules. First of all we provide in Lemma 4.3 of Subsection 4.1 a way to construct exceptional vectors in $V(\lambda)$. Using this construction we show in Proposition 4.6 of Subsection 4.2 that, if a G-module $V(\lambda)$ is 2-tame, then it is either a fundamental G-module (i.e. $h(\lambda) = 1$) or is a Cartan component in a tensor product of two fundamental G-modules (i.e. $h(\lambda) = 2$). Further on, in Proposition 4.7, we shall show that, in the latter situation, each of the fundamental modules satisfies a very strict condition, related to the notion of HW-density introduced in Definition 4.2. Using the explicit description of all HW-dense modules which we present in Corollary 4.5, we are able to complete in Proposition 4.8 a classification of non-fundamental tame G-modules $V(\lambda)$ (i.e. modules $V(\lambda)$ such that $h(\lambda) = 2$).

4.1 The varieties $\sigma_2(\mathbb{X}(\lambda))$ and $\mathbb{X}_2(\lambda)$

In some sense, in this article we study the difference between the variety $\sigma_2(\mathbb{X}(\lambda))$ and its open subset $\mathbb{X}_2(\lambda)$. Here we collect some of their basic features. First we recall that, if a generic rank of $V(\lambda)$ is greater than 1 (i.e. if $\mathbb{P}(\lambda) \neq \mathbb{X}(\lambda)$), then

$$\sigma_2(X(\lambda)) = \overline{X_2(\lambda)}.$$

We provide a kind of explicit description of elements of $\mathbb{X}_2(\lambda)$ (Lemma 4.1) and exhibit some set of elements of $\sigma_2(\mathbb{X}(\lambda))$, which tend to be exceptional (Lemma 4.3).

Lemma 4.1. Let $V(\lambda)$ be an irreducible representation of a reductive group G. Then
a) any pair $([v_1],[v_2]) \in \mathbb{X}(\lambda) \times \mathbb{X}(\lambda)$ is G-conjugate to a pair $([v^\lambda],[v^{w\lambda}])$ for some $w \in \mathcal{W}$,
b) any element of $\mathbb{X}_2(\lambda)$ is conjugate to $[v^\lambda + v^{w\lambda}]$ for some $w \in \mathcal{W}$.

Proof. Fix $v_1, v_2 \in X(\lambda)$ such that $[v_1] \neq [v_2]$. Let B_1, B_2 be Borel subalgebras of G such that v_1, v_2 are the corresponding to B_1, B_2 highest weight vectors. It is known that $B_1 \cap B_2$ contains a maximal torus T_{12} of G and there exists $w \in \mathcal{W}_{12} := N_G(T_{12})/T_{12}$ such that $B_1 = wB_2$. Thus $([v_1],[v_2])$ is conjugate to $([v^\lambda],[v^{w\lambda}])$ for some $w \in \mathcal{W}$. This completes the proof of part a).

To prove part b) we observe that any element of $X_2(\lambda)$ is a sum $v_1 + v_2$ for some $v_1, v_2 \in X(\lambda)$ such that $v_1 \neq v_2$. According to part a) the pair $([v_1],[v_2])$ is conjugate to $([v^\lambda],[v^{w\lambda}])$ for some $w \in \mathcal{W}$. Thus $[v_1 + v_2]$ is conjugate to $[av^\lambda + bv^{w\lambda}]$ for some non-zero $a, b \in \mathbb{F}$. As $[v_1] \neq [v_2]$, $\lambda \neq w\lambda$. Hence $[v^\lambda + v^{w\lambda}]$ is conjugate to $[av^\lambda + bv^{w\lambda}]$ for any non-zero $a, b \in \mathbb{F}$. This completes the proof of b). □

Corollary 4.2. The varieties $\sigma_2(\mathbb{X}(\lambda))$ and $\mathbb{X}_2(\lambda)$ have open G-orbits. (see also [Zak93, Ch. III, Thm 1.4])

Lemma 4.3. Fix $x \in X(\lambda)$ and $t \in \mathfrak{g}$. Then $[x + tx] \in \sigma_2(\mathbb{X}(\lambda))$.

Proof. By definition $\mathbb{X}_2(\lambda) \cup \mathbb{X}(\lambda)$ is the union of lines going through pairs of points of $\mathbb{X}(\lambda)$. Thus the tangent space $\mathrm{T}_x X(\lambda) \subset V$ to $X(\lambda)$ in x belongs to $\overline{X_2(\lambda)} = \sigma_2(X(\lambda))$. On the other hand, $x + tx$ belongs to $\mathrm{T}_x X(\lambda)$ as tx is tangent to $X(\lambda)$. This completes the proof. □

4.2 HW-density and 2-tameness

In this subsection, we analyze the notion of 2-tameness via the notion of HW-density given in Definition 4.2. This analysis allows to find out all tame modules which are not fundamental. Notice that 2-tameness is, a priori, weaker than tameness, but is much simpler to check. A posteriori, it turns out that tameness and 2-tameness are equivalent for the class of homogeneous projective varieties considered in this paper.

We proceed in the following way. We prove that, if an irreducible G-module V is 2-tame, then it is either a fundamental module of G (i.e. only one mark of the highest weight of V is distinct from zero and this mark equals 1) or is a Cartan component of the tensor product of two fundamental modules of G, see Proposition 4.6. In the latter case we prove that both fundamental modules in the product have to be HW-dense, see Proposition 4.7. It turns out that HW-density is a very strict condition as we show in Corollary 4.5. This result leads to Proposition 4.8, which lists all 2-tame G-modules, which are not fundamental.

We start with the definition of HW-density followed by the statements of the results. The proofs of these results are given below until the end of the section.

Definition 4.1. Let $\mathbb{X}$ be a smooth subvariety of a projective space $\mathbb{P}$. We say that $\mathbb{X}$ is *HW-dense*, if for any point $x_1 \in \mathbb{X}$ there exists an open subset U of the tangent space to $\mathbb{X}$ at x_1 such that for all $v \in U$ there exists $x_2 \in \mathbb{X}$ such that $v \in \langle x_1, x_2 \rangle$.

Definition 4.2. We say that a simple G-module $V(\lambda)$ is *HW-dense*, if $\mathbb{X}(\lambda)$ is HW-dense in $\mathbb{P}(\lambda)$.

We shall prove below in this subsection the following criterion of HW-density.

Lemma 4.4. The G-module $V(\lambda)$ is HW-dense if and only if one of the following equivalent conditions holds:

(a) The set $X(\lambda)$ of highest weight vectors is dense in $V(\lambda)$.

(b) All non-zero vectors of V are highest weight vectors with respect to some choice of a Borel subgroup of G,

(c) G acts transitively on $\mathbb{P}(V)$.

Corollary 4.5. Let V be an effective fundamental HW-dense G-module. Then (G, V) is isomorphic to $(Sp(V), V)$, $(SL(V), V)$ or $(SL(V), V^*)$.

Now, we formulate Propositions 4.6, 4.7 announced at the beginning of Section 4.

Proposition 4.6. Assume that $V(\lambda)$ is tame. Then $h(\lambda) < 3$.

Proposition 4.7. Let $\lambda_1, \lambda_2 \in \Lambda$ be non-zero weights. Assume that $V(\lambda_1 + \lambda_2)$ is 2-tame. Then both $V(\lambda_1), V(\lambda_2)$ are HW-dense.

The proof of Propositions 4.6, 4.7 is presented below in this subsection. Corollary 4.5 and Proposition 4.7 immediately imply the following proposition.

Proposition 4.8. Assume that $V(\lambda)$ is an effective 2-tame module and $h(\lambda) = 2$. Then $(G, V(\lambda))$ appears in the following list:

$$1)\ (SL(V_1) \times SL(V_2), V_1 \otimes V_2);\quad 2)\ (SL(V_1) \times Sp(V_2), V_1 \otimes V_2);\quad 3)\ (Sp(V_1) \times Sp(V_2), V_1 \otimes V_2);$$
$$4)\ (SL(V),\ \mathrm{S}^2 V);\quad 5)\ (SL(V), \mathfrak{sl}(V));\quad 6)\ (Sp(V), \mathfrak{sp}(V)) \cong (Sp(V),\ \mathrm{S}^2 V)$$

(here $\mathfrak{sl}(V), \mathfrak{sp}(V)$ denote the adjoint modules of the corresponding groups).

All modules listed in Proposition 4.8 are tame. Cases 1) and 4) of Lemma 4.8 are known to be tame. Cases 2), 3) have the same secant varieties as case 1), and hence are also tame. Cases 5) and 6) are tame due to [Kaji98]. Therefore Proposition 4.8 explicitly lists all non-fundamental tame modules.

The rest of the current subsection is dedicated to the proofs of Propositions 4.6, 4.7 and Lemma 4.4. We need the following lemma for the $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \mathrm{GL}(V_3)$-module $V_1 \otimes V_2 \otimes V_3$, where V_1, V_2, V_3 are finite-dimensional vector spaces.

Lemma 4.9. Let x_i, y_i be linearly independent vectors in V_i for $i = 1, 2, 3$. Then

$$T := x_1 \otimes x_2 \otimes x_3 + y_1 \otimes x_2 \otimes x_3 + x_1 \otimes y_2 \otimes x_3 + x_1 \otimes x_2 \otimes y_3 \neq v_1 \otimes v_2 \otimes v_3 + w_1 \otimes w_2 \otimes w_3$$

for all $v_i, w_i \in V_i$ $(i = 1, 2, 3)$. In other words, $\mathrm{rk}\, T > 2$.

Proof. Assume on the contrary that

$$T = v_1 \otimes v_2 \otimes v_3 + w_1 \otimes w_2 \otimes w_3$$

for some $v_i, w_i \in V_i, i = 1, 2, 3$. We have $V_1 \otimes V_2 \otimes V_3 \cong \mathrm{Hom}(V_2^* \otimes V_3^*, V_1)$. Therefore for any $x \in V_1 \otimes V_2 \otimes V_3$ we can define $\mathrm{Im}_1(x) \subset V_1$ as the image of the corresponding homomorphism from $\mathrm{Hom}(V_2^* \otimes V_3^*, V_1)$. Similarly we define $\mathrm{Im}_2(x)$ and $\mathrm{Im}_3(x)$. We have

$$\mathrm{Im}_i(T) = \langle x_i, y_i \rangle, i = 1, 2, 3.$$

On the other hand, if $v_1 \otimes v_2 \otimes v_3 + w_1 \otimes w_2 \otimes w_3 \neq 0$,

$$\mathrm{Im}_i(v_1 \otimes v_2 \otimes v_3 + w_1 \otimes w_2 \otimes w_3) \subset \langle v_i, w_i \rangle (i = 1, 2, 3).$$

Therefore, $\langle x_i, y_i\rangle = \langle v_i, w_i\rangle, i = 1, 2, 3$. Hence, without loss of generality, we may assume that

$$V_i = \langle x_i, y_i\rangle = \langle v_i, w_i\rangle (i = 1, 2, 3),$$

i.e. that $\dim V_i = 2$ $(i = 1, 2, 3)$. For two-dimensional spaces V_1, V_2, V_3 this lemma is well known; see e.g. [L12]. □

Note that for $(G, V(\lambda)) = (SL(V_1) \times SL(V_2) \times SL(V_3), V_1 \otimes V_2 \otimes V_3)$ we have $h(\lambda) = 3$. Moreover, Lemma 4.9 is essentially a particular case of Proposition 4.6.

Proof of Proposition 4.6. Assume on the contrary that $h(\lambda) \geq 3$. Then there exist non-zero weights $\lambda_1, \lambda_2, \lambda_3 \in \Lambda^+$ such that $\lambda = \lambda_1 + \lambda_2 + \lambda_3$. Then

$$v^{\lambda_1} \otimes v^{\lambda_2} \otimes v^{\lambda_3} \in V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3)$$

is a highest weight vector of weight $\lambda = \lambda_1 + \lambda_2 + \lambda_3$. Therefore the smallest G-submodule of $V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3)$ containing $v^{\lambda_1} \otimes v^{\lambda_2} \otimes v^{\lambda_3}$ is isomorphic to $V(\lambda)$. We identify $V(\lambda)$ with this submodule of $V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3)$ and set

$$v^\lambda := v^{\lambda_1} \otimes v^{\lambda_2} \otimes v^{\lambda_3}.$$

By Lemma 4.3 we have

$$v^\lambda + tv^\lambda \in \overline{X_2(\lambda)} (= \sigma_2(X(\lambda)))$$

for any $t \in \mathfrak{g}$. Therefore

$$\underline{\mathrm{rk}}_{\mathbb{X}(\lambda)}(v^\lambda + tv^\lambda) \leq 2 \tag{2}$$

for any $t \in \mathfrak{g}$. By the Leibnitz rule we have

$$T_\lambda := v^\lambda + tv^\lambda = v^{\lambda_1} \otimes v^{\lambda_2} \otimes v^{\lambda_3} + tv^{\lambda_1} \otimes v^{\lambda_2} \otimes v^{\lambda_3} + v^{\lambda_1} \otimes tv^{\lambda_2} \otimes v^{\lambda_3} + v^{\lambda_1} \otimes v^{\lambda_2} \otimes tv^{\lambda_3}$$

(note that T_λ is of the form T of Lemma 4.9). We claim that

$$v^\lambda + tv^\lambda \notin X_2(\lambda) \cup X(\lambda) \cup \{0\}$$

for all $t \in U$ from some open subset U of $\mathfrak{g}$. As $\lambda_i \neq 0$, there exists some open subset $U \subset \mathfrak{g}$ such that $[tv^{\lambda_i}] \neq [v^{\lambda_i}]$ $(i = 1, 2, 3)$ for all $t \in U$. We fix $t \in U$. We claim that

$$\mathrm{rk}(v^\lambda + tv^\lambda) \geq 3. \tag{3}$$

Assume on the contrary that $v^\lambda + tv^\lambda \in X_2(\lambda) \cup X(\lambda) \cup 0$, then

$$v^\lambda + tv^\lambda = g_1(v^{\lambda_1}) \otimes g_1(v^{\lambda_2}) \otimes g_1(v^{\lambda_3}) + g_2(v^{\lambda_1}) \otimes g_2(v^{\lambda_2}) \otimes g_2(v^{\lambda_3})$$

for some $g_1, g_2 \in G$ and thus

$$T_\lambda = v^\lambda + tv^\lambda = v_1 \otimes v_2 \otimes v_3 + w_1 \otimes w_2 \otimes w_3$$

for some $v_1, v_2, v_3, w_1, w_2, w_3 \in V(\lambda)$. This contradicts the statement of Lemma 4.9. Comparing (2) and (3) we see that $v^\lambda + tv^\lambda$ is an exceptional vector of $V(\lambda)$ and thus $V(\lambda)$ is 2-wild. □

Proof of Proposition 4.7. We use notation analogous to the one in Proposition 4.6. Fix $\lambda := \lambda_1 + \lambda_2$. Set

$$v^\lambda := v^{\lambda_1} \otimes v^{\lambda_2} \in V(\lambda_1) \otimes V(\lambda_2).$$

This defines a canonical embedding $V(\lambda) \to V(\lambda_1) \otimes V(\lambda_2)$. As $\lambda_i \neq 0$, there exists some open subset $U \subset \mathfrak{g}$ such that $[tv^{\lambda_i}] \neq [v^{\lambda_i}]$ $(i = 1, 2, 3)$ for all $t \in U$. We fix $t \in U$. Repeating the argument preceding (2) we show that (2) holds in new notation.

As $V(\lambda)$ is 2-tame we have

$$v^\lambda + tv^\lambda = g_1 v^\lambda + g_2 v^\lambda \text{ or } v^\lambda + tv^\lambda = g_1 v^\lambda$$

for some $g_1, g_2 \in G$. Then

$$\mathrm{Im}_i(v^\lambda + xv^\lambda) = \langle v^{\lambda_i}, tv^{\lambda_i}\rangle \ (i = 1, 2),$$

and, if $g_1 v^\lambda + g_2 v^\lambda \neq 0$,

$$\mathrm{Im}_i(g_1 v^\lambda + g_2 v^\lambda) = \langle g_1 v^{\lambda_i}, g_2 v^{\lambda_i}\rangle \ (i = 1, 2) \text{ or } \mathrm{Im}_i(g_1 v^\lambda + g_2 v^\lambda) = \langle g_1 v^{\lambda_i}\rangle.$$

Hence $g_1 v^{\lambda_i}, g_2 v^{\lambda_i} \in \langle v^{\lambda_i}, xv^{\lambda_i}\rangle$ and either

$$[g_1 v^{\lambda_i}] \neq [v^{\lambda_i}] \text{ or } [g_2 v^{\lambda_i}] \neq [v^{\lambda_i}] \ (i = 1, 2).$$

Therefore both $V(\lambda_1)$ and $V(\lambda_2)$ are HW-dense. This completes the proof. □

To prove Lemma 4.4 we need two technical lemmas. The first gives a reformulation of Definition 4.2.

Lemma 4.10. A simple G-module $V(\lambda)$ is HW-dense if and only if there exists an open subset $U \subset \mathfrak{g}$ such that, for all $t \in U$, there exists an element

$$v \in X(\lambda) \cap \langle v^\lambda, tv^\lambda\rangle$$

such that $[v] \neq [v^\lambda]$ (note that if such an element v exists, then $[tv^\lambda] \neq [v^\lambda]$).

Proof. This is fairy easy exercise for Lie algebras-Lie groups formalism. We omit it. □

Lemma 4.11. Fix $v_1 \in X(\lambda)$ and $t \in \mathfrak{g}$. Assume that there exists $v_2 \in X(\lambda)$ such that $v_2 \in \langle v_1, \mathfrak{g}v_1\rangle$. Then all non-zero vectors of $\langle v_1, v_2\rangle$ belongs to $X(\lambda)$.

Proof. Without loss of generality we assume that $[v_1] \neq [v_2]$. Then, by Lemma 4.1, a), the pair (v_1, v_2) is conjugate to the pair $(v^\lambda, v^{w\lambda})$ for some $w \in \mathcal{W}$ and thus we can assume that

$$v_1 = v^\lambda \text{ and } v_2 = v^{w\lambda}$$

for the fixed maximal torus $T \in G$. The space $\mathfrak{g}v_1$ is clearly T-invariant and the weights of this space is a subset of the set $\lambda + \Delta$ (this is a point-wise sum). As $v^{w\lambda} \in \mathfrak{g}v^\lambda$, we have

$$w\lambda = \lambda + \beta \text{ for some } \beta \in \Delta$$

(note that $w\lambda \neq \lambda$ as $[v_1] \neq [v_2]$). Let $SL_2(\beta)$ be the T-stable SL_2-subalgebra corresponding to the root $\beta \in \Delta$. Then the space $\langle v^\lambda, v^{w\lambda}\rangle$ is a two-dimensional simple $SL_2(\beta)$-module and thus any two non-zero elements of $\langle v_1, v_2\rangle$ are $SL_2(\beta)$-conjugate, and hence G-conjugate. Therefore all non-zero elements of $\langle v_1, v_2\rangle$ belong to $X(\lambda)$. □

Proof of Lemma 4.4. The equivalence of conditions (a), (b), (c) is clear. It is also immediate to verify that each of these conditions implies HW-density. It remains to show that, if the module $V(\lambda)$ is HW-dense, then it satisfies condition (a).

Assume that $V(\lambda)$ is HW-dense. Then according to Lemma 4.10 and Lemma 4.11 there exists and open subset $U \subset \mathfrak{g}$ such that for any non-zero $a \in \mathbb{F}$ we have

$$v^\lambda + atv^\lambda \in X(\lambda),$$

i.e. we have that $X(\lambda) \cap \langle v^\lambda, \mathfrak{g}v^\lambda\rangle$ is a dense subset of $\langle v^\lambda, \mathfrak{g}v^\lambda\rangle$. Note that $v^\lambda \in \mathfrak{g}v^\lambda$, as $\lambda \neq 0$, and hence

$$\langle v^\lambda, \mathfrak{g}v^\lambda\rangle = \langle \mathfrak{g}v^\lambda\rangle.$$

We have

$$\dim X(\lambda) = \dim Gv^\lambda = \dim \mathfrak{g}v^\lambda$$

and therefore

$$\overline{X(\lambda)} = \mathfrak{g}v^\lambda.$$

On the other hand

$$V(\lambda) = \langle X(\lambda) \rangle$$

and hence

$$V(\lambda) = \overline{X(\lambda)} = \mathfrak{g}v^\lambda.$$

□

5 Restriction to a Levi subgroup

The main result of this section is Proposition 5.1. This proposition provides a strong sufficient condition for wildness of a representation. We shall apply this proposition to study tameness/wildness of fundamental representations in the subsequent section of this article.

Proposition 5.1. Let V be an irreducible G-module and $\underline{V}$ be a $\underline{G}$-module, which is a chopping of V. If $\underline{V}$ is wild, then V is wild.

This proposition is an immediate corollary of Proposition 5.2. To state Proposition 5.2 we need more notation.

Recall that we have fixed Cartan and Borel subgroups $H \subset B \subset G$ and that Π denotes the corresponding set of simple roots. Let $\underline{\Pi} \subset \Pi$ be a subset. Then $\underline{\Delta} = \Delta \cap \langle \underline{\Pi} \rangle$ is a root system having $\underline{\Pi}$ as a set of simple roots and $\underline{\Delta}^{\pm} = \underline{\Delta} \cap \Delta^{\pm}$ as sets of positive and negative roots. Further, let $\underline{\mathfrak{g}} = \mathfrak{h} \oplus (\oplus_{\alpha \in \underline{\Delta}})\mathfrak{g}^\alpha$. Then $\underline{\mathfrak{g}}$ is a reductive subalgebra of $\mathfrak{g}$; we call subalgebras of this form (reductive) Levi subalgebras. Let $\underline{G} \subset G$ be the corresponding Levi subgroup. We shall add underline to denote the attributes of $\underline{G}$ with the notational conventions already introduced for G.

Note that G and $\underline{G}$ have a common Cartan subgroup H and hence have the same weight lattice Λ. However, the dominant Weil chambers do not coincide, unless $\underline{\Pi} = \Pi$, a case which is of no use for us. We have an inclusion $\Lambda^+ \subset \underline{\Lambda}^+$, so a weight $\lambda \in \Lambda^+$ can be regarded as a dominant weight for both G and $\underline{G}$. Furthermore, since $\underline{B} = B \cap \underline{G}$, the B-highest weight vectors are also $\underline{B}$-highest weight vectors.

Fix $\lambda \in \Lambda^+$. There is a $\underline{G}$-equivariant inclusion of the corresponding representations

$$\underline{V} = \underline{V}(\lambda) = \mathfrak{U}(\underline{\mathfrak{g}})v^\lambda \subset V(\lambda) = V \ ,$$

where v^λ denotes the B-highest weight vector in $V(\lambda)$. Let $\underline{\mathbb{X}}$ denote the unique closed $\underline{G}$-orbit in $\mathbb{P}(\underline{V})$ and, as before, let $\mathbb{X}$ denote the unique closed G-orbit in $\mathbb{P}(V)$. We have

$$\underline{\mathbb{X}} = \underline{G}[v^\lambda] \subset G[v^\lambda] = \mathbb{X}.$$

For points in $\mathbb{P}(\underline{V})$, we have two well defined rank functions $\mathrm{rk}_{\underline{\mathbb{X}}}$ and $\mathrm{rk}_{\mathbb{X}}$ (notice that $\underline{V}$ is a chopping of V according to Definition 2.4). We would like to compare these functions and prove the following.

Proposition 5.2. Let $\underline{G} \subset G$ and $\underline{V} \subset V$ be as above. If $[\psi] \in \mathbb{P}(\underline{V})$, then $\mathrm{rk}_{\underline{\mathbb{X}}}[\psi] = \mathrm{rk}_{\mathbb{X}}[\psi]$.

Proof. First, observe that the multiplicity of the $\underline{G}$-module $\underline{V}(\lambda)$ in $V(\lambda)$ is 1. This holds because $\underline{V}(\lambda)$ has a weight vector with weight λ and this weight has multiplicity 1 in $V(\lambda)$. Consequently, there is a well-defined $\underline{G}$-equivariant projection

$$\pi : V \twoheadrightarrow \underline{V} \ .$$

Let $P \subset G$ be the parabolic subgroup containing B and having $\underline{G}$ as a Levi component; the roots of P are $\Delta^+ \sqcup \underline{\Delta}^-$. Let N_P be the unipotent radical of P; the roots of N_P are $\Delta^+ \setminus \underline{\Delta}^+$. Then N_P acts trivially on $\underline{V}$.

Lemma 5.3. We have $\pi(X \cup 0) = \underline{X} \cup 0$.

Proof. Let N_P^- be the nilradical of the parabolic P^- opposite to P, with respect to the given Cartan subgroup H. In other words, N_P^- is the regular unipotent subgroup of N^- with roots $\Delta(N_P^-) = -\Delta(N_P)$. We have

$$X = \overline{Gv^\lambda} = \overline{P^- v^\lambda} = \overline{N_P^- (\underline{G} v^\lambda)} = \overline{N_P^- \underline{X}} \, .$$

Thus, to prove the lemma it is sufficient to show that for all $g \in N_P^-$ and all $v \in \underline{X}$ we have $\pi(gv) \in \underline{X}$. Let $g \in N_P^-$ and $v \in \underline{X}$. Since the exponential map $\exp : \mathfrak{n}_\mathfrak{p}^- \to N_P^-$ is surjective, we can write $g = \exp(\xi)$ with $\xi \in \mathfrak{n}_\mathfrak{p}^-$. Viewing ξ as an element of $\mathfrak{gl}(V)$ we can write

$$gv = (1 + \xi + \frac{1}{2}\xi^2 + ...)v = v + \xi v + \frac{1}{2}\xi^2 v + \cdots \, .$$

Let $\underline{V}' = \ker(\pi)$, so that $V = \underline{V} \oplus \underline{V}'$ as $\underline{G}$-modules. Then, for $\xi \in \mathfrak{n}_\mathfrak{p}^-$, we have $\xi(\underline{V}) \subset \underline{V}'$. Hence $\pi(gv) = v$. □

Now, let $[\psi] \in \mathbb{P}(\underline{V})$. The inequality $\underline{\mathrm{rk}}[\psi] \geq \mathrm{rk}[\psi]$ is immediate. Let $r = \mathrm{rk}[\psi]$ and

$$\psi = v_1 + ... + v_r$$

be a minimal expression, with $[v_j] \in \mathbb{X}$. Then we have

$$\psi = \pi(\psi) = \pi(v_1) + ... + \pi(v_r)$$

and, according to the above lemma, $[\pi(v_j)] \in \underline{\mathbb{X}} \cup 0$ (this is a set). Hence $\underline{\mathrm{rk}}[\psi] \leq \mathrm{rk}[\psi]$ and so

$$\underline{\mathrm{rk}}[\psi] = \mathrm{rk}[\psi].$$

□

6 Fundamental representations (classical groups)

In this subsection we prove Theorem 1.1 for fundamental modules of classical groups, i.e. we prove Theorem 6.1. The result follows directly from Propositions 6.2, 6.4, 6.5 of Subsections 6.1, 6.2, 6.3, where we consider the cases of SL_n, SO_n, Sp_{2n}, respectively.

Theorem 6.1. Let $V(\lambda)$ be a fundamental module of a simple classical group G. Then $V(\lambda)$ is tame if and only if the pair $(G, V(\lambda))$ appears in the following table.

Group G	Representation V	Highest weight of V
Classical groups		
SL_n	$\mathbb{F}^n, (\mathbb{F}^n)^*, (\Lambda^2\mathbb{F}^n), (\Lambda^2\mathbb{F}^n)^*$	$\pi_1, \pi_{n-1}, \pi_2, \pi_{n-2}$
SO_n	$\mathbb{F}^n, RSpin_n (n \leq 10)$	$\pi_1, \pi_{\frac{n}{2}} (2 \mid n)$, $\pi_{\frac{n}{2}-1} (2 \mid n), \pi_{\frac{n-1}{2}} (2 \nmid n)$
SP_{2n}	$\mathbb{F}^{2n}, \Lambda_0^2\mathbb{F}^{2n}$	π_1, π_2

, (4)

where the notation is the same as in Theorem 1.1.

Moreover, all wild fundamental representations of classical groups are 2-wild.

Our approach for classical groups is tensor-based and we often use symmetric/antisymmetric bilinear forms. To prove Theorem 6.1 we also need some sufficient condition of wildness for representations. We provide such a condition in Proposition 5.1 of Section 5. In a similar way Proposition 5.1 will be very useful in Section 7, where we consider the fundamental representations of the exceptional groups.

6.1 $G = SL_n$

Recall that the fundamental representations of $G = SL_n$ are obtained as exterior tensor powers of the natural representation, i.e. $V(\pi_k) = \Lambda^k\mathbb{F}^n$, $k = 1, ..., n-1$. Furthermore, we have

$$(\Lambda^k\mathbb{F}^n)^* = \Lambda^{n-k}\mathbb{F}^n$$

as SL_n-modules.

Proposition 6.2. The fundamental representations of SL_n which are tame are exactly

$$\mathbb{F}^n, (\mathbb{F}^n)^*, \Lambda^2\mathbb{F}^n, (\Lambda^2\mathbb{F}^n)^*.$$

Moreover, all wild fundamental representations of SL_n are 2-wild.

Proof. The closed G-orbit $\mathbb{X} \subset \mathbb{P}(\Lambda^k\mathbb{F}^n)$ is the Grassmann variety $\mathrm{Gr}_k(\mathbb{F}^n)$ under its Plücker embedding. It is well known that a suitable isomorphism between $\Lambda^k\mathbb{F}^n$ and $\Lambda^{n-k}\mathbb{F}^n$ induces an isomorphism between the respective projective spaces, which carries $\mathrm{Gr}_k(\mathbb{F}^n)$ to $\mathrm{Gr}_{n-k}(\mathbb{F}^n)$. Hence, for our purposes, it is sufficient to consider $k \leq n/2$.

The fact that $V(\pi_1) = \mathbb{F}^n$ and $V(\pi_2) = \Lambda^2\mathbb{F}^n$ are tame is well known. In fact, in the first case we have $\mathbb{X} = \mathbb{P}(\mathbb{F}^n)$, so all vectors have rank 1. In the second case, $\Lambda^2\mathbb{F}^n$ can be identified with the space of skew-symmetric $n \times n$ matrices. Such a matrix has even rank (in the usual sense) and the SL_n-orbit X through a highest weight vector in $\Lambda^2\mathbb{F}^n$ consists of all matrices of rank 2. A skew-symmetric matrix ψ of rank $2r$ can be written as a sum of r skew-symmetric matrices of rank 2, and so $\mathrm{rk}_{\mathbb{X}}[\psi] = r$. We can now see that the set

$$\{[\psi] \in \mathbb{P}(\Lambda^2\mathbb{F}^n) : \mathrm{rk}_{\mathbb{X}}\psi \leq r\}$$

is closed for every r. This completes the argument in this case.

We now turn to the remaining cases. Due to the duality

$$\Lambda^k\mathbb{F}^n \leftrightarrow \Lambda^{n-k}(\mathbb{F}^n)^*,$$

it suffices to consider $n \geq 6$. Proposition 5.2 implies that, to show that $\Lambda^k\mathbb{F}^n$ $(3 \leq k \leq n/2)$ is 2-wild, it is sufficient to show that $\Lambda^3\mathbb{F}^6$ is 2-wild.

Lemma 6.3. The representation of SL_6 on $\Lambda^3\mathbb{F}^6$ is 2-wild.

Proof. It is shown in [Zak93, Ch. III, Thm 1.4], that $\sigma_2(X(\Lambda^3\mathbb{F}^6)) = \Lambda^3\mathbb{F}^6$ and therefore it is enough to show that $\Lambda^3\mathbb{F}^6$ contains a vector of rank 3 or more. To do this we count the number of orbits of vectors of rank 0, 1, 2 and compare this number with the known number of orbits for the action of SL_6 on $\Lambda^3\mathbb{F}^6$, see [Gu48] or [R07].

By definition there is one orbit of vectors of rank 0 and one orbit of vectors of rank 1. Let us consider the vectors of rank 2 in V. We show that there are two orbits of such vectors. Any vector of rank 2 can be written as

$$\psi = v_1 \wedge v_2 \wedge v_3 + v_4 \wedge v_5 \wedge v_6 ,$$

with some $v_j \in V$. The first possibility is that $v_1, ..., v_6$ form a basis of $\mathbb{F}^6$. This is indeed the generic situation. If suitable Borel and Cartan subgroups of SL_6 are chosen, the two summands of ψ are, respectively, the highest and lowest weight vectors in V. The group GL_6 acts transitively on the set of all bases of $\mathbb{F}^6$; the group SL_6 acts transitively on the set of their projective images. Thus the points of the first type form a single G-orbit $\mathbb{X}_2'$ which is open in $\mathbb{P}(\Lambda^3\mathbb{F}^6)$. We denote by Z the complement to this orbit in $\mathbb{P}(\Lambda^3\mathbb{F}^6)$.

The second possibility is to have

$$\dim(\langle v_1, v_2, v_3\rangle \cap \langle v_4, v_5, v_6\rangle) = 1 .$$

If this is the case, by changing the vectors if necessary, we may reduce to the situation where $v_1 = v_4$ and

$$\psi = v_1 \wedge (v_2 \wedge v_3 + v_5 \wedge v_6) \ , \quad \text{with} \quad \langle v_2, v_3 \rangle \cap \langle v_5, v_6 \rangle = 0 \ .$$

Since $v_2 \wedge v_3 + v_5 \wedge v_6$ has rank 2 in $\Lambda^2 \mathbb{F}^6$ (with respect to $\mathrm{Gr}_2(\mathbb{F}^6)$), we deduce that ψ has indeed rank 2 in V. The point ϕ does not belong to $\mathbb{X}_2'$, because the action of GL_6 respects linear dependencies. On the other hand, it is also clear that GL_6 acts transitively on the set $\mathbb{X}_2''$ of points of this second type, and hence SL_6 acts transitively on the set of their images in $\mathbb{P}$. Note that

$$\text{if } \dim(\langle v_1, v_2, v_3 \rangle \cap \langle v_4, v_5, v_6 \rangle) > 1, \text{ then } \mathrm{rk}[\psi] = 1.$$

We can conclude that there are exactly two G-orbits consisting of points of rank 2, namely

$$\mathbb{X}_2' = \mathbb{P} \setminus Z, \quad \mathbb{X}_2'' = Z \cap \mathbb{X}_2 \ .$$

Thus there are four SL_6-orbits of vectors of rank 0, 1, 2. It is known that SL_6 has five orbits in $\Lambda^3 \mathbb{F}^6$. Therefore $\Lambda^3 \mathbb{F}^6$ has a unique SL_6-orbit of vectors of rank 3 or more and $\Lambda^3 \mathbb{F}^6$ is 2-wild. An example of a vector of rank 3 is (see [Gu48])

$$\Lambda 3 = v_1 \wedge v_2 \wedge v_4 + v_1 \wedge v_5 \wedge v_3 + v_6 \wedge v_2 \wedge v_3 \ .$$

□

□

6.2 $G = SO_n$

Let $\ell = rank(G) = \lfloor \frac{n}{2} \rfloor$. In this section we prove the following proposition.

Proposition 6.4. The natural representation $V(\pi_1)$ is tame.

1) If n is even, then, for $j = 2, ..., \ell - 2$, the representation $V(\pi_j)$ is wild.

2) If n is odd, then, for $j = 2, ..., \ell - 1$, the representation $V(\pi_j)$ is wild.

3) The spin representations ($V(\pi_{\ell-1})$ and $V(\pi_\ell)$ for even n and $V(\pi_\ell)$ for odd n) are tame if and only if $n \leq 10$.

Moreover, all fundamental representations of SO_n which are wild are 2-wild.

Proof. The first statement is well known. Indeed, the group SO_n has exactly two orbits in $\mathbb{P}(\pi_1)$, namely, the quadric and its complement. The first one consists, by definition, of vectors of rank 1. The second one consists necessarily of points of rank 2.

The second and third statement in the proposition are concerned with fundamental representations of SO_n, which are not the natural nor the spin representation. We handle the two statements at once. In the case $j = 2$, the representation is actually the adjoint representation, i.e. $V(\pi_2) = \mathfrak{so}_n$. Here results of [KY00] show that $\sigma_2(\mathbb{X}) \neq \mathbb{X} \sqcup \mathbb{X}_2$. Thus the rank function is wild. The remaining cases, $j \geq 3$, are reduced to the case $j = 2$ via Proposition 5.2.

Now, we turn to the last statement of the proposition, concerning the spin representations. First, recall that, for even n, the geometric properties we are concerned with are the same for the two spin representations $V(\pi_{\ell-1})$ and $V(\pi_\ell)$. Also, either one of these representations remains irreducible when restricted to $Spin_{n-1}$, and furthermore, $Spin_{n-1}$ acts transitively on the closed orbit of $Spin_n$) in $\mathbb{P}(\pi_\ell)$. Thus, it is enough to check statement 3) for the representations $V(\pi_\ell)$ of $Spin_{2\ell}$. Let $\mathbb{X}$ denote the closed orbit of $Spin_{2\ell}$ in $\mathbb{P}(\pi_\ell)$.

It is shown, in [Car97] Section 3.5, that for $2\ell = 12$ the secant variety $\sigma_2(\mathbb{X})$ contains elements of rank 3. Thus the representation $V(\pi_6)$ of $Spin_{12}$ is 2-wild. Using Proposition 5.1, we deduce that the representation $V(\pi_\ell)$ of $Spin_{2\ell}$ is wild for all $\ell \geq 6$. So, according to the remarks made earlier in this proof, the spin representations of $Spin_n$ are wild for $n \geq 11$.

It remains to verify that the spin representations are tame for even $n \leq 10$. We consider only $n \geq 8$, the other cases are covered by our results for SL_m. For $n = 8$, the geometric properties of the representations $V(\pi_1)$ and $V(\pi_4)$ are the same, so the result follows from the first statement of the proposition. For $n = 10$, the representation $V(\pi_5)$ of $Spin_{10}$ is subminuscule and hence tame by [BL13]. This completes the proof of the proposition. □

6.3 $G = Sp_{2n}$

Proposition 6.5. The fundamental representations of Sp_{2n} which are tame are exactly $V(\pi_1)$ and $V(\pi_2)$. All other fundamental representations of Sp_{2n} are 2-wild.

Proof. The representation $V(\pi_1)$ is simply the natural representation of Sp_{2n} on $\mathbb{F}^{2n}$. The action of Sp_{2n} on $\mathbb{P}(\mathbb{F}^{2n})$ is transitive, i.e. $\mathbb{X} = \mathbb{P}(\mathbb{F}^{2n})$ and there is nothing more to prove here. The representations $V(\pi_2)$ and $V(\pi_k), k \geq 3$ are considered in Lemmas 6.8 and 6.7, respectively. □

Lemma 6.6. The representation $V(\pi_3)$ of Sp_{2n} is 2-wild for $n \geq 3$.

Proof. Let $n \geq 3$ and $G = Sp_{2n}$ with respect to the skew-symmetric form

$$(z_1 \wedge z_6 + z_2 \wedge z_5 + z_3 \wedge z_4) + (z_7 \wedge z_8 + ... + z_{2n-1} \wedge z_{2n})$$

on $\mathbb{F}^{2n}$, where $z_1, ..., z_{2n}$ are the dual basis corresponding to the basis $v_1, ..., v_{2n}$ of $\mathbb{F}^{2n}$.

Let X be the set of points $[w_1 \wedge w_2 \wedge w_3]$ such that $w_1, w_2, w_3 \in \mathbb{F}^{2n}$ span a 3-dimensional isotropic subspace $\mathbb{F}^{2n}$. We set

$$\Lambda^3_0\mathbb{F}^{2n} := \langle X \rangle.$$

The set X is Sp_{2n}-stable and thus $\Lambda^3_0\mathbb{F}^{2n}$ is an Sp_{2n}-module. There is a unique up to scaling Sp_{2n}-isomorphism between $V(\pi_3)$ of Sp_{2n} and $\Lambda^3_0\mathbb{F}^{2n}$. We identify $V(\pi_3)$ with $\Lambda^3_0\mathbb{F}^{2n}$. The varieties X and $X(\pi_3)$ coincides under this identification.

Note that if $\psi \in V(\pi_3)$ has rank 3 as a vector of the SL_{2n}-module $\Lambda^3\mathbb{F}^{2n}$, then ψ has rank 3 or more as a vector of the Sp_{2n}-module $V(\pi_3)$.

Consider the tensor $\Lambda 3 \in \Lambda^3\mathbb{F}^6$ given at the end of the proof of Lemma 6.3. One have

$$[v_1 \wedge v_2 \wedge v_4], [v_1 \wedge v_5 \wedge v_3], [v_6 \wedge v_2 \wedge v_3] \in X(\pi_3)$$

and thus $\Lambda 3 \in \Lambda^3_0\mathbb{F}^6 \subset \Lambda^3_0\mathbb{F}^{2n}$. Moreover the rank of $\Lambda 3$ is 3 or less. As $\Lambda 3$ has rank 3 as an element of SL_6-module $\Lambda^3\mathbb{F}^6$, $\Lambda 3$ has rank 3 as an element of $\Lambda^3\mathbb{F}^{2n}$ (see Proposition 5.2). Hence

$$\operatorname{rk} \Lambda 3 = 3, \tag{5}$$

where $\Lambda 3$ considered as an element of $\Lambda^3_0\mathbb{F}^{2n}$.

It is shown in [Zak93, Ch. III, Thm 1.4], that $\sigma_2(\mathbb{X}) = \mathbb{P}(\Lambda^3_0\mathbb{F}^6)$. Thus $\Lambda 3$ has border rank 2 or less as an element of $\Lambda^3_0\mathbb{F}^6$ and hence

$$\underline{\operatorname{rk}} \Lambda 3 \leq 2,$$

here $\Lambda 3$ considered as an element of $\Lambda^3_0\mathbb{F}^{2n}$. Therefore the Sp_{2n}-module $V(\pi_3) = \Lambda^3_0\mathbb{F}^{2n}$ is 2-wild. □

Lemma 6.7. Fix $n \geq k \geq 3$. The representation $V(\pi_k)$ of Sp_{2n} is 2-wild.

Proof. If $n \geq k \geq 3$, the Dynkin diagram C_n of Sp_{2n} has a subdiagram $\underline{C}_{n,k} := C_{n-k+3}$. The chopping of π_k to this diagram equals π_3. By Lemma 6.6, $V(\pi_3)$ is not tame for $\underline{G} = Sp_{2(n-k+3)}$ (this group corresponds to the Dynkin diagram $\underline{C}_{n,k}$) and thus $V(\pi_k)$ is not a tame Sp_{2n}-module by Proposition 5.1. □

We are now going to prove that $V(\pi_2)$ is tame for $Sp_{2n}(n \geq 2)$. To do this we set $V := \mathbb{F}^{2n}$ and fix a non-degenerate antisymmetric bilinear form ω on V. Note that the second fundamental module of $\mathfrak{sp}(V)$ is isomorphic to the set of vectors in $\Lambda^2 V$, which are annihilated by ω (here we consider ω as an element of $(\Lambda^2 V)^*$). We denote this space by $\Lambda_0^2 V$. To complete the proof of Proposition 6.5 we prove the following lemma.

Lemma 6.8. a) For any $\underline{\omega} \in \Lambda_0^2 V$ the rank of $\underline{\omega}$ as a bilinear coform is twice the rank of $\underline{\omega}$ as a vector in an $Sp(V)$-module.
b) The $Sp(V)$-module $\Lambda_0^2 V$ is tame.

To prove Lemma 6.8 we introduce a notion related to bilinear coforms $\underline{\omega} \in \Lambda^2 V$. A bilinear coform $\underline{\omega}$ defines a map $V^* \to V$ by $v \to \underline{\omega}(v, \cdot)$. We denote the image of this map by $\operatorname{Supp} \underline{\omega}$. We have natural inclusions

$$\Lambda^2 \operatorname{Supp} \underline{\omega} \to \Lambda^2 V, \qquad \Lambda_0^2 \operatorname{Supp} \underline{\omega} \to \Lambda_0^2 V,$$

and $\underline{\omega} \in \Lambda_0^2 \operatorname{Supp} \underline{\omega} \subset \Lambda_0^2 V$. Note that $\underline{\omega}$ is nondegenerate as an element of $\Lambda^2 \operatorname{Supp} \underline{\omega}$ and, in particular, defines a bilinear form $\underline{\omega}^*$ on $\operatorname{Supp} \underline{\omega}$ (there is no canonical way to extend $\underline{\omega}^*$ to the whole V).

Lemma 6.8 follows from Lemma 6.9 below; a proof of Lemma 6.8 is presented after the proof of Lemma 6.9.

Lemma 6.9. Let $\underline{\omega} \in \Lambda_0^2 V$ be a bilinear coform of rank $2r$. Then there exist a set of elements $x_1, ..., x_r, y_1, ..., y_r \in \operatorname{Supp} \underline{\omega}$ such that

$$\underline{\omega} = x_1 \wedge y_1 + ... + x_r \wedge y_r,$$

and $\omega(x_i, y_i) = 0$ for all i.

In turn, Lemma 6.9 follows from Lemma 6.10 below; a proof of Lemma 6.9 is presented after the proof of Lemma 6.10.

Lemma 6.10. Let $\underline{\omega} \in \Lambda_0^2 V$ be a bilinear coform of rank $2r$. If $r > 0$, then there exist elements $x_1, y_1 \in \operatorname{Supp} \underline{\omega}$ such that

$$\operatorname{rk}(\underline{\omega} - x_1 \wedge y_1) = 2r - 2,$$

and $\omega(x_1, y_1) = 0$.

In turn, Lemma 6.10 follows from Lemma 6.11 below; a proof of Lemma 6.10 is presented after the proof of Lemma 6.11.

Lemma 6.11. Let $\underline{\omega} \in \Lambda_0^2 V$ be a bilinear coform of rank $2r$. If $r > 0$, then there exist an open subset $U \subset \operatorname{Supp} \underline{\omega}$ such that for any $x_1 \in U$ there exists $y_1 \in \operatorname{Supp} \underline{\omega}$ such that $\underline{\omega}^*(x_1, y_1) \neq 0$ and $\omega(x_1, y_1) = 0$.

Proof. If a form ω is zero on $\operatorname{Supp} \underline{\omega}$, then for any non-zero $x_1 \in \operatorname{Supp} \underline{\omega}$ there exists $y_1 \in \operatorname{Supp} \underline{\omega}$ such that $\underline{\omega}^*(x_1, y_1) \neq 0$, because the form $\underline{\omega}^*$ is non-degenerate on $\operatorname{Supp} \underline{\omega}$. In this case $\omega(x_1, y_1) = 0$, because $\omega = 0$.

We assume that ω is non-zero on $\operatorname{Supp} \underline{\omega}$. Since $\underline{\omega} \in \Lambda_0^2 V$, the pairing of $\underline{\omega}$ with ω equals 0. Thus $[\underline{\omega}^*] \neq [\omega|_{\Lambda^2 \operatorname{Supp} \underline{\omega}}]$. Hence, for some open subset $U \subset \operatorname{Supp} \underline{\omega}$, and any $x_1 \in U$ both $\omega(x_1, \cdot), \underline{\omega}^*(x_1, \cdot)$ are non-zero and

$$[\omega(x_1, \cdot)] \neq [\underline{\omega}^*(x_1, \cdot)].$$

Therefore for any $x_1 \in U$ there exists $y_1 \in \operatorname{Supp} \underline{\omega}$ such that

$$\underline{\omega}^*(x_1, y_1) \neq 0 \text{ and } \omega(x_1, y_1) = 0.$$

□

Proof of Lemma 6.10. Let (x_1, y_1) be a pair as in Lemma 6.11. We denote by W_2 the space spanned by x_1, y_1 and by W_{2r-2} the orthogonal complement to W_2 in $\mathrm{Supp}\,\underline{\omega}$. Thanks to the choice of x_1, y_1, the form $\underline{\omega}^*$ is non-degenerate on W_2 and therefore $\mathrm{Supp}\,\underline{\omega} = W_2 \oplus W_{2r-2}$. Then $\underline{\omega} = \underline{\omega}^2 + \underline{\omega}^{2r-2}$ for uniquely determined coforms $\underline{\omega}^2 \in \Lambda^2 W_2$ and $\underline{\omega}^{2r-2} \in \Lambda^2 W_{2r-2}$. We have $\underline{\omega}^2 = \lambda(x_1 \wedge y_1) = x_1 \wedge (\lambda y_1)$ for some $\lambda \in \mathbb{F}^\times$. Therefore

$$\mathrm{rk}(\underline{\omega} - x_1 \wedge (\lambda y_1)) = 2r - 2,$$

and $\omega(x_1, (\lambda y_1)) = 0$. □

Proof of Lemma 6.9. To prove Lemma 6.9 we use induction.

The r-th statement of the induction is: Let $\underline{\omega} \in \Lambda_0^2 V$ be a bilinear coform of rank $2r$. Then there exist a set of elements $x_1, ..., x_r, y_1, ..., y_r \in \mathrm{Supp}\,\underline{\omega}$ such that

$$\underline{\omega} = x_1 \wedge y_1 + ... + x_r \wedge y_r,$$

and $\omega(x_i, y_i) = 0$ for all i.

Basis of the induction, for $r = 1$: Let $\underline{\omega} \in \Lambda_0^2 V$ be a bilinear coform of rank 2. Then there exist elements $x_1, y_1 \in \mathrm{Supp}\,\underline{\omega}$ such that $\underline{\omega} = x_1 \wedge y_1$, and $\omega(x_1, y_1) = 0$.

First, we check the basis of the induction. Let x_1, y_1 be basis of $\mathrm{Supp}\,\underline{\omega}$. Then $\underline{\omega} = \lambda x_1 \wedge y_1$ for some $\lambda \in \mathbb{F}^\times$. As $\underline{\omega} \in \Lambda_0^2 V$, we have $\omega(\underline{\omega}) = \omega(\lambda x_1 \wedge y_1) = \omega(x_1, \lambda y_1) = 0$. Then $\underline{\omega} = x_1 \wedge (\lambda y_1)$ and $\omega(x_1, \lambda y_1)$=0. Therefore we finish with the basis of induction.

Now we prove that the r-th statement of the induction follows from the $(r-1)$-th statement. We assume that the $(r-1)$-th statement holds. According to Lemma 6.9 there exists x_r, y_r such that

$$\mathrm{rk}(\underline{\omega} - x_r \wedge y_r) = 2r - 2,$$

and $\omega(x_r, y_r) = 0$. Note that $\omega(x_r \wedge y_r) = \omega(x_r, y_r) = 0$ and therefore $\underline{\omega} - x_r \wedge y_r \in \Lambda_0^2 V$. By hypothesis, there exist $x_1, ..., x_{r-1}, y_1, ..., y_{r-1} \in \mathrm{Supp}(\underline{\omega} - x_r \wedge y_r) \subset \mathrm{Supp}\,\underline{\omega}$ such that

$$\underline{\omega} - x_r \wedge y_r = x_1 \wedge y_1 + ... + x_{r-1} \wedge y_{r-1},$$

and $\omega(x_i, y_i) = 0$ for all i. This completes the proof of Lemma 6.9. □

Proof of Lemma 6.8. First note that a highest weight vector of the $Sp(V)$-module $\Lambda_0^2 V$ is a wedge product of two ω-orthogonal vectors of V. Fix a coform $\underline{\omega} \in \Lambda_0^2 V$. A sum of r vectors from the $Sp(V)$-orbit of a heighest weight vector has rank at most $2r$ as a bilinear coform. Hence the rank of $\underline{\omega}$ as a vector of an $Sp(V)$-module is not less than half the rank of $\underline{\omega}$ as a bilinear coform. On the other hand, Lemma 6.9 implies that the rank of $\underline{\omega}$ as a vector of an $Sp(V)$-module is not larger than half the rank of $\underline{\omega}$ as a bilinear coform. Therefore the rank of $\underline{\omega}$ as a vector of an $Sp(V)$-module is equal to half the rank of $\underline{\omega}$ as a bilinear coform. This proves part a) of Proposition 6.8.

The set of coforms of rank r or less is closed for all r and this finishes part b). □

7 Fundamental representations (exceptional groups)

In this subsection we prove Theorem 1.1 for fundamental modules of exceptional groups, i.e. we prove Theorem 7.1. Essentially, we consider case-by-case all 27 fundamental representations of the 5 exceptional groups and provide some arguments for each case, by which the corresponding fundamental module is wild or tame. The result is presented below.

Theorem 7.1. Assume that $V(\lambda)$ is a fundamental effective G-module. Then $V(\lambda)$ is tame if and only if the pair $(G, V(\lambda))$ appears in the following table.

G	Representation V	Highest weight of V
E_6	$\mathbb{F}^{27}$, $(\mathbb{F}^{27})^*$	π_1, π_5
F_4	$\mathbb{F}^{26}$	π_1
G_2	$\mathbb{F}^7$	π_1

, (6)

where the notation is the same as in Theorem 1.1.

Moreover, all wild fundamental representations of exceptional groups are 2-wild.

The types of arguments are presented in the following tables.

Symbol	Argument for being **tame**	References
SM	the representation is reduced to a subminuscule representation	Section 1, [BL13]
F4T	the representation is equivalent to $V(\pi_1)$ of F_4	Subsection 7.1

Symbol	Argument for being **wild**	References
CC	the representation is chopable to a wild representation of some classical group	—
Ad	the representation is adjoint	Section 1, [Kaji98]
AdC	the representation is chopable to the adjoint representation of some exceptional group	—
E7W	the representation is equivalent to the E_7-representation V(π_1)	Subsection 7.3
F4W	the representation is equivalent to $V(\pi_2)$ of F_4	Subsection 7.2

.

In the following tables, we provide, for each fundamental representation of each exceptional group, an argument by which it is tame or wild.

F. weights of E$_6$	π_1	π_2	π_3	π_4	π_5	π_6
Arguments	SM	CC	CC	CC	SM	CC or Ad

F. weights of E$_7$	π_1	π_2	π_3	π_4	π_5	π_6	π_7
Arguments	E7W	CC	CC	CC	CC	Ad	CC

F. weights of E$_8$	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8
Arguments	Ad	CC	CC	CC	CC	CC	AdC	CC

F. weights of F$_4$	π_1	π_2	π_3	π_4
Arguments	F4T	F4W	CC	Ad

F. weights of G$_2$	π_1	π_2
Arguments	SM	Ad

For the representations $V(\pi_k)$ of exceptional groups $E_n (n = 6, 7, 8)$, for which argument CC is applicable, chopping of V in the vertex with number $n-1$ is a 2-wild representation of a classical group of type D_{n-1}. To apply argument AdC one should chop vertex with number 1. For all representations of the exceptional group F_4, to apply argument CC one can always chop the vertex with number 1.

Let us first justify arguments SM, CC, Ad, AdC.

SM) It was shown [BL13] that any subminuscule representation is tame, i.e. that rank and border rank coincide for such representations.

CC) According to Proposition 5.1, if some chopping $\underline{V}$ of a representation V is 2-wild, then V is 2-wild.

Ad) According to [Kaji98], all adjoint representations of exceptional groups are 2-wild.

AdC) According to Proposition 5.1 and Ad), if some chopping $\underline{V}$ of a representation V is an adjoint representation of an exceptional group, then V is 2-wild.

The rest of this section is devoted to the justification of arguments F4T, F4W and E7W, done in Subsections 7.1, 7.2 and 7.3, respectively. The results are as follows.

F4T) In Proposition 7.2 we show that the representation $V(\pi_1)$ is tame.

F4W) In Proposition 7.9 we show that the representation $V(\pi_2)$ of F_4 is 2-wild.

E7W) In Proposition 7.17 we show that the representation $V(\pi_1)$ of E_7 is 2-wild.

7.1 Tameness of $V(\pi_1)$ for F_4

In this section we prove the following.

Proposition 7.2. The fundamental representation $V(\pi_1)$ of F_4 is tame.

Proof. It is known, [Zak93, p. 59], that the generic rank of $V(\pi_1)$ is three, so that

$$\sigma_3(\mathbb{X}(\pi_1)) = \mathbb{P}(V(\pi_1)).$$

In Lemmas 7.3 and 7.4 below, we show that $V(\pi_1)$ is 2- and 3-tame, respectively, which implies that this module is tame. □

Lemma 7.3. The F_4-module $V(\pi_1)$ is 2-tame.

Lemma 7.4. The F_4-module $V(\pi_1)$ is 3-tame.

The first fundamental representation $V(\pi_1)$ of F_4 is 26-dimensional and is the (nontrivial) representation of smallest possible dimension for this group. The discussion which follows involves several representations of various groups this would make the notation $V(\lambda)$ ambiguous. We have chosen to denote the representations spaces by indices corresponding to their dimension. The set of highest weight vectors, previously denoted by $X(\lambda)$, will be denoted by $X(V)$. We let V_{26} denote the representation space of $(F_4, V(\pi_1))$ and $X(V_{26})$ be the set of highest weight vectors. To study V_{26} we use the fact that it can be obtained as a generic hyperplane in the smallest, 27-dimensional representation of E_6, which we denote by $V_{27} = (E_6, V(\pi_1))$. We summarize some known results in the following lemma.

Lemma 7.5. (i) The algebra of E_6-invariant polynomials on V_{27} is polynomial on one generator of degree 3, i.e. $\mathbb{F}[V_{27}]^{E_6} = \mathbb{F}[DET]$, where $DET \in \mathrm{S}^3(V_{27}^*)$.

(ii) The orbits of E_6 in V_{27} are the following:

$$0,\ X(V_{27}),\ \{DET = 0\} \setminus \overline{X(V_{27})},\ \{DET = a\},\ a \in \mathbb{F}^\times;$$

their dimensions are, respectively, $0, 17, 26, 26$. The orbits of $E_6 \times \mathbb{F}^\times$ in V_{27} are the following (lower indices indicate dimension):

$$\mathcal{O}_0 = \{0\}\ ,\quad \mathcal{O}_{17} = X(V_{27})\ ,\quad \mathcal{O}_{26} = \{DET = 0\} \setminus \overline{X(V_{27})}\ ,\quad \mathcal{O}_{27} = \{DET \neq 0\}\ .$$

(iii) There are exactly three E_6 orbits in the projective space $\mathbb{P}(V_{27})$ and they are exactly the rank subsets with respect to $\mathbb{X}(V_{27})$, namely,

$$\mathbb{X}(V_{27})\ ,\quad \mathbb{X}_2(V_{27}) = \{DET = 0\} \setminus \mathbb{X}(V_{27})\ ,\quad \mathbb{X}_3(V_{27}) = \{DET \neq 0\}\ ;$$

their dimensions are, respectively, $16, 25, 26$. The secant varieties of $\mathbb{X}(V_{27})$ are exactly the closures of the E_6-orbits in $\mathbb{P}(V_{27})$.

(iv) The stabilizer of any vector $v \in \{DET \neq 0\}$ is isomorphic to F_4. The orth-complement $v^\perp \subset V_{27}$ is an irreducible F_4-module isomorphic to V_{26}, i.e. $V_{27} = \langle v \rangle \oplus V_{26}$ as F_4-modules.

(v) The secant varieties of $\mathbb{X}(V_{26})$ are obtained as intersections of the secant varieties of $\mathbb{X}(V_{27})$ with the hyperplane $\mathbb{P}(V_{26})$, i.e.

$$\mathbb{X}(V_{26}) = \mathbb{P}(V_{26}) \cap \mathbb{X}(V_{27})\ ,\quad \sigma_2(\mathbb{X}(V_{26})) = \mathbb{P}(V_{26}) \cap \sigma_2(\mathbb{X}(V_{27}))\ ,\quad \sigma_3(\mathbb{X}(V_{26})) = \mathbb{P}(V_{26})\ .$$

Proof. Since the results are known, but are a compilation of the work of many authors, we confine ourselves to giving references (not necessarily the original ones) for the various parts of the lemma. Part (i) can be found in Table II in [Kac80]. As for part (ii), the fact that

$$\{DET = a\}, a \neq 0$$

is a single E_6-orbit, is proven in [Kac80, Proposition 1.1], while the enumeration of the orbits in the nulcone $\{DET = 0\}$ is given in [Zak93, p. 59]. Part (iii) can also be deduced from the discussion on p. 59 of [Zak93] or can be seen to follow directly from the fact that V_{27} is a subcominuscule representation and for such representations the rank sets are exactly the group orbits in the projective space, cf. [BL13, §4]. Parts (iv) and (v) are also quoted from [Zak93, p. 59-60]. □

The above proposition and, specifically, parts (iv) and (v) allow us to practically forget about the group F_4 and use only properties of V_{27} and a generic hyperplane inside it. We shall need to understand the structure of V_{27} with respect to a subgroup of E_6 of type D_5. Let $H \subset E_6$ be the regular subgroup whose root system is generated by the set of simple roots $S := \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ (we use the numbering of simple roots as in [VO90, p. 292]). It turns our that $H \cong Spin_{10}$.

Lemma 7.6. The decomposition, as an H-module, of the simple E_6-module V_{27} is

$$V_{27} \cong V_1 \oplus RSpin_{10} \oplus V_{10} \ ,$$

where V_1 is a one-dimensional trivial H-module, $RSpin_{10}$ is the spinor H-module, and V_{10} is the natural representation of SO_{10} (recall that $Spin_{10}$ is a cover of SO_{10}).

Proof. The result is obtained by a straightforward consideration of the weights of the modules involved, using the fact that H is a regular subgroup of E_6. □

Lemma 7.7. The nonzero isotropic vectors in V_{10} belong to $X(V_{27})$ and have rank 1 as elements of the E_6-module V_{27}. The non-isotropic vectors in V_{10} belong to $\mathcal{O}_{26}$ and have rank 2 as elements of the E_6-module V_{27}.

Proof. Since H is a regular subgroup of E_6, the weight spaces for E_6 in V_{27} are also weight spaces for H. Thus V_{10} is a span of some of these weight spaces. The E_6-weights of V_{27} are

$$\varepsilon_i \pm \varepsilon, -\varepsilon_i - \varepsilon_j \ (i \neq j).$$

The weights appearing in V_{10} are

$$\varepsilon_i - \varepsilon, -\varepsilon_1 - \varepsilon_i \ (i \neq 1).$$

The weight $-\varepsilon_6 - \varepsilon$ is the lowest weight of V_{27} and thus any element of the corresponding weight space belongs to $\mathcal{O}_{17}$. On the other hand, any element of the weight space of weight $-\varepsilon_6 - \varepsilon$ is isotropic. Since all isotropic vectors of V_{10} are conjugate by SO_{10}, all isotropic vectors of V_{10} belong to $\mathcal{O}_{17}$.

It remains to show that all non-isotropic vectors of V_{10} (they are all $SO_{10} \times \mathbb{F}^\times$-conjugate) belong to $\mathcal{O}_{26}$. To this end, we note that the weights of V_{10}

$$\varepsilon_2 - \varepsilon, -\varepsilon_1 - \varepsilon_2$$

are, respectively, the highest and the lowest weight of V_{27} with respect to the set of simple roots of E_6

$$\Pi' = \{\varepsilon_2 - \varepsilon_1, \varepsilon_1 + \varepsilon_4 + \varepsilon_5 - \varepsilon, \varepsilon_6 - \varepsilon_4, \varepsilon_4 - \varepsilon_5, -\varepsilon_4 - \varepsilon_3 - \varepsilon_6 + \varepsilon, \varepsilon_3 - \varepsilon_6\}.$$

Thus $v^{\varepsilon_2 - \varepsilon} + v^{-\varepsilon_1 - \varepsilon_2} \in \mathcal{O}_{26}$ [Zak93, Ch. III, Thm 1.4]. Since all non-isotropic vectors of V_{10} are $SO_{10} \times \mathbb{F}^\times$-conjugate, all non-isotropic vectors of V_{10} belong to $\mathcal{O}_{26}$. □

Proof of Lemma 7.3. According to Lemma 7.5, to prove that V_{26} is 2-tame it suffices to show that for any $x \in V_{26} \cap \mathcal{O}_{26}$ there exist $x_+, x_- \in V_{26} \cap \mathcal{O}_{17}$ such that $x = x_+ + x_-$. We fix $x \in V_{26} \cap \mathcal{O}_{26}$. First, we note that there exists a Borel subalgebra $\mathfrak{b} \subset E_6$ with a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$ such that $x \in V_{10}$ (we use the notation of Lemma 7.7). Then $x \in V_{10} \cap \mathcal{O}_{26}$ and thus x is a non-isotropic vector of V_{10} by Lemma 7.7. Let $x^\perp$ be the orthogonal complement to x in V_{10}. Note that a non-degenerate symmetric bilinear form $(\cdot, \cdot)$ of V_{10} is still non-degenerate after restriction to $x^\perp$.

Since V_{26} is a 26 dimensional subspace of the 27 dimensional space V_{27}, we have

$$\dim(V_{26} \cap x^\perp) \geq \dim(x^\perp) - 1 = 9.$$

As $9 > \frac{1}{2} \dim V_{10}$, the restriction of $(\cdot, \cdot)$ to $x^\perp \cap V_{26}$ is non-zero. Hence there exists $y \in x^\perp \cap V_{26}$ such that $(y, y) = \frac{-(x,x)}{4}$. Set

$$x_+ = \tfrac{x}{2} + y, \quad x_- = \tfrac{x}{2} - y.$$

We have

$$(x_+, x_+) = \tfrac{1}{4}(x, x) + (y, y) = 0 = (x_-, x_-) \text{ and } x_+ + x_- = x,$$

i.e. the vectors $x_\pm$ are isotropic and their sum is equal to x. Thus $x_\pm \in \mathcal{O}_{17}$. Since $y \in V_{26}$, we have $x_\pm \in V_{26}$. Hence $x_\pm \in X(V_{26})$. Therefore V_{26} is 2-tame. □

Before proceeding with the proof of Lemma 7.4, we need the following auxilary result.

Lemma 7.8. Let V be a finite-dimensional vector space. Let Z_f be a hypersurface determined as the zero-locus of a non-zero homogeneous polynomial $f \in \mathbb{F}[V]$ and let $X \in V$ be any subset spanning V. Then, $V = Z_f + X$, i.e. for every $v \in V$ there exist $v_2 \in Z_f$ and $x \in X$ such that $v = v_2 + x$.

Proof. Assume on the contrary, that there exists $v \in V$ such that $v \neq v_2 + x$ for any $v_2 \in Z_f$ and $x \in X$. Then $f(v + tx) \neq 0$ for any $x \in X$ and any $t \in \mathbb{F}$. The function $f(v + tx)$ is polynomial and thus $f(v + tx) \neq 0$ for any $t \in \mathbb{F}$ if and only if $f(v + tx)$ is a non-zero constant as a polynomial of t. Hence the first derivative of $f(v + tx)$ with respect to t is zero for $t = 0$, i.e. the value of df in the direction x at the point v is zero. As X spans V, $df = 0$ at v.

We claim that $f(w) = 0$ for all points $w \in V$ such that $df = 0$ at w (equivalent to: all partial derivatives of f vanish at w). Indeed, the set of equations $df = 0$ determines some subvariety Z_{df} of V and it suffices to show that $f = 0$ at any smooth point of any irreducible component of Z_{df}. Obviously $df = 0$ on the smooth locus of any irreducible component of Z_{df}. Thus f is constant on any irreducible component of Z_{df}. Since f is homogeneous, $f(0) = 0$, and any irreducible component of Z_{df} contains 0. Thus $f|_{V_{df}} = 0$.

Compiling the previous two paragraphs, we obtain $f(v) = 0$. Thus $v = v + 0$, where $v \in Z_f$ and $0 \in X$. This completes the proof. □

Proof of Lemma 7.4. The secant variety $\sigma_2(X(V_{26}))$ is the zero-locus of some homogeneous function DET of degree 3 and $X(\pi_1)$ spans V_{26}. Thus, according to Lemma 7.8, any vector

$$x \in V_{26} \backslash \sigma_2(X(V_{26}))$$

may be represented as $x = x_1 + x_2$, for some $x_1 \in X(\pi_1)$ and $x_2 \in \sigma_2(X(V_{26}))$. Therefore, by Lemma 7.3, any vector in $V_{26} \backslash \sigma_2(X(V_{26}))$ has rank 3. □

7.2 Wildness of $V(\pi_2)$ for F_4

Proposition 7.9. The fundamental representation $V(\pi_2)$ of F_4 is 2-wild.

The proof is presented at the end of this subsection, after the necessary preliminary results. Our approach is based on the following two statements. First, the F_4-module $V(\pi_1)$ has an invariant non-degenerate symmetric bilinear form $(\cdot,\cdot)$ and thus $F_4 \subset \mathrm{SO}(V(\pi_1))$. Second, the decomposition of $\Lambda^2 V(\pi_2)$ as an F_4-module is

$$\Lambda^2 V(\pi_1) \cong V(\pi_2) \oplus V(\pi_4),$$

see [VO90, Table 5 on p. 305]. This allows us to represent the elements of $V(\pi_2)$ as anti-symmetric tensors and perform calculations. We end up finding an element, laying on a tangent line to $X(V(\pi_2))$, whose rank is 3 (see Lemma 4.3).

From now on, we consider $V(\pi_2)$ as a subspace of $\Lambda^2 V(\pi_1)$. We start with some statements about $\Lambda^2 V(\pi_1)$. To any $\omega \in \Lambda^2 V(\pi_1)$ we associate two integers $\dim \mathrm{Im}\,\omega, \mathrm{rk}\,\mathrm{Im}\,\omega$ defined as follows. These numbers are invariants of ω and we shall use them to distinguish the rank three element we are searching for. First, we consider $V := V(\pi_1)$ as a vector space, ignoring the F_4-action on it. Any element $\omega \in \Lambda^2 V$ can be interpreted as a linear map $V^* \to V$. We denote the image of this map by $\mathrm{Im}\,\omega$ (note that any tensor in $V \otimes V$ defines, in a natural way, two such maps, but since $\omega \in \Lambda^2 V$, these two maps differ by a sign and thus they have the same image). The number $\dim \mathrm{Im}\,\omega$ is always an even integer. Further, for any $\omega \in \Lambda^2 V$, the space $\mathrm{Im}\,\omega$ carries a symmetric bilinear form, which is the restriction of $(\cdot,\cdot)$; we denote the rank of this form by $\mathrm{rk}\,\mathrm{Im}\,\omega$. In the following lemma, we list the values of $\dim \mathrm{Im}\,\omega, \mathrm{rk}\,\mathrm{Im}\,\omega$ for a class of elements ω which are of interest for us.

Lemma 7.10. Let x_1, x_2, y_1, y_2 be vectors such that the subspaces $\langle x_1, x_2\rangle$ and $\langle y_1, y_2\rangle$ of V are two-dimensional and isotropic. Set

$$\omega = x_1 \wedge x_2 + y_1 \wedge y_2.$$

Then one of the following possibilities holds:

Cases	$\dim \mathrm{Im}\,\omega$	$\mathrm{rk}\,\mathrm{Im}\,\omega$
(a)	4	4
(b)	4	2
(c)	4	0
(d)	2	1
(e)	2	0
(f)	0	0

(7)

Proof. We consider two cases: the vectors x_1, x_2, y_1, y_2 are either linearly dependent or independent.

Case 1). Assume that the vectors x_1, x_2, y_1, y_2 are linearly independent. Then $\dim \mathrm{Im}\,\omega = 4$ and $\mathrm{rk}\,\mathrm{Im}\,\omega$ is equal to the rank of the space $\langle x_1, x_2, y_1, y_2\rangle$. The rank of this space is equal to the rank of the matrix

$$\left(\begin{array}{cc|cc} (x_1,x_1) & (x_1,x_2) & (x_1,y_1) & (x_1,y_2) \\ (x_2,x_1) & (x_2,x_2) & (x_2,y_1) & (x_2,y_2) \\ \hline (y_1,x_1) & (y_1,x_2) & (y_1,y_1) & (y_1,y_2) \\ (y_2,x_1) & (y_2,x_2) & (y_2,y_1) & (y_2,y_2) \end{array}\right) = \left(\begin{array}{cc|cc} 0 & 0 & (x_1,y_1) & (x_1,y_2) \\ 0 & 0 & (x_2,y_1) & (x_2,y_2) \\ \hline (y_1,x_1) & (y_1,x_2) & 0 & 0 \\ (y_2,x_1) & (y_2,x_2) & 0 & 0 \end{array}\right) \quad (8)$$

It is easy to see that the matrix (8) can be presented as

$$(8) = \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix}, \quad A = \begin{pmatrix} (x_1,y_1) & (x_1,y_2) \\ (x_2,y_1) & (x_2,y_2) \end{pmatrix}.$$

Therefore the matrix (8) has even rank, i.e. ω is of type (a), (b), or (c).

Case 2). Assume that the vectors x_1, x_2, y_1, y_2 are linearly dependent. Then the spaces $\langle x_1, x_2\rangle$ and $\langle y_1, y_2\rangle$ must intersect. They may intersect along a one-dimensional subspace or coincide.

Suppose first that they intersect along a one-dimensional subspace. Without loss of generality we assume that they intersect along the subspace generated by $x_2 = y_2$. Then

$$\omega = x_1 \wedge x_2 + y_1 \wedge y_2 = (x_1 + y_1) \wedge x_2.$$

Hence $\dim \operatorname{Im} \omega = 2$. Since

$$0 = (x_2, x_1) = (y_2, x_1), (x_2, y_1) = (y_2, y_1) = 0,$$

we have $(x_2, x_1 + y_1) = 0$, but it is not clear whether or not $(x_1 + y_1, x_1 + y_1) = 0$. Thus $\operatorname{rk} \operatorname{Im} \omega \leq 1$. This corresponds to cases (d) and (e).

Suppose now that the spaces $\langle x_1, x_2\rangle$ and $\langle y_1, y_2\rangle$ coincide. Then ether $[\omega] = [x_1 \wedge y_1]$ (case (e)) or $\omega = 0$ (case (f)). □

We shall need some information about the $SO(V(\pi_1))$-orbits in $\Lambda^2 V(\pi_1)$. To obtain this information, we use the canonical identification of $\Lambda^2 V$ with $\mathfrak{so}(V)$, for any vector space V with a non-degenerate symmetric bilinear form $(\cdot, \cdot)$. Recall that, for any two vectors $x_1, x_2 \in V$, to $x_1 \wedge x_2 \in \Lambda^2 V$ corresponds the linear operator

$$(x_1 \wedge x_2)(\cdot) : V \to V (v \to ((x_2, v)x_1 - (x_1, v)x_2)).$$

The image of a linear operator corresponding to a form $\omega \in \Lambda^2 V$ coincides with $\operatorname{Im} \omega$ [1].

Now, we study some nilpotent $SO(V)$-orbits in $\mathfrak{so}(V)$. For any nilpotent element $e \in \mathfrak{so}(V)$ there exist elements $h, f \in \mathfrak{so}(V)$ such that $\{e, h, f\}$ is an $\mathfrak{sl}_2$-triple (this is a well-known corollary of the Jacobson-Morozov theorem). Thus the classification of the conjugacy classes of nilpotent elements in $\mathfrak{so}(V)$ is reduced to the classification of the conjugacy classes of $\mathfrak{sl}_2$-subalgebras in $\mathfrak{so}(V)$. Any $\mathfrak{sl}_2$-subalgebra of $\mathfrak{so}(V)$ determines a structure of $\mathfrak{sl}_2$-module on V. Therefore the the classification of the conjugacy classes of nilpotent elements in $\mathfrak{so}(V)$ is reduced to the classification of the equivalence classes of $\mathfrak{sl}_2$-modules of dimension $n := \dim V$. Such equivalence classes are in a one-to-one correspondence with the partitions of n (in some sense such partitions corresponds to the Jordan normal forms). For nilpotent orbits such partitions can be encoded by Hasse diagrams, e.g. the Hasse diagram $(3^1 2^2 1^{n-7})$ corresponds to the partition $3, 2, 2, 1, 1, ..., 1$ (unit appears $n-7$ times). The following lemma provides an elementary way to compute $\dim \operatorname{Im} x$ and $\operatorname{rk} \operatorname{Im} x$ for a nilpotent element x with known partition. The proof of it is left to the reader.

Lemma 7.11. Let $\bar{d} = d_1, ..., d_k$ be a partition of $n := \dim V$. Let $x \in \mathfrak{so}(V)$ be a nilpotent element with partition $\bar{d}$. Then

$$\dim \operatorname{Im} x = \Sigma_i \max\{d_i - 1, 0\}, \quad \operatorname{rk} \operatorname{Im} x = \Sigma_i \max\{d_i - 2, 0\}.$$

The space $\Lambda^2 V$ is a simple $SO(V)$-module (isomorphic to the adjoint module) and the set of highest weight vectors of it is

$$X(\Lambda^2 V) := \{x_1 \wedge x_2 \mid \langle x_1, x_2\rangle \text{ is an isotropic 2-dimensional space }\}.$$

The secant variety $\sigma_2(X(\Lambda^2 V))$ is described in [Kaji98]. According to this description $\sigma_2(X(\Lambda^2 V))$ consists of one $\mathrm{SO}(V) \times \mathbb{F}^\times$-orbit of some semisimple element h (we denote the corresponding form ω_h), 5 non-zero nilpotent orbits (here we use $\dim V \geq 9$) with Hasse diagrams

$$(3^2 1^{n-6}), (3^1 2^2 1^{n-7}), (3^1 1^{n-3}), (2^4 1^{n-8}), (2^2 1^{n-2})$$

and one zero-orbit. We denote some representatives of these orbits by

[1] There is a third way to describe $\operatorname{Im} \omega$: $(\cdot, \cdot)$ defines a map $V \to V^*$ and ω defines a map $V^* \to V$, $\operatorname{Im} \omega$ is the image of the composition of these maps.

$$\omega_h, \omega_{3^2 1^{n-6}}, \omega_{3^1 2^2 1^{n-7}}, \omega_{3^1 1^{n-3}}, \omega_{2^4 1^{n-8}}, \omega_{2^2 1^{n-2}} \text{ and } 0,$$

respectively. Using Lemma 7.11 we obtain the following data:

Form ω	$\dim \operatorname{Im} \omega$	$\operatorname{rk} \operatorname{Im} \omega$
ω_h	4	4
$\omega_{3^2 1^{n-6}}$	4	2
$\omega_{3^1 2^2 1^{n-7}}$	4	1
$\omega_{2^4 1^{n-8}}$	4	0
$\omega_{3^1 1^{n-3}}$	2	1
$\omega_{2^2 1^{n-4}}$	2	0

. (9)

Observe that (9) looks very similar to (7).

Now we start working with a root-weight approach on the F_4-module $V(\pi_1)$ (we use the description of the corresponding roots and weights given in [VO90, p. 294–295]). The weights of the $V(\pi_1)$ are

$$\pm\varepsilon_i, \frac{\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4}{2}. \tag{10}$$

For any weight α from (10) we pick a vector $v^\alpha \in V(\pi_1)$ of weight α. The highest weight vector of $\Lambda^2 V(\pi_1)$ (with respect to the order given in [VO90, p. 294]) is

$$\varepsilon_1 + \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4}{2} = \frac{3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4}{2} = \pi_2.$$

Thus the highest weight vector of $V(\pi_2) \subset \Lambda^2 V(\pi_1)$ is

$$v^{\pi_2} := v^{\varepsilon_1} \wedge v^{\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2}}. \tag{11}$$

We set

$$x := v^{\varepsilon_1}, \quad y := v^{\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2}}. \tag{12}$$

Since the weights $\varepsilon_1, \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4), \varepsilon_1 + \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$ are all non-zero, the vectors $x, y \in V(= V(\pi_1))$ are isotropic and pairwise orthogonal. We shall need the following lemma.

Lemma 7.12. Let Δ_{F_4} be the root system of F_4 and let $G \subset F_4$ be the regular subgroup of type SL_2 with root system $\{\pm\frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\} \subset \Delta_{F_4}$. Then the space $\langle x, y\rangle$ is a two-dimensional simple G-module. Furthermore, for any non-zero vector $z \in \langle x, y\rangle$ we have

$$\dim F_4 z = \dim F_4 x = \dim F_4 y = 16.$$

Proof. The first statement of the lemma follows from the facts that all non-zero weight spaces of $V(\pi_1)$ are one-dimensional, the weights

$$\varepsilon_1, \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4}{2} \tag{13}$$

differ by the root $\frac{\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4}{2}$ and the line going through the weights (13) contains no other weights of $V(\pi_1)$. Since $\langle x, y\rangle$ is a two-dimensional module of $G \cong SL_2$, all non-zero vectors of $\langle x, y\rangle$ are conjugate by $G \subset F_4$. Thus for any non-zero vectors $z_1, z_2 \in \langle x, y\rangle$ we have

$$\dim F_4 z_1 = \dim F_4 z_2.$$

In particular,

$$\dim F_4 z = \dim F_4 x = \dim F_4 y.$$

It remains to mention that

$$\dim F_4 x = \dim X(\pi_1) = 1 + \tfrac{\dim F_4 - \dim B_3}{2} = 16.$$

□

We shall now make a digression and prove two auxillary lemmas needed further on. The first is a technical statement from linear algebra.

Lemma 7.13. Let V and W be finite-dimensional vector spaces and let V_2 be a space of dimension 2 with basis x, y. Let $\phi : V_2 \otimes V \to W$ be a linear map. Assume that for any $v \in V$ we have

$$[\phi(x \otimes v)] = [\phi(y \otimes v)]. \tag{14}$$

Then one of the following possibilities holds:

(a) there exist $\phi_1 \in V_2^*$ and a linear map $\phi_W : V \to W$ such that

$$\phi(l \otimes v) = \phi_1(l)\phi_W(v)$$

for all $v \in V, l \in V_2$;

(b) there exists a function $\phi_1 : V_2 \otimes V \to \mathbb{F}$ and a vector $w \in W$ such that

$$\phi(v \otimes t) = \phi_1(v \otimes t)w.$$

Proof. A linear map $\phi : V_2 \otimes V \to W$ can be identified with an element $\phi^* \in V_2^* \otimes V^* \otimes W$. Such an element ϕ^* can be (non-canonically) represented by a matrix

$$\begin{pmatrix} a_1 & a_2 & ... & a_n \\ b_1 & b_2 & ... & b_n \end{pmatrix}, \tag{15}$$

where $a_1, ..., a_n, b_1, ..., b_n \in V^*$ and n is the dimension of W. Then, according to (14), we have

$$a_i b_j - a_j b_i = 0, \tag{16}$$

for all i, j. There are two types of solutions for this system, which we consider next in two cases.

Case 1) Assume that for all i we have

$$[a_i] = [b_i], \text{ or } a_i = 0, \text{ or } b_i = 0.$$

Then there exist non-zero vectors $c_1, ..., c_n \in V^*$ and numbers $\lambda_1, ..., \lambda_n, \mu_1, ..., \mu_n \in \mathbb{F}$ such that

$$a_i = \lambda_i c_i \text{ and } b_i = \mu_i c_i$$

for all i. Then for all i, j we have

$$0 = a_i b_j - a_j b_i = (\lambda_i \mu_j - \lambda_j \mu_i) c_i c_j. \tag{17}$$

Since $c_i, c_j \neq 0$ for all i, j, the set of equations (17) is equivalent to the statement that the matrix

$$\begin{pmatrix} \lambda_1 & \lambda_2 & ... & \lambda_n \\ \mu_1 & \mu_2 & ... & \mu_n \end{pmatrix} \tag{18}$$

has rank 1, i.e. that after some change of basis in V_2 the matrix (18) has the form

$$\begin{pmatrix} \nu_1 & \nu_2 & ... & \nu_n \\ 0 & 0 & ... & 0 \end{pmatrix}$$

for some numbers $\nu_1, ..., \nu_n \in \mathbb{F}$, i.e. that after some change of basis in V_2 the matrix (15) has the form

$$\begin{pmatrix} \nu_1 c_1 & \nu_2 c_2 & ... & \nu_n c_n \\ 0 & 0 & ... & 0 \end{pmatrix}.$$

This statement is equivalent to the statement (a) of Lemma 7.13.

Case 2) Assume that for some i we have

$$[a_i] \neq [b_i], \quad a_i \neq 0, b_i \neq 0.$$

Then equations (16) imply that there exist numbers $\lambda_1, ..., \lambda_n$ such that

$$a_j = \lambda_j a_i, \quad b_j = \lambda_j b_i$$

for all j. Then, after some change of basis in W, the matrix (15) takes the form

$$\begin{pmatrix} a_i & 0 & ... & 0 \\ b_i & 0 & ... & 0 \end{pmatrix}.$$

This statement is equivalent to the statement (b) of Lemma 7.13. $\square$

We also need the following simple statement.

Lemma 7.14. Let P be a homogeneous polynomial on a vector space V. Then there exists $t_0 \in V$ such that for all $t \in V$ there exists $\lambda \in \mathbb{F}$ such that $P(t + \lambda t_0) = 0$.

Proof. The statement of the lemma holds for $P = 0$ for trivial reasons. Assume that $P \neq 0$. Pick t_0 such that $P(t_0) \neq 0$. Then the coefficient attached to the highest power of λ in $P(t + \lambda t_0)$ equals to $P(t_0)$, and hence is non-zero. Thus for any $t \in V$ the equation on λ, $P(t + \lambda t_0) = 0$ has a solution. This completes the proof. $\square$

For the proof of Proposition 7.9 we also need the following lemma.

Lemma 7.15. There exists $t \in \mathfrak{f}_4$ such that

$$\omega_t := x \wedge y + t(x \wedge y) = x \wedge y + tx \wedge y + x \wedge ty$$

satisfies the following conditions

$$(19a)\ \operatorname{rk}\operatorname{Im}\omega_t = 1\ , \quad (19b)\ \dim\operatorname{Im}\omega_t = 4. \tag{19}$$

To prove Lemma 7.15 we need the following auxiliary lemma.

Lemma 7.16. Let

$$P_{xx}(t) := (tx, tx), \quad P_{yy}(t) := (ty, ty) \tag{20}$$

for any $t \in \mathfrak{f}_4$. Then

a) both polynomials P_{xx}, P_{yy} are non-zero and irreducible,

b) $[P_{xx}] \neq [P_{yy}]$.

Proof. Assume that $P_{xx} = 0$. Then $\mathfrak{f}_4 x$ is an isotropic subspace of V_{26} and hence

$$16 = \dim \mathfrak{f}_4 x \leq \frac{1}{2} \dim V_{26} = 13.$$

This is not true and thus $P_{xx} \neq 0$. By similar reasons $P_{yy} \neq 0$.

Assume that P_{xx} is reducible, i.e. is a product of two monomials. Then there is a hypersurface $H \subset \mathfrak{f}_4$ such that $P_{xx}|_H = 0$. Therefore Hx is an isotropic subspace of V_{26} and thus $\dim Hx \leq \frac{1}{2}V_{26}$. Hence we have

$$15 \leq \dim Hx \leq \frac{1}{2} \dim V_{26} = 13.$$

This is not true and thus P_{xx} is irreducible. By similar reasons P_{yy} is irreducible and this completes part a).

According to (12) and (20), P_{xx} is a weight vector of weight $-2\varepsilon_1$ and P_{xx} is a weight vector of weight $-(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$. Hence $[P_{xx}] \neq [P_{yy}]$ and the proof is complete. $\square$

Proof of Lemma 7.15. First we deal with condition (19a). We claim that there exists a polynomial P on $\mathfrak{f}_4$ of degree 4 such that $P(t) = 0$ if and only if the matrix

$$\left(\begin{array}{cc|cc} (x,x) & (x,y) & (x,tx) & (x,ty) \\ (y,x) & (y,y) & (y,tx) & (y,ty) \\ \hline (tx,x) & (tx,y) & (tx,tx) & (tx,ty) \\ (ty,x) & (ty,y) & (ty,tx) & (ty,ty) \end{array}\right)$$

has rank 1 or less. To check this we start by recalling that x and y are isotropic and pairwise orthogonal (see the text preceeding Lemma 7.12), i.e.

$$(x,x) = (x,y) = (y,x) = (y,y) = 0.$$

Also, if Δ_{F_4} denotes the root system of F_4, we have (see [VO90, Table 1, p. 294])

$$\begin{array}{cc} -\varepsilon_1 \notin \varepsilon_1 + \Delta_{F_4} & -\varepsilon_1 \notin \frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2} + \Delta_{F_4} \\ -\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2} \notin \varepsilon_1 + \Delta_{F_4}, & -\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2} \notin \frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2} + \Delta_{F_4}, \end{array}$$

where "+" denotes the pointwise sum. As $(\cdot,\cdot)$ is F_4-invariant, $(v_\alpha, v_\beta) \neq 0$ implies $\alpha + \beta = 0$. Hence

$$(tx,x) = (x,tx) = (ty,y) = (y,ty) = (x,ty) = (ty,x) = (tx,y) = (y,tx) = 0,$$

for all $t \in \mathfrak{f}_4$. Therefore we have

$$\left(\begin{array}{cc|cc} (x,x) & (x,y) & (x,tx) & (x,ty) \\ (y,x) & (y,y) & (y,tx) & (y,ty) \\ \hline (tx,x) & (tx,y) & (tx,tx) & (tx,ty) \\ (ty,x) & (ty,y) & (ty,tx) & (ty,ty) \end{array}\right) = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & (tx,tx) & (tx,ty) \\ 0 & 0 & (ty,tx) & (ty,ty) \end{array}\right). \tag{21}$$

Now we see that the matrix (21) has rank 1 or less if and only if

$$P(t) := (tx,tx)(ty,ty) - (tx,ty)^2 = 0.$$

By definition P is a homogeneous polynomial of degree 4 (possibly $P = 0$). This proves the claim made in the beginning of the proof.

Now we discuss condition (19b). We denote by Z_P the zero-locus of P in $\mathfrak{f}_4$. We claim that there exists an open subset $U_P \subset Z_P$ such that for any $t \in U_P$ condition (19b) is satisfied. The statement of (19b) is equivalent to the statement that the vectors x, y, tx, ty are linearly independent. Let us check that there exists a non-zero open subset $U_P \subset Z_P$ such that the vectors

$$x, y, tx, ty \tag{22}$$

are linearly independent for every $t \in U_P$. By Lemma 7.14 there exists $t_0 \in \mathfrak{f}_4$ such that for every $t \in \mathfrak{f}_4$ there exists $\lambda \in \mathbb{F}$ such that $P(t + \lambda t_0) = 0$. We set

$$\tilde{V} := V/\langle x, y, t_0x, t_0y\rangle.$$

The statement related to (22) is implied by the following statement: there exists an open subset $U_0 \subset \mathfrak{f}_4$ such that for all $t \in U_0$

$$\tilde{tx}, \tilde{ty} \tag{23}$$

span a two-dimensional subspace in $\tilde{V}$, where $\tilde{tx}, \tilde{ty}$ are the images respectively of tx, ty in $\tilde{V}$.

There is a natural map

$$\phi : \mathfrak{f}_4 \otimes \langle x, y\rangle \to \tilde{V}$$

defined on simple tensors by the formula

$$t \otimes v \to \tilde{tv}.$$

Assume that the vectors $\tilde{tx}, \tilde{ty}$ are linearly dependent for all t. Lemma 7.13 implies that one of the following two situations occurs:

(I) there exist $\phi_1 \in \langle x, y\rangle^*$ and a linear map $\phi_V : \mathfrak{f}_4 \to V$ such that

$$\phi(t \otimes z) = \phi_1(z)\phi_V(t)$$

for all $z \in \langle x, y\rangle, t \in \mathfrak{f}_4$, or

(II) there exists a function $\phi_1 : \langle x, y\rangle \otimes \mathfrak{f}_4 \to \mathbb{F}$ and a vector $v \in V$ such that

$$\phi(t \otimes z) = \phi_1(t \otimes z)v.$$

First, we consider possibility (I). Let $z \in \langle x, y\rangle$ be a non-zero vector such that $\phi_1(z) = 0$ (such vector always exists!). Then

$$\tilde{tz} = \phi(t \otimes z) = \phi_1(z)\tilde{\phi}(t) = 0$$

for all $t \in \mathfrak{f}_4$. Thus for all $t \in \mathfrak{f}_4$ we have

$$tz \in \langle x, y, t_0x, t_0y\rangle.$$

In particular, $\dim F_4z \leq 4$. This contradicts Lemma 7.12.

Second, we consider possibility (II). Then

$$\mathfrak{f}_4x \subset \langle x, y, t_0x, t_0y, \tilde{v}\rangle,$$

where $\tilde{v}$ is any preimage of v in V. In this case we have

$$\dim X(\pi_1) = \dim F_4x \leq 5.$$

But we know that $\dim X(\pi_1) = 16$ and thus the possibility (II) also can be true.

Compiling the previous paragraphs together, we conclude that there exists a non-zero open subset $U_P \subset Z_P$ such that for every $t \in U_P$ we have

$$\operatorname{rk} \operatorname{Im} \omega_t = \operatorname{rk}\langle x, y, tx, ty\rangle \leq 1, \quad \dim \operatorname{Im} \omega_t = \dim\langle x, y, tx, ty\rangle = 4. \tag{24}$$

Now we show that there exists $t \in U_P$ such that

$$\operatorname{rk}\langle x, y, tx, ty\rangle = 1. \tag{25}$$

Assume on the contrary that

$$\operatorname{rk}\langle x, y, tx, ty\rangle = 0 \tag{26}$$

for any $t \in U_P$. This is equivalent to the condition that the matrix (21) equals 0 for all $t \in U_P$ and hence equals 0 for all $t \in Z_P$. Thus from $P(t) = 0$ follows that $P_{xx}(t) = P_{yy}(t) = 0$ (we use notation (20)), i.e.

$$P \mid P_{xx}^n, P_{yy}^n$$

for some $n \in \mathbb{Z}_{\geq 0}$. As P_{xx}, P_{yy} both are irreducible (Lemma 7.16) and non-zero,

$$[P] = [P_{xx}]^k = [P_{yy}]^l$$

for some $k, l \in \mathbb{Z}_{\geq 1}$. We have $\deg P_{xx} = \deg P_{yy} = 2$ and therefore $[P_{xx}] = [P_{yy}]$. This is false by Lemma 7.16b) and thus the assumption (26) is false. This proves statement (25) and completes the proof of Lemma 7.15. □

Proof of Proposition 7.9. Pick $t \in \mathfrak{f}_4$ such that

$$\omega_t := x \wedge y + t(x \wedge y) = x \wedge y + tx \wedge y + x \wedge ty$$

satisfies conditions (19) (such t exists according to Lemma 7.15). The vector $x \wedge y$ is the highest weight vector of $V(\pi_2)$ (see the statement related to (11) and (12)). Thus

$$\omega_t = ((x \wedge y) + t(x \wedge y)) \in \sigma_2(X(\pi_2))$$

according to Lemma 4.3. Assume that $\omega_t = \omega_1 + \omega_2$ for some $\omega_1, \omega_2 \in X(\pi_2)$. Equivalently, there exist $g_1, g_2 \in F_4$ such that

$$\omega_t = g_1 x \wedge (g_1 y) + g_2 x \wedge (g_2 y).$$

Since the spaces

$$\langle g_1 x, g_1 y \rangle \text{ and } \langle g_2 x, g_2 y \rangle$$

are two-dimensional and isotropic, the pair

$$\dim \operatorname{Im} \omega_t, \quad \operatorname{rk} \operatorname{Im} \omega_t$$

corresponding to ω_t has to appear in cases (a)-(e) of (7). This is not the case, due to (19), and thus ω_t is an exceptional vector for $V(\pi_2)$. Therefore $V(\pi_2)$ is wild and we have completed the proof of Proposition 7.9. □

7.3 Wildness of $V(\pi_1)$ for E_7

This subsection is devoted to the proof of the following proposition.

Proposition 7.17. The representation $V(\pi_1)$ of E_7 is 2-wild.

Our proof goes through the adjoint module of E_8 and this should somehow justify notation of this subsection. We set $G = E_8$ and $\underline{G} = E_7$. By $\mathfrak{g}, \mathfrak{h}, \pi_1$... and so on we denote attributes of E_8 and by $\underline{\mathfrak{g}}, \underline{\mathfrak{h}}, \underline{\pi_1}$... we denote the corresponding attributes of $\underline{G} = E_7$.

The idea of the proof of Proposition 7.17 is to identify $V(\underline{\pi_1})$ of E_7 with some subspace of $\mathfrak{e}_8$ ($\cong V_{\pi_1}$ of E_8) and then prove the following two lemmas.

Lemma 7.18. a) For any $x \in X(\underline{\pi_1})$ we have $\dim E_8 x = 58$.
b) For any $x \in X_2(\underline{\pi_1})$ we have $\dim E_8 x \in \{58, 92, 114\}$.

Lemma 7.19. There exists $x \in V(\underline{\pi_1})$ such that $\dim E_8 x = 112$.

It is known that $\sigma_2(X(\underline{\pi_1})) = V(\underline{\pi_1})$ [Zak93, Ch. III, Thm. 1.4]. Therefore $V(\underline{\pi_1})$ is wild if and only if there exists $x \in V(\underline{\pi_1})$ such that

$$x \notin X_2(\underline{\pi_1}) \cup X(\underline{\pi_1}) \cup 0.$$

According to Lemma 7.18 and Lemma 7.19 such elements $x \in V(\underline{\pi_1})$ exists and hence Lemmas 7.18 and 7.19 imply Proposition 7.17. First we make sense for Lemma 7.18 and Lemma 7.19 and then we present a proof of both this lemmas. To do this we need more notation.

We note the amazing fact that the $V(\underline{\pi_1})$ has only finitely many $\underline{G}$-orbits and we wish to say some words about it (see e.g. [V76]). The description of the E_7-orbits on $\mathbb{F}^{56}$($\dim V(\underline{\pi_1}) = 56$) appears in [H71]. The idea of the of a piece of description which is used here comes from [BC76] and is related to the description of $\mathfrak{sl}_2$-triples in exceptional groups due to [D52]. There is a recently developed software, which allows, in principle, to solve such problems [GVY12].

In our proof of 2-wildness of $V(\underline{\pi_1})$ we use that $V(\underline{\pi_1})$ is the 1-grading component of some grading of $\mathfrak{e}_8$ (representations which arises in such a way are called θ-representations, see [V76], [Kac80]). We need more notation related to θ-representations.

For any $t \in \mathfrak{h}^*$ we denote by $\mathfrak{g}_t \subset \mathfrak{g}$ the corresponding weight space (we note that $\mathfrak{g}_t \neq 0$ if and only if $t \in \Delta \cup 0$). We identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via the Cartan-Killing form and thus consider fundamental weights π_i as elements of $\mathfrak{h}^*$. We set

$$\Delta_i := \{\alpha \in \Delta \cup 0 \mid (\alpha, \pi_1) = i\} \ (i \in \mathbb{F}),$$
$$\mathfrak{g}_i := \bigoplus_{t \in \Delta_i} \mathfrak{g}_t \ (i \in \mathbb{F}).$$

The spaces $\{\mathfrak{g}_i\}_{i\in\mathbb{F}}$ form a grading of $\mathfrak{g}$. The space $\mathfrak{g}_0$ is a Lie algebra and it acts in a natural way on $\mathfrak{g}_i$ for any $i \in \mathbb{F}$. By definition, a *θ-representation* is the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$.

We have

$$\mathfrak{g}_i = 0 \text{ if } i \notin \{-2,-1,0,1,2\}, \quad \mathfrak{g}_0 \cong \mathfrak{e}_7 \oplus \mathbb{F},$$
$$\dim \mathfrak{g}_2 = \dim \mathfrak{g}_{-2} = 1, \quad \dim \mathfrak{g}_1 = \mathfrak{g}_{-1} = 56, \quad \dim \mathfrak{g}_0 = 134.$$

We identify $\underline{\mathfrak{g}}$ with $[\mathfrak{g}_0, \mathfrak{g}_0]$. As $\mathfrak{e}_7$-modules ($\mathfrak{e}_7 = \underline{\mathfrak{g}} = [\mathfrak{g}_0, \mathfrak{g}_0]$) both $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are isomorphic to $V(\underline{\pi_1})$. Further we identify $V(\underline{\pi_1})$ with $\mathfrak{g}_1$.

We note that π_1 of E_8 is a positive root (see [VO90, Table 1 on p. 293-295] for roots and fundamental weights of E_8).

The following Lemma plays a key role in the proof of Lemma 7.18.

Lemma 7.20. Let α_1, α_2 be roots of E_8 such that $\alpha_1 \neq -\alpha_2$. Then $v^{\alpha_1} + v^{\alpha_2}$ is a nilpotent element and $\dim E_8(v^{\alpha_1} + v^{\alpha_2}) \in \{58, 92, 114\}$.

Proof. If $\alpha_1 = \alpha_2$, then $v^{\alpha_1} + v^{\alpha_2}$ is conjugate to v^{α_1}. The nilpotent element v^{α_1} is a generic nilpotent element of the corresponding Levi subalgebra with semisimple part isomorphic to A_1. Therefore $\dim E_8(v^{\alpha_1} + v^{\alpha_2}) = 58$.

If $\alpha_1 \neq \alpha_2, -\alpha_2$, the vector $v^{\alpha_1} + v^{\alpha_2}$ is a nilpotent element of the Lie algebra $\mathfrak{l}_{\alpha_1,\alpha_2}$ corresponding to the root system generated by α_1, α_2. We have three possibilities: $(\alpha_1, \alpha_2) = 1$, $(\alpha_1, \alpha_2) = 0$, $(\alpha_1, \alpha_2) = -1$. In the first and third cases, we have $\mathfrak{l}_{\alpha_1,\alpha_2} \cong \mathfrak{sl}_3 = A_2$. In the second case, we have $\mathfrak{l}_{\alpha_1,\alpha_2} \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 = 2A_1$. For any of these Lie algebras of rank 2 it is easy to check that:

1) if $(\alpha_1, \alpha_2) = 1$, then $v^{\alpha_1} + v^{\alpha_2}$ is conjugate in $\mathfrak{l}_{\alpha_1,\alpha_2}$ to v^{α_1} (and therefore to v^{α_2}), and thus is a distinguished nilpotent element for some root subalgebra A_1,

2) if $(\alpha_1, \alpha_2) = 0$, then $v^{\alpha_1} + v^{\alpha_2}$ is a distinguished nilpotent element of $\mathfrak{l}_{\alpha_1,\alpha_2} \cong 2A_1$,

3) if $(\alpha_1, \alpha_2) = -1$, then $v^{\alpha_1} + v^{\alpha_2}$ is a distinguished nilpotent element of $\mathfrak{l}_{\alpha_1,\alpha_2} \cong A_2$.

Hence $\dim E_8(v^{\alpha_1} + v^{\alpha_2}) = 58, 92, 114$, respectively, for cases 1, 2, 3, see [CM93, 8.4, Table: nilpotent elements for E_8]. □

Now we are ready to prove Lemma 7.18.

Proof of Lemma 7.18. For any weight $\alpha \in \Delta_1$ and any $v^\alpha \in \mathfrak{g}_\alpha$ we have

$$v^\alpha \in \mathrm{V}(\underline{\pi_1})$$

and v^α is a highest weight vector with respect to some choice of Borel subalgebra of $E_7 = \underline{\mathfrak{g}}$, i.e.

$$v^\alpha \in X(\underline{\pi_1}).$$

On the other hand $v^\alpha \in X(\pi_1)$ and thus

$$\dim E_8 v^\alpha = \dim X(\pi_1) = 58.$$

This completes part a).

We proceed to part b). By Lemma 4.1, any element of $X_2(\underline{\pi_1})$ is $\underline{G}$-conjugate to the sum of two weight vectors. In our case this means that any $x \in X_2(\underline{\pi_1})$ is $\underline{G}$-conjugate to

$$v^{\alpha_1} + v^{\alpha_2}$$

for some $\alpha_1, \alpha_2 \in \Delta_1$. From this statement and Lemma 7.20 part b) of Lemma 7.18 follows immediately. □

We are now ready to prove Proposition 7.17.

Proof of Lemma 7.19. We now construct an element x, with $\dim E_8 x = 112$.

The Dynkin diagram of E_8 has a subdiagram D_4 of E_8 and all roots of E_8 are conjugate. Hence there exists roots $\alpha_1, \alpha_2, \alpha_3$ such that the quadruple

$$(-\pi_1, \alpha_1, \alpha_2, \alpha_3)$$

is a system of simple roots of Dynkin type D_4, i.e.

1) $(-\pi_1, \alpha_i) = -1$ for $i = 1, 2, 3$,

2) $(\alpha_i, \alpha_j) = 0$ for $i, j \in \{1, 2, 3\}$, $i \neq j$.

Condition 1) means that $\alpha_1, \alpha_2, \alpha_3 \in \Delta_1$. The element $v^{\alpha_1} + v^{\alpha_2} + v^{\alpha_3}$ is a distinguished element of $3A_1$, where by $3A_1$ we denote a subgroup of G corresponding to the root subsystem

$$\cup_i \{-\alpha_i, \alpha_i\} \subset \Delta.$$

Therefore $\dim E_8(v^{\alpha_1} + v^{\alpha_2} + v^{\alpha_3}) = 112$, see [CM93, 8.4, Table: nilpotent elements for E_8]. Hence, for $x = (v^{\alpha_1} + v^{\alpha_2} + v^{\alpha_3}) \in V(\underline{\pi_1})$, we have $\dim E_8 x = 112$. This completes the proof. □

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