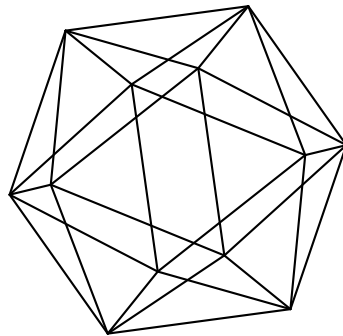# Max-Planck-Institut für Mathematik Bonn

Differential equations and monodromy

by

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DIFFERENTIAL EQUATIONS AND MONODROMY

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ABSTRACT. In these expository notes, we describe results of Cauchy, Fuchs and Pochhammer on differential equations. We then apply these results to hypergeometric differential equation of type $_nF_{n-1}$ and describe Levelt's theorem determining the monodromy representation explicitly in terms of the hypergeometric equation. We also give a brief overview, without proofs, of results of Beukers and Heckman, on the Zariski closure of the monodromy group of the hypergeometric equation. In the last section, we recall some recent results on thin-ness and arithmeticity of hypergeometric monodromy groups

1. INTRODUCTION

In these notes, we recall (in sections 1 to 3) the basic theory of differential equations on the unit disc and on the punctured unit disc. For references see [Cod-Lev] and [Le].

In sections 4 and 5 we apply the theory developed in sections 1 through 3 to (state and) prove a result of Levelt [Le] on the monodromy of hypergeometric differential equations of type $_nF_{n-1}$.

In the next few sections we use this description to prove some results (some are not proved completely because the proofs are lengthy) of Beukers and Heckman [Beu-Hec] on the Zariski closure of the monodromy of the foregoing hypergeometric equation. In particular, they completely determine when the monodromy is finite.

2. Monodromy Groups

The concept of monodromy arises in many seemingly different situations. We will deal with some of the simplest ones, namely the monodromy associated to linear differential equations on open subsets in the complex plane.

2.1. **Differential Equations on Open Sets in the Plane.** Let U be a connected open subset of the complex plane. Fix holomorphic functions $f_i : U \to \mathbb{C}$ with $0 \leq i \leq n-1$. Consider the differential equation

$$\frac{d^n y}{dz^n} + \sum_{i=0}^{n-1} f_i(z) \frac{d^i y}{dz^i} = 0.$$

If y_1, y_2 are solutions, then so is $c_1 y_1 + c_2 y_2$ with $c_1, c_2 \in \mathbb{C}$. That is, the space of solutions is a vector space. A fundamental result of Cauchy says that when U is the unit disc, there are holomorphic functions $y_1, \cdots, y_n$ on the disc which are solutions of this differential equation, which are linearly independent and such that all solutions are linear combinations of these solutions. These solutions are called **fundamental solutions**.

Theorem 1. *(Cauchy) Let $f_0, \cdots, f_{n-1} : \Delta \to \mathbb{C}$ be holomorphic functions on the open unit disc Δ and consider the differential equation*

$$\frac{d^n y}{dz^n} + f_{n-1}(z)\frac{d^{n-1} y}{dz^{n-1}} + \cdots + f_1(z)\frac{dy}{dz} + f_0(z)y = 0.$$

Suppose that $z_0, \cdots, z_{n-1}$ are arbitrary complex numbers. Then there exists a solution y to the differential equation, which is holomorphic in the whole of the disc, such that $\frac{d^j y}{dz^j}(0) = z_j$ for all j with $0 \leq j \leq n-1$.

In particular, the differential equation has n fundamental solutions.

Proof. We first prove this when $n = 1$. Suppose then that we have the equation

$$\frac{dy}{dz} + f_0(z)y = 0,$$

where $f_0(z) = \sum_{k=0}^{\infty} a_k z^k$ a power series which converges in $\mid z \mid < 1$. Suppose $y(z) = \sum_{k=0}^{\infty} x_k z^k$ is a formal power series with x_k a sequence of elements of $\mathbb{C}$. By looking at the coefficient of z^{k-1} on both sides (which are formal power series) of the differential equation, it follows that, if the formal power series y is to be a solution of the differential equation, then the x_k (for $k \geq 1$) must satisfy the recursive relation

(1) $$-kx_k = x_{k-1}a_0 + x_{k-2}a_1 + \cdots + x_0 a_{k-1}.$$

Let $R < 1$; then the convergence of $f_0(z)$ in $| z | < 1$ implies that there is a constant $M \geq 1$ such that $| a_k | R^k < M$ for all $k \geq 0$. Suppose $r < R$ is fixed. Let, for each j, M_j denote the supremum

$$M_j = sup\{| x_j | R^j, | x_{j-1} | R^{j-1}, \cdots, | x_1 | R, | x_0 |\}.$$

The equation (1) shows that for each $k \geq 1$ we have

$$k | x_k | \leq \frac{M_{k-1}}{R^{k-1}} M + \frac{M_{k-1}}{R^{k-2}} \frac{M}{R} + \cdots + \frac{M_{k-1}}{R} \frac{M}{R^{k-2}} + M \frac{M}{R^{k-1}} = k \frac{M_{k-1} M}{R^{k-1}}.$$

Therefore, $| x_k | R^k \leq M_{k-1} M$ for all $k \geq 0$. In particular, since by assumption $M \geq 1$, we have $M_k \leq M_{k-1} M$ and hence $\frac{M_k}{M^k}$ is a decreasing sequence, and hence a bounded sequence. We may assume (increasing M if necessary), that $M_k \leq M M^k$ for all k. Therefore, $| x_k | r^k \leq \frac{M M^k}{R^k} r^k$. Therefore, if $\frac{Mr}{R} < 1$ then $\sum | x_k | r^k$ is dominated by the convergent geometric series $M \sum (\frac{Mr}{R})^k$. Hence the formal power series $\sum x_k z^k$ converges in the smaller disc $| z | < \frac{R}{M}$.

We may similarly solve the differential equation in every small disc inside the unit disc Δ (as a convergent power series around $z_0 \in \Delta$) and by the uniqueness of the power series - thanks to the recursion (1) - the two power series coincide as functions on the intersections of the smaller discs. Therefore, by the principle of analytic continuation, there is a holomorphic function y on all of the disc which is a solution of the differential equation $\frac{dy}{dz} + f_0(z)y = 0$.

Exactly the same proof shows that if now $y : \Delta \to \mathbb{C}^n$ has values in the vector space $\mathbb{C}^n$, $v \in \mathbb{C}^n$ and f_0 is replaced by a matrix valued *holomorphic* function $A(z) : \Delta \to M_n(\mathbb{C})$ (i.e. an $M_n(\mathbb{C})$-valued convergent holomorphic function on Δ) then there is a power series y with coefficients x_k in $\mathbb{C}^n$ which is a solution of the differential equation $\frac{dy}{dz} + A(z)(y(z)) = 0$ and which converges on all of Δ with $y(0) = v$.

Observe also that if $v = 0$, then the solution y has the property that all its derivatives at 0 vanish, in view of the differential equation that y satisfies. Since the solution y is holomorphic in a neighbourhood of 0, it follows that $y = 0$ in a neighbourhood of 0 and hence y is identically zero on the disc D by the principle of analytic continuation.

Suppose, successively, that $v = \varepsilon_1, \cdots, \varepsilon_n$. We get, from the preceding two paragraphs, solutions $y_1, y_2, \cdots, y_n$ of the differential equation

$$(2) \qquad \frac{dy}{dz} + A(z)y = 0,$$

with $y_i(0) = \varepsilon_i$ for each i. Moreover, if y is any solution of the differential equation (2), with $y(0) = v = \sum c_i \varepsilon_i$, then $y - \sum c_i y_i = 0$. Thus the solutions $y_1, \cdots, y_n$ are linearly independent and every solution is a linear combination of $y_1, y_2, \cdots, y_n$.

We now choose

$$A(z) = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & -1 \\ f_0(z) & f_1(z) & f_2(z) & \cdots & f_{n-1}(z) \end{pmatrix} \quad and \quad y = \begin{pmatrix} w_1(z) \\ w_2(z) \\ \cdots \\ w_n(z) \end{pmatrix},$$

where y is viewed as a column vector. Then the vector valued equation $\frac{dy}{dz} + A(z)y(z) = 0$ yields the n scalar valued equations $w_1'(z) = w_2(z), \cdots w_{n-1}'(z) = w_n(z)$, and $w_n'(z) + f_0(z)w_1(z) + \cdots + f_{n-1}(z)w_n(z) = 0$. In other words, $w = w_1(z)$ is the solution to the scalar valued differential equation

$$\frac{d^n w}{dz^n} + f_{n-1}(z)\frac{d^{n-1}w}{dz^{n-1}} + \cdots f_1(z)\frac{dw}{dz} + f_0(z)w = 0.$$

This proves Cauchy's theorem in all cases.

$\square$

If U is now taken to be any connected open set in $\mathbb{C}$, then the foregoing result of Cauchy says that at each point p of the open set, there are holomorphic functions $y_1, \cdots, y_n$ defined in an open neighborhood of p which are fundamental solutions to the above differential equation. If Γ denotes a closed loop in the open set U starting and ending at p, then analytic continuation along the path of solutions is possible and when we return to the original point, we get new fundamental solutions $w_1, \cdots, w_n$. This means that there is a matrix $M = M(\gamma)$ depending on the path, such that $w = My$ in a neighborhood of p. One can check that the matrix M depends only on the homotopy class of the path γ based at p and not on the path itself.

Moreover, if γ_1, γ_2 are *two* paths based at p, and γ is the composition of these paths, then one can check that $M(\gamma) = M(\gamma_1)M(\gamma_2)$. Thus, the association $\gamma \to M(\gamma)$ yields a group homomorphism from the fundamental group of the open set U based at p, into $GL_n(\mathbb{C})$. This homomorphism is called the "monodromy representation" and the image is called the "monodromy group".

2.2. **Finiteness.** Let $U \subset \mathbb{C}$ be a connected open set. Then the space of holomorphic functions on U is an integral domain and the corresponding field of of fractions, i.e. ratios of holomorphic functions, is a field, called the field of meromorphic functions on U. Let $U^* \to U$ (given by $\tau \mapsto z$) be the universal cover U^* of U and let Γ be the deck transformation group.

We say that a function $f : U^* \to \mathbb{C}$ is **algebraic** if it satisfies a polynomial relation $f^n(\tau) + \sum_{i=0}^{n-1} \phi_j(z) f^i(\tau) = 0$ with coefficients ϕ_j in the field K of meromorphic functions on U.

Lemma 2. *A function f on U^* is algebraic if and only if its orbit under the deck transformation group Γ is finite.*

Proof. The polynomial relation holds if f is replaced by any translate under an element $\gamma \in \Gamma$. But since there are only finitely many roots to any polynomial, it follows that the orbit of f under Γ is finite.

On the other hand, if a function f on U^* is invariant under Γ, then it defines a holomorphic function on the base U. Therefore, if the orbit under Γ is finite, then the polynomial $P(t) = \prod_{\gamma \in \Gamma / \Gamma_f} (t - \gamma(f))$ has coefficients in K. Hence f is algebraic. $\square$

Corollary 1. *Suppose*

$$\frac{d^n y}{dz^n} + f_{n-1}(z) \frac{d^{n-1} y}{dz^{n-1}} + \cdots + f_1 \frac{dy}{dz} + f_0 y = 0$$

is a differential equation with coefficients f_i holomorphic on U. Suppose that the monodromy representation is irreducible. Then a nonzero solution to the equation is algebraic if and only if the monodromy is finite.

Proof. If f is a solution, then so is $\gamma(f)$ for $\gamma \in \Gamma$. If the monodromy is finite, it means that the orbit of f under Γ is finite, in particular, and hence it is algebraic by the lemma.

On the other hand, if some solution f is algebraic, then by the lemma the orbit is finite. It means that the Γ translates of f span a subspace which is Γ stable and these translates are algebraic. By irreducibility, this is the whole space. This means that for some basis of the space of solutions, the orbit of Γ is finite for every element of the basis. This means that the image under the monodromy representation of Γ is finite. $\square$

3. Punctured Disc

Now consider the open set $U = \Delta^*$, obtained by removing the point 0 from the unit disc Δ.

Example 1. Let us look at the differential equation

$$\frac{dy}{dz} - \frac{\alpha}{z} y = 0,$$

where $\alpha \in \mathbb{C}$ is fixed. Solving, we get $y = z^\alpha$. This function is not "single valued". We view $z = e^{2\pi i \tau} = \phi(\tau)$ with $\tau \in \mathfrak{h}$ the upper half plane, with ϕ being a covering map. Consider the path $\omega : [0,1] \to \mathfrak{h}$ starting at i and ending at $i + 1$. Its composite $\gamma = \phi \circ \omega$ is a closed loop in Δ^* based at $p = e^{-2\pi}$; the effect of traversing along this path on the solution y is to multiply it by $e^{2\pi i \alpha}$. Thus $M(\gamma)$ is the 1×1 matrix $e^{2\pi i \alpha}$.

Example 2. As another example, consider the equation

$$\frac{d^2 y}{dz^2} = -\frac{1}{z} \frac{dy}{dz}.$$

Clearly the constant function $y_1 = 1$ is a solution; it is invariant under the action of the loop γ.

It is easily checked that $y_2 = \frac{1}{2\pi i} \log z$ is another solution; to view this solution as a function, we write $y = \frac{1}{2\pi i} \log(e^{2\pi i \tau}) = \frac{1}{2\pi i} 2\pi i \tau = \tau$; hence the action of the loop γ of the preceding example, is to take y_2 into the new solution $y_2 + \frac{1}{2\pi i} 2\pi i = y_2 + y_1$. Hence

$$M(\gamma) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

3.1. **Regular Singular Points and a Theorem of Fuchs.** We now look at a more general case; suppose we have a differential equation of the form

$$(3) \qquad \frac{d^n y}{dz^n} + f_{n-1}(z) \frac{d^{n-1} y}{dz^{n-1}} + \cdots + f_1(z) \frac{dy}{dz} + f_0(z) y,$$

where each $f_i(z)$ has at most a pole of order $n - i$ at $z = 0$. Then the monodromy M (i.e. the action of the generator of $\pi_1(\Delta^*) \simeq \mathbb{Z}$) acts on the $\mathbb{C}^n$ space of solutions. Write $\theta = z \frac{d}{dz}$; using the relation $\theta^2 = z^2 \frac{d^2}{dz^2} + \theta$ one can show, by induction, that

$$z^k \frac{d^k}{dz^k} = \theta(\theta - 1) \cdots (\theta - k + 1)$$

for every $k \geq 1$. Then the differential equation (after multiplying throughout by z^n), takes the form

$$\theta(\theta - 1) \cdots (\theta - n + 1)y + z f_{n-1}(\theta(\theta - 1) \cdots (\theta - n + 1)y+$$

$$+ \cdots + z^{n-2} f_2 \theta(\theta - 1)y + z^{n-1} f_1 \theta y + z^n f_0 y = 0.$$

Rewriting this yields

$$\theta^n y + F_{n-1}(z)\theta^{n-1} + F_{n-2}(z)\theta^{n-2}y + \cdots + F_1(z)\theta y + F_0(z)y = 0$$

where now the functions F_i are holomorphic on all of the disc (including the puncture). Write $a_i = F_i(0)$ and $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 = \prod_{j=1}^{n}(t - \alpha_j)$. The equation $f(t) = 0$ is called the **indicial equation** and the roots of the indicial equation (i.e. roots of the polynomial f) are called the indicial roots.

Theorem 3. *(Fuchs) With the preceding notation, assume that 0 is a regular singular point of the differential equation (3). Then, the characteristic polynomial of the monodromy matrix M of the differential equation (3) is the polynomial*

$$\prod_{j=1}^{n}(t - e^{2\pi i \alpha_j}).$$

Moreover, every solution of the differential equation (3) is a linear combination of functions of the form $\phi(z)z^\alpha P(\log z)$ where ϕ is a holomorphic function on all of the disc, α is a complex number and P is a polynomial.

Theorem 3 will be recast in terms of matrix valued solutions and the differential equation (3) will be rewritten as a *first order* equation.

3.2. **First order Matrix Valued Differential Equations.** Suppose now that $A : \Delta \to M_n(\mathbb{C})$ is a holomorphic map on all of the disc. Then $A(z)$ is represented by a convergent power series $A(z) = \sum A_k z^k$ where $A_k \in M_n(\mathbb{C})$. We look for local solutions $Y : \Delta^* \to M_n(\mathbb{C})$ to the first order equation $z\frac{dY}{dz} = A(z)Y(z)$.

Notation. If $T \in M_n(\mathbb{C})$ and $z \in \Delta^*$ we write z^T for the matrix represented by the (convergent) exponential power series in the matrix variable $(\log z)T$:

$$z^T = \exp((\log z)T) = \sum_{k=0}^{\infty} \frac{(\log z)^k T^k}{k!}$$

We list some properties of the matrix exponent.

[1] If A and B are commuting square matrices of size n, then $z^{A+B} = z^A z^B$.

[2] If $A \in M_n(\mathbb{C})$ and $g \in GL_n(\mathbb{C})$, then $z^{gAg^{-1}} = gz^A g^{-1}$.

[3] If $N \in M_n(\mathbb{C})$ is nilpotent, then z^N is a polynomial in $logz$.

[4] If $A \in M_n(\mathbb{C})$ is a diagonal matrix whose diagonal entries are $a_1, a_2, \cdots, a_n$ then z^A is also a diagonal matrix whose diagonal entries are $z^{a_1}, z^{a_2}, \cdots, z^{a_n}$.

[5] These properties imply that if $A \in M_n(\mathbb{C})$ is any matrix, then the entries of the matrix z^A are linear combinations of functions of the form $z^\alpha P(logz)$ where P is a polynomial and $\alpha \in \mathbb{C}$ is a fixed complex number.

[6] The derivative of z^A satisfies: $z\frac{dz^A}{dz} = z^A A$.

[7] The monodromy operator on the multivalued function z^A is simply $e^{2\pi i A}$, since $z^A = e^{2\pi i \tau A}$ and the generator of the Deck transformation group takes τ to $\tau + 1$.

For a reference to the following see [Cod-Lev], Theorem (4.1) and Theorem (4.2).

Theorem 4. *(Fuchs) Suppose $A : \Delta \to M_n(\mathbb{C})$ is a holomorphic function. Let $\mathfrak{h} \to \Delta^*$ be the exponential covering map as before. Consider the differential equation in $Y(z) = Y^*(\tau) \in M_n(\mathbb{C})$:*

$$(4) \qquad \qquad \frac{dY}{dz} = \frac{A(z)}{z}Y.$$

The monodromy of the equation acts on the space of solutions Y^* by the formula $Y^*(\tau + 1) = Y^*(\tau)M$ where $M \in GL_n(\mathbb{C})$. Moreover, the semi-simple part of the matrix M is conjugate to the exponential $e^{2\pi i A_s}$ of A_s, where A_s is the semi-simple part of the matrix $A_0 = A(0)$.

The solution Y is of the form $Y(z) = X(z)z^{A_0}$ where $X(z)$ is a function which is holomorphic on all of the disc including the puncture 0.

Proof. First assume that if λ, μ are *distinct* eigenvalues of the matrix A_0, then they do not differ by an integer. This means that no eigenvalue of the adjoint transformation adA_0 can be a nonzero integer. We then show that there is a holomorphic function $X : \Delta \to GL_n(\mathbb{C})$ such that

$$Y(z) = X(z)z^{A_0}$$

is a solution of the differential equation (4). Write $X(z) = \sum_{k=0}^{\infty} X_k z^k$, and solve for the coefficients X_k. Write, as before, $\theta = z\frac{d}{dz}$. Then the differential equation for Y is $\theta Y(z) = A(z)Y(z)$; moreover, by the

formula for the differentiation for a product, we get

$$\theta Y(z) = \theta(X(z)z^{A_0}) = \theta(X(z))z^{A_0} + X(z)z^{A_0}A_0 =$$

$$= A(z)Y(z) = A(z)X(z)z^{A_0}.$$

We now cancel z^{A_0} on both sides of the preceding equation and obtain

$$\theta(X(z)) + X(z)A_0 = A(z)X(z),$$

where now A is holomorphic on all of the disc and X is assumed to be holomorphic on all of the disc. Writing the power series for X and A, we then get, for $k \geq 1$, the recursion

$$kX_k + X_kA_0 = A_0X_k + \sum_{j=0}^{k-1} A_{k-j}X_j,$$

and for $k = 0$, the equation $X_0A_0 = A_0X_0$. We can solve for X_0 by taking X_0 to be identity. The recursion for the coefficients is

$$(k - adA_0)X_k = \sum_{j=0}^{k-1} A_{k-j}X_j.$$

This can be solved for all $k \geq 1$ since, by assumption, *non-zero integers k cannot be eigenvalues of the operator adA_0*; therefore, $k - adA_0$ is an invertible operator and hence X_k may be written as a combination of the $X_j : j \leq k - 1$.

[We now check that the formal power series $\sum X_k z^k$ converges in a small enough neighborhood of 0. Consider the sequence $1 - \frac{adA_0}{k}$ for $k \geq 1$. For k large enough, the k-Th term of this sequence is close to the identity matrix; by assumption, all the terms of this sequence are non-singular. Hence the sequence $(1 - \frac{adA_0}{k})^{-1}$ is bounded from above by a constant $M > 1$ say. Since the sequence $A_{k-j}R^{k-j}$ $(k \geq j)$ is bounded, we may assume that $\mid A_{k-j} \mid R^{k-j} \leq M$ for all k, j. Let, as in the proof of Cauchy's theorem, M_k be the supremum of the matrix norms $\mid X_j \mid R^j$ for $j \leq k$. The recursive relation for the X_k now implies that

$$k \mid X_k \mid \leq MMk_{k-1}.$$

Therefore, $\mid X_k \mid R^k \leq M^2 M_{k-1}$. Since $M \geq 1$, we also have $\mid X_j \mid R^j \leq M^2 M_{k-1}$ for all $j \leq k - 1$. Hence $M_k \leq M^2 M_{k-1}$ and therefore, $M_k m^{-2k}$ is a decreasing sequence and is bounded. We may assume then that $\mid X_k \mid R^k \leq M_k \leq MM^{2k}$ and hence $\sum X_k z^k$ converges if $\mid z \mid < \frac{R}{M^2}$.]

Thus the monodromy action on $Y(z) = X(z)z^{A_0}$ is simply right multiplication by the exponential matrix $e^{2\pi i A_0}$ of A_0 since the solution $X(z)$ is holomorphic also at the puncture and is invariant under the monodromy action. This proves the Theorem in the case when distinct eigenvalues of A_0 remain distinct modulo 1.

The proof of the general case of the Theorem can be reduced to this case. Fix an eigenvalue λ of the linear transformation A_0. Write $\mathbb{C}^n = E \oplus F$ where E is the generalized λ airspace for A_0, and F an A_0 stable supplement to E. If $\varepsilon_1, \cdots, \varepsilon_r$ is a basis of E, and $\varepsilon_{r+1}, \cdots, \varepsilon_n$ a basis of F, then with respect to the basis $\varepsilon_1, \cdots, \varepsilon_n$ of $\mathbb{C}^n$, the matrix of the transformation which is z times the identity on E and identity on F is given by $\begin{pmatrix} zI_r & 0 \\ 0 & I_{n-r} \end{pmatrix}$, where I_k is the identity matrix of size k. Moreover, $A_0 = \begin{pmatrix} \lambda I_r + N_r & 0 \\ 0 & \delta_0 \end{pmatrix}$ where δ_0 acts on F and N_r is a *nilpotent* matrix of size r. Write $Y(z) = \begin{pmatrix} zI_r & 0 \\ 0 & I_{n-r} \end{pmatrix} W(z) = M(z)W(z)$. Then it is easily seen that $W(z)$ satisfies the equation

$$\theta W(z) = B(z)W(z)$$

where $B(z)$ is holomorphic on Δ and $B_0 = B(0) = \begin{pmatrix} (\lambda - 1)I_r + N_r & \beta_0 \\ 0 & \delta_0 \end{pmatrix}$. Thus the semi-simple part of the exponential of B_0 is conjugate to that of A_0. Moreover, the monodromy of W and of Y are the same. Consequently, Y may be replaced by W in the statement of the theorem without altering the conclusion.

We now apply the preceding repeatedly to ensure that if λ and λ' are two distinct eigenvalues of A_0 which differ by an integer, the A_0 is replaced by B_0 such that these eigenvalues become equal. That is, suppose $\lambda = \lambda' + m$ for some positive integer m say. As above, we replace λ by $\lambda - 1$ without altering the monodromy; we do this m times until λ is replaced by λ', with monodromy unchanged.

Applying this procedure repeatedly to all eigenvalues which differ by an integer, we can thus ensure that all the distinct eigenvalues of A_0 remain distinct modulo 1. Then we are in the special case where adA_0 does not have non-zero integers as eigenvalues. In that case, the Theorem has already been proved.

$\square$

Theorem 3 is deduced from Theorem 4 in the same way Cauchy's theorem was proved: we convert the n-th order scalar valued diferential equation into a matrix valued first order equation.

3.3. Complex Reflections.
For a reference to the following, see Theorem 3.1.2 of [Beu].

Theorem 5. (*Pochhammer*) *If as before, we have a differential equation*

$$\frac{d^n y}{dz^n} + \sum_{i=0}^{n-1} f_i(z)\frac{d^i y}{dz^i} = 0,$$

on the punctured disc Δ^*, *and we assume that the functions* f_i *have at most a simple pole at* $z = 0$, *then there are* $n-1$ *solutions which extend holomorphic ally to the puncture, and one solution which (possibly) has singularities at the puncture. Moreover, the monodromy matrix is of the form*

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & * \\ 0 & 1 & 0 & \cdots & * \\ \cdots & \cdots & \cdots & & \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}$$

for some $c \neq 0$.

The number c is called the *exceptional eigenvalue* (it can even be 1) and the matrix M is called a *complex reflection* (it is identity on a co dimension one subspace of $\mathbb{C}^n$).

Proof. By the same procedure as before, the differential equation of order n can be converted to a differential equation of order 1 but with solutions in the vector space $\mathbb{C}^n$:

$$\frac{dy}{dz} = A(z)y(z) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ -f_0(z) & -f_1(z) & -f_2(z) & \cdots & -f_{n-1}(z) \end{pmatrix}(y(z)).$$

We write the vector valued formal power series expansion $y(z) = \sum x_k z^k$, with x_k in $\mathbb{C}^n$. If we write $A(z) = \frac{A_{-1}}{z} + A_0 + A_1 z + \cdots + A_k z^k + \cdots$, then $A(z) - \frac{A_{-1}}{z}$ is a convergent power series (with values in $M_n(\mathbb{C})$) in $| z |< 1$. Solving term by term, for each $k \geq 0$ comparing the coefficient of z^k we get (cf 1)

$$(5) \qquad kx_k = A_{-1}x_k + A_0 x_{k-1} + A_1 x_{k-2} + \cdots + A_k x_0.$$

Now the first $n-1$ rows of the matrix A_{-1} are all zero. the only other eigenvalue d of A_{-1} is the residue of $-f_{n-1}$ at 0. If d is never a positive integer, then $A_{-1} - k$ is invertible for *all* positive integers k. Hence the above recursion shows that all the $x_k; k \geq 1$ are uniquely determined by x_0; the equation for $k = 0$ shows that x_0 satisfies the equation $A_{-1}x_0 = 0$. This is a co-dimension one subspace of $\mathbb{C}^n$ and hence the space of holomorphic solutions of the differential equation is of dimension at least $n-1$. This proves the first part.

If a solution is holomorphic, then analytic continuation along a loop around 0 does not change the function and hence the monodromy element acts trivially. This proves the second part of the Theorem.

Slightly more work is needed when the eigenvalue d is a positive integer. The equation (5) may be applied to all $k \neq d$; in particular, if x_0 is a fixed vector such that $A_{-1}x_0$ is zero, then $x_1, \cdots, x_{d-1}$ are uniquely determined from equation (5) and depend linearly on x_0. However, the equation (5) applied to $k = d$ shows that there exists a linear transformation B_d on $\mathbb{C}^n$ such that for each $x_0 \in Ker(A_{-1})$ we have

$$(d - A_{-1})(x_d) = B_d(x_0).$$

The image W of $d - A_{-1}$ has dimension $n-1$ and hence if $B_d(x_0)$ lie in this image W, then the recursion (5) still applies to locate an x_d. The other $x_j(j \geq k+1)$ are now uniquely determined by (5) and hence the space of solutions which are holomorphic at 0 has dimension at least $n-2$: this is the dimension of the space of x_0 satisfying $A_{-1}x_0 = 0$ and $B_k(x_0) \in W = Image(d - A_{-1})$.

Now we take $x_0 = 0$. Then by (5), $x_1 = \cdots = x_{d-1} = 0$. Moreover, x_d satisfies $(A_1 - d)(x_d) = 0$. Since the kernel of $A_1 - d$ has dimension one, there does exist a non-zero x_d with this property. Then by (5), all x_j with $j \geq d+1$ are determined and hence there exists an extra holomorphic solution w of the form $w(z) = z^d x_d + z^{d+1} x_{d+1} + \cdots + x_k z^k + \cdots$. Hence the space of holomorphic solutions is again of dimension at least $n-1$. This proves the first part when d is a positive integer. The statement about the monodromy matrix follows as before. $\square$

3.4. **The Plane with Two Punctures.** Consider the twice punctured plane $\mathcal{U} = \mathbb{C} \setminus \{0, 1\}$, and a differential equation

$$\frac{d^n y}{dz^n} + \sum_{i=0}^{n-1} f_i(z) \frac{d^i y}{dz^i} = 0,$$

where $f_i : \mathbb{C} \backslash \{0, 1\} \to \mathbb{C}$ are holomorphic. Now the fundamental group of $\mathcal{U}$ is the free group F_2 on two generators h_0, h_1, given by small loops going counterclockwise once around 0 and 1 respectively. Thus the monodromy representation of the differential equation is a homomorphism from F_2 into $GL_n(\mathbb{C})$; it is completely described by specifying what the images of h_0 and h_1 are. Thus the monodromy is described by giving two matrices in $GL_n(\mathbb{C})$.

The open set $\mathcal{U} = \mathbb{C} \setminus \{0, 1\}$ may also be viewed as $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Thus, the fundamental group of $\mathcal{U}$ can also be thought of as the free group on the small loops h_∞, h_0, h_1 going around $\infty, 0, 1$ modulo the relation $h_\infty h_1 h_0 = 1$. Denote, by A the image of h_∞ and by B^{-1} that of h_0. Then $C = A^{-1}B$.

Since the universal cover of $\mathbb{C} \setminus \{0, 1\}$ is the upper half plane, it follows that the solutions to the foregoing equations are functions on the upper half plane and that the fundamental group of $\mathcal{U}$ is the deck transformation group.

4. The Hypergeometric Differential Equation

Suppose that $\mathcal{U} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Put $\theta = z\frac{d}{dz}$ and let $\alpha_1, \cdots, \alpha_n$, and $\beta_1, \cdots, \beta_n$ be complex numbers. Write

$$(6) \qquad D = (\theta + \beta_1 - 1) \cdots (\theta + \beta_n - 1) - z(\theta + \alpha_1) \cdots (\theta + \alpha_n).$$

This is a differential operator on $\mathcal{U}$. The equation $Dy = 0$ is called the "hypergeometric differential equation" and the solutions are called "hypergeometric functions". These are functions on the upper half plane.

Theorem 6. *Under the monodromy representation considered in the preceding subsection, the monodromy of the generator h_0 around the puncture 0 has characteristic polynomial $\prod(t - e^{2\pi i(1-\beta_j)})$ and the monodromy action of h_∞ has characteristic polynomial $\prod(t - e^{2\pi i\alpha_j})$. Moreover, the element h_1 acts by a complex reflection.*

Proof. Since the hypergeometric differential equation is already written in the "θ" form, and the coefficients of powers of θ are linear polynomials in z, it follows that 0 is a regular singular point of the differential equation $Du = 0$ where D is the operator in (6). The indicial equation at 0 is thus $\prod_{i=1}^n (t + \beta_j - 1)) = 0$. By the Theorem of Fuchs (Theorem 3), it follows that the monodromy of h_0 has characteristic polynomial $\prod(t - e^{2\pi i(1-\beta_j)})$.

We now consider the point ∞; by changing the variable z to the variable $w = \frac{1}{z}$, the operator $\theta_z = z\frac{d}{dz}$ changes to $-\theta_w = -w\frac{d}{dw}$. Multiplying throughout by w, the operator D changes to

$$w(-\theta_w + \beta_1 - 1) \cdots (-\theta_w + \beta_n - 1) - (-\theta_w + \alpha_1) \cdots (-\theta_w + \alpha_n),$$

which is just a constant multiple of the hypergeometric operator

$$D' = (\theta_w - \alpha_1) \cdots (\theta_w - \alpha_n) - w(\theta_w + 1 - \beta_1) \cdots (\theta_w + 1 - \beta_n).$$

Therefore, ∞ is also a regular singular point of the equation $Du = 0$ and the monodromy statement follows as in the preceding paragraph.

Consider now the point $z = 1$. We write out the operator D of (6) (which is in "θ" form) in terms of powers of $\frac{d}{dz}$: this is of the form

$$D = z^n \frac{d^n}{dz^n} + P_{n-1}(z)\frac{d^{n-1}}{dz^{n-1}} + \cdots + P_0(z)$$

$$-z(z^n \frac{d^n}{dz^n} + Q_{n-1}(z)\frac{d^{n-1}}{dz^{n-1}} + \cdots + Q_0(z))$$

where P_i, Q_i are polynomials in z. Therefore,

$$D = z^n(1-z)\frac{d^n}{dz^n} + R_{n-1}(z)\frac{d^{n-1}}{dz^{n-1}} + \cdots + R_0(z),$$

where the $R_i(z)$ are polynomials. Hence the hypergeometric equation $Dy = 0$ at $z = 1$ (after normalising the highest coefficient to be 1), has the property that all its coefficients $\frac{R_i(z)}{z^n(1-z)}$ at $z = 1$ have at most a simple pole at $z = 1$. By Theorem 5 it follows that h_1 maps to a complex reflection. $\qquad\square$

The following theorem says that these facts suffice to characterise the monodromy action.

4.1. **Statement of Levelt's Theorem.** The monodromy representation is very simply described. Suppose that $\alpha_j - \beta_k$ is not an integer for any two suffices j, k. Write $f(x) = \prod_{j=1}^{n}(x - e^{2\pi i \alpha_j}), g(x) = \prod_{k=1}^{n}(x - e^{2\pi i \beta_k})$. These are monic polynomials of degree n. Write $f = x^n + a_{n-1}x^{n-1} + \cdots + a_0$. The quotient ring $R = \mathbb{C}[x]/(f(x))$ is a vector space of dimension n and has as basis the vectors $1, x, \cdots, x^{n-1}$. Write A for the linear operator on the ring R given by multiplication by x. With respect to the foregoing basis, the matrix of A is

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & -a_0 \\ 1 & 0 & 0 & \cdots & -a_1 \\ 0 & 1 & 0 & \cdots & -a_2 \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

and is called the "companion matrix" of f. Similarly, let B be the companion matrix of g. Note that $C = A^{-1}B$ is identity on the first $n - 1$ basis vectors. Hence C is a complex reflection.

Theorem 7. *(Levelt,1960) There exists a basis $\varepsilon_1, \cdots, \varepsilon_n$ of the space of solutions to the hypergeometric equation such that the monodromy representation sends h_0 to B^{-1} and h_∞ to A.*

Moreover, if ρ is any representation of the free group h_0, h_∞ into $GL_n(\mathbb{C})$ such that the images of h_∞, h_0^{-1} have characteristic polynomials f, g, and such that h_1 goes to a complex reflection, then ρ is the above monodromy representation.

The representation described in Levelt's theorem is called the "hypergeometric representation" and the monodromy group is called a "hypergeometric".

We prove Levelt's theorem in the next section.

5. LEVELT'S THEOREM

5.1. **Notation.** Denote by R_0 the ring $\mathbb{Z}[x_i^{\pm 1}, y_i^{\pm 1}]$ of Laurent polynomials in the variables $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ with integral coefficients, and by K_0 its quotient field. Let R denote the sub-ring of R_0 generated by the elementary symmetric functions σ_i in x_i and the elementary symmetric functions τ_j in y_j together with the inverses $\sigma_n^{-1}, \tau_n^{-1}$. Denote by K the quotient field of R; then $K \subset K_0$. Put

$$f = f(t) = \prod_{i=1}^{n}(t - x_i) = t^n + \sum_{i=1}^{n-1} A_i t^{n-i},$$

$$g = g(t) = \prod_{j=1}^{n}(t - y_j) = t^n + \sum_{i=1}^{n} B_i t^{n-i}.$$

Then f, g are polynomials in t with coefficients in R. F_2 denotes the free group on two generators (which in the sequel, are often written h_0, h_∞). Define a representation ρ on $F_2 = < h_0, h_\infty >$ by $h_0 \mapsto A$ and $h_\infty^{-1} \mapsto B$ where A, B are companion matrices of f, g respectively. Denote by $\Gamma = \Gamma(f, g)$ the group generated by A, B in $GL_n(R)$, and by G the Zariski closure of Γ in GL_n. As usual, G^0 denotes the connected component of identity of G; it is a normal subgroup in G of finite index. Denote by Γ^0 the intersection of Γ with G^0.

The group Γ is called the *hypergeometric group* corresponding to the parameters x_i, y_j. Sometimes, it is simply called the **hypergeometric** corresponding to the polynomials f, g above.

Let $\pi : R \to S$ a ring homomorphism with S an integral domain whose quotient field is denoted K_S. Denote by $a_i, b_i \in S$ the images of x_i, y_i under the map $\pi : R \to S$. Denote by f_S, g_S the monic polynomial in t given by

$$f_S(t) = \prod_{i=1}^{n}(t - a_i), \quad g_S(t) = \prod_{i=1}^{n}(t - b_i).$$

We view the free S module S^n as the quotient ring $S[t]/(f_S(t))$. With respect to the basis $1, t, \cdots, t^{n-1}$ of S^n, A_S is simply the matrix of the "multiplication by t" operator. Denote by k the g.c.d. of f_S and g_S. B_S is the operator which acts as follows: $B_S(t^i) = A_S(t^i) = t^{i+1}$ if $i \le n-2$ and $B_S(t^{n-1}) = t^n - g_S(t)$. Let W be the ideal of $S^n = S[t]/(f_S(t))$ generated by the polynomial k; then W is A_S stable. Hence so is $W \otimes K_S$.

Lemma 8. *The subspace $W_S = W \otimes K_S$ is also B_S stable and under the action of A_S, B_S, the subspace W_S is irreducible.*

Moreover, on the quotient V_S/W_S the operators A_S, B_S coincide and as a module over A (multiplication by t), the quotient V_S/W_S is the ring $K_S[t]/(k(t))$.

In particular, if f, g are co-prime (i.e. $a_i \neq b_j$ for any i, j), then $V_S = W_S$ is irreducible for the action of F_2.

Proof. We temporarily write $A_S = A, B_S = B$. Put $D = A - B$. Then, the image of D on V_S is the line generated by $f_S - g_S$. Moreover, D is zero on the monomials $1, t, \cdots, t^{n-2}$ and is $f - g$ on t^{n-1}. Therefore, the image under D of a polynomial of degree exactly $n-1$ is a non-zero multiple of $f - g$.

Any subspace of V_S which is stable under A contains an eigenvector for A; these eigenvectors are of the form $\varepsilon_i = \frac{f(t)}{x - a_i}$ for some i. This a polynomial of degree exactly $n-1$. Hence $D(\varepsilon_i)$ is a non-zero multiple of $f - g$; thus any subspace of V_S which is stable under A, D contains $f - g$ and hence contains $W_S = K_S[t](f - g) + K_S[t]f = K_S[t]k(t) = (k(t))$. This proves the first part of the Lemma.

On the quotient V_S/W_S, the operator D is zero, since the image of D lies in W_S. Hence $A = B$ on the quotient $V_S/W_S = K_S[t]/(k(t))$. This proves the second part.

The third part is a corollary of the first part. $\qquad\square$

Theorem 9. *(Levelt) Suppose that $a_i \neq b_j$ for any i, j. Suppose $h_0 \mapsto a$ and $h_\infty^{-1} \mapsto b$ is any other irreducible representation ρ' of F_2 into $GL_n(\overline{K}_S)$ such that the following two conditions hold. (1) the characteristic polynomial of a is $f_S(t) = \prod(t - a_i)$ and the characteristic polynomial of b is $g_S(t) = \prod(t - b_j)$. (2) $a^{-1}b$ is identity on a co-dimension one subspace of K^n.*

Then ρ' is equivalent to ρ.

Proof. Put $D' = a - b$ and Let W be the kernel of D'. By assumption, W has co-dimension one in V. Write

$$X = \cap_{i=0}^{n-2} a^{-i}W.$$

Since X is an intersection of $n-1$ hyperplanes in V, X is non-zero. Let $v \in X$, with $v \neq 0$.

We claim that v is cyclic for the action of a. Suppose not. Then, $v, av, \cdots, a^{n-1}v$ are linearly dependent. We then claim that $a^{n-1}v$ is a linear combination of $v, av, \cdots, a^{n-2}v$:

Suppose $v, av, \cdots, a^{n-2}v$ are already linearly dependent. By applying a suitable power of a to a linear dependence relation, we see that $a^{n-1}v$ is a linear combination of the vectors $v, av, \cdots, a^{n-2}v$.

Suppose $v, av, \cdots, a^{n-2}v$ are linearly independent. Since the vectors $v, av, \cdots, a^{n-1}v$ are linearly dependent, it follows that $a^{n-1}v$ is a linear combination of $v, av, \cdots, a^{n-2}v$.

Since $f_S(a) = 0$, it follows that the span E of $v, av, \cdots, a^{n-2}v$ is a stable. Since all the vectors $v, av, \cdots, a^{n-2}v$ lie in W by the definition of X, it follows that $a = b$ on E and hence E is stable under Γ. Since $E \neq 0$, it follows that the characteristic polynomial of $a = b$ on E are equal and have a common eigenvalue, contradicting the assumption that $a_i \neq b_j$ for any i, j. Therefore, v is a cyclic vector for the action of a.

Hence $v, av, \cdots, a^{n-1}v$ is a basis for V. It follows that with respect to this basis, the matrix of A is the companion matrix of f_S.

By the construction of X, we have $a^i v \in W$ for $i \leq n-2$. Therefore, $ba^i v = aa^i v = a^{i+1}v$ for $i \leq n-2$. Induction on $i \leq n-2$ shows that $a^i v = b^i v$ for all $i \leq n-1$. With respect to the basis $v, av, \ldots, a^{n-1}v$ of V, the matrix of a is the companion matrix of $f_S(t)$, and that of b is the companion matrix of $g_S(t)$. This completes the proof of the Theorem. $\square$

6. Results of Beukers-Heckman

The Zariski closure of the hypergeometric also has a pleasant description. This is described by Beukers and Heckman [Beu-Hec]. For ease of exposition, we assume that the roots of $f(x), g(x)$ are roots of unity (i.e. α_j, β_k are rational numbers), and that f, g are products of cyclotomic polynomials. Then $f(x), g(x) \in \mathbb{Z}[x]$. Moreover, $f(0) = \pm 1$ and $g(0) = \pm 1$. We recall from the previous section that the monodromy group $H(f, g)$ is generated by the companion matrices A, B of the polynomials f, g respectively.

6.1. **The Finite Case.** We may assume that the numbers α_j and β_k lie in the closed open interval $[0, 1)$. We say that the numbers α_j and β_k "interlace" if between any two α_j there is a β_k and conversely. [Beu-Hec] give a criterion for the monodromy group to be finite in terms of the parameters α, β:

Theorem 10. *(Beukers-Heckman) The hypergeometric group corresponding to the parameters α_j, β_k is finite if and only if the parameters α_j and β_k interlace.*

6.2. **Imprimitivity.** We say that $f(X), g(X)$ are "imprimitive" if there exists an integer $k \geq 2$ and polynomials f_1, g_1 such that $f(x) = f_1(x^k)$, $g(x) = g_1(x^k)$. Otherwise, we say that f, g are a "primitive" pair.

We assume henceforth that f, g form a primitive pair and that α_j, β_k do not satisfy the interlacing condition. Let G be the Zariski closure of the hypergeometric. Write $c = \frac{f(0)}{g(0)}$. Then $c = \pm 1$.

Theorem 11. *(Beukers-Heckman) Suppose that the roots of f, g do not interlace, $f, g \in \mathbb{Z}[x]$ are primitive.*

If $c = -1$ then the Zariski closure of the hypergeometric is isomorphic to $O(n)$, the orthogonal group on n variables.

If $c = 1$, then the Zariski closure is the symplectic group Sp_n (under our assumptions, n will necessarily be even).

We do not prove this theorem, since that would take us too far afield. We refer to [Beu-Hec] for a proof of a more general result from which Theorem 11 follows.

7. Symplectic Case

In this section, we will assume that the Zariski closure of the hypergeometric is a symplectic group; i.e. assume that $f, g \in \mathbb{Z}[x]$, f, g form

a primitive pair and that the roots of f, g do not interlace. Assume that $f(0) = g(0) = 1$ so that in Theorem 11 $c = 1$. By the result of Beukers and Heckman (Theorem 11) the hypergeometric $H(f, g)$ is a Zariski dense subgroup of $Sp_\Omega(\mathbb{Z})$ for a non-degenerate symplectic form Ω on $\mathbb{Q}^n$. It is then an interesting question to ask when $H(f, g)$ has finite index (i.e. when is $H(f, g)$ an arithmetic symplectic group). There is no complete characterisation but some cases are now known.

7.1. Arithmetic Groups. See [Sin-Ven] for the following result.

Theorem 12. *suppose that f, g are as in the beginning of this subsection and that the difference $f - g = c_0 + \cdots + c_d X^d$ with leading coefficient $c = c_d \neq 0$; assume that $\mid c \mid \leq 2$. Then $H(f, g)$ has finite index in $Sp_\Omega(\mathbb{Z})$; thus the hypergeometric group is an arithmetic group.*

As a family of examples, consider, for an even integer n,

$$f(X) = \frac{X^{n+1} - 1}{X - 1} = X^n + x^{n-1} + \cdots + X + 1,$$

$$g(X) = (X + 1)\frac{X^n - 1}{X - 1} = X^n + 2X^{n-1} + 2X^{n-2} + \cdots + 2X + 1.$$

The difference $f - g = -(X^{n-1} + X^{n-2} + \cdots + X)$ has leading coefficient $c = -1$ and hence the hypergeometric $H(f, g)$ has finite index in Sp_Ω, by Theorem 12.

7.2. Thin Groups. We recall the following definition (see [Sar] for details)

Definition 1. Let $\Gamma \subset SL_n(\mathbb{Z})$ be a subgroup and G its Zariski closure in SL_n. Then G is defined over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{Z}) = G \cap SL_n(\mathbb{Z})$. We say that Γ is **thin** if Γ has infinite index in $G(\mathbb{Z})$. Otherwise, we say that Γ is arithmetic.

Note that the notion of thinness and of arithmeticity depends on the embedding $\Gamma \subset SL_n(\mathbb{Z})$.

It is widely believed that most hypergeometric groups in Theorem 11 are thin. Theorem 12 says however, that not all the hypergeometrics are thin. There is no general criterion as to when the hypergeometric are thin, except when the Zariski closure is $O(n, 1)$ (see [FMS]). In the next subsection, we will see examples of thin hypergeometrics with Zariski closure Sp_4 (constructed by Brav and Thomas [Br-Th]).

7.3. **Fourteen Families.** Of special interest are the hypergeometrics corresponding to $f = (X - 1)^4$ (i.e. when the monodromy around infinity is *maximally unipotent*. The number of choices for g are limited: $g \in \mathbb{Z}[X]$ must be a product of cyclotomic polynomials, and must have degree 4; moreover, $g(1) \neq 0$. With these constraints there are exactly 14 choices for g. It is known that the hypergeometric $H(f, g)$ is also the monodromy group associated to a family of *Calabi-Yau threefolds* fibering over the thrice punctured projective line. Moreover, these threefolds turn up in mirror symmetry.

In [AESZ] (see also [CEYY]), the question of thinness or arithmeticity of these groups was first raised. Theorem 11 and its proof enables us to prove that the monodromy group is arithmetic in three of these cases; later Singh [Sin] adapted the method to prove arithmeticity in four more cases. On the other hand, Brav and Thomas [Br-Th] have proved that 7 of these hypergeometric groups are thin. In particular , they prove

Theorem 13. *(Brav and Thomas) Suppose* $f(X) = (X - 1)^4$ *and* $g(X) = \frac{X^5 - 1}{X - 1}$. *Then the hypergeometric group* $H(f, g) \subset Sp_\Omega(\mathbb{Z})$ *(is Zariski dense in* Sp_Ω *and) has infinite index in* $Sp_\Omega(\mathbb{Z})$; *in particular, it is a thin monodromy group.*

To sum up, out of these fourteen families, 7 are arithmetic ([Sin-Ven], [Sin]) and 7 are thin [Br-Th]. Theorem 13 is the first example of a "higher-rank" thin monodromy group whose Zariski closure is a simple group.

7.4. **Questions.** As was mentioned before, there is no general criterion to determine when a group is thin or not. Consider the hypergeometric $H(f, g)$ associated to

$$f(X) = (X - 1)^n, \quad g(X) = \frac{X^{n+1} - 1}{X - 1},$$

say, with even n. It is easy to deduce from [Beu-Hec] that $H(f, g)$ is Zariski dense in Sp_n. When $n = 4$, this is the group considered in Theorem 13 and is thin. However, for $n \geq 6$ and even, it is not known if the group $H(f, g)$ has infinite index in the integral symplectic group. In particular, let us consider the subgroup Γ of $Sp_6(\mathbb{Z}) \simeq Sp_\Omega(\mathbb{Z})$ generated by the companion matrices of $(X - 1)^6$, $\frac{X^7 - 1}{X - 1}$. It is not known if Γ has finite index or not.

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