On infinitesimal deformations of rational surface singularities

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SFB/MPI 85-35

0. Introduction

This paper is concerned with the computation of the space T^1_X of first order infinitesimal deformations of a two-dimensional rational singularity $(X,0)$. For cyclic resp. dihedral quotient singularities the dimension of this space was determined in [Rie],[Pi 1] resp. [Be, Rie]. In these cases one obtains the formula

$$0.1. \qquad \dim T^1_X = \dim T^1_{\tilde{X}} + \mathrm{emb}(X) - 4$$

unless X is a rational double point. Here $\mathrm{emb}(X)$ denotes the embedding dimension of X, $\pi: \tilde{X} \to X$ the minimal resolution of X, and $T^1_{\tilde{X}} \cong H^1(\tilde{X}, \Theta_{\tilde{X}})$ is the space of first order infinitesimal deformations of $\tilde{X}$. The data at the right hand side of (0.1.) can for many rational surface singularities be computed in terms of the resolution graph (see e.g. [Ar], Cor. 6 and [La]).

For arbitrary two-dimensional quotient singularities a (computer aided) proof of (0.1.) was recently given by K. Behnke, C. Kahn and O. Riemenschneider, using methods of invariant theory ([Ka], [Be, Ka, Rie]).

On the other hand J. Wahl had found an example of a (non Gorenstein) rational surface singularity for which $\dim T^1_X > \dim T^1_{\tilde{X}} + \mathrm{emb}(X) - 4$ (see [Be, Rie], p. 4 and example 4.21. below). In a letter he also gave a proof of the inequality

$$0.2. \qquad \dim T^1_X \geq \dim T^1_{\tilde{X}} + \mathrm{emb}(X) - 4$$

for all rational surface singularities. We give his proof in an appendix to our paper.

In this article we prove 0.1. for a large class of two-dimensional rational singularities (see Theorem 4.10. below). We briefly sketch the method applied.

From Schlessinger's description of T^1_X (cf. [Schl] or Theorem 1.1. below) one concludes by local duality that the dual space $(T^1_X)^*$ can be computed as follows:
Let $i: X \hookrightarrow \mathbb{C}^n$ be a closed embedding of a Stein representative, and let Ω^1_X and $\Omega^1_{\mathbb{C}^n}$ be the sheaves of Kähler differentials and ω_X the canonical sheaf. By $X' = X - \{0\}$ we denote the smooth part of X. Then $(T^1_X)^*$ is isomorphic to the cokernel of the natural map

$$H^0(X', i^*\Omega^1_{\mathbb{C}^n} \otimes \omega_X) \xrightarrow{(\mu' \otimes 1)} H^0(X', \Omega^1_X \otimes \omega_X),$$

induced by the epimorphism $\mu': i^*\Omega^1_{\mathbb{C}^n} \to \Omega^1_X$.

Let $f_1, \ldots, f_n$ be a system of generators for the maximal ideal of $\mathcal{O}_{X,0}$. For a suitable trivialization $i^*\Omega^1_{\mathbb{C}^n} \cong \mathcal{O}^n_X$ $\mu': \mathcal{O}^n_X \to \Omega^1_X$ is defined by $\mu'(g_1, \ldots, g_n) = g_1 df_1 + \ldots + g_n df_n$. This map can be studied using the resolution $\pi: \tilde{X} \to X$. Let $E = \pi^{-1}(0)$ be the exceptional set, Z the fundamental cycle, and let $\Omega^1_X\langle\log E\rangle$ be the sheaf of meromorphic 1-forms with at most logarithmic poles along E. As above we have a map

$$\mu : \mathcal{O}_{\tilde{X}}^{\oplus n} \longrightarrow \Omega^1_{\tilde{X}}\langle \log E\rangle(-Z),$$

$(g_1,\ldots,g_n) \to g_1 df_1 + \ldots + g_n df_n$, where now $f_1,\ldots,f_n$ are considered as holomorphic functions on $\tilde{X}$. As $(X,0)$ is a rational singularity and ω_X is reflexive there is a natural isomorphism between $H^0(X', \mathcal{O}_{X'}^{\oplus n} \otimes \omega_{X'})$ and $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}^{\oplus n} \otimes \omega_{\tilde{X}})$. Using this isomorphism one sees that T^1_X is dual to the cokernel of the following composite map

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}^{\oplus n} \otimes \omega_X) \xrightarrow{(\mu \otimes 1)^*} H^0(\tilde{X}, \Omega^1_{\tilde{X}}\langle \log E\rangle(-Z)) \hookrightarrow H^0(X', \Omega^1_{X'} \otimes \omega_{X'}).$$

The cokernel of the inclusion $H^0(\tilde{X}, \Omega^1_{\tilde{X}}\langle \log E\rangle(-Z)) \hookrightarrow H^0(X', \Omega^1_{X'} \otimes \omega_{X'})$ can be computed using results of J. Wahl [Wa, 1] (see Ch. 2). For the discussion of $(\mu \otimes 1)^*$ we have to make more restrictive assumptions (e.g. that the fundamental cycle is reduced) in order to be able to control the kernel and the cokernel of μ. This discussion is performed in Ch. 3 and Ch. 4 and leads to the proof of 0.1. for a large class of rational surface singularities. The precise results are stated in Theorem 4.8. and example 4.13.

We want to thank Jonathan Wahl for letting us include his proof of the inequality 0.2 in this article.

1. Schlessinger's description of T_X^1 and duality

Let (X,o) be a normal surface singularity. We recall a result of M. Schlessinger [Schl] which gives a cohomological description of the space T_X^1 of infinitesimal deformations of X. Then we apply duality to obtain the description of $(T_X^1)^*$ which is basic for our paper.

Let $i: X \hookrightarrow \mathbb{C}^n$ be an embedding of a small Stein space representing the singularity (X,O). Denote by $X' = X - \{O\}$ the smooth part of X, by $\Omega^1_{\mathbb{C}^n}$ resp. Ω^1_X the sheaves of Kähler differentials on $\mathbb{C}^n$ resp. X, and by $\Theta_{\mathbb{C}^n}$ resp. Θ_X their duals.

Theorem 1.1 (Schlessinger [Schl] §1, Lemma 2):

The module T_X^1 of first order infinitesimal deformations of (X,O) is the kernel of the map

$$H^1(X',\Theta_{X'}) \to H^1(X',\Theta_{\mathbb{C}^n}\big|_{X'})$$

which is induced by the natural inclusion of tangent sheaves $\Theta_{X'} \hookrightarrow \Theta_{\mathbb{C}^n}|_{X'}$.

To apply local duality we remark that $H^1(X',\Theta_{X'})$ is canonically isomorphic to $H^2_{\{O\}}(X,\Theta_X)$, the second local cohomology group with support in the singular point O. Similarly $H^1(X',\Theta_{\mathbb{C}^n}\big|_{X'})$ is canonically isomorphic to $H^2_{\{O\}}(X,\Theta_{\mathbb{C}^n}|_X)$. Then we see by local duality that T^1_X is dual to the cokernel of

$$\mathrm{Hom}_{\mathcal{O}_X}(\Theta_{\mathbb{C}^n}|_X, \omega_X) \to \mathrm{Hom}_X(\Theta_X, \omega_X).$$

As all sheaves are reflexive we finally get

<u>Corollary 1.2</u>:

$(T^1_X)^*$ is isomorphic to the cokernel of the map

$$H^0(X', \Omega^1_{\mathbb{C}^n}|_{X'} \otimes \omega_{X'}) \to H^0(X', \Omega^1_{X'} \otimes \omega_{X'})$$

induced by the restriction map $\Omega^1_{\mathbb{C}^n} \otimes \mathcal{O}_X \to \Omega^1_X$.

<u>Remark 1.3</u>:

We can make this result a little more explicit: Observe that the restriction $\Omega^1_{\mathbb{C}^n|X}$ is generated as an $\mathcal{O}_X$-module by the differentials $df_1, \ldots, df_n$ of the coordinate functions f_i on $\mathbb{C}^n$. Equivalently we can take for $f_1, \ldots, f_n$ any set of generators for the maximal ideal of $\mathcal{O}_{X,0}$. Let $\mu': \mathcal{O}^n_X \to \Omega^1_X$ be the surjection defined by $\mu'(g_1, \ldots, g_n) = \sum_{i=1}^n g_i df_i$. Then $(T^1_X)^*$ is isomorphic to the cokernel of the map

$$\mu' \otimes 1: H^0(X', \omega^{\oplus n}_{X'}) \to H^0(X', \Omega^1_{X'} \otimes \omega_{X'}).$$

In an invariant way the image of $(\mu' \otimes 1)$ can be characterized as the subspace of $H^0(X', \Omega^1_{X'} \otimes \omega_{X'})$ generated by all elements of the form $\Sigma g_i \otimes dh_i$, $g_i \in H^0(X', \omega_{X'})$, $h_i \in H^0(X', \mathcal{O}_{X'})$.

2. The case of rational singularities

We keep our previous hypotheses and assume moreover that X is a rational singularity. Let $\pi: \tilde{X} \to X$ be the minimal good resolution of X, and let $E = \pi^{-1}(0)$ be the exceptional set. The irreducible components $E_1,\dots,E_r$ of E are nonsingular rational curves of selfintersection number $-b_i = E_i \cdot E_i \leq -2$.

Let Ω^1_X resp. ω_X be the sheaves of holomorphic 1- resp. 2-forms on $\tilde{X}$. Observe that by rationality $H^0(\tilde{X},\omega_{\tilde{X}}) \cong H^0(X',\omega_{X'})$ (see e.g. [Pi, 2], §15). We denote the pull backs to $\tilde{X}$ of the functions f_i of Remark 1.3 also by f_i. Their differentials are sections of $\Omega^1_{\tilde{X}}\langle\log E\rangle(-Z)$, where $\Omega^1_X\langle\log E\rangle$ denotes the sheaf of meromorphic 1-forms on $\tilde{X}$ with logarithmic poles along E and Z is the fundamental cycle of $\tilde{X}$. Again we define a sheaf map

$$\mu: \mathcal{O}_{\tilde{X}}^{\oplus n} \to \Omega^1_{\tilde{X}}\langle\log E\rangle(-Z)$$

by $\mu(g_1,\dots,g_n) = \sum_{i=1}^{n} g_i df_i$. This induces a map

$$(\mu \otimes 1)^*: H^0(\tilde{X},\omega_{\tilde{X}}^{\oplus n}) \to H^0(X,\Omega^1_{\tilde{X}}\langle\log E\rangle(-Z) \otimes \omega_{\tilde{X}}).$$

Let ρ be the inclusion

$$\rho: H^0(X,\Omega^1_{X}\langle\log E\rangle(-Z) \otimes \omega_{X}) \to H^0(X',\Omega^1_{X'} \otimes \omega_{X'}).$$

By Remark 1.3 we get

Lemma 2.1:

$(T^1_X)^*$ is isomorphic to the cokernel of the composite map

$$\rho\cdot(\mu\otimes 1)^*:\ H^0(\tilde{X},\omega_{\tilde{X}}^{\oplus n}) \to H^0(\tilde{X},\Omega^1_{X'}\otimes\omega_{X'}).$$

The main result of this section is

Proposition 2.2:

The cokernel of the inclusion map $\rho:\ H^0(\tilde{X},\Omega^1_{\tilde{X}}\text{<log E>}(-Z)\otimes\omega_{\tilde{X}}) \to H^0(X',\Omega^1_{X'}\otimes\omega_{X'})$ has dimension

$$\dim H^1(\tilde{X},\Theta_{\tilde{X}}) + \sum_{i=1}^{r}(b_i-3) + \dim H^0(|Z-E|,\Omega^1_{\tilde{X}}\text{<log E>}(-E)\otimes \otimes\omega_{\tilde{X}}\otimes\mathcal{O}_{Z-E})$$

As ρ is injection this - together with Lemma 2.1 - implies

Corollary 2.3:

Let $\mathcal{C}$ resp. R be the kernel resp. cokernel of $\mu:\mathcal{O}_{\tilde{X}}^{\oplus n} \to \Omega^1_{\tilde{X}}\text{<log E>}(-Z)$. Then

$$\dim T^1_X = \dim T^1_{\tilde{X}} + \sum_{i=1}^{r}(b_i-3) + \dim H^0(\tilde{X},\mathcal{C}\otimes\omega_{\tilde{X}}) + \dim (H^1(\tilde{X},R\otimes\omega_{\tilde{X}}) + \dim H^0(|Z-E|,\Omega^1_{\tilde{X}}\text{<log E>}(-E)\otimes\omega_{\tilde{X}}\otimes\mathcal{O}_{Z-E})$$

Remark 2.4:

We are mainly interested in the case that the fundamental cycle is reduced (i.e. Z = E), in this case the last term in formula 2.2 resp. 2.3 does not appear.

For the proof of Proposition 2.2 we factor ρ in the sequence of inclusions

$$H^0(\tilde{X},\Omega^1_{\tilde{X}}\langle\log E\rangle(-Z)\otimes\omega_{\tilde{X}})\hookrightarrow H^0(\tilde{X},\Omega^1_{\tilde{X}}\langle\log E\rangle(-E)\otimes\omega_{\tilde{X}})\hookrightarrow$$

(2.5)

$$\hookrightarrow H^0(X,\Omega^1_{\tilde{X}}\otimes\omega_{\tilde{X}})\hookrightarrow H^0(X',\Omega^1_{X'}\otimes\omega_{X'}).$$

To study these inclusions we use the well-known exact sequences

(2.6) $$0\to\Omega^1_{\tilde{X}}\to\Omega^1_{\tilde{X}}\langle\log E\rangle\to\bigoplus_{i=0}^{r}\mathcal{O}_{E_i}\to 0$$

(2.7) $$0\to\Omega^1_{\tilde{X}}\langle\log E\rangle(-E)\to\Omega^1_{\tilde{X}}\to\bigoplus_{i=1}^{r}\omega_{E_i}\to 0$$

(where ω_{E_i} is the canonical sheaf of the curve E_i) and the following vanishing result, which is derived from [Wa, 1], Theorem C,D by applying Serre duality:

<u>Theorem 2.8</u> (J. Wahl):

(i) Let $\tilde{X}$ be the minimal good resolution of a normal surface singularity X. Then $H^1(\tilde{X},\Omega^1_{\tilde{X}}\langle\log E\rangle\otimes\omega_{\tilde{X}})=0$

(ii) If X is a rational singularity then also $H^1(\tilde{X},\Omega^1_{\tilde{X}}\langle\log E\rangle(-E)\otimes\omega_{\tilde{X}})=0$.

Proposition 2.2 is a direct consequence of (2.5) and

<u>Lemma 2.9</u>:

(i) The cokernel of the inclusion $H^0(\tilde{X},\Omega^1_{\tilde{X}}\otimes\omega_{X})\hookrightarrow H^0(X',\Omega^1_{X'}\otimes\omega_{X'})$ has dimension $\dim H^1(\tilde{X},\Theta_{\tilde{X}})-\dim H^1(\tilde{X},\Omega^1_{\tilde{X}}\otimes\omega_{\tilde{X}})$

(ii) The cokernel of the inclusion $H^0(\tilde{X},\Omega^1_{\tilde{X}}\langle\log E\rangle(-E)\otimes\omega_{\tilde{X}})\hookrightarrow$ $\hookrightarrow H^0(\tilde{X},\Omega^1_{\tilde{X}}\otimes\omega_{\tilde{X}})$ has dimension $\dim H^1(\tilde{X},\Omega^1_{\tilde{X}}\otimes\omega_{\tilde{X}})+\sum_{i=1}(b_i-3)$

(iii) The cokernel of the inclusion $H^0(\tilde{X},\Omega^1_{\tilde{X}}<\log E>(-Z)\otimes \omega_{\tilde{X}}) \hookrightarrow$

$\to H^0(\tilde{X}, \Omega^1_{\tilde{X}} <\log E> (-E) \otimes \omega_{\tilde{X}})$ has dimension

$$\dim H^0(|Z-E|, \Omega^1_{\tilde{X}} < \log E > (-E) \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{Z-E})$$

<u>Proof:</u> (i) The long exact sequence for local cohomology gives

$$0 \to H^0(\tilde{X}, \Omega^1_{\tilde{X}} \otimes \omega_{\tilde{X}}) \to H^0(X', \Omega^1_{X'} \otimes \omega_{X'}) \to H^1_E(\tilde{X}, \Omega^1_{\tilde{X}} \otimes \omega_{\tilde{X}}) \to$$
$$\to H^1(\tilde{X}, \Omega^1_{\tilde{X}} \otimes \omega_{\tilde{X}}) \xrightarrow{(*)} H^1(X',\Omega^1_{X'} \otimes \omega_{X'}) \to \dots$$

We claim that the map $(*)$ is zero. To prove this consider the commutative diagramm

$$\begin{array}{ccc} H^1(\tilde{X}, \Omega^1_{\tilde{X}} <\log E >\otimes \omega_{\tilde{X}}) & \longrightarrow & H^1(X', \Omega^1_{X'} < \log E > \otimes \omega_{X'}) \\ \uparrow & & \uparrow \cong \\ H^1(\tilde{X}, \Omega^1_{\tilde{X}} \otimes \omega_{\tilde{X}}) & \xrightarrow{(*)} & H^1(X', \Omega^1_{X'} \otimes \omega_{X'}) \end{array}$$

By Theorem 2.5 the cohomology group in the upper left hand corner vanishes. The statement now follows from the fact that $H^1_E(\tilde{X},\Omega^1_{\tilde{X}} \otimes \omega_{\tilde{X}})$ and $H^1(\tilde{X},\Theta_{\tilde{X}})$ are dual.

(ii) From (2.7) and Theorem 2.8.ii we get the exact sequence

$$0 \to H^0(\tilde{X},\Omega^1_{\tilde{X}}<\log E>(-E) \otimes \omega_{\tilde{X}}) \to H^0(\tilde{X}, \Omega^1_{\tilde{X}} \otimes \omega_{\tilde{X}}) \to \bigoplus_{i=1}^{r} H^0(E_i,\omega_{\tilde{X}} \otimes \omega_{E_i}) \to 0$$

and an isomorphism

$$H^1(\tilde{X}, \Omega^1_{\tilde{X}} \otimes \omega_{\tilde{X}}) \cong \bigoplus_{i=1}^{r} H^1(E_i,\omega_{\tilde{X}} \otimes \omega_{E_i}) .$$

By the acjunction formula we see that $\omega_{\tilde{X}} \otimes \omega_{E_i}$ has degree b_i-4 on E_i, hence

$$\sum_{i=1}^{r} \dim H^0(E_i,\omega_{\tilde{X}} \otimes \omega_{E_i}) - \sum_{i=1}^{r} \dim H^1(E_i,\omega_{\tilde{X}} \otimes \omega_{E_i}) = \sum_{i=1}^{r} (b_i-3) .$$

This proves part (ii) of the Lemma.

(iii) follows from the exact sequence

$$0 \to \Omega^1_{\tilde{X}} < \log E > (-Z) \to \Omega^1_{\tilde{X}} < \log E > (-E) \to \Omega^1_{\tilde{X}} < \log E > (-E) \otimes \mathcal{O}_{Z-E} \to 0$$

and the vanishing of $H^1(\tilde{X}, \Omega^1_{\tilde{X}} < \log E > (-Z) \otimes \omega_{\tilde{X}})$ (which is easy, since $\mathcal{O}_{\tilde{X}}(-Z)$ is generated by global sections).

3. Computation of $H^o(\tilde{X}, \mathcal{C} \otimes \omega_{\tilde{X}})$.

Recall that $\mathcal{C}$ was defined as the cokernel of

$$\mu : \mathcal{O}_{\tilde{X}}^{\oplus n} \longrightarrow \Omega^1_{\tilde{X}} < \log E > (-Z)$$

Let $\mathcal{F}$ be the image of μ . Then $\mathcal{F}$ is a torsion free sheaf, and $\mathcal{C}$ is concentrated on E.
In this section we assume that the fundamental cycle is reduced (i.e. Z = E) and that it meets every irreducible component of E - except possibly (-2)-curves-strictly negatively.

In order to compute $\mathcal{C}$ we will construct holomorphic functions on $\tilde{X}$ with prescribed divisors. We use the following observation of M.Artin ([Ar], proof of Theorem 4):

Lemma 3.1.:

Let $\pi : \tilde{X} \to X$ be the minimal good resolution of a rational surface singularity. Let D be an effective divisor on $\tilde{X}$ such that $D \cdot E_i = 0$ for every irreducible component E_i of E. Then there is an open neighbourhood U of E in $\tilde{X}$ and a holomorphic function f on U such that $(f) = D \cap U$.

Corollary 3.2.:

Let $\tilde{X}$ be the minimal resolution of a rational surface singularity with reduced fundamental cycle E. Let $E = E'+E''$ be a decomposition into effective divisors with connected E'. Denote by F the sum of irreducible components of E' which meet E", and write $E' = E'_o + F$. Let D' be an effective divisor with support in E', and let Δ be an effective divisor on a small neighbourhood U of E'_o which has no components in common with E. Put $D := D' + \Delta$. Suppose that

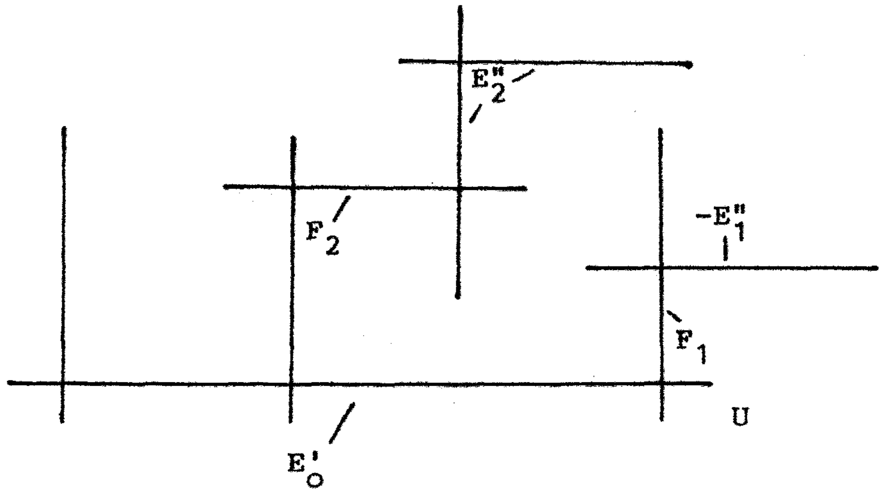
(i) $D \cdot E_i = 0$ for all components E_i of E'_o

(ii) the multiplicity of a component E_i of F in D is greater or equal to

$$\frac{D \cdot E_i}{b_i - E_i \cdot E''}$$

Then there exists a holomorphic function f on $\tilde{X}$ such that $(f) \cap U = D$.

<u>Proof:</u> Let $E_1'',\dots,E_k''$ be the connected components of E''; let F_i be the component of F meeting E_i'' , and let m_i be its multiplicity in D.



<u>Fig.1</u>

We put $C := D + \sum_{i=1}^{k} m_i E_i''$. Since $E_i'' + F_i$ is the exceptional set of a rational singularity with reduced fundamental cycle it follows from (ii) that $C \cdot E_i \le 0$ for all irreducible components of E. Obviously $C \cdot E_i = 0$ for all E_i contained in U, so we can modify C outside U to obtain an effective divisor $\tilde{C}$ with $\tilde{C} \cdot E_i = 0$ for $i = 0,\dots,k$. Applying (3.1) to $\tilde{C}$ we obtain the desired function f.

The next two lemmata give our description of $\mathcal{C}$. First we investigate $\mathcal{C}$ near curves with "high self-intersection number". Recall that we assume Z = E throughout this chapter.

<u>Lemma 3.3:</u>

(i) Let p be a smooth point of E, and assume that $E \cdot E_i < 0$ for the unique irreducible component E_i of E containing p. Then $\mathcal{C}_p = 0$.

(ii) Let p be a point where two components E_i, E_j of E intersect, and assume that $E \cdot E_i < 0$, $E \cdot E_j < 0$. Then $\mathcal{C}$ is a skyskraper sheaf near p and $\dim \mathcal{C}_p = 1$.

Proof:

(i) Let (u,v) be a holomorphic coordinate system near p such that $E_i = \{v = 0\}$ locally. Then $\Omega^1_{\tilde{X}} \langle \log E \rangle (-E)$ is generated by dv and $v \cdot du$ locally.

As the fundamental cycle is reduced, there is a global holomorphic function f_1 on $\tilde{X}$ which gives a local equation for E, i.e. $f_1 = \varepsilon_1 \cdot v$ for a unit ε_1. Let Δ_o be the curve $\{u = 0\}$. As $E \cdot E_i < 0$ we can choose other curves $\Delta_1, \ldots, \Delta_\ell$ which are disjoint from Δ_o and such that $E + \sum_{i=o}^{\ell} \Delta_i$ intersects $E_o, \ldots, E_r$ trivially. By Lemma 3.1 there is a holomorphic function f_2 on $\tilde{X}$ with divisor $E + \sum_{i=o}^{\ell} \Delta_i$. Locally near p the function f_2 is of the form $f_2 = \varepsilon_2 \cdot uv$ for another unit ε_2. Obviously df_1 and df_2 generate $\Omega^1_{\tilde{X}} \langle \log E \rangle (-E)$ near p.

(ii) We proceed as before and choose smooth curves Δ_1, Δ_2 through p such that E_1, E_2, Δ_1, Δ_2 are pairwise transversal in p. There are local coordinates u,v with $E_i = \{v = 0\}$, $E_j = \{u = 0\}$, and holomorphic functions f, g_1, g_2 on $\tilde{X}$ such that $f = uv$, $g_k = uv(a_k u + b_k v + \text{higher order terms})$ with $a_1 : b_1 \neq a_2 : b_2$. Locally at p the sheaf $\Omega^1_{\tilde{X}} \langle \log E \rangle (-E)$ is generated by vdu and udv, while $\mathcal{F}$ is generated by df, dg_1, dg_2. A simple calculation now shows that $\dim \mathcal{C}_p = 1$.

It remains to see how $\mathcal{C}$ looks like on a linear chain of curves of self-intersection (-2) which have intersection number 0 with E. So let $E_o, \ldots, E_{t+1}$ be irreducible components of E such that

$$E_1 \cdot E_1 = \dots = E_t \cdot E_t = -2$$

$$E_o \cdot E_1 = E_1 \cdot E_2 = \dots = E_t \cdot E_{t+1} = 1$$

E_i meets no other component of E but E_{i-1} and E_{i+1} $(i = 1,\dots,t)$ $E \cdot E_o < 0$ and $E \cdot E_{t+1} < 0$.

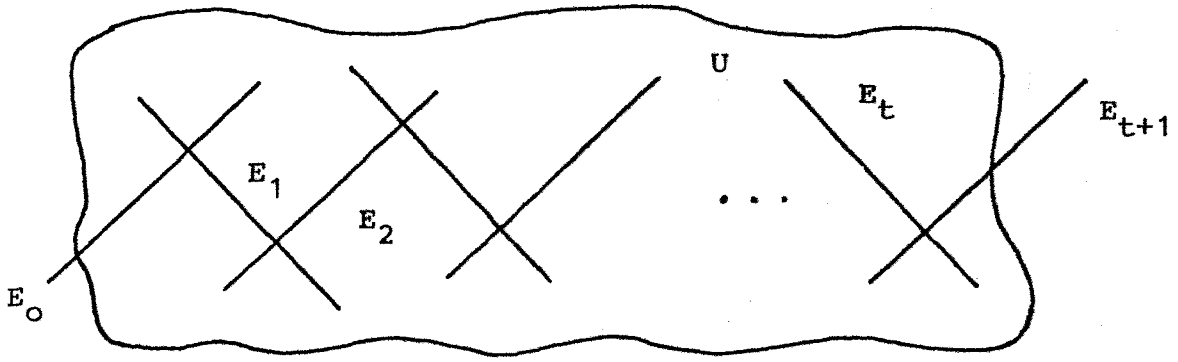


Fig.2

Let U be a small neighbourhood of $E_1 \cup \dots \cup E_t$. Since $E \cap U$ intersects $E_1,\dots,E_t$ trivially, E is a principal divisor on U (cf.[Ar]). The ideal sheaf $J_E|_U$ is generated by a single holomorphic function, say f_1. It vanishes to first order along $E \cap U$.

Blowing down $E_1 \cup \dots \cup E_t$ yields a rational double point A_t. So f_1 can be extended to a minimal set f_1, f_2, f_3 of generators of the algebra of holomorphic functions on U. It is well-known that f_2 and f_3 can be chosen such that $f_1^{t+1} = f_2 f_3$ and such that they have the divisors

$$(f_2) = \sum_{i=1}^{t} i \cdot E_i + (t+1)(E_{t+1} \cap U)$$

$$(f_3) = \sum_{i=1}^{t} (t-i+1) \cdot E_i + (t+1)(E_o \cap U)$$

Remark 3.4.:

$\mathcal{F}|_U$ is generated by df_1, $d(f_1f_2)$, $d(f_1f_3)$.

Proof: By Corollary 3.2 we see that f_1, f_1f_2, f_1f_3 can be chosen as restrictions of holomorphic functions on $\tilde{X}$. Conversely any holomorphic function on U which vanishes along $E \cap U$ is of the form $h \cdot f_1$, where h is in the ideal generated by f_1, f_2, f_3.

We put

$$D := \sum_{i=1}^{t} \max(i, t-i+1)\, E_i$$

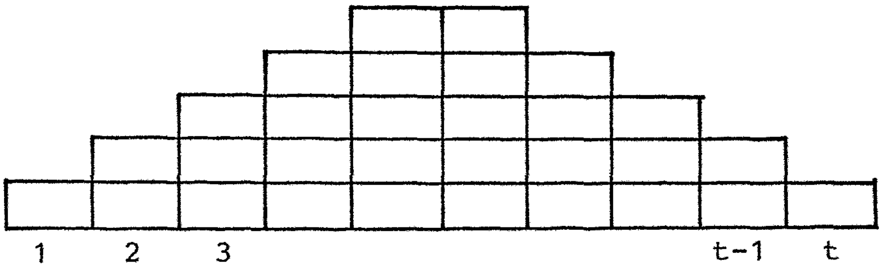


Fig.3

Lemma 3.5.:

(i) If t is odd, then $\mathcal{C}|_U \cong \mathcal{O}_D$

(ii) If t is even, say $t = 2k$, then $\mathcal{C}|_U$ has a torsion subsheaf τ of length 1, concentrated at $E_k \cap E_{k+1}$, and there is an exact sequence

$$0 \to \tau \to \mathcal{C}|_U \to \mathcal{O}_D \to 0$$

Proof: One easily checks that $\Omega^1_{\tilde{X}} < \log E > (-E)|_U$ is free with generators $f_1 \frac{df_2}{f_2}$ and $f_1 \frac{df_3}{f_3}$. Since $(t+1)df_1 = f_1 \frac{df_2}{f_2} + f_1 \frac{df_3}{f_3}$,

we see that $\mathcal{C}|_U$ is cyclic with generator $f_1 \frac{df_2}{f_2} =$
$= -f_1 \frac{df_3}{f_3}$. The claim now follows from (3.4) by a simple calculation in local coordinates.

For later use we note

Lemma 3.6.:

$$\dim H^0(|D|, \mathcal{O}_D) = [\tfrac{t+1}{2}]$$

Proof: For an effective cycle C supported on the exceptional locus of a rational surface singularity one has $H^1(|C|, \mathcal{O}_C) = 0$ (cf. [Ar]). So it is sufficient to compute the holomorphic Eulercharacteristic $\chi(\mathcal{O}_D)$ of $\mathcal{O}_D$.

Consider the sequence of divisors

$$D_1 = E_1 \;,\; D_2 = E_1 + E_2, \ldots \;\ldots \;,\; D_{t-1} = E_1 + \ldots + E_{t-1}, D_t = E_1 + \ldots + E_t$$
$$D_{t+1} = E_1 + 2E_2, \ldots \;\ldots, \; D_{2t-2} = 2E_1 + \ldots + 2E_{t-1}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

ending with D (cf. fig.3). Let E_{i_ℓ} be the curve which is added to D_ℓ to obtain $D_{\ell+1}$. Then the intersection number $D_\ell \cdot E_{i_\ell}$ is 1, if E_{i_ℓ} does not start a new row, and it is 0 otherwise.

From the exact sequence

$$0 \to \mathcal{O}_{E_{i_\ell}}(-D_\ell) \to \mathcal{O}_{D_{\ell+1}} \to \mathcal{O}_{D_\ell} \to 0$$

we obtain $\chi(\mathcal{O}_{D_{\ell+1}}) = \chi(\mathcal{O}_{D_\ell}) + (1 - D_\ell \cdot E_{i_\ell})$. So

$$\chi(\mathcal{O}_D) = \sum_\ell (1 - D_\ell \cdot E_{i_\ell}) \quad .$$

By the discussion above this sum has precisely $[\frac{t+1}{2}]$ summands 1, and all other summands are zero.

4. Computation of $H^1(\tilde{X}, R \otimes \omega_{\tilde{X}})$

The most difficult part in formula (2.3) for dim T_X^1 seems to be $H^1(\tilde{X}, R \otimes \omega_{\tilde{X}})$. Recall that we have the exact sequence

$$0 \to R \to \mathcal{O}_{\tilde{X}}^{\oplus n} \xrightarrow{\mu} \Omega_{\tilde{X}}^1 < \log E > (-Z) \to \mathcal{C} \to 0 ,$$

so by Hilbert's syzygy theorem R is locally free of rank n-2.

We will apply the results of Chapter 3, so we assume again that the fundamental cycle is reduced and meets all non-(-2)-curves strictly negatively. In other words: If an irreducible component E_i of E meets t_i other curves, then its self-intersection number $-b_i$ fulfills

$$b_i \geq t_i \quad \text{for} \quad i=1,\dots,r$$

$$b_i \geq t_i+1 \text{ if } \quad b_i \neq 2 \qquad .$$

The restriction of the locally free sheaf R to E_i is a direct sum of line bundles (cf. [GR]VII, Satz 5). We now give estimates for the degrees of these bundles.

Proposition 4.1:

Let E_i be an irreducible component of E.

(i) If $b_i \geq t_i+2$, then $R \otimes \mathcal{O}_{E_i}$ decomposes into line bundles of degree at least -2 .

(ii) If $b_i = t_i+1$, then all direct summands of $R \otimes \mathcal{O}_{E_i}$ have degree at least -1 .

(iii) If $b_i = 2$, $t_i = 1$ and E_i meets a (-2)-curve, then $R \otimes \mathcal{O}_{E_i}$ is trivial.

Proof: Consider the exact sequences

$$0 \to R \to \mathcal{O}_{\tilde{X}}^{\oplus n} \xrightarrow{\mu} \mathcal{F} \to 0$$

$$0 \to \mathcal{F} \to \Omega^1_{\tilde{X}} < \log E > (-E) \to \mathcal{C} \to 0$$

The first one remains exact, when restricted to E_i:

$$(4.2) \quad 0 \to R \otimes \mathcal{O}_{E_i} \to \mathcal{O}_{E_i}^{\oplus n} \xrightarrow{\mu} \mathcal{F} \otimes \mathcal{O}_{E_i} \to 0$$

But $\mathcal{F} \otimes \mathcal{O}_{E_i}$ is no longer torsion free, the second sequence gives

$$(4.3) \quad 0 \to \underset{\sim}{\mathrm{Tor}}_1^{\mathcal{O}_{\tilde{X}}} (\mathcal{C}, \mathcal{O}_{E_i}) \to \mathcal{F} \otimes \mathcal{O}_{E_i} \to \Omega^1_{\tilde{X}} < \log E > (-E) \otimes$$
$$\otimes \mathcal{O}_{E_i} \to \mathcal{C} \otimes \mathcal{O}_{E_i} \to 0 .$$

So the torsion subsheaf of $\mathcal{F} \otimes \mathcal{O}_{E_i}$ is concentrated in the points , where $\mathcal{C}$ is a skyscraper sheaf, and it has length 1 there (cf.(3.3) and (3.5)).

First we prove (iii) : In this case $\mathrm{Tor}_1^{\mathcal{O}_{\tilde{X}}} (\mathcal{C}, \mathcal{O}_{E_i}) = 0$, while $\mathcal{C} \otimes \mathcal{O}_{E_i}$ is a skyscraper sheaf of length 1 (see Lemma 3.5). Hence by (4.2) the Chern class of $\mathcal{F} \otimes \mathcal{O}_{E_i}$ is zero. By (4.3) we see that $R \otimes \mathcal{O}_{E_i}$ has Chern class zero. But a subsheaf of $\mathcal{O}_{E_i}^{\oplus n}$ has trivial Chern class, if and only if it is trivial.

We now concentrate on (i) and (ii). If we want to show that $R \otimes \mathcal{O}_{E_i}$ splits into direct summands of degree at least -1, it is sufficient to show the surjectivity of $H^0(E_i, \mathcal{O}_{E_i}^{\oplus n}) \xrightarrow{\mu} H^0(E_i, \mathcal{F} \otimes \mathcal{O}_{E_i})$. This follows from the cohomology sequence of (4.2) and the observation that $H^1(E_i, R \otimes \mathcal{O}_{E_i})$ is never zero, if

$R \otimes \mathcal{O}_{E_i}$ has a line bundle summand of degree -2 or less. Similarly for the estimate -2 in (i) it suffices to prove the surjectivity of $H^o(E_i, \mathcal{O}_{E_i}(1)^{\oplus n}) \xrightarrow{\mu} H^o(E_i, \mathcal{F} \otimes \mathcal{O}_{E_i}(1))$.

We will discuss the torsion part and the non-torsion part of $\mathcal{F} \otimes \mathcal{O}_{E_i}$ separately. For the torsion part we use

Lemma 4.4:

Let E_i, E_j be two components of E which meet in a point p and for which $E \cdot E_i < 0$, $E \cdot E_j < 0$. Let f be a holomorphic function on $\tilde{X}$ whose zero divisor contains E_i with multiplicity 2, E_j with multiplicity 1, and no other curve passing through p. Then df represents a generator of the torsion part of $(\mathcal{F} \otimes \mathcal{O}_{E_i})_p$.

Proof: Let (u,v) be local coordinates around p with $E_i = \{v = 0\}$, $E_j = \{u = 0\}$. The computation in the proof of (3.3.ii) shows that locally $\Omega^1_{\tilde{X}} \langle \log E \rangle (-E)$ is generated by vdu and udv, while $\mathcal{F}$ is generated by $vdu+udv$, u^2dv, $uvdv$, $uvdu$, v^2du. So the kernel of the map $\mathcal{F}/v\mathcal{F} \to \Omega^1_{\tilde{X}} \langle \log E \rangle(-E)/$ $/v.\Omega^1_{\tilde{X}} \langle \log E \rangle(-E)$ is generated by $uvdv$.

Corollary 4.5:

Let E_i be a component of E such that $b_i \geq t_i+1$. Then there are holomorphic functions on $\tilde{X}$ which vanish of order 2 along E_i and whose differentials generate the torsion of $\mathcal{F} \otimes \mathcal{O}_{E_i}$.

Proof: For each non-(-2)-curve E_j meeting E_i we find by (3.2) a holomorphic function on $\tilde{X}$ which vanishes of order 2 along E_i and all the curves $E_k \neq E_j$ that meet E_i.

The non-torsion part of $\mathcal{F}\otimes\mathcal{O}_{E_i}$ is the image $\tilde{\mathcal{F}}_i$ of $\mathcal{F}\otimes\mathcal{O}_{E_i}$.
in $\Omega^1_{\tilde{X}}\langle\log E\rangle(-E)\otimes\mathcal{O}_{E_i}$). It is clear that the differential of a holomorphic function on $\tilde{X}$ has a non-vanishing image in $\tilde{\mathcal{F}}_i$ only if it vanishes of order 1 along E_i. In view of (4.5) it suffices for the proof of (i) resp. (ii) to show that the maps $H^0(E_i,\mathcal{O}_{E_i}^{\oplus n}(1)) \xrightarrow{\mu} H^0(E_i,\tilde{\mathcal{F}}_i(1))$ resp. $H^0(E_i,\mathcal{O}_{E_i}^{\oplus n}) \xrightarrow{\mu} H^0(E_i,\tilde{\mathcal{F}}_i)$ are surjective. Before doing this we note

<u>Lemma 4.6:</u>

$\tilde{\mathcal{F}}_i$ has Chern class $2(b_i-t_i)-2$ on E_i, and $H^1(E,\tilde{\mathcal{F}}_i)=0$.

<u>Proof:</u> Observe that $\Omega^1_{\tilde{X}} < \log E > (-E) \otimes \mathcal{O}_{E_i} \cong (\omega_{E_i}(t_i) \oplus \mathcal{O}_{E_i})(-E\cdot E_i)$.

So the claim on the degree of $\tilde{\mathcal{F}}_i$ follows from the exact sequence

$$0 \to \tilde{\mathcal{F}}_i \to \Omega^1_{\tilde{X}} < \log E> (-E) \otimes \mathcal{O}_{E_i} \to \mathcal{C}\otimes\mathcal{O}_{E_i} \to 0 .$$

The sequence (4.2) shows that $H^1(E_o,\mathcal{F}\otimes\mathcal{O}_{E_i}) = 0$, hence also $H^1(E_i,\tilde{\mathcal{F}}_i) = 0$.

We now prove (4.1.i): As mentioned above it suffices to prove the surjectivity of $H^0(E_i,\mathcal{O}_{E_i}^{\oplus n}(1)) \to H^0(E_i,\tilde{\mathcal{F}}_i(1))$. The latter space has dimension $2(b_i-t_i+1)$ by Lemma 4.6. Now choose a small curve Δ transversal to E_i which does not meet any other component of E. By Corollary 3.2 we find for $0 \le k < b_o-t_o$ holomorphic functions f_k on $\tilde{X}$ whose zero divisor contains E_i and all components of E adjacent to E_i with multiplicity 1, and Δ with multiplicity k.
Choose local coordinates (u,v) around the point of $E_i \cap \Delta$ such that $E_i = \{v = 0\}$, $\Delta = \{u = 0\}$. Then $f_k = \varepsilon_k \cdot u^k \cdot v$ with some unit ε_k. So

$$df_k = k \cdot u^{k-1}v\,du + u^k dv + \text{higher terms.}$$

If we take all linear combinations of $df_o, \dots, df_{b_o - t_o}$ with coefficients in $H^o(E_o, \mathcal{O}_{E_o}(1))$ (which means that we allow constants and $\frac{1}{u}$ as coefficients), we get $2(b_o - t_o + 1)$ linearly independent sections of $\tilde{\mathcal{F}}_i(1)$.

Finally we prove (4.1.ii): In this case $\dim H^o(E_i, \tilde{\mathcal{F}}_i) = 2$, and as above one constructs two independent holomorphic functions which vanish of first order along E_i. This shows that $H^o(E_i, \mathcal{O}^{\oplus n}_{E_i}) \xrightarrow{\mu} H^o(E_i, \tilde{\mathcal{F}}_i)$ is surjective.

As in chapter 3 we also have to consider chains of (-2)-curves .

Proposition 4.7:

Let $E_o, E_1, \dots, E_t, E_{t+1}$ be irreducible components of E such that $E_1, \dots, E_t$ from a chain of (-2)-curves, E_o meets E_1, E_{t+1} meets E_t, and there is no intersection of $E_1, \dots, E_t$ with other components. Also assume that $E \cdot E_o < 0$ and $E \cdot E_{t+1} < 0$.

Then on a sufficiently small neighbourhood U of $E_1 \cup \dots \cup E_t$ the vector bundle R splits into a trivial summand of rank $n-3$ and a line bundle $\mathcal{L}$. The restrictions of $\mathcal{L}$ to the irreducible components are

$$\mathcal{L} \otimes \mathcal{O}_{E_i} \cong \begin{cases} \mathcal{O}_{E_i} & \text{if } 1 \le i \le t, \quad i \neq k,k+1 \quad \text{for } t = 2k \text{ even} \\ & \phantom{\text{if } 1 \le i \le t, \quad} i \neq k \quad\quad\;\; \text{for } t = 2k-1 \text{ odd} \\ \mathcal{O}_{E_k}(-2) & \text{if } i = k;\ t = 2k-1 \\ \mathcal{O}_{E_i}(-1) & \text{if } i = k,k+1;\ t = 2k \end{cases}$$

Proof: The splitting of $R|_U$ into a trivial summand and a line bundle follows from Remark 3.4. It remains to compute the Chern classes of $R \otimes \mathcal{O}_{E_i}$ $(1 \le i \le t)$. By (4.2) and (4.3) we have $c_1(R \otimes \mathcal{O}_{E_i}) = -c_1(\mathcal{F} \otimes \mathcal{O}_{E_i}) = c_1(\mathcal{C} \otimes \mathcal{O}_{E_i}) - c_1(\underset{\sim}{\mathrm{Tor}}_1^{\mathcal{O}_X}(\mathcal{C}, \mathcal{O}_{E_i}))$.

The claim is that this number is equal to $E_i \cdot D$, where D is the divisor of Lemma 3.5.

Let $\mathcal{T}$ be the torsion subsheaf of $\mathcal{C}$. By Lemma 3.5. we have an exact sequence

$$0 \to \mathcal{T} \to \mathcal{C} \to \mathcal{O}_D \to 0 .$$

Tensoring this sequence with $\mathcal{O}_{E_i}$ we obtain

$$c_1(\mathcal{C} \otimes \mathcal{O}_{E_i}) - c_1(\mathrm{Tor}_1^{\mathcal{O}_{\tilde{X}}}(\mathcal{C}, \mathcal{O}_{E_i}) = c_1(\mathcal{O}_D \otimes \mathcal{O}_{E_i}) - c_1(\mathrm{Tor}_1^{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_D, \mathcal{O}_{E_i})) .$$

But $\mathcal{O}_D \otimes \mathcal{O}_{E_i} \cong \mathcal{O}_{E_i}$, $\mathrm{Tor}_1^{\tilde{X}}(\mathcal{O}_D, \mathcal{O}_{E_i}) \cong \mathcal{O}_{E_i}(-D)$.

The following theorem contains the main result of this paper:

Theorem 4.8:

Let $\pi : \tilde{X} \to X$ be the minimal resolution of a rational surface singularity $(X,0)$, let $E = \bigcup_{i=1}^{r} E_i$ be the decomposition of the exceptional set $E = \pi^{-1}(0)$ into irreducible components, and let $-b_i$ be the self-intersection number of E_i. Denote by t_i the number of components of E different from E_i which meet E_i, and by s_i the number of chains of curves of self-intersection number -2 and trivial intersection with E that meet E_i. Assume that

(a) $b_i \geq t_i+1$ for $b_1 > 2$, $b_i \geq t_i$ for $b_i = 2$.

(b) $s_i \leq b_i - t_i - 2$ if $b_i - t_i \geq 2$

(c) $s_i = 0$ if $b_i = t_i+1$

Furthermore assume that inequality (b) is strict for at least one E_i. Then

$$\dim T^1_X = \dim T^1_{\tilde{X}} + \mathrm{emb}(X) - 4 .$$

Proof: From Corollary 2.3 we get

$$\dim T^1_X = \dim T^1_{\tilde{X}} + \sum_{i=1}^{r} (b_i - 3) + \dim H^0(\tilde{X}, \mathcal{C} \otimes \omega_{\tilde{X}}) + \dim H^1(\tilde{X}, R \otimes \omega_{\tilde{X}})$$

By our assumptions the formula for the embedding dimension in [Ar] gives $\mathrm{emb}(X) = 1 - E \cdot E = 1 + \sum_{i=1}^{r} (b_i - t_i)$. Hence

(4.9) $$\dim T^1_X - (\dim T^1_{\tilde{X}} + \mathrm{emb}\, X - 4) = \dim H^0(\tilde{X}, \mathcal{C} \otimes \omega_{\tilde{X}}) + \dim H^1(\tilde{X}, R \otimes \omega_{\tilde{X}}) - (r-1)$$

Let $L_1, \ldots, L_p$ be the maximal chains of (-2)-curves

$$L_j = E_1^{(j)} \cup \ldots \cup E_{t_j}^{(j)} \quad \text{such that } E \cdot E_\tau^{(j)} = 0 \text{ for } 1 \leq \tau \leq t_j .$$

To each L_j we associate the divisor

$$D_j = \sum_{\tau=1}^{t_j} \max(\tau, t_j - \tau + 1) \cdot E_\tau^{(j)}$$

as in (3.5). Then we have the exact sequence

(4.10) $$\ldots \to H^1(\tilde{X}, R \otimes \omega_{\tilde{X}}(-D_1 - \ldots - D_p)) \to H^1(\tilde{X}, R \otimes \omega_{\tilde{X}}) \to H^1(\tilde{X}, R \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{D_1 + \ldots + D_p}) \to 0$$

Then Theorem 4.8 follows from (4.9), (4.10) and

Lemma 4.11:

Under the assumptions of Theorem 4 we have

(i) $\dim H^{O}(\tilde{X}, \mathcal{C} \otimes \omega_{\tilde{X}}) + \dim H^{1}(\tilde{X}, R \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{D_1+\ldots+D_p}) = r-1$

(ii) $\dim H^{1}(\tilde{X}, R \otimes \omega_{\tilde{X}}(-D_1 \ldots - D_p)) = 0$.

<u>Proof:</u> (i) Every point, where two curves E_i, E_j with $b_i > t_i$, $b_j > t_j$ meet, gives a onedimensional contribution to $H^{O}(\tilde{X}, \mathcal{C} \otimes \omega_{\tilde{X}})$, and all other contributions to the sum above come from the chains of (-2)-curves.

By Serre-duality and the adjunction formula $H^{1}(\tilde{X}, R \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{D_1+\ldots+D_p})$ has the same dimension as $\bigoplus_{j=1}^{p} H^{O}(|D_j|, R^* \otimes \mathcal{O}_{D_j}(D_j))$. Recall from (4.7) that on D_j the bundle R^* decomposes into a trivial bundle and a line bundle, say $\mathcal{L}_j$, with $\mathcal{L}_j \otimes \mathcal{O}_{E_\tau(j)} \cong \mathcal{O}_{E_\tau(j)}(-D_j)$. By the negativity of the intersection matrix $\mathcal{O}_{D_j}(D_j)$ has no sections, hence

$$H^{O}(|D_j|, R^* \otimes \mathcal{O}_{D_j}(D_j)) \cong H^{O}(|D_j|, \mathcal{L}_j \otimes \mathcal{O}_{D_j}(D_j)) \cong H^{O}(|D_j|, \mathcal{O}_{D_j})$$

has dimension $[\frac{t_j+1}{2}]$ by Lemma 3.6. On the other hand

$\dim H^{O}(|D_j|, \mathcal{C} \otimes \omega_{\tilde{X}}) = [\frac{t_j+2}{2}]$ by (3.5) and (3.6).

So each chain L_j contributes t_j+1 to the sum on the right hand side of (4.11).

Using the fact that the resolution graph of X is a tree, one easily sees that the number of intersection points of curves not contained in $\bigcup_{j=1}^{p} L_j$ and the numbers t_j+1 for every chain L_j sum up to r.

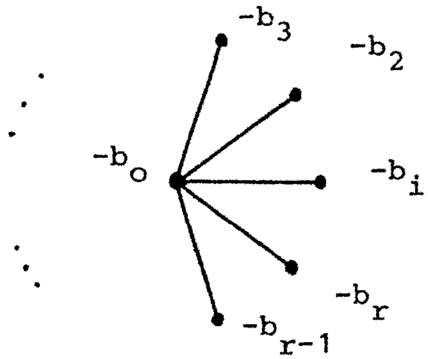
ii) Since the fundamental cycle is reduced, it suffices to show that

$H^1(|E|, R \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_E(-D_1-\ldots-D_p)) = 0$, and by Serre duality this means that $H^0(|E|, R^* \otimes \mathcal{O}_E(E+D_1+\ldots+D_p)) = 0$.

By our hypothesis and the Proposition 4.1, 4.7 the restriction $R^* \otimes \mathcal{O}_{E_i}(E+D_1+\ldots+D_p)$ to E_i is a direct sum of line bundles of degree at most 0, and for one index i it is a direct sum of line bundles of degree at most -1. Hence $R^* \otimes \mathcal{O}_E(E+D_1+\ldots+D_p)$ has no nontrivial global sections.

<u>Example 4.12:</u>

Consider the weighted dual graph



$\bullet = \mathbb{P}_1(\mathbb{C})$, $b_o,\ldots,b_r \geq 2$

If $b_o \geq r+1$ this is the resolution graph of a rational surface singularity X. Its embedding dimension is $\mathrm{emb}(X) = 3 + \sum_{i=o}^{r}(b_i-2)$ (cf.[Ar]).Theorem 4.8 gives

$$\dim T^1_X = \dim T^1_{\tilde{X}} + \sum_{i=o}^{r}(b_i-2)-1$$

if $b_o \geq r+3$ or $b_o = r+2$ and at least one b_i, $i = 1,\ldots,3$ is greater than 3. For $\dim T^1_{\tilde{X}}$ one computes from the exact sequence

$$0 \to \mathcal{D}er_E(\tilde{X}) \to \Theta_{\tilde{X}} \to \bigoplus_{i=o}^{r} \mathcal{O}_{E_i}(E_i) \to 0$$

that $\dim T^1_{\tilde{X}} = \sum_{i=o}^{r}(b_i-1)+\dim H^1(\tilde{X},\mathcal{D}er_E(\tilde{X}))$. Here $\mathcal{D}er_E(\tilde{X})$ is the dual of $\Omega^1_{\tilde{X}}<\log E>$, i.e. the sheaf of vectorfields parallel to E.

$H^1(\tilde{X}, \mathcal{D}er_E(\tilde{X}))$ parametrises the infinitesimal deformations of $\tilde{X}$ to which all the E_i lift, so it has dimension at least r-3. One can check that equality holds, if $b_o \geq 2r-2$.

Example 4.13:

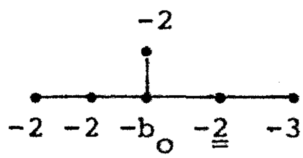
Let X be twodimensional quotient singularity of type $\mathbb{T}_m$, $\mathbb{O}_m$, $\mathbb{I}_m$ (cf. [Br] 2.9), and assume that the selfintersection number of the central curve of the exceptional set is at least 6+p, where p denotes the number of chains of (-2)-curves E_i with $E \cdot E_i = 0$. Then the equality

$$\dim T^1_X = \dim T^1_{\tilde{X}} + \mathrm{emb}(X) - 4 = \sum_{i=o}^{r} (2b_i - 3) - 1$$

holds.

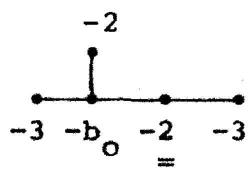
Proof: Theorem 4.8 applies to all cases of quotient singularities as listed in [Br] 2.11, apart from the following two types:

$\mathbb{I}_m$, $m = 30(b_o-2)+7$ with resolution graph



$b_o \geq 8$

$\mathbb{I}_m$, $m = 30(b_o-2)+17$ with resolution graph



In both cases there is a chain (of length one) of (-2)-curves which meets a (-3)-curve. Let $L_1 = E_1$ be the (-2)-curve and E_2 the (-3)-curve in question. We replace the divisor D_1 in the proof of Theorem 4.8 by $D_1' := E_1+E_2$. Put $D' := D_1'+D_2$ in the first case, and $D' := D_1$ in the second case. In analogy to Lemma 4.11 we have

(4.14) Claim: $H^1(\tilde{X}, R^* \otimes \omega_{\tilde{X}}(-D')) = 0$.

Proof: As in 4.13 we have to show that $H^0(E, R^* \otimes \mathcal{O}_E(E+D')) = 0$. The restriction of $R^* \otimes \mathcal{O}_E(E+D')$ to the central curve E_0 and to E_2 is a direct sum of line bundles of negative degree (cf.4.1), and it has degree ≤ 0 on all components but E_1. On E_1 it is a direct sum of a line bundle of degree one and of line bundles of degree -1. This shows that the vectorbundle $R^* \otimes \mathcal{O}_E(E+D')$ cannot have any global sections on E.

(4.15) Claim: $\dim H^1(\tilde{X}, R \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{D_1'}) = 1$.

Proof: As in the proof of Lemma 4.11 (i) it suffices to show that $\dim H^0(E_1 \cup E_2, R^* \otimes \mathcal{O}_{E_1+E_2}(E_1+E_2)) = 1$.

Let g_1, g_2, g_3 be the global functions on $\tilde{X}$ of remark 3.4, whose differentials generate $\mathcal{F}$ in a neighbourhood of E_1. We may assume that g_1 vanishes with multiplicity 1 along E_1 and E_2, g_2 vanishes with mulitiplicity 3 along E_1 and multiplicity 1 along E_2, and g_3 vanishes with mulitiplicity 3 both along E_1 and E_2 .

Call $\mathcal{F}' \subset \Omega_{\tilde{X}} \langle \log E \rangle (-E)$ the subsheaf generated by dg_1, dg_2, dg_3 and let $\mathcal{L}$ be the sheaf of relations between them:

$$(4.16) \qquad 0 \to \mathcal{L} \to \mathcal{O}_{\tilde{X}}^{\oplus 3} \to \mathcal{F}' \to 0 \quad .$$

One easily sees that $(\Omega^1_{\tilde{X}} \langle \log E \rangle (-E)/\mathcal{F}') \otimes \mathcal{O}_{E_2}$ is a torsion sheaf of length at least one, so $c_1(\mathcal{F}' \otimes \mathcal{O}_{E_2}) \leq 1$.

Hence by (4.16)

$$\deg \mathcal{L}|_{E_2} \geq -1 \quad \text{, while by Prop. 4.7}$$

$$\deg \mathcal{L}|_{E_1} = -2 \quad .$$

Now by Proposition 4.7 the restriction of $R^* \otimes \mathcal{O}_{E_1+E_2}(E_1+E_2)$ to E_1 is a sum of line bundles of negative degrees and one line bundle of degree one, namely $\mathcal{L}^* \otimes \mathcal{O}_{E_1}(E_1+E_2)$. By Proposition 4.1 and (4.6) the vectorbundle $R^* \otimes \mathcal{O}^1_{E_2}(E_1+E_2)$ has at most one line bundle summand of non-negative degree, which then is trivial. This summand does ~~does~~ not agree with $\mathcal{L}^* \otimes \mathcal{O}_{E_2}(E_1+E_2)$ (which has degree ≤ -1), so a holomorphic section of $R^* \otimes \mathcal{O}_{E_1+E_2}(E_1+E_2)$ has to vanish on E_2. This proves claim (4.15).

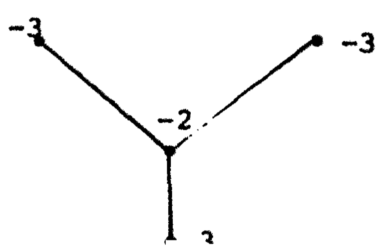
The rest of the proof for the equality $\dim T^1_X = \dim T^1_{\tilde{X}} + \mathrm{emb}(X)-4$ for the singularities under consideration is analoguous to the proof of (4.8).

<u>Remark 4.17:</u> There are 63 individual quotient singularities of type $\mathbb{T}$, $\mathbb{O}$, $\mathbb{I}$ that are not covered by example 4.15.

Finally we want to give a partial analysis of the example of J.Wahl mentioned in the introduction.

<u>Example 4.18:</u>

Let X be the rational surface singularity with resolution graph



$\bullet \cong \mathbb{P}_1(\mathbb{C})$

The fundamental cycle is $Z = 2E_o+E_1+E_2+E_3$, where E_o denotes the central curve. We have $\mathrm{emb}(X) = 6$, $\dim T^1_{\tilde{X}} = 7$, so formula (0.1) would give 9 for $\dim T^1_X$. We want to show that $\dim T^1_X \geq 10$. We apply Corollary 2.3.

As $Z - E = E_o$ and $\Omega^1_{\tilde{X}} < \log E > (-E) \otimes \omega_{\tilde{X}} \otimes {}_{E_o} \cong \omega_{E_o}(2) \oplus {}_{E_o}(-1)$

we see that $\dim H^o(|Z-E|, \Omega^1_{\tilde{X}} < \log E > (-E) \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{Z-E}) = 1$.

On the other hand $\mathcal{C}$ is concentrated in the points of intersections of E_o with the other components of E and has length one there, so $\dim H^o(\tilde{X}, \mathcal{C} \otimes \omega_{\tilde{X}}) = 3$. Therefore by Corollary 2.3

$$\dim T^1_X \geq 6 + 3 + 1 + \dim H^1(\tilde{X}, R \otimes \omega_{\tilde{X}}) \geq 10 \ .$$

Remark 4.19: In this example one can compute the map$(\mu \otimes 1)^*$ of chapter 2 quite explicitely, and using the methods of this paper one obtains that actually $\dim T^1_X = 11$.

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Appendix

In this appendix we give a proof, due to Jonathan Wahl, of the following result:

Theorem (Wahl): Let X be a rational singularity of embedding dimension emb(X), and let $\pi : \tilde{X} \to X$ be the minimal good resolution.
Then $\dim T^1_{\tilde{X}} \geq \dim T^1_X + \mathrm{emb}(X) - 4$.

Proof: Let R_Z be the functor from the category of Artin rings to the category of sets, defined by

$$R_Z(A) := \{\text{equivalence classes of deformations } \tilde{\mathfrak{X}} \to \operatorname{Spec} A \text{ of } \tilde{X} \text{ to which } Z \text{ lifts}\}$$

(cf.[Wa 1], § 2). R_Z has a formal versal deformation space (ibid.,Prop.2.2) which by the natural blowing down map $\phi: R_Z \to \mathcal{D}ef_X$ maps injectively to the base space of the versal deformation of X.

By a result of Karras Z is always smoothable [Ka], hence Theorem 2.12 of [Wa 1] applies:

The versal deformation space of R_Z has irreducible components of dimension $\dim H^1(\tilde{X},\Theta_{\tilde{X}}) - \dim H^1(\tilde{X}, N_Z)$, N_Z the normal bundle of Z, and for a general point of such a component the fibre $\tilde{\mathfrak{X}}_t$ has smooth rational curve of selfintersection number $Z \cdot Z$ as exceptional divisor.

Let S be the base space of the formal versal deformation of X, and consider the point t as a point of S.
By openess of versality [Pou] the dimension of the tangent space of S at t, which is at most $\dim T^1_X$, is the sum of the

dimensions of the tangent space of the versal deformation of the singularity of X_t and the number of directions for which the given family induces trivial deformation of X_t .

We have $-2\cdot Z^2-4$ for the first summand, and for the second observe that R_Z induces only trivial deformations of X_t. (There are no equisingular deformations of a cone over a rational curve).

Hence

$$\dim T^1_X = \dim T^1_{S,o} \geq \dim T^1_{S,t} \geq \dim T^1_{X_t} + \dim R_Z ,$$

and it is an easy exercise to compute

$\dim H^1(\tilde{X},N_Z) = -1 - Z^2$. Putting every thing together yields the desired estimate. □

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