Holomorphic Curves into Algebraic Subvarieties of an Abelian Variety

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Abstract

We apply a technique introduced in [KR] to holomorphic curves into an algebraic variety whose irregularity is greater than its dimension and establish the Second Main Theorem. This gives a new geometric proof of Bloch's conjecture proved by Ochiai, Kawamata, Green-Griffiths in the late 70's.

1 Introduction

In this paper, we study the value distribution theory of holomorphic curves into a smooth projective algebraic variety whose irregularity is greater than its dimension. This was first studied by Bloch [B] in 1926. Bloch [B] made the following assertion with a sketchy proof.

Let X be a smooth projevtive variety whose irregularity $q(X)$ is greater than its dimension. Then any holomorphic curve $f : \mathbf{C} \to X$ is algebraically degenerate, i.e., its image is contained in a proper algebraic subvariety of X.

Since Bloch's proof contained serious gaps, the above assertion was called Bloch's conjecture. In 1977, Ochiai [O] revived this conjecture and verified the assertion in many important cases (see [O, Conjecture C and Remark

4.2]). Subsequently, Green (see, for example [N-O]) and Kawamata [K] completed the arguments of Bloch-Ochiai in its full generality. Green-Griffiths [G-G] gave a new proof from the point of view of jet differentials and curvature. Ochiai's proof was a proof by contradiction, establishing a virtual version of the Second Main Theorem for "non-degenerate" holomorphic curves $f : \mathbf{C} \to X$. This point of view was followed by Noguchi [No1,2] and he established a Second Main Theorem type estimate

$$T_{f,D}(r) \le CN_{f^*D}(r) + O_{exc}(\log r + \log T_{f,D}(r))^1$$

for algebraically non-degenerate holomorphic curves into algebraic varieties X with abundant logarithmic 1-forms along D, where C is a positive constant independent of f. The constant C stems from a combinatorial argument (see [O], [No1,2] and [N-O]) and is not best possible. The best possible form of the Second Main Theorem was conjectured by Lang [L] in 1986. This is stated as follows.

Conjecture 1 *Let X be a smooth projective algebraic variety, D a divisor with at worst simple normal crossings, E the hyperplane section and K the canonical divisor of X. Then there exists a proper analytic subset D' which depends only on D such that for any holomorphic curve $f : \mathbf{C} \to X$ with the non-degeneracy condition $f(\mathbf{C}) \not\subset D'$ the following inequality holds:*

$$T_{f,D}(r) + T_{f,K}(r) \le N_{f^*D}(r) + O_{exc}(\log r + \log T_{f,E}(r)).$$

At this stage, this conjecture is still open in its full generality. In 1991, the author [KR] established the Second Main Theorem of this form for holomorphic curves into an Abelian variety A. Thus the advantage of our method is that we are able to get estimates arbitrarily close to the best possible estimate. To explain the new technique in [KR], we here give an outline of the proof of the Second Main Theorem for holomorphic curves in Abelian varieties ([KR]). We first approximate the Abelian variety A by a collection $\{S_i\}_{i=1}^M$ of algebraically equivalent embedded algebraic curves and reduce the problem to the collection of equidimensional value distribution problems of maps f_i from a covering space $\pi_i : X_i \to \mathbf{C}$ into a compact Riemann surface S_i. This

[1] The symbol O_{exc} means that the inequality is valid for r outside a Borel set of finite linear measure. See [L-C] for details. Hereafter we do not mention this explicitly.

reduction is made possible by the existence of the parallel translation with respect to the flat metric of A. We then apply Griffiths-King-Noguchi's Second Main Theorem (see, for example, [G-K], [No3], [L-C]) for holomorphic maps $f_i : X_i \to S_i$ and take the "average" over the collection $\{S_i\}_{i=1}^{M}$ of curves. Note that the coefficients involved in Griffiths-King-Noguchi's Second Main Theorem are best possible. At this stage, we know that the original problem is reduced to the estimate of the "average" of the collection of differences of the counting functions which come from the ramification loci of π_i and f_i $(i = 1, \cdots, M)$:

$$N_{\tilde{f}^*\pi^*D_i}(r) - (N_{\tilde{f}^*\pi^*mD}(r) + N_{\tilde{f}^*\widetilde{D}_i}(r)),$$

where $D_i \subset A$ and $\widetilde{D}_i \subset P(TA)$ are certain divisors defined canonically from D and S_i, which count the ramification divisors of π_i and f_i, respectively, and $\tilde{f}$ is the 1-st jet lifting of f to the projectivized tangent bundle $P(TA)$ ([KR]). Second, we make most of the freedom which we find in the choice of the collection $\{S_i\}_{i=1}^{M}$ of curves and use Cauchy-Crofton type averaging argument for each jet maps of f up to order $\dim A$. Then the problem is inductively reduced to estimating the proximity functions of higher jet maps of f, with respect to the analytic sets obtained by taking the projectivized tangent bundles successively of D in $(PT)^i A$'s, where T and P mean the operations of taking the tangent bundle and the projectivization of fibers, respectively. (Note that the proximity function can be defined not only with respect to divisors but also for higher codimensional analytic sets.) Namely, we look at higher order tangential approaches of the holomorphic curve f to the divisor D. For deatils, see [KR, Lemma 6]. At $\dim A$-th step, we are able to prove the desired estimate, by using a natural generalization of [KR, Lemma 4] to higher derivatives (cf. "Lemma on Logarithmic derivatives", see, for example, [L-C], [No3]).

Our new proof of Bloch-Ochiai's theorem is a generalization of the proof of [KR, Theorem 9] and is outlined as follows. Now let X be a smooth projective algebraic variety of dimension m and suppose $q(X) = n > m$. Standard arguments in the theory of Albanese varieties (see, for example, [G-H]) imply that there is a holomorphic map (Albanese map)

$$\tau : X \to A$$

of X into some Abelian variety A of dimension n such that
(i) if ω is a holomorphic 1-form of A, then the pull back $\tau^*\omega$ of ω to X is not identically zero,
(ii) the connected algebraic subgroup of A which leaves $\tau(X)$ invariant consists of the identity only.
By [U], the condition (ii) is equivalent to saying that the proper subvariety $\tau(X)$ of A is of general type. Let $g : \mathbf{C} \to X$ be a holomorphic curve and set $f = \tau \circ g$. Then f fulfills the non-degeneracy condition in [KR, Proposition 1], if f is not contained in a proper subvariety of Y whose Kodaira dimension is smaller than its complex dimension. Suppose that f is non-degenerate in this sense. We get a non-degenerate holomorphic curve

$$f : \mathbf{C} \to A$$

into the Abelian variety A with the constraint

$$f(\mathbf{C}) \subset \tau(X).$$

Set $Y = \tau(X)$. We study this holomorphic curve using the method in [KR]. First of all, we have freedom in choosing an ample divisor in A. We choose D so that $D_Y = D \cap Y$ is a divisor in Y of the form, say,

$$D_Y = 2D_{Y,1} + D_{Y,2} + \cdots + D_{Y,s},$$

and $D_{Y,1}$ an ample divisor of Y, namely, D and Y are tangent along $D_{Y,1}$ with multiplicity ≥ 2. The main point in the proof then lies in the estimate of counting functions coming from the ramification loci of $\pi_i : X_i \to \mathbf{C}$ and $f_i : X_i \to S_i$. Namely, the ramification term $N_{\tilde{f}^*\tilde{D}_i}(r)$ includes contributions of $D_{Y,1}$. Since the non-degeneracy condition in [KR, Proposition 1] is fulfilled, the constraint $g(\mathbf{C}) \subset Y$ and the special position property of D and Y imply

$$\begin{aligned} & N_{\tilde{f}^*\pi^*D_i}(r) - (N_{\tilde{f}^*\pi^*mD}(r) + N_{\tilde{f}^*\tilde{D}_i}(r) \\ \leq \ & -\tfrac{1}{2}N_{f^*D_{Y,1}}(r) + \varepsilon' T_{f,D}(r) + O(\log T_{f,D}(r)) \end{aligned}$$

for any small $\varepsilon' > 0$, if we choose $\{S_i\}$ appropriately, as in the proof of [KR, Theorem 9]. It follows from the Second Main Theorem ([KR, Theorems 1 and 4]) for holomorphic curves in Abelian varieties that there exists a positive constant C independent of f such that

$$N_{f^*D_{Y,1}}(r) \geq CT_{f,D}(r) = CT_{f,D_Y}(r).$$

(Note that $T_{f,D}(r) = T_{f,D_Y}(r)$, where f is a map into A in the left hand side while f is a map to Y in the right hand side.) The effect of this estimate on our argument in the proof of [KR, Theorems 1 and 4] is very strong. Indeed, we have the following estimate:

$$T_{f,D_Y}(r) \leq O_{exc}(\log T_{f,D_Y}(r))$$

if $f(\boldsymbol{C})$ is not contained in a proper subvariety of Y whose Kodaira dimension is smaller than its complex dimension. We note that the above argument remains valid for holomorphic curve segment f from a disk of radius R in $\boldsymbol{C}$ into Y. So the above estimate is the Second Main Theorem for holomorphic curve segment f whose image is not contained in a proper subvariety not of general type, as well as giving the contradiction if we assume that the domain of the definition of f is the entire $\boldsymbol{C}$.

2 Notations

In this paper we shall use freely the standard notations from the Nevanlinna theory. For reader's convenience, we gather basic definitions of Nevanlinna's characteristic function, counting function and proximity function. For more precise information, see, for example, [L], [N-O], [L-C], *etc.*. Let $f : \boldsymbol{C} \to X$ be a holomorphic curve into a smooth projective variety X. Let D be an effective divisor on X and we fix a Hermitian metric $\|\cdot\|$ for the corresponding line bundle $O_X(D)$. Let σ be a canonical section of $O_X(D)$, namely the zero divisor of σ coincides with D. Set

$$\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\frac{1}{\|\sigma\|^2}.$$

Then ω is the curvature form of the Hermitian metric $\|\cdot\|$ and represents the Chern class of $O_X(D)$. Define the *proximity function* of f with respect to D by

$$m_{f,D}(r) = \int_{|z|=r} \log\frac{1}{\|\sigma \circ f\|}\eta$$

where $\eta = \frac{d\theta}{2\pi}$ in polar coordinates. Hereafter we assume, for simplicity, that $f(0) \notin D$. The *counting function* is, by definition

$$N_{f^*D}(r) = \int_0^r \frac{dt}{t} n_{f^*D}(t),$$

where $n_{f^*D}(t)$ is the number of solutions of the equation $f(z) \in D$ in $\{| z |< t\}$ counted with multiplicities. By $N_{f,Ram}(r)$, we denote the counting function of the ramification divisor on $\boldsymbol{C}$ of a holomorphic map f. The *characteristic function* is defined purely differential geometrically by

$$T_{f,D,\omega}(r) = \int_0^r \frac{dt}{t} \int_{|z|<t} f^*\omega.$$

If we change the Hermitian metric we get a new curvature form ω'. Then Jensen's formula implies

$$T_{f,D,\omega'}(r) = T_{f,D,\omega}(r) + O(1).$$

This justifies the notation $T_{f,D}(r)$ up to $O(1)$ terms. Thus the transform

$$D \longmapsto T_{f,D}(r)$$

is a homomorphism (up to $O(1)$ terms). Lang and Cherry [L-C] call this the *height transform.* This name comes from the formal analogy in Nevanlinna theory and Diophantine approximations observed by Vojta [V]. From the point of view of noncompact intersection theory, the First Main Theorem in Nevanlinna theory

$$T_{f,D}(r) = N_{f^*}(r) + m_{f,D}(r) + O(1)$$

is considered to be an analogue of the Gauss-Bonnet theorem. If $\omega \geq 0$ then $T_{f,D}(r)$ is a convex function of $\log r$. So if $T_{f,D}(r)$ is not a constant function, then $T_{f,D}(r) \to \infty$ as $r \to \infty$. Suppose now $O_X(D)$ is semiample, i.e., the linear system $| D |$ is base point free. Then we have Cauchy-Crofton's averaging formula ([N-O,(5.4.11)])

$$T_{f,D}(r) = \int_{D' \in |D|} N_{f^*D'}(r) + O(1)$$

where the measure on $| D |$ is induced from the Fubini-Study measure normalized so that the total measure is 1.

3 A New Proof of Bloch-Ochiai's Theorem

Let A be an Abelian variety of dimension n and Y a proper subvariety of A such that
(i) if ω is a holomorphic 1-form of A, the restriction of ω to Y is not identically zero,
(ii) the connected algebraic subgroup of A which leaves Y invariant consists of the identity only.
By [U], the condition (ii) is equivalent to
(ii)' the proper subvariety Y is of general type.
Write $\boldsymbol{C}(R)$ for the open disk in $\boldsymbol{C}$ centered at the origin and of radius R, possibly $R = \infty$. Let

$$f : \boldsymbol{C} \longrightarrow Y$$

be a non-constant entire holomorphic curve into $Y \subset A$. Suppose D is a reduced ample smooth divisor on A such that $f(\boldsymbol{C}) \not\subset D$ and

$$D \cap Y = 2D_{Y,1} + D_{Y,2} + \cdots + D_{Y,s},$$

where each $D_{Y,i}$ is a reduced irreducible divisor on Y. This means that D and Y are tangent at least along a divisor $D_{Y,1}$ with multiplicity ≥ 2. We further assume that $D_{Y,1}$ is an ample divisor on Y. Such divisor on A certainly exists. We consider f as a holomorphic curve into an Abelian variety A and apply the argument in the proof of [KR, Theorems 1 and 4], with divisor $D \subset A$. Let p be the origin of A and let $\{S_i\}_{i=1}^{M}$ be a collection of mutually algebraically equivalent smooth algebraic curves in A through p, which is chosen generically as in the proof of [KR, Theorems 1 and 4]. Set $k = S_i \cdot D$ and $m = c_1(K_{S_i}) > 1$. Define a divisor D_i on A by

$$D_i = \phi_i^{-1}(\Delta),$$

where

$$\phi_i : A \longrightarrow S^k(A), \ \phi_i(a) = \{D \cap (S_i + a)\}$$

which is a holomorphic map to the k-th symmetric product of A and Δ is the coinsidence divisor on $S^k(A)$. Let X_i be an open Riemann surface in $\boldsymbol{C} \times D$ defined by the equation

$$X_i = \{(z, q) \in \boldsymbol{C} \times D\, ;\ q - f(z) \in S_i\}.$$

Then X_i is a parabolic covering space of $\boldsymbol{C}$ in the sense of [S]. Namely

$$\pi_i : X_i \longrightarrow \boldsymbol{C}, \ \pi_i(z,q) = z$$

is a finite surjective holomororphic mapping of covering index k. Define for $i = 1, \cdots, M$, a holomorphic map of a covering space X_i of $\boldsymbol{C}$ into a compact Riemann surface S_i by

$$f_i : X_i \longrightarrow S_i, \ f_i(z,q) = q - f(z).$$

We take a Hermitian metric $\| \cdot \|$ on $O_A(D)$ and Hermitian metrics $\| \cdot \|_i$ on $L_i = O_{S_i}(p)$ in a uniform way. Let σ_i be a canonical section of $L_i = O_{S_i}(p)$. Taking a positive number $c << M$ so that $\frac{M}{M-c}$ is very close to 1. Then taking the "average" over $\{S_i\}$, we have

$$\begin{aligned} m_{f,D}(r) &= \int_{\partial \boldsymbol{C}(r)} \log(\tfrac{1}{\|\sigma \circ f\|})\eta + O_{c,M}(1) \\ &\leq \tfrac{1}{M-c}\textstyle\sum_{i=1}^{M} \int_{\partial \pi_i^{-1}(\boldsymbol{C}(r))} \log(\tfrac{1}{\|\sigma_i \circ f_i\|_i})\pi^*\eta + O_{c,M}(1) \\ &= \tfrac{1}{M-c}\textstyle\sum_{i=1}^{M}(\tfrac{1}{m}T_{f_i,K_{S_i}}(r) - N_{f_i^*p}(r)) + O_{c,M}(1), \end{aligned}$$

where $O_{c,M}(1)$ is an $O(1)$ term which may depend on c and M but independent of f. Now we recall Griffiths-King-Noguchi's Second Main Theorem ([G-K], [No3], [S], [L-C]). The following form is sufficient for our purpose:

$$T_{f_i,K_{S_i}}(r) + T_{f_i,L_i}(r) - N_{f_i^*p}(r) + N_{f_i,Ram}(r) - N_{\pi_i,Ram}(r)$$

$$\leq O_{exc}(\log r + \log T_{f_i,L_i}(r)).$$

Here we notice

$$N_{\pi_i,Ram}(r) \leq N_{f^*D_i}(r)$$

and

$$T_{f,D}(r) \leq \frac{1}{M-c}\sum_{i=1}^{M} T_{f_i,L_i}(r) + O_{c,M}(1).$$

Then all the above estimates and the concavity of the logarithm imply

$$\begin{aligned} m_{f,D}(r) \leq\ & \tfrac{1}{M-c}\textstyle\sum_{i=1}^{M}\{\tfrac{1}{m}N_{f^*D_i}(r) - (1-\tfrac{1}{m})N_{f^*D}(r) \\ & -\tfrac{1}{m}N_{f_i,Ram}(r)\} - \tfrac{1}{m}T_{f,D}(r) \\ & +O_{exc}(\log r + \log T_{f,D}(r)) + O_{c,M}(1). \end{aligned}$$

We now apply a trick in the proof of [KR, Theorem 1]. Write $D_i = mD + (D_i - mD)$. Then we have

$$
\begin{aligned}
m_{f,D}(r) \ \le \ & \tfrac{1}{m}(\tfrac{M}{M-c}T_{f,mD}(r) - T_{f,D}(r)) \\
& -\tfrac{M}{M-c}(1-\tfrac{1}{m})N_{f^*D}(r) \\
& +\tfrac{1}{m}\tfrac{1}{M-c}\textstyle\sum_{i=1}^{M}\{N_{f^*D_i}(r) - (N_{f^*[mD]}(r) + N_{f_i,Ram}(r))\} \\
& +O_{exc}(\log r + \log T_{f,D}(r)) + O_{c,M}(1).
\end{aligned}
$$

From the First Main Theorem, we have

$$
\begin{aligned}
T_{f,D}(r) \ \le \ & C_{M,c,m}N_{f^*D}(r) \\
& +\tfrac{1}{m}\tfrac{1}{1-\frac{M}{M-c}+\frac{1}{m}}\tfrac{1}{M-c}\textstyle\sum_{i=1}^{M}\{N_{f^*D_i}(r) \\
& -(N_{f^*[mD]}(r) + N_{f_i,Ram}(r))\} \\
& +O_{exc}(\log r + \log T_{f,D}(r)) + O_{c,M}(1)
\end{aligned}
$$

with

$$
\begin{aligned}
C_{M,c,m} \ = \ & \frac{1-\frac{M}{M-c}+\frac{M}{M-c}\frac{1}{m}}{1-\frac{M}{M-c}+\frac{1}{m}} \\
= \ & 1 + (\text{small positive number}).
\end{aligned}
$$

Let $\tilde{f}$ be the first jet lifting of f:

$$\tilde{f} : \boldsymbol{C} \longrightarrow P(TA) = A \times P_{n-1}(\boldsymbol{C}),$$

where $P(TA)$ is the projectivized tangent bundle of A. Let $\pi : P(TA) \to A$ be the natural projection. We define a divisor $\widetilde{D_i}$ on $P(TA)$ by

$$\widetilde{D_i} = \cup_{a \in A} D_a,$$

where D_a stands for the divisor of degree $k = S_i \cdot D$ in $P_{n-1}(\boldsymbol{C})$ defined by the images in $P_{n-1}(\boldsymbol{C})$ of the tangent hyperplanes of D at $D \cap (S_i + a)$. Since f_i ramifies at (z, q) if $\tilde{f}$ hits $\widetilde{D_i}$, we have

$$N_{f_i,Ram}(r) \ge N_{\tilde{f}^*\widetilde{D_i}}(r).$$

We thus reduce the problem to the estimate of the averaged differences of counting functions

$$\sum_{i=1}^{M}\{N_{\widetilde{f}^*\pi^*D_i}(r)-(N_{\widetilde{f}^*\pi^*[mD]}(r)+N_{\widetilde{f}^*\widetilde{D}_i}(r))\}.$$

We now have from [KR, Lemma 2]

Lemma 1 *Let H be a hyperplane in $P_{n-1}(C)$. Then there exists an ample hypersurface $E_i \subset A$ such that*

$$O_{A\times P_{n-1}(C)}(\widetilde{D}_i)=O_{A\times P_{n-1}(C)}(\pi^*E_i+k(A\times H))$$

and

$$c_1(O_A(D_i-(mD+E_i)))=0.$$

This formula is the reason why we used the trick $D_i = mD + (D_i - mD)$. Lemma 1 means that, if we can replace the counting functions by the corresponding characteristic functions, then we are done. But of course we cannot do so, except $N_{f^*[mD]}(r)$, by Cauchy-Crofton's formula, since $[mD]$ may be replaced by any other member of the linear system $\mid mD \mid$ and we may assume by choosing $\{S_i\}$ suitably that $\mid mD \mid$ is semiample. We now observe that we can take the freedom to choose a collection $\{S_i\}$ of curves in A and that the divisors $D_i \subset A$ and $\widetilde{D}_i \subset A\times P_{n-1}(C)$ deform if S_i deforms. Thus we find the possibility of using Cauchy-Crofton type averaging argument over a collection $\{S_i\}$ of curves in A. Now define a codimension 2 subvariety $\widetilde{D}$ of $P(TA)$ by

$$\widetilde{D}=P(TD)=\bigcup_{a\in D}[\text{the projectivization of } T_aD].$$

Let E be the exceptional divisor in the blow up along $\widetilde{D}\subset P(TA)$ and let $\overline{f}$ be the canonical extension of f to the blown up target. We denote the proper transforms (in the blown up space) of divisors in $P(TA)$ by the same symbol. We see that every $\widetilde{D}_i \subset P(TA)$ contains $\widetilde{D}$ as a codimension 1 subvariety. Therefore, Lemma 1 and Cauchy-Crofton type averaging argument (see discussions after [KR, Lemma 4] and the proof of [KR, Theorem 9]) implies that for any small positive number ε' we can choose a collection $\{S_i\}_{i=1}^M$ of curves (this choice may vary according as r varies) such that

$$\begin{aligned} N_{\tilde{f}^*\tilde{D}_i}(r) &= T_{\tilde{f},\tilde{D}_i}(r) + N_{\tilde{f}^*E}(r) - \varepsilon' T_{\tilde{f},\tilde{D}_i}(r) \\ &= T_{\tilde{f},\tilde{D}_i}(r) - T_{\tilde{f},E}(r) + N_{\tilde{f}^*E}(r) - \varepsilon' T_{\tilde{f},\tilde{D}_i}(r). \end{aligned}$$

We now recall our special situation. The holomorphic curve $f : \boldsymbol{C} \to A$ has its image in Y and the special position property of D and Y. Since Y is tangent to D along $D_{Y,1}$, we have the following essential estimate (cf. the proof of [KR, Theorem 9]):

$$N_{\tilde{f}^*E}(r) \geq \frac{1}{2} N_{f^*D_{Y,1}}(r).$$

Although f is algebraically degenerate in the sense $f(\boldsymbol{C}) \subset Y$, we have from [KR, Lemma 6]

Lemma 2 *Let the notations be as above. Then for any small $\varepsilon > 0$, we have*

$$m_{\tilde{f},\tilde{D}}(r) \leq \varepsilon T_{f,D}(r) + O_{exc}(\log r + \log T_{f,D}(r)) + O_\varepsilon(1),$$

where $O_\varepsilon(1)$ is an $O(1)$ term depending on ε and $O_{exc}(\log r + \log T_{f,D}(r))$ term is essentially explicit by the error estimate in [L-C].

Since we have assumed that the non-degeneracy condition in [KR, Proposition 1] for the entire holomorphic curve f is fulfilled, we have

Lemma 3 *Notations as above. We have, for any small $\varepsilon > 0$,*

$$T_{\tilde{f},E}(r) \leq \varepsilon T_{f,D}(r) + O_{exc}(\log r + \log T_{f,D}(r)) + O_\varepsilon(1).$$

We note that in the proof of Lemmas 2 and 3 ([KR, Lemma 6 and Proposition 1]), consideration of higher order *jet differentials* is essential. This is analogous to the fact that both Ochiai and Green-Griffiths considered jet differentials. Thus, we get a desired estimate for the difference of counting functions:

$$\begin{aligned} &N_{\tilde{f}^*\pi^*D_i}(r) - (N_{\tilde{f}^*\pi^*[mD]}(r) + N_{\tilde{f}^*\tilde{D}_i}(r)) \\ &\leq -\tfrac{1}{2} N_{f,D_{Y,1}}(r) + \varepsilon T_{f,D}(r) + O_{exc}(\log r + \log T_{f,D}(r)). \end{aligned}$$

Since $D_{Y,1}$ is an ample divisor on Y, we have from [KR, Theorem 1]

$$\begin{aligned} N_{f^*D_{Y,1}}(r) &= T_{f,D_{Y,1}}(r) - m_{f,D_{Y,1}}(r) \\ &\geq CT_{f,D_Y}(r) - m_{f,D_{Y,1}}(r) \\ &\geq CT_{f,D}(r) - m_{f,D}(r) \\ &\geq (C-\varepsilon)T_{f,D}(r) + O_{exc}(\log r + \log T_{f,D}(r)) \end{aligned}$$

for some positive constant C which depends on the choice of D only. (Although Y is not necessarily smooth, $T_{f,D_Y}(r)$ is well defined by taking a desingularization of Y and pull back of the Chern form of $O_A(D)$.) We thus have

$$\begin{aligned} T_{f,D}(r) &\leq C_{M,c,m}N_{f,D}(r) - \frac{1}{m}\frac{1}{1-\frac{M}{M-c}+\frac{1}{m}}\frac{1}{M-c}\frac{C-\varepsilon}{2}T_{f,D}(r) \\ &\quad + O_{exc}(\log r + \log T_{f,D}(r)) + O_{c,M}(1) + O_{\varepsilon}(1). \end{aligned}$$

If we choose M sufficiently big and c sufficiently small and take $\varepsilon > 0$ sufficiently small, we finally have

$$T_{f,D}(r) \leq O_{exc}(\log r + \log T_{f,D}(r)).$$

Since f is a holomorphic curve into an Abelian variety, we have $T_{f,D}(r) \geq cr^2$ for some positive constant c (depending on $f'(0)$ only) (cf. [N-O,(5.2.33)]. Note that our definition of the characteristic function uses $\int_0^r \frac{dt}{t}\cdots$ while Noguchi-Ochiai's definition uses $\int_1^r \frac{dt}{t}\cdots$.). Therefore, if f is not a constant map, we get a contradiction. Therefore, $f(\mathbf{C})$ must be contained in a proper subvariety of Y whose Kodaira dimension is less than its dimension. This completes the proof of Bloch-Ochiai's Theorem:

Theorem 1 *Let X be a smooth projective variety whose irregularity $q(X)$ is greater than its dimension. Then any holomorphic curve $f : \mathbf{C} \to X$ is algebraically degenerate, i.e., its image is contained in a proper subvariety.* □

It is easy to show the following Corollary (see [N-O,(6.4.11)]).

Corollary 1 *Let f be an entire holomorphic curve into an Abelian variety. Then the Zariski closure of f is a translation of an Abelian subvariety.*

From Theorem 1 and Corollary 1, we have

Theorem 2 *Let X be a smooth projective variety whose irregularity $q(X)$ is greater than its dimension. Then any holomorphic curve $f : \boldsymbol{C} \longrightarrow X$ is algebraically degenerate and its image is contained in a proper algebraic subvariety V of X such that $\tau(V)$ is a translated Abelian subvariety of the Albanese variety of X, where τ is the Albanese map.*

Suppose now that we start with a holomorphic curve segment

$$f : \boldsymbol{C}(R) \to Y \subset A$$

instead of an entire holomorphic curve. The problem is that Lemma 3 ([KR, Proposition 1]) is valid only for entire holomorphic curves, in general. On the other hand, it is clear from the condition (i) (the restriction of any holomorphic 1-form of A to Y is not identically zero) that the holomorphic curve segment is not contained in a possibly non-closed linear subspace of A, it is contained in a translated proper Abelian subvariety. So we can use Lemma 3 in this case also. We thus get the Second Main Theorem for holomorphic curve segment

$$T_{f,D}(r) \leq O_{exc}(\log r + \log T_{f,D}(r))$$

from which we are able to estimate the Landau-Schottky radius for holomorphic curve segment in $Y(\subset A)$ which is not contained in a translation of a proper Abelian subvariety. Thus we recover Green's criterion on hyperbolicity of smooth subvarieties of an Abelian variety [G]:

Corollary 2 (Green) *A smooth proper subvariety Y of an Abelian variety is Kobayashi hyperbolic if and only if Y contains no translated Abelian subvariety.*

4 Proof of Lemma 3

This section is devoted to the proof of Lemma 3. Since the proof of Lemma 3 is a reproduction of the proof of [KR, Proposition 1], the reader can skip this section. From Lemma 2 ([KR, Lemma 6]), to prove Lemma 4, it suffices to show the following

Lemma 4 *Let A be an Abelian variety and D an ample divisor. Let $f : \mathbf{C} \to A$ be a holomorphic curve such that $f(\mathbf{C})$ is not contained in a (not necessarily closed) complex subvariety V which is not Zariski dense and is foliated by translations of a proper Abelian subvariety (e.g., $f(\mathbf{C})$ is not contained in a closed subvariety whose Kodaira dimension is less than its complex dimension.). Let $\overline{N}_{f^*D}(r)$ denote the counting function without counting multiplicities. Then we have*

$$0 \leq N_{f^*D}(r) - \overline{N}_{f^*D}(r) \leq O(\log r + \log T_{f,D}(r)).$$

Proof. From the proof of Lemma 2, which is identical to that of [KR, Lemma 6], we have

$$\overline{N}_{\overline{f}^*E(r)}(r) \geq \frac{1}{n-1} N_{\overline{f}^*E}(r) + O(\log r + \log T_{f,D}(r)).$$

On the other hand, the middle term $N_{f^*D}(r) - \overline{N}_{\overline{f}^*D}(r)$ of the consequence of Lemma 4 is dominated by $N_{\overline{f}^*E}(r) = N_{\tilde{f}^*\tilde{D}}(r)$. Therefore, to prove Lemma 4, it suffices to show

$$\overline{N}_{\overline{f}^*E}(r) \leq O(\log r + \log T_{f,D}(r)).$$

Let $\mathcal{D}$ be an irreducible algebraic variety parametrizing curves (one dimensional algebraic subvarieties) in A. We shall choose $\mathcal{D}$ so that the curves parametrized by $\mathcal{D}$ have sufficiently large degree with respect to $c_1(O_A(D))$. At each $z \in \mathbf{C}$, we consider a jet $j(f)(z)$ of f up to order, say, d. We choose $\mathcal{D}$ so that at each $z \in \mathbf{C}$, the number of curves in $\mathcal{D}$ which pass through $f(z)$ and which have $j(f)(z)$ as their d-th jet at $f(z)$ is finite, say b. Let $C_1(z), \cdots, C_b(z)$ be such curves for $z \in \mathbf{C}$. If $\zeta \in \mathbf{C}$ varies around a neighborhood of $0 \in \mathbf{C}$, we have a collection of holomorphic curve segments $B_1(z+\zeta), \cdots, B_b(\zeta)$ in D by cutting $C_1(z+\zeta), \cdots, C_b(z+\zeta)$ by D. We then consider the $(n-1)$-st osculating space $B_i'(z) \wedge B_i''(z) \wedge \cdots \wedge B_i^{(n-1)}(z)$, $(1 \leq i \leq b)$ in A of (irreducible components of) these curve segments in D. This is not necessarily well-defined, because the above wedge product may vanish identically. But, for the time being, we assume that it is well-defined. Now suppose that $z \in \mathbf{C}$ is such that $\tilde{f}(z)$ hits $\tilde{D}$. If d is chosen sufficiently large, $d \geq d_0$ (such d_0 depends on the extrinsic geometry of D in A only), then (the projectivization of) the $(n-1)$-st osculating space of one of $B_i(z+\zeta)$ at $f(z)$ contains (the projectivization of) the tangent vector of f at $f(z)$. We thus have, generically, for each $z \in \boldsymbol{C}$, a finite number, say d', of points in D such that the tangent hyperplanes coincides with those $(n-1)$-st osculating spaces of $B_1(z+\zeta), \cdots, B_b(z+\zeta)$ at $\zeta = 0$. This gives rise to a holomorphic curve g into $P(H^0(P_{n-1}(\boldsymbol{C}), O(d)))$. From [KR, Lemma 4] and its natural generalization to higher derivatives, we see

$$T_{g,O(1)}(r) \leq (\log r + \log T_{f,D}(r)).$$

We consider g as a moving target in the sense of [R-S]. The First Main Theorem for moving target [R-S] states

$$T_{[f'],d'H}(r) + T_{g,O(1)}(r) = N_{[f'],g}(r) + m_{[f'],g}(r) + O(1),$$

where $N_{[f'],g}(r)$ and $m_{[f'],g}(r)$ are counting and proximity functions of $[f']$ with respect to the moving target g (see [R-S]). [KR, Lemma 4] implies

$$T_{[f'],d'H}(r) \leq O(\log r + \log T_{f,D}(r))$$

if f is not constant. We thus have

$$N_{[f'],g}(r) \leq O(\log r + \log T_{f,D}(r)).$$

Clearly, we have

$$\overline{N}_{\overline{f}^* E}(r) \leq N_{[f'],g}(r).$$

Thus, we have the desired estimate:

$$\overline{N}_{\overline{f}^* E}(r) \leq O(\log r + \log T_{f,D}(r)),$$

if every $B_i'(z) \wedge B_i''(z) \wedge \cdots \wedge B_i^{(n-1)}(z) \not\equiv 0$ $(1 \leq i \leq b)$ (i.e., the moving target g is well-defined) and if $f(\boldsymbol{C})$ is not contained in the moving target g. So the above argument breakes down if
(i) $B_i'(z) \wedge B_i''(z) \wedge \cdots \wedge B_i^{(n-1)}(z) \not\equiv 0$ for all irreducible components and the image of $[f']$ is contained in the moving target, or
(ii) $B_i'(z) \wedge B_i''(z) \wedge \cdots \wedge B_i^{(n-1)}(z) \equiv 0$ for an irreducible component for some i, say $i = 1$.

Note that we have a freedom in choosing $\mathcal{D}$. So, if the above argument breakes down for all admissible choice of $\mathcal{D}$, we see that eifher
(i) $f(C)$ is contained in a proper Abelian subvariety of dimension $n-1$ in A, or
(ii) some curve segment $B_i(z+\zeta)$ in D is contained in a proper Abelian subvariety of dimension $\leq n-2$.
But this will be the case only if the image $f(C)$ is contained in a (not necessarily closed) subvariety $V \subset A$ which is not Zariski dense and such that
(a) for some étale covering of A, the pull back of V splits a proper Abelian subvariety A',
(b) V is tangent to D along a divisor (codimension 1 analytic set) W in V and V is foliated by translations of the above Abelian subvariety A',
(c) (after taking a suitable étale covering) W is foliated by translations of a divisor in A'.
If V is closed, it is shown in [U] that any proper subvariety of an Abelian variety whose Kodaira dimension is smaller than its complex dimension is foliated by the translations of a proper Abelian subvariety. This completes the proof of Lemma 4. □

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