A NEW PROOF THAT TEICHMÜLLER SPACE IS A COMPLEX STEIN MANIFOLD

by

A.J. Tromba

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 3

Dept. of Mathematics
Univ. of California
Santa Cruz CA 95064
USA

MPI/87-39

A NEW PROOF THAT TEICHMÜLLER SPACE IS A COMPLEX STEIN MANIFOLD

A.J. Tromba

Abstract

We show that Dirichlet's energy is a proper pluri-subharmonic function on Teichmüller space with respect to its natural complex structure.

Let M be an oriented compact surface without boundary and with genus greater than one. Let $\mathcal{A}$ be the space of almost complex structures on M compatible with its orientation and let $\mathcal{D}_0$ be the space of all diffeomorpisms of M homotopic to the identity. Then [4], [5], [6] Teichmüller space is defined to be the quotient $\mathcal{A}/\mathcal{D}_0$, where $\mathcal{D}_0$ acts on $\mathcal{A}$ by pull back. In [4] it is shown that $\mathcal{T}(M)$ has the structure of a 6(genus M) - 6 C^∞ smooth manifold. If $\mathcal{M}_{-1}$ denotes the infinite dimensional Fréchet manifold of Riemannian metrics of constant curvature -1, then $\mathcal{D}_0$ acts naturally on $\mathcal{M}_{-1}$ and $\mathcal{T}(M)$ is diffeomorphic to $\mathcal{M}_{-1}/\mathcal{D}_0$.

This diffeomorphism is described as follows (for details see [4], [8]: There is a natural $\mathcal{D}$-invariant diffeomorphism $\Phi : \mathcal{M}_{-1} \to \mathcal{A}$ given by

$$\Phi(g) = -g^{-1}\mu_g$$

where $\mathcal{M}_g$ is the volume element of g. Φ then passes to a diffeomorphism $\bar{\Phi}$ from $\mathcal{M}_{-1}/\mathcal{D}_0$ to $\mathcal{A}/\mathcal{D}_0$. Let $\theta : \mathcal{A} \to M_{-1}$ be the inverse of Φ. For $J \in \mathcal{A}$, $\theta(J)$ is the unique Poincaré metric associated to J. Denote by $\bar{\theta}$ the induced diffeomorphism from $\mathcal{A}/\mathcal{D}_0$ to $\mathcal{M}_{-1}/\mathcal{D}_0$. We also have a natural $\mathcal{D}_0$ invariant metric on A given by

$$<<H,K>> = \frac{1}{2}\int_M tr(HK)d\mu_{\Phi(J)}$$

and a natural L_2 splitting [8] of $T_J\mathcal{A}$, namely each $H \in T_J\mathcal{A}$ can be uniquely decomposed as

(1.1) $$H = H^{TT} + L_X J$$

where $L_X J$ is the Lie derivative of J w.r.t. the vector field X on M, and H^{TT} denotes a (1,1) tensor which is trace free and divergence free w.r.t. $\theta(J)$. The decomposition (1.1) is L_2-orthogonal. Since $\mathcal{D}_0$ acts as a group of isometries $<<,>>$ passes to a metric $<,>$ on $\mathcal{T}(M) = \mathcal{A}/\mathcal{D}_0$ described as follows. The term $L_X J$ is always tangent to the orbit of $\mathcal{D}_0$ through J. We say that $L_X J$ is the <u>vertical</u> part of $H \in T_J A$ in the decomposition (1.1). Similarly we say that H^{TT} represents the <u>horizontal</u> part of H. Let $\pi : \mathcal{A} \to \mathcal{A}/D_0$ be the natural projection map. Given $H,K \in T_{[J]}\mathcal{A}/\mathcal{D}_0$ there are unique horizontal vectors $\tilde{H},\tilde{K} \in T_J\mathcal{A}$ such that $D\pi(J)K = K$. Then

(1.2) $$<H,K>_{[J]} = <<\tilde{H},\tilde{K}>>_J .$$

Let us now consider the model $\mathcal{M}_{-1}/\mathcal{D}_0$ of $\mathcal{T}(M)$. The tangent space of $\mathcal{M}_{-1}$ at a metric, $g \in T_g\mathcal{M}_{-1}$ consists of those (0,2) tensors h on M satisfying the equation

(1.3) $$-\Delta(tr_g h) + \delta_g\delta_g h + \frac{1}{2}(tr_g h) = 0$$

where $tr_g h = g^{ij}h_{ij}$ is the trace of h w.r.t. the metric tensor g_{ij}, $\delta_g\delta_g h$ is the double covariant divergence of h w.r.t. g and Δ is the Laplace-Beltrami operator on functions. For example see [8] for details. The L_2-metric on $\mathcal{M}_{-1}$ is given by the inner product

$$\langle\langle h,k \rangle\rangle_g = \frac{1}{2} \int_M \text{trace } (HK) d\mu_g \tag{1.4}$$

where $H = g^{-1}h$, $K = g^{-1}k$ are the $(1,1)$ tensors on M obtained from h and k via the metric g, or "by raising an index", i.e.

$$H^i_j = g^{ik} h_{kj}$$

and similarly for K.

The inner product (1.4) is $\mathcal{D}_0$ invariant. Thus $\mathcal{D}_0$ acts smoothly on $\mathcal{M}_{-1}$ as a group of isometries with respect to this metric, and consequently we have an induced metric on $\mathcal{T}(M)$ in such a way that the projection map $\pi : \mathcal{M}_{-1} \to \mathcal{M}_{-1}/\mathcal{D}_0$ becomes a Riemannian submersion [4]. In [5] it is shown that this induced metric is precisely the metric originally introduced by Weil.

Let $\langle , \rangle$ be the induced metric on $\mathcal{T}(M)$. We can characterize $\langle , \rangle$ as follows. From [3] we can show that given $g \in \mathcal{M}_{-1}$ every

$$h = h^{TT} + L_X g \tag{1.5}$$

where $L_X g$ is the Lie derivative of g w.r.t. some (unique X) and h^{TT} is a trace free, divergence free, symmetric tensor. Moreover the decomposition (1.5) is L_2-orthogonal. Recall that a conformal coordinate system (where $g_{ij} = \lambda \delta_{ij}$, λ some smooth positive function) is also a complex holomorphic coordinate system. In this system

$$h^{TT} = \text{Re}(\xi(z) dz^2)$$

where Re is "real part" and $\xi(z)dz^2$ is a holomorphic quadratic differential. In fact, trace free, divergence free symmetric two tensors are precisely the real parts of holomorphic quadratic differentials.

Now $L_X g$ is always tangent to the orbit of $\mathcal{D}_0$ through g. We say that $L_X g$ is the <u>vertical</u> part of h in decomposition 1.4. Similarly we say that h^{TT} represents the <u>horizontal</u> part of h. Given $h,k \in T_{[g]}\mathcal{T}(M)$ there are unique horizontal vectors $\tilde{h},\tilde{k} \in T_g\mathcal{M}_{-1}$ such that $D\pi(g)\tilde{h} = h$ and $D\pi(g)\tilde{k} = k$. Then

$$\langle h,k\rangle_{[g]} = \langle\langle\tilde{h},\tilde{k}\rangle\rangle_g .$$

The map $\bar{\theta} : \mathcal{A}/\mathcal{D}_0 \to \mathcal{M}_{-1}/\mathcal{D}_0$ has a derivative which can be described as follows. Let $H \in T_{[J]}\mathcal{A}/\mathcal{D}_0$ and $\tilde{H}$ its horizontal lift. Then

$$D\bar{\theta}([J])H = D\pi \cdot D\theta(J)\tilde{H} \tag{1.6}$$

where $D\theta(J)\tilde{H} = -(J\tilde{H})_\#$ and $(J\tilde{H})_\#$ is the $(0,2)$ tensor obtained from $(J\tilde{H})$ by lowering an index via the metric g, i.e.

$$(J\tilde{H})_{\#ij} = g_{ik}(JH)^k_j$$

Suppose now that $g_0 \in \mathcal{M}_{-1}$ is fixed and that $s : (M,g) \to (M,g_0)$ is a smooth C^1 map homotopic to the identity and is viewed as a map from M with some aribtrary metric $g \in \mathcal{M}_{-1}$ to M with its g_0 metric.

Define the Dirichlet energy of s by the formula

$$E_g(s) = \frac{1}{2}\int_M |ds|^2 d\mu_g \tag{1.7}$$

where $ds^2 = \text{trace } ds \otimes ds$ depends on both g and g_0.

By the embedding theorem of Nash-Moser we may assume that (M,g_0) is isometrically embedded in some Euclidean R^K. Thus we can think of

$s : (M,g) \to (M,g_0)$ 'as a map into R^K and Dirichlet's functional takes the equivalent form

$$(1.8) \qquad E_g(s) = \frac{1}{2} \sum_{i=1}^{k} \int g(x) < \nabla_g s^i(x), \nabla_g s^i(x) > d\mu_g .$$

There is another, equivalent, and useful way to express (1.5) and (1.8) using local conformal cordinate systems $g_{ij} = \lambda\delta_{ij}$ and $(g_0)_{ij} = \rho\delta_{ij}$ on (M,g) and (M,g_0) respectively, namely

$$(1.9) \qquad E_g(s) = \frac{1}{4} \int_M [\rho(s(z))|s_z|^2 + \rho(s(z))|s_{\bar{z}}|^2] dz d\bar{z}$$

For fixed g, the critical points of E_g are then said to be <u>harmonic maps</u>. The following result is due to Eells-Sampson, Hartman and Schoen-Yau [3], [10].

<u>Theorem</u> (1.10) Given metrics g and g_0 with $g_0 \in M_{-1}$ there exists a unqiue harmonic map $s(g) : (M,g) \to (M,g_0)$ which is homotopic to the identity, and is the absolute minimum for E_g. Moreover $s(g)$ depends differentialy on g in any H^r topology, $r > 2$, and is a C^∞ diffeomorphism.

Consider now the function

$$g \to E_g(s(g)).$$

This function on $\mathcal{M}_{-1}$ is $\mathfrak{D}$-invariant and thus can be viewed as a function on Teichmüller space. To see this one must show that

$$E_{f*g}(s(f^*(g))) = E_g(s(g)).$$

Let $c(g)$ be the complex structure associated to g, and induced by a conformal coordinate system for g. For $f \in \mathfrak{D}_0$, $f : (M,f^*c(g)) \to (M,c(g))$

is holomorphic and consequently since the composition of harmonic maps and holomorphic maps is still harmonic we may conclude, by uniqueness that

$$s(f^*g) = s(g) \circ f .$$

Since Dirichlet's functional is invariant under complex holomorphic changes of coordinates it follows immediately that

$$E_{f^*(g)}(s(g) \circ f) = E_g(s(g)).$$

Consequently for $[g] \in \mathcal{M}_{-1}/\mathcal{D}_0$ define the C^∞ smooth function

$$\tilde{E} : \mathcal{M}_{-1}/\mathcal{D}_0 \to R$$

by

$$\tilde{E}[g] = E_g(s(g)).$$

In [9] we prove the following

Theorem 1.1. If $s : (M,g) \to (M,g_0)$ is harmonic the form $\xi(z)dz^2$ is a holomorphic quadratic differential on the complex curve $(M,c(g_0))$, and thus $\mathrm{Re}\ \xi(z)dz^2$ represents a trace free, divergence free symmetric two tensor on (M,g). Hence $\mathrm{Re}\ \xi(z)dz^2$ is a horizontal tangent vector to $\mathcal{M}_{-1}$ at g. In addition

$$(1.12)\quad D\tilde{E}[g]h = -\frac{1}{2} \langle\langle \mathrm{Re}\ \xi(z)dz^2, \tilde{h}\rangle\rangle_g = -\frac{1}{2} \sum_{\ell} \int_M g(x)(\tilde{H}\nabla_g s^\ell, \nabla_g s^\ell)d\mu_g$$

where $\tilde{h}$ is the horizontal lift of $h = T_{(g)}\mathcal{T}(M)$ and $\tilde{H} = (\tilde{h})^\#$ is obtained from h by raising an index via g.

Finally $[g_0]$ is the only critical point of $\tilde{E}$. The Hessian of $\tilde{E}$ at $[g_0]$ is given by

(1.13) $$D^2\tilde{E}[g_0](h,k) = \langle h,k\rangle$$

$h,k \in T_{[g_0]}\mathcal{T}(M)$. That is, the second variation of Dirichlet's energy function is (up to a positive constant) Weil-Petersson metric.

Suppose we look at the first derivative 1.12 in conformal coordinates $(g)_{ij} = \lambda\delta_{ij}$. Then if $\tilde{h}$ is horizontal

$$2\,\frac{\partial \dot{E}}{\partial g}(g,s)\tilde{h} = -\int\langle h^{\#}\nabla s^{\ell},\nabla s^{\ell}\rangle_{R^2}dxdy$$

$$= -\int\frac{1}{\lambda}\{\tilde{h}_{11}(\frac{\partial s^{\ell}}{\partial x})^2 + 2\tilde{h}_{12}(\frac{\partial s^{\ell}}{\partial x})(\frac{\partial s^{\ell}}{\partial y}) + \tilde{h}_{22}(\frac{\partial s^{\ell}}{\partial x})^2\}dxdy \quad .$$

where $h^{\#} = \frac{1}{\lambda}\{h_{ij}\}$. Since $\tilde{h}_{11} = -\tilde{h}_{22}$ this is equal to

$$-\int\frac{1}{\lambda}\{\tilde{h}_{11}[(\frac{\partial s^{\ell}}{\partial x})^2 - (\frac{\partial s^{\ell}}{\partial y})^2 + 2\tilde{h}_{12}(\frac{\partial s^{\ell}}{\partial x})\;(\frac{\partial s^{\ell}}{\partial y})\}dxdy.$$

Now

$$(\frac{\partial s^{\ell}}{\partial y} - i\,\frac{\partial s^{\ell}}{\partial y})^2(dx + dy)^2 = \xi(z)dz^2$$

is a quadratic differential. But

$$Re(\xi(z)dz^2) = [(\frac{\partial s^{\ell}}{\partial x})^2 - (\frac{\partial s^{\ell}}{\partial y})^2]dx^2 + [(\frac{\partial s^{\ell}}{\partial y})^2 - (\frac{\partial s^{\ell}}{\partial y})^2]dy^2 + 4(\frac{\partial s^{\ell}}{\partial x})(\frac{\partial s^{\ell}}{\partial y})dxdy$$

If s is harmonic $Re(\xi(z)dz^2)$ is a trace free divergence free tensor. In general the second derivative of $\tilde{E}$ at an arbitrary $[g]$ will not be intrinsic. However we can ask for the second derivative of the function $g \mapsto E_g(s(g)) = \hat{E}(g)$. (For $g \in \mathcal{M}$, the space of all Riemannian metrics it still follows from [3], [10] that E_g has a unique minimum $s(g)$ which depends differentiably on g). This was computed in [9]. Thus we have

<u>Theorem 1.14</u> If h is not trace free

$$D\hat{E}(g)h = -\frac{1}{2}\sum_{\ell}\int_{M} g(x)(H_T\nabla s^{\ell},\nabla s^{\ell})\,d\mu_g$$

where H_T is the trace free part of $(h)^{\#}$ moreover if h and k are trace free:

$$D^2\hat{E}(g)(h,k) = \frac{1}{2}\sum_{\ell}\int_{M} (h\cdot k)g(x)(\nabla_g s^{\ell},\nabla_g s^{\ell})\,d\mu_g$$

$$-\sum_{\ell}\int_{M} g(x)(h^{\#}\cdot\nabla_g s^{\ell},\nabla w^{\ell}(k))\,d\mu_g$$

where

(1.5)
$$h\cdot k = g^{ab}g^{cd}h_{ac}k_{bd}$$

$$= tr(HK)$$

$H = h^{\#}$, $K = k^{\#}$ the (1.1) tensors obtained from h and k by raising an index and

$w^{\ell}(k) = Ds^{\ell}(g)k$, the derivative of $s(g)$ in the direction k.

§2 The complex structure on $\mathcal{T}(M)$

In this section we describe the explicite complex coordinates on $\mathcal{T}(M)$ discovered by Uwe Abresch and Arthur Fischer. We shall use the description $\mathcal{A}/\mathcal{D}_0$ for Teichmüller space. The tangent space of the Fréchet manifold $\mathcal{A}$ of almost complex structures $J \in C^\infty(T_1,(M))$, $J^2 = -I$ at a point $J \in T_J\mathcal{A}$ consists of all those C^∞ (1.1) tensors H such that $HJ = -JH$. $\mathcal{A}$ has a natural almost complex structure Φ [6] , where $\Phi_j : T_J\mathcal{A} \to T_J\mathcal{A}$ is defined by

$$\Phi_J(H) = JH$$

We then have

<u>Theorem 2.1</u> The Fréchet manifold $\mathcal{A}$ of almost complex structures can be given an explicite complex structure.

<u>Proof.</u> (a sketch) Let $J_0 \in \mathcal{A}$ be fixed, and let $\mathcal{U}$ be an open neighborhood of $0 \in T_{J_0}\mathcal{A}$ consisting of those (1,1) tensors H such that $(I + H)$ is invertible. Define the mapping

$$\psi : U \to \mathcal{A}$$

by

$$\psi(H) = (I + H)J_0(I + H)^{-1} = J. \tag{2.2}$$

It is clear that $J^2 = -I : TM \to TM$ iff $J_0^2 = -I$, which says that the range of ψ is in $\mathcal{A}$. A straightforward algebraic excercise shows that the inverse of ψ, $\psi^{-1} : \psi(U) \to T_J\mathcal{A}$ is given by

$$\psi^{-1}(J) = (J - J_0)(J + J_0)^{-1} \tag{2.3}$$

A short calculation shows that

(2.4) $$\psi^*(\Phi) = \Phi_{J_0},$$

(2.5) $$D\psi_J^{-1}(\Phi_J D\psi_H(\dot{J})) = \Phi_{J_0}(\dot{J}) = J_0\dot{J}$$

where Φ is the almost complex structure on $\mathcal{A}$ $(\Phi_J(K) = JK)$ and $\Phi_{J_0} : T_{J_0}\mathcal{A} \to T_{J_0}\mathcal{A}$ the fixed linear almost complex struture on $T_{J_A}\mathcal{A}$.

Relation 2.4 says that ψ is a complex coordinate that for Φ.

<u>The Explicit Complex Coordinates on $\mathcal{T}(M)$</u>

We shall now describe how this complex structure on $\mathcal{A}$ induces a complex structure for $\mathcal{T}(M)$. For $J \in \mathcal{A}$ let $\theta(J)$ be the unique Poincaré metric associated to J and $\bar{\theta} : \mathcal{A}/\mathcal{D}_0 \to \mathcal{M}_{-1}/\mathcal{D}_0$ the induced map. We know that the tangent space to $\mathcal{A}/\mathcal{D}_0$ at $[J]$ can be identified with $\mathcal{H}^{TT}(J)$ the space of trace free divergence free (1,1) tensors w.r.t. $\theta(J)$.

Let $H \in T_{[J]}\mathcal{A}/\mathcal{D}_0$ be a tangent vector, and let $\tilde{H}$ be the $\mathcal{D}$-invariant horizontal lift of H. Thus for $f \in \mathcal{D}_0$, $f^*\tilde{H} = \tilde{H}$, $D\pi(J)\tilde{H}(J) = H([J])$ for all $J \in \pi^{-1}([J])$, . Let $\mathcal{W}$ be the open neighborhood of $0 \in \mathcal{H}^{TT}(J_0)$ consisting of those $\tilde{H}(J_0)$ with $(I + \tilde{H}(J_0))$ invertible, and $\psi : \mathcal{W} \to A$ the map defined in (2.2). If $\pi : \mathcal{W} \to A$ denotes the bundle projection, define $\psi : \mathcal{W} \to \mathcal{T}(M)$ by

(2.6) $$\bar{\psi} = \pi \circ \psi .$$

Thus

(2.7) $$\bar{\psi}(\tilde{H}(J_0)) = \pi\{(I + \tilde{H}(J_0))J_0(I + \tilde{H}(J_0)^{-1}\}.$$

Then the set of all such $\tilde{\psi}$'s is a complex structure for $\mathcal{T}(M)$.

We can identify an open neighborhood of $[J_0]$ in $\mathcal{T}(M)$ in $\mathcal{T}(M)$ with an open neighborhood of $\psi(\mathscr{H})$. The first derivative $D\psi_H(\dot{J}), \dot{J} \in T_{J_0}\mathscr{A}$ is easily calculated to be

$$(2.8) \qquad D\psi_H(\dot{J}) = \dot{J}J_0(I + H)^{-1} - (I + H)J_0(I + H)^{-1}\dot{J}(I + H)^{-1}.$$

Thus

$$(2.9) \qquad D\psi_0(\dot{J}) = \dot{J}J_0 - J_0\dot{J} = -2J_0\dot{J}.$$

As a map into $C^\infty(T_1^1(M))$ we can compute the second derivative:

$$(2.10) \qquad D^2\psi_0(\dot{J}_1, \dot{J}_2) = 2J_0(\dot{J}_1\dot{J}_2 + \dot{J}_2\dot{J}_1).$$

§3 Teichmüller space is a Stein manifold

This theorem was first proved by Bers and Ehrenpreis [1].

Our main result is

Theorem 3.1 The map $\tilde{E} : \mathcal{T}(M) \to R$ is proper and

$$\frac{\partial^2 \tilde{E}}{\partial z\, \partial \bar{z}}[g] > 0 \tag{3.2}$$

where the derivative is taken w.r.t. the natural complex structure on $\mathcal{T}(M)$ introduced in the last section.

Proof. That $\tilde{E}$ is proper is proved in [7]. It clearly suffices to show that

$$D^2(\tilde{E} \circ \bar{\varphi})[g](h,h) > 0 \tag{3.3}$$

for any $h \in T_{[g]}(\mathcal{M}_{-1}/\mathcal{D}_0)$ and where $\bar{\psi}$ is a complex coordinate system for $\mathcal{T}(M)$.

For g a Riemannian metric on M, let $\hat{E}(g) = E_g(s(g))$. Let $\theta : \mathcal{A} \to \mathcal{M}_{-1}$ be the Poincaré maps and ψ a complex coordinate system for $\mathcal{A}$ about $J_0 = \theta^{-1}(g)$. Therefore (3.3) is clearly equivalent to

$$D^2(\hat{E} \circ \varphi)_0(H,H) > 0 \tag{3.4}$$

for all $H \in \mathcal{H}^{TT}(J_0)$ where $\varphi = \theta \circ \psi$. However

$$D^2(\hat{E} \circ \varphi)_0 = D^2\hat{E}(D\varphi_0, D\varphi_0) + D\hat{E} \circ D^2\varphi_0 \tag{3.5}$$

We would like to compute $D\varphi_0(H)$ and $D^2\varphi_0(H,H)$. Let S_2 be the space of C^∞ (0,2) tensors and $S_2^{TT}(g)$ denote the trace for divergence free symmetric two tensors with respect to g. Then from [8] we know that $D\theta(J) : T_J\mathscr{A} \to T_g\mathscr{M}_{-1} \subset S_2$ is given by

$$D\theta(J)\dot{J} = \rho g + h \tag{3.6}$$

where $g = \theta(J)$, $h = -(J\dot{J})_\#$ and

$$\Delta\rho - \rho = \delta_g\delta_g h$$

Δ the Laplace-Beltrami operator on functions (see for example 1.6).

Let $L_g = \Delta - I$, I the identity. Then

$$\rho = L_g^{-1}(\delta_g\delta_g h).$$

If h is divergence free then $\rho = 0$.

From 2.8 we know that

(i) $$D\psi_H(\dot{J}_1) = \dot{J}_1J_0(I + H)^{-1} - (I + H)J_0(I + H)^{-1}\dot{J}_1(I + H)^{-1}$$

(ii) $$D\psi_0(\dot{J}_1) = -2J_0\dot{J}_1$$

(iii) $$D^2\psi_0(\dot{J}_1\dot{J}_2) = 2J_0(\dot{J}_1\dot{J}_2 + \dot{J}_2\dot{J}_1)$$

Therefore

$$(3.7) \qquad D\varphi_H(\dot{J}_1) = D\theta(J) \circ D\psi_H(\dot{J}_1)$$

$$= \{JD\psi_H(\dot{J}_1)\}_\# + \rho g$$

$$= \{JD\psi_H(\dot{J}_1)\}_\# + \rho(J) \cdot \theta(J)$$

$g = \theta(J)$, $J = (I + H)J_0(I + H)^{-1}$ and $\rho(J) = L_g^{-1}(\delta_g\delta_g(JD\psi_H(\dot{J}_1))$ where, as usual, # denotes lowering an index via the metric g.

Now $\psi(0) = J_0$ and $D\psi_0(\dot{J}_1)$ is a trace free divergence free tensor, whence it follows that $\rho(J_0) = 0$.

Let us first consider the term

$$H \mapsto \{-JD\psi_H(\dot{J}_1)\}_\#$$

in expression (3.7) for which we would like to compute the derivative in the direction J_2. But

$$\{-JD\psi_H(\dot{J}_1)\}_\# = -\{(I + H)J_0(I + H)^{-1}D\psi_H(\dot{J}_1)\}_\#$$

$$= -\{(I + H)J_0(I + H)^{-1}\dot{J}_1J_0(I + H)^{-1} + \dot{J}_1(I + H)^{-1}\}_\#$$

For $H = 0$ this is equal to $-2\dot{J}_1 = D\varphi_0(\dot{J}_1)$. The derivative of

$$H \mapsto -\{(I + H)J_0(I + H)^{-1}\dot{J}_1J_0(I + H)^{-1} + \dot{J}_1(I + H)^{-1}\}$$

at 0, in the direction of $\dot{J}_2$ is easily computed to be

(3.8) $$2(\dot{J}_1\dot{J}_2 - \dot{J}_2\dot{J}_1).$$

Consider now the map

(3.9) $$H \mapsto \theta(J)_{i\ell}A^{\ell}_{j} - (A^{\ell}_{j})^{\#}$$

where A is a fixed (1,1) tensor. The derivative of this at 0 in the direction $\dot{J}_2$ is

(3.10) $$(D\theta(J_0)D\psi_0(\dot{J}_2))_{i\ell}A^{\ell}_{j} - ((-2\dot{J}_2)_{\#})_{i\ell}A^{\ell}_{j}$$

In the case $A^{\ell}_{j} = -2\dot{J}_1$ we see that this is equal to

$$- 4(J_2J_1)_{\#}$$

Adding this and (3.8) together we find that the derivative of

$$H \mapsto (-JD\psi_H(\dot{J}_1))_{\#}$$

at 0 is the bilinear map

$$(\dot{J}_1\dot{J}_2) \mapsto 2(\dot{J}_1\dot{J}_2 + \dot{J}_2\dot{J}_1).$$

Thus in order to complete our computation of the derivative of

$$H \mapsto D\varphi_H(\dot{J}_1)$$

we must consider the second term in the final expression (3.7) on the derivative of the map

$$J \mapsto \rho(J)\theta(J)$$

at the point J_0. Since $\rho(J_0) = 0$ we need only calculate $D\rho(J_0)\dot{J}_2$. Let $X = JD\psi_0(\dot{J}_1)$, $Y = JD\psi_0(\dot{J}_2)$ and $g = \theta(J_0)$. Then since X and Y are trace free divergence free it follows that

$$D\rho(J_0)Y = L_g^{-1}(\delta_g D_g \delta_g(Y)X) \tag{3.11}$$

where $D_g\delta_g(Y)$ is the derivative of the divergence operator δ_g with respect to g in the direction Y. Thus we have our formula for $D^2\varphi_0$ namely

$$D^2\varphi(\dot{J}_1,\dot{J}_2) = 2(\dot{J}_1\dot{J}_2 + \dot{J}_2\dot{J}_1) + L_g^{-1}(\delta_g(D_g\delta_g)(Y)X)$$

where $X = JD\psi(\dot{J}_1)$, $Y = JD\psi_0(\dot{J}_2)$.

Lemma 3.13

$$\delta_g(D_g\delta_g)(X)X = 0 \ .$$

Proof By corollary 4A of [8]

$$(D_g\delta_g)(X)X = \frac{1}{2} \ {*d\mu}$$

μ, a real valued function on M. Thus

$$\delta_g\{D_g\delta_g)(X)X\} = \frac{1}{2}\delta_g{*d\mu} = 0 \ .$$

This gives us

Theorem 3.14

$$D^2\varphi_0(H,H) = 4(H^2)_{\#}$$

We are now ready to complete the proof of theorem 3.1. By formula (3.5) we must show that the sum of $D^2\hat{E}(D\varphi_0 H, D\varphi_0 H)$ and $D\hat{E} \circ D^2\varphi_0$ is strictly positive.

Now for $\tilde{h} \in S_2^{TT}(g)$, $h \in T_{[g]}\mathcal{T}(M)$, $D\hat{E}(g)(\tilde{h}) = D\tilde{E}[g]h$. By 1.12 we see that for k arbitrary

$$D\hat{E}(g)k = -\frac{1}{2}\sum_{\ell}\int_M g(x)(K_T\nabla_g s^{\ell}, \nabla_g s^{\ell})d\mu_g$$

where $K = (k)^{\#}$ and K_T is the trace free part of K. Therefore

$$D\hat{E}(g)D^2\varphi_0(H,H) = -2\sum_{\ell}\int_M g(x)((H^2)_T\nabla s^{\ell}, \nabla s^{\ell})d\mu_g.$$

Lemma 3.15

If $H \in T_J\mathcal{A}$ is divergence free then $H^2 = \mu I$ where μ is a non-negative function which vanishes at, at most finitely many points of M.

Proof Write H in conformal coordinates $g_{ij} = \lambda\delta_{ij}$ as $H = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$. Then $\lambda a - i\lambda h$ is a holomorphic quadratic differential on M and thus has $4(\text{genus } M) - 4$ zeros (genus $M > 1$). $H^2 = (a^2 + b^2)I = \mu I$, $\mu = \frac{1}{2}\text{ trace } H^2$, which concludes the proof of the lemma. Consequently we see that:

(3.16) $$D\hat{E}(g)D^2\varphi_0(H,H) = 0$$

Therefore (c.f. 3.4 and 3.5)

$$D^2\hat{E}(g)(D\varphi_0(H),\ D\varphi_0(H)) = D^2(\hat{E} \circ \varphi)_0(H,H)$$

If $\tilde{k} = D\varphi_0(H) = (-2H)_\#$ then

(3.17) $$D^2\hat{E}(g)(\tilde{k},\tilde{k}) = \frac{1}{2}\sum_{\ell}\int_M (\tilde{k}\cdot\tilde{k})g(x)(\nabla_g s^\ell,\nabla_g s^\ell)d\mu_g$$

$$- \int_M g(x)(\tilde{k}^\#\nabla_g s^\ell,\nabla_g w^\ell(k))d\mu_g$$

If the second term of 3.17 were positive we would be done, i.e. theorem 3.1 would be proved. The next lemma shows that this is not the case

<u>Lemma 3.18</u>

$$-\sum_{\ell}\int_M g(x)(\tilde{k}^\#\nabla_g s^\ell,\nabla w^\ell(\tilde{k}))d\mu_g \le 0$$

<u>Proof</u> Consider the map $g \mapsto E_g(s(g))$. Since $s(g)$ is a critical point of E_g we have the relation

(3.19) $$\frac{\partial E_g}{\partial s} \circ Ds(g)\tilde{k} = 0$$

where $W(\tilde{k}) = Ds(g)(\tilde{k})$, for all g. Therefore the derivative of (3.19) with respect to g, must be identically zero, or consequently we see that

$$0 = \frac{\partial^2 E_g}{\partial g \partial s}(\tilde{k}, Ds(g)\tilde{k}) + \frac{\partial}{\partial s}\{\frac{\partial E_g}{\partial s} \circ Ds(g)\tilde{k}\} \circ Ds(g)\tilde{k}.$$

The second term is precisely the second variation of Dirichlet's energy E_g $D^2E_g(w,w)$, at the critical point $s(g)$ in the direction w where $w(\tilde{k}) = Ds(g)(\tilde{k})$. Since $s(g)$ is an absolute minimum it follows that $D^2E_g(w,w) \geq 0$. Thus (c.f. 1.12)

$$\frac{\partial^2 E_g}{\partial g \partial s}(\tilde{k}, w) = - D^2E_g(w,w) = -\int_M g(x)(\tilde{k}^{\#}\nabla_g s^{\ell}, \nabla w^{\ell}(\tilde{k}))d\mu_g \leq 0$$

which completes 3.18.

By this last lemma, our only chance to show that $\frac{\partial^2 \tilde{E}}{\partial z \partial \bar{z}} > 0$ is to show

$$\text{(3.19)} \qquad I = D^2E_g(s)(w,w) < \frac{1}{2}\sum_{\ell}\int_M (\tilde{k}\cdot\tilde{k})g(x)(\nabla_g s^{\ell}, \nabla_g s^{\ell})d\mu_g$$

which is what we now proceed to do we, fortunately have an explicit formula for the second variation of E_g at a minimum s, namely [4,p. 139] in conformal coordinates $g_{ij} = \lambda\delta_{ij}$ with local coordinates $(x,y) = (x^1,x^2)$ we have

$$I = D^2E_g(s)(w,w) = \int_M \{\langle\nabla_{\frac{\partial}{\partial x}} w, \nabla_{\frac{\partial}{\partial x}} w\rangle + \nabla_{\frac{\partial}{\partial y}} w, \nabla_{\frac{\partial}{\partial y}} w\rangle\} dx \wedge dy$$

$$-\int_M \{\langle\mathcal{R}(w,\frac{\partial s}{\partial x})\frac{\partial s}{\partial x}, w\rangle + \langle\mathcal{R}(w,\frac{\partial s}{\partial y}), \frac{\partial s}{\partial y}, \frac{\partial w}{\partial y})\} dx \wedge dy$$

where $\langle,\rangle : R^k \times R^k \to R$ is the Euclidean inner product and $\mathcal{R}$ is the curvature tensor of $(M,g_0) \subset R^k$.

Since the curvature of (M,g_0) is -1 we see that

$$I \geq \int_M (<\nabla_{\frac{\partial}{\partial x}} w, \nabla_{\frac{\partial}{\partial x}} w> + \nabla_{\frac{\partial}{\partial y}} w, \nabla_{\frac{\partial}{\partial y}} w>) dx \wedge dy$$

where strict inequality holds if $w \neq 0$. We are assuming that (M,g_0) is isometrically embedded in R^k. For $p \in (M,g_0) \subset R^k$ let $\Pi_{(p)} : R^k \to T_pM$ be the orthogonal projection of R^k onto the tangent space to M at p. Then the condition that s, $s : (M,g) \to (M,g_0)$ be harmonic can be written (in conformal coordinates) as

$$\Pi(s)\Delta s = \Pi(s)\left(\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2}\right) = 0$$

$$s = (s^1, \ldots, s^k).$$

This can be written in terms of the metric g as

$$\Pi(s)(\sqrt{g}\Delta_g s) = 0$$

Δ_g = Laplace-Beltrami on the coordinate functions $(s^1, \ldots, s^k)$. We know that the unique harmonic map s depends on g, so let us write this as

$$\Pi(s(g))(\sqrt{g}\Delta_g s(g)) = 0 \tag{3.20}$$

and this holds for all g.

Differentiating (3.20) w.r.t. g in the direction of a trace free (w.r.t. g) tensor h we obtain

$$D\Pi(s)(\sqrt{g}\Delta_g s(g)) + \Pi(s(g))(\sqrt{g}\Delta_g w) \tag{3.21}$$

$$\Pi(s(g)\{\frac{\partial}{\partial g}[\sqrt{g}\Delta_g](h)\}s = 0$$

$$= D\Pi(s)w(\Delta s) + \Pi(s)\Delta w$$

$$+ \Pi(s)\{\frac{\partial}{\partial g}[\sqrt{g}\Delta_g](h)\}s$$

$w = Ds(g)h$.

Now

$$\sqrt{g}\Delta_g s = \frac{\partial}{\partial x^j}\ (\sqrt{g}\ g^{ij}\frac{\partial s}{\partial x^i})$$

Therefore

$$\frac{\partial}{\partial g}\{\sqrt{g}\Delta_g(h)\}(s) =$$

$$- \frac{\partial}{\partial x^i}\ (\sqrt{g}h^{ij}\frac{\partial s}{\partial x^i})$$

But necessarily the second variation $I = D^2E_g(s)(w,w)$ equals

$$I = \int_M \langle D\Pi(s)(w)(\Delta s) + \Pi(s)(\Delta s),w\rangle\} dx\wedge dy$$

From this and 3.21 we see that

$$I = \int_M \langle\frac{\partial}{\partial x^j}(\sqrt{g}h^{ij}\frac{\partial}{\partial x^i}s),w\rangle dx\wedge dy$$

Integrating by parts we get

$$-I = \int_M h^{11}\ \frac{\partial s}{\partial x} \bullet \frac{\partial w}{\partial x}\ \sqrt{g}\ dx\wedge dy + \int_M h^{22}\ \frac{\partial s}{\partial y} \bullet \frac{\partial w}{\partial y}\ \sqrt{g}\ dx\wedge dy$$

$$\int_M h^{12} \frac{\partial s}{\partial x} \bullet \frac{\partial w}{\partial x} \sqrt{g}\; dx \wedge dy + \int_M h^{21} \frac{\partial s}{\partial y} \bullet \frac{\partial w}{\partial y} \sqrt{g}\; dx \wedge dy$$

where $\bullet$ denotes the R^k inner product. Thus

$$-I = \int_M (h^{22} \frac{\partial s}{\partial y} + h^{12} \frac{\partial s}{\partial x}) \bullet \frac{\partial w}{\partial x} \sqrt{g}\; dx \wedge dy$$

$$+ \int_M (h^{21} \frac{\partial s}{\partial y} + h^{21} \frac{\partial s}{\partial x}) \bullet \frac{\partial w}{\partial y} \sqrt{g}\; dx \wedge dy$$

Since

$$\nabla_{\frac{\partial}{\partial y}} w = \Pi(s) \frac{\partial w}{\partial y}$$

$$\nabla_{\frac{\partial}{\partial x}} w = \Pi(s) \frac{\partial w}{\partial x}$$

we see that this is equal to

$$\int_M (h^{22} \frac{\partial s}{\partial y} + h^{12} \frac{\partial s}{\partial x}) \bullet (\nabla_{\frac{\partial}{\partial x}} w) \sqrt{g}\; dx \wedge dy$$

$$+ \int_M (h^{21} \frac{\partial s}{\partial y} + h^{21} \frac{\partial s}{\partial x}) \bullet (\nabla_{\frac{\partial}{\partial x}} w) \sqrt{g}\; dx \wedge dy$$

Applying the Schwartz inequality and using the fact that $g_{ij} = \lambda \delta_{ij}$, $\sqrt{g} = \lambda$ we obtain

$$(3.22) \qquad I \le \left(\sqrt{\int_M \| h^{22} \frac{\partial s}{\partial y} + h^{12} \frac{\partial s}{\partial x}) \|^2 \lambda^2 dx \wedge dy}\right) \left(\sqrt{\int_M \| \nabla_{\frac{\partial}{\partial x}} w \|^2 dx \wedge dy}\right)$$

$$+ \left(\sqrt{\int_M \| h^{11} \frac{\partial s}{\partial x} + h^{21} \frac{\partial s}{\partial y}) \|^2 \lambda^2 dx \wedge dy}\right) \left(\sqrt{\int_M \| \nabla_{\frac{\partial}{\partial x}} w \|^2 dx \wedge dy}\right)$$

where

$$\|\nabla_{\frac{\partial}{\partial x}} w\|^2 = \Pi(s)\frac{\partial w}{\partial x} \bullet \Pi(s)\frac{\partial w}{\partial x}$$

Now write the right hand side of 3.22 as

$$\sqrt{A_1}\ \sqrt{B_1} + \sqrt{A_2}\ \sqrt{B_2}\ .$$

Using the fact that

$$\sqrt{A_1}\ \sqrt{B_1} + \sqrt{A_2}\ \sqrt{B_2} \le \sqrt{(A_1+A_2)(B_1+B_2)}$$

and that $h^{11} = -h^{22}$,
we see that I is less than or equal to

$$\left(\sqrt{\int_M ((h^{11})^2 + (h^{12})^2)(\|\tfrac{\partial s}{\partial x})\|^2 + \|\tfrac{\partial s}{\partial y})\|^2)\lambda^2 dx\wedge dy}\right)\left(\sqrt{\int_M (\|\nabla_{\frac{\partial}{\partial x}} w\|^2 + \|\nabla_{\frac{\partial}{\partial y}} w\|^2 dx\wedge dy}\right)$$

Since $h^{ij} = \frac{1}{\lambda^2}h_{ij}$ this is equal to

$$\left(\sqrt{\frac{1}{2}\sum_{\ell}\int_M (h\cdot h)\ \ g(s)(\nabla_g s^{\ell},\nabla_g s^{\ell})d\mu_g}\right)\ \sqrt{\int_M (\|\nabla_{\frac{\partial}{\partial x}} w\|^2 + \|\nabla_{\frac{\partial}{\partial y}} w\|^2 dx\wedge dy)}$$

If $w \neq 0$ this is strictly less than

$$\sqrt{\sum_{\ell}\int_M (h\cdot h)\ \ g(s)(\nabla_g s^{\ell},\nabla_g s^{\ell})d\mu_g}\ \ \sqrt{I}$$

or

$$\sqrt{I} < \sqrt{\sum_{\ell}\frac{1}{2}\int_M (h\cdot h)\ \ g(s)(\nabla_g s^{\ell},\nabla_g s^{\ell})d\mu_g}$$

whence

$$(3.23) \qquad I < \frac{1}{2} \sum_{\ell} \int_{M} (h \cdot h)\; g(s)(\nabla_g s^{\ell}, \nabla_g s^{\ell}) d\mu_g$$

If $w = 0$ the inequality 3.23 clearly holds. This establishes 3.19 and thus 3.1, the holomorphic convexity of Dirichlet's energy.

REFERENCES

[1] BERS, L. and EHRENPREIS L., Holomorphic conxitiy of Teichmüller Spaces. Bull AMS (70) 1964 pp. 761-764.

[4] EARLE, C. and EELLS, J., Deformations of Riemannian Surfaces, Lecture Notes in Mathematics 103 Springer-Verlag.

[3] EELLS, J. and SAMPSON, J.H., Harmonic mappings of Riemannian manifolds, Amer. J. Math. 86 (1964) 109-160.

[4] FISCHER, A.E. and TROMBA, A.J., On a purely Riemannian proof of the structure and dimension of the unramified moduli space of a compact Riemann surface. Math. Ann. 267 (1984) 311-345.

[5] FISCHER, A.E. and TROMBA, A.J., On the Weil-Petersson metric on Teichmüller space, Trans. AMS 284 (1984), 319-335.

[6] FISCHER, A.E. and TROMBA, A.J., Almost complex principle bundles and the complex structure on Teichmüller space, Crelles J. Band 252, pp. 151-160 (1984).

[7] FISCHER, A.E. and TROMBA, A.J., A new proof that Teichmüller space is a cell, Trans. AMSS (to appear).

[8] TROMBA, A.J., On a natural algebraic affine connection on the space of almost complex structures and the curvature of Teichmüller space with respect to its Weil-Petersson metric, Manuscripta Math. vol. 56, Fas. 4, 475-497 (1986).

[9] TROMBA, A.J., On an enery function for the Weil-Petersson metric; Manuscripta Math. (to appear).

[10] SHOEN, R. and YAU, S.T., On univalent harmonic maps between surfaces, Inventiones mathematics 44, 265-278 (1978).