Abelian Integral in Chiral Potts Model

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Abstract. We show how the ground state energy per site for the phase I regime in the superintegrable chiral Potts model can be expressed as abelian integrals of second kind of the hyperelliptic curve for rapidity variables. The Prym varieties related to this curve are explicitly described for the splitting of Jacobian variety of the rapidity curve.

Introduction

Considerable progress has been made in understanding chiral Potts model of statistical mechanics [1–9] [11]. This model gives the solutions of Yang-Baxter equation (or the star-triangle equation) with Boltzmann weights depending on "rapidities" which lie on high genus curves. For the N-state chiral Potts model, these "rapidity" curves have the algebraic expression:

$$W_{N,k'}: \qquad t^N = \frac{(1-k'\lambda)(1-k'\lambda^{-1})}{k^2} \quad , (t,\lambda) \in \mathbf{C}^2,$$

depending on a complex parameter k', $k'^2 \neq 0,1$, here $k^2 + k'^2 = 1$. For the eigenvalue problem of quantum chiral Potts spin chain associated to this statistical mechanical model, many important results have been derived from the physical consideration, e.g., the ground state energy, the phenomena of level crossing transition to a new ground state [1, 4, 5, 11]. All those results are obtained with no uniformization substitutions for the Riemann surface $W_{N,k'}$. However we have seen many tantalizing clues that the theory of abelian integrals have much involved in the solution of chiral Potts model and would like to make these structure explicit. Using the abelian integral of first kind, Baxter [6] has obtained the hyperelliptic theta function parametrization of the curve $W_{N,k'}$. The mathematical treatment of these hyperelliptic theta functions is given in [12], which is based on symmetries of the curves $W_{N,k'}$. In this paper, we would like to indicate the abelian integrals of second kind also appear in the ground state energy per site found by Baxter, McCoy et al. [1, 4].

The ground state energy per site for the (phase I) regime $0 \leq k' \leq 1$ of the superintegrable chiral Potts model [1] is expressed by

$$e_0^{\mathrm{I}}(k') = -(1+k') \sum_{\ell=1}^{N-1} F\left(-\frac{1}{2}, \frac{\ell}{N}; 1; \frac{4k'}{(1+k')^2}\right) \qquad (1)$$

and at $k' = 1$,

$$e_0^{\mathrm{I}}(1) = -4\pi^{-\frac{1}{2}} \sum_{\ell=1}^{N-1} \frac{\Gamma(\frac{3}{2} - \frac{\ell}{N})}{\Gamma(\frac{N-\ell}{N})} \quad ,$$

where $F(a,b;c;\kappa)$ is the hypergeometric function and has the expression

$$F(a,b;c;\kappa) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \xi^{b-1}(1-\xi)^{c-b-1}(1-\kappa\xi)^{-a} d\xi \qquad (2)$$

$$(\mathrm{Re}c > \mathrm{Re}b > 0, \ |\kappa| < 1).$$

We shall show the above expression is an abelian integral of second kind of the "rapidity" curve $W_{N,k'}$. For our purpose, we shall restrict our attention to only the regime

$$0 < k' < 1$$

unless otherwise specified except Sect. 4.

The organization of this paper is as follows. In Sect. 1 we shall list the basic geometrical properties of the curve $W_{N,k'}$ needed for later discussion and state the main result of this paper (Theorem 1) which is the relation of (1) and the abelian integral of second kind for the curve $W_{N,k'}$. The proof of this result will be given in Sect. 2 (for odd N) and Sect. 3 (for even N). The structure of Prym varieties related to $W_{N,k'}$ inevitably emerges from the discussion of energy per site for even N in Sect. 3.. In Sect. 4, we shall study the other Prym varieties related to $W_{N,k'}$. We have found the splitting of Jacobian of $W_{N,k'}$ into a dual pair of abelian subvarieties, which are the Jacobians of different quotients of $W_{N,k'}$. The conclusions are stated in Theorem 2 and 3. The important aspect for the splitting of Jacobian related to their hyperelliptic theta functions has not been discussed in this paper. Its understanding should be useful for the application to the solvable statistical system. Also for the non-superintegrable chiral Potts model, the relation between abelian integrals and ground energy per site is not clear at this moment. Work along this line is under consideration.

I would like to thank Professor B. M. McCoy for introducing me the research area of chiral Potts models. It is believed that the geometry underlying the rapidity curves should somehow explain the nature of the solution of this statistical model and this work is out of one of the attempts. We are able to relate the result obtained by McCoy et al [1] to the geometry of Riemann surface for rapidity variables. I am most grateful to Professor F. Hirzebruch for his kind invitation and warm hospitality of Max-Planck-Institut für Mathematik where this work was completed.

Section 1. Chiral Potts curves

The algebraic curves for the "rapidity" variables of chiral Potts N-state model are characterized as hyperelliptic curves of genus $N-1$ having an order N automorphism with exactly 4 fixed points. Here we shall always assume

$$N \geq 3 ,$$

and call them the CP ($N-$) curves. The geometry of such curves have been studied in [12] . In this section we shall describe the basic properties needed for the discussion of this paper.

For a fixed N, the CP curves are defined by

$$W_{N,k'}: \quad t^N = \frac{(1-k'\lambda)(1-k'\lambda^{-1})}{k^2} \quad , (t,\lambda) \in \mathbf{C}^2, \qquad (3)$$

depending on a a complex parameter k', $k'^2 \neq 0,1$, here $k^2 + k'^2 = 1$. They can also expressed by

$$W_{N,k'}: \quad \mathrm{w}^2 = \left(t^N - \frac{1-k'}{1+k'}\right)\left(t^N - \frac{1+k'}{1-k'}\right) \quad , (t,\mathrm{w}) \in \mathbf{C}^2, \qquad (4)$$

with the relations

$$\mathrm{w}^2 = \frac{k'}{k^2}(\lambda - \lambda^{-1}) \, ,$$
$$\lambda = \frac{1}{2k'}\left\{k^2\left(\mathrm{w} - t^N\right) + k'^2 + 1\right\}.$$

The automorphism group $\mathrm{Aut}(W_{N,k'})$ of $W_{N,k'}$ has the order $4N$ with the following generators

$$\begin{aligned} \theta: & \ (t,\lambda) \rightsquigarrow (\omega t, \lambda), \\ \sigma: & \ (t,\lambda) \rightsquigarrow (t, \lambda^{-1}), \\ \iota: & \ (t,\lambda) \rightsquigarrow \left(\frac{1}{t}, \frac{1-k'\lambda}{k'-\lambda}\right), \end{aligned} \qquad (5)$$

here $\omega = e^{\frac{2\pi i}{N}}$. In the coordinate of (4), the above generators are expressed by

$$\begin{aligned} \theta: & \ (t,\mathrm{w}) \rightsquigarrow (\omega t, \mathrm{w}), \\ \sigma: & \ (t,\mathrm{w}) \rightsquigarrow (t, -\mathrm{w}), \\ \iota: & \ (t,\mathrm{w}) \rightsquigarrow \left(\frac{1}{t}, \frac{\mathrm{w}}{t^N}\right). \end{aligned} \qquad (6)$$

σ is the hyperelliptic involution, and θ is the order N element with 4 fixed points. We have

$$\iota\theta = \theta^{-1}\iota \, ,$$
$$\text{Center of } \mathrm{Aut}(W_{N,k'}) = \left\{ \begin{array}{ll} <\sigma> & \text{for odd } N \\ <\sigma, \theta^{\frac{N}{2}}> & \text{for even } N \end{array} \right. . \qquad (7)$$

For the rest of this note, t, λ, w will always be the coordinates (3) (4) of the CP curve $W_{N,k'}$ and θ, σ, ι the automorphisms defined by (5) or (6). It is known that that the abelian differential of first kind of $W_{N,k'}$ is given by

$$\Gamma(W_{N,k'}, \Omega) = \left\{ p(t)\frac{dt}{\mathrm{w}} \mid p(t) : \text{polynomial of } t \text{ with degree } \leq N-2 \right\}.$$

For the expression of ground state energy per site (1), it involves the abelian differentials of second kind which we are going to describe.

Let $\mathbf{b}_j, \mathbf{b}'_j$, $1 \leq j \leq N$, be the elements of $W_{N,k'}$ fixed by σ with the t coordinate given by

$$\mathbf{b}_j \leftrightsquigarrow \omega^{-j}\left(\frac{1+k'}{1-k'}\right)^{\frac{1}{N}}, \quad \mathbf{b}'_j \leftrightsquigarrow \omega^{-j}\left(\frac{1-k'}{1+k'}\right)^{\frac{1}{N}},$$

and $\mathbf{p},\mathbf{p}',\mathbf{q},\mathbf{q}'$ be the elements fixed by θ with the λ coordinate given by

$$\mathbf{p} \leftrightsquigarrow 0, \quad \mathbf{p}' \leftrightsquigarrow \infty, \quad \mathbf{q} \leftrightsquigarrow k', \quad \mathbf{q}' \leftrightsquigarrow k'^{-1}.$$

Define the following meromorphic forms on the Riemann surface $W_{N,k'}$,

$$\Upsilon_\ell = \left(\frac{t^N - \frac{1+k'}{1-k'}}{t^N - \frac{1-k'}{1+k'}}\right)\frac{t^{\ell-1}dt}{\mathrm{w}} \quad , \ \ell = 1,\ldots,N-1. \qquad (8)$$

The divisor of Υ_ℓ has the expression:

$$\mathrm{div}(\Upsilon_\ell) = (\ell-1)(\mathbf{q}+\mathbf{q}') + (N-\ell-1)(\mathbf{p}+\mathbf{p}') + 2\sum_{j=1}^{N}(\mathbf{b}_j - \mathbf{b}'_j).$$

Hence Υ_ℓ, $1 \leq \ell \leq N-1$, are $N-1$ linear independent abelian differentials of second kind of the CP curve $W_{N,k'}$. Denote

$\widetilde{\mathbf{qp}}$ = the path in $W_{N,k'}$ from $\mathbf{q}$ to $\mathbf{p}$ with t coordinate along $\arg(t) = \frac{\pi}{N}$.

We now state the main result of this paper:

Theorem 1 . With Υ_ℓ, $\widetilde{\mathbf{qp}}$ as above, the ground state energy per site (1) for the (phase I) regime of $0 < k' < 1$ is

$$e_0^{\mathrm{I}}(k') = N(1+k')\sum_{\ell=1}^{N-1}\frac{(-1)^{\frac{\ell-1}{N}}}{\Gamma(\frac{\ell}{N})\Gamma(1-\frac{\ell}{N})}\left(\frac{1-k'}{1+k'}\right)^{2-\frac{\ell}{N}}\int_{\widetilde{\mathbf{qp}}}\Upsilon_\ell$$

The proof of the above theorem will be given in the next 2 sections.

Section 2 . Energy per site for odd N

First we state an easy lemma and omit the proof:

Lemma 1 . The automorphism T of P^1 with

$$\mathrm{T}(0) = 0, \ \mathrm{T}(\infty) = 1, \ \mathrm{T}\left(\frac{1-k'}{1+k'}\right) = \infty, \ \mathrm{T}\left(\frac{1+k'}{1-k'}\right) = \frac{(1+k')^2}{4k'}$$

is given by by the transformation

$$\xi = \mathrm{T}(\zeta) = \frac{\zeta}{\zeta - \frac{1-k'}{1+k'}} \quad , \; \zeta \in \mathbf{C} \; ,$$

and the following relations hold:

$$\zeta = -\left(\frac{1-k'}{1+k'}\right)\frac{\xi}{1-\xi} \; ,$$

$$d\xi = -\frac{\frac{1-k'}{1+k'}}{(\zeta - \frac{1-k'}{1+k'})^2} \, d\zeta \quad . \tag{9}$$

We now introduce the variables

$$\mathrm{v} = \frac{t^{N-1}}{\mathrm{w}} \; , \qquad \zeta = t^N \; ,$$

$$\eta = \left(\frac{1+k'}{1-k'}\right)^{1-\frac{1}{N}} \mathrm{v}, \quad \xi = \frac{\zeta}{\zeta - \frac{1-k'}{1+k'}} \quad . \tag{10}$$

By (3), (4), (9), (10), , we have

$$\mathrm{v}^{2N} = \frac{\zeta^{2N-2}}{(\zeta - \frac{1-k'}{1+k'})^N (\zeta - \frac{1+k'}{1-k'})^N} \; , \qquad (\zeta, \mathrm{v}) \in \mathbf{C}^2, \tag{11}$$

$$\eta^{2N} = \frac{\xi^{2N-2}(1-\xi)^2}{\left(1 - \frac{4k'}{(1+k')^2}\xi\right)^N} \; , \qquad (\xi, \eta) \in \mathbf{C}^2 \; . \tag{12}$$

The above curves are birational equivalent, hence with the same Riemann surface for their non-singular models. In this section we are going to discuss the case for odd N. The even N case will be discussed in the next section.

For odd N, the curve (11) is irreducible. The corresponding Riemann surface is a cyclic $2N$-fold covering of the Riemann sphere with the projection map

$$(\zeta, \mathrm{v}) \rightsquigarrow \zeta \; ,$$

and the covering transform is induced by

$$(\zeta, \mathrm{v}) \rightsquigarrow \left(\zeta, e^{\frac{\pi i}{N}} \mathrm{v}\right) .$$

Proposition 1 . For odd N, $W_{N,k'}$ is birational to the curve (11) (or (12)), and the projection

$$W_{N,k'} \rightarrow W_{N,k'} / < \theta\sigma > \; (= \mathbf{P}^1)$$

corresponds to the map

$$(\zeta, \mathrm{v}) \rightsquigarrow \zeta \quad , (\text{or } (\xi,\eta) \rightsquigarrow \xi) \, .$$

Proof. The automorphism group of $W_{N,k'}$ generated by θ and σ is cyclic with the generator $\theta\sigma$. From the definition of θ and σ, one knows the variable ζ is the coordinate of $W_{N,k'}/ < \theta\sigma >$. By the relation of (10), we have the commutative diagram:

$$\begin{array}{ccc} W_{N,k'} & \rightarrow & \{(\zeta, \mathrm{v}) \in \text{curve } (11)\} \\ \downarrow & & \downarrow \\ W_{N,k'}/ < \theta\sigma > & = & \mathbf{P}^1 = \{\zeta \in \mathbf{C}\} \cup \{\infty\}. \end{array}$$

Both the vertical morphisms are of degree $2N$, therefore the conclusion follows immediately. q.e.d.

Lemma 2 . The expression

$$\left(\frac{\xi}{1-\xi}\right)^{\frac{\ell-1}{N}} \frac{d\xi}{\eta} \quad , 1 \leq \ell \leq N-1,$$

defines the meromorphic form of the curve (12) and equals to the differential of second kind $N(-1)^{1+\frac{\ell-1}{N}} \left(\frac{1-k'}{1+k'}\right)^{2-\frac{\ell+1}{N}} \Upsilon_\ell$ of $W_{N,k'}$ under the isomorphism in Proposition 1.

Proof. By (9), (10),

$$\begin{aligned} \left(\frac{\xi}{1-\xi}\right)^{\frac{\ell-1}{N}} \frac{d\xi}{\eta} &= N(-1)^{1+\frac{\ell-1}{N}} \left(\frac{1-k'}{1+k'}\right)^{2-\frac{\ell}{N}} \frac{\mathrm{w} t^{\ell-1} dt}{\left(t^N - \frac{1-k'}{1+k'}\right)^2} \\ &= N(-1)^{1+\frac{\ell-1}{N}} \left(\frac{1-k'}{1+k'}\right)^{2-\frac{\ell}{N}} \left(\frac{t^N - \frac{1+k'}{1-k'}}{t^N - \frac{1-k'}{1+k'}}\right) \frac{t^{\ell-1} dt}{\mathrm{w}} \\ &= N(-1)^{1+\frac{\ell-1}{N}} \left(\frac{1-k'}{1+k'}\right)^{2-\frac{\ell}{N}} \Upsilon_\ell \quad . \end{aligned}$$

q.e.d.

Lemma 3 . Let C be the path in the curve (12) from (ξ,η)=(0,0) to (1,0) with the real coordinates ξ, η and $0 \leq \xi \leq 1$. Then for $1 \leq \ell \leq N-1$,

$$F\left(-\frac{1}{2}, \frac{\ell}{N}; 1; \frac{4k'}{(1+k')^2}\right) = \frac{1}{\Gamma(\frac{\ell}{N})\Gamma(1-\frac{\ell}{N})} \int_C \left(\frac{\xi}{1-\xi}\right)^{\frac{\ell-1}{N}} \frac{d\xi}{\eta} \quad .$$

Proof. By (2), we have

$$F\left(-\frac{1}{2},\frac{\ell}{N};1;\frac{4k'}{(1+k')^2}\right)$$

$$=\frac{\Gamma(1)}{\Gamma(\frac{\ell}{N})\Gamma(1-\frac{\ell}{N})}\int_0^1 \xi^{\frac{\ell}{N}-1}(1-\xi)^{-\frac{\ell}{N}}\left(1-\frac{4k'}{(1+k')^2}\xi\right)^{\frac{1}{2}}d\xi$$

$$=\frac{1}{\Gamma(\frac{\ell}{N})\Gamma(1-\frac{\ell}{N})}\int_0^1 \xi^{\frac{\ell-1}{N}}(1-\xi)^{\frac{1-\ell}{N}}\frac{d\xi}{\frac{\xi^{\frac{N-1}{N}}(1-\xi)^{\frac{1}{N}}}{\left(1-\frac{4k'}{(1+k')^2}\xi\right)^{\frac{1}{2}}}}$$

$$=\frac{1}{\Gamma(\frac{\ell}{N})\Gamma(1-\frac{\ell}{N})}\int_C \left(\frac{\xi}{1-\xi}\right)^{\frac{\ell-1}{N}}\frac{d\xi}{\eta}\ .$$

q.e.d.

By (1) and Lemma 2, 3, we have obtain Theorem 1 for odd N.

Section 3. Energy per site for even N

In this section we shall discuss the case of even N. In this situation, (11) (or (12)) defines a reducible curve with 2 irreducible components. The component in (11), (12) which the functions $\mathrm{v},\zeta,\eta,\xi$ of (10) for $W_{N,k'}$ satisfy is given by

$$\mathrm{v}^N=\frac{\zeta^{N-1}}{(\zeta-\frac{1-k'}{1+k'})^{\frac{N}{2}}(\zeta-\frac{1+k'}{1-k'})^{\frac{N}{2}}}\,,\qquad (\zeta,\mathrm{v})\in\mathbf{C}^2, \tag{13}$$

$$\eta^N=\frac{-\xi^{N-1}(1-\xi)}{\left(1-\frac{4k'}{(1+k')^2}\xi\right)^{\frac{N}{2}}}\,,\qquad (\xi,\eta)\in\mathbf{C}^2\ . \tag{14}$$

The curves (13) and (14) are birational equivalent and they define a cyclic N-fold covering of the Riemann sphere under the projection map

$$(\zeta,\mathrm{v})\rightsquigarrow\zeta,\ (\text{ or }(\xi,\eta)\rightsquigarrow\xi).$$

It relates to the CP curve $W_{N,k'}$ through the following proposition.

Proposition 2. For even N, denote

$$\overline{W_{N,k'}}:=W_{N,k'}/<\theta^{\frac{N}{2}}\sigma>. \tag{15}$$

Then $\overline{W_{N,k'}}$ is birational to the curve (13) (or (14)), and the projection

$$\overline{W_{N,k'}}\rightarrow W_{N,k'}/<\theta,\sigma>\ (=\mathbf{P}^1)$$

corresponds to the map

$$(\zeta, \mathrm{v}) \rightsquigarrow \zeta \quad , (\text{or } (\xi, \eta) \rightsquigarrow \xi) \, .$$

Proof. The automorphism group $< \theta \; , \; \sigma >$ of $W_{N,k'}$ is isomorphic to $< \theta^{\frac{N}{2}} \sigma > \times < \theta >$. As the function v is invariant under $\theta^{\frac{N}{2}} \sigma$ and the variable ζ is the coordinate of $W_{N,k'} / < \theta, \sigma >$, we have the commutative diagram:

$$\begin{array}{ccc} W_{N,k'} / < \theta^{\frac{N}{2}} \sigma > & \to & \{ (\zeta, \mathrm{v}) \in \text{curve } (13) \} \\ \downarrow & & \downarrow \\ W_{N,k'} / < \theta, \sigma > & = & \mathrm{P}^1 = \{ \zeta \in \mathrm{C} \} \cup \{ \infty \}. \end{array}$$

Both the vertical morphisms are of degree N, therefore the conclusion follows immediately. q.e.d.

It is easy to see that the order 2 automorphism $\theta^{\frac{N}{2}} \sigma$ of $W_{N,k'}$ has no fixed points, hence the genus of $\overline{W_{N,k'}}$ equals to $\frac{N}{2}$. Consider the degree 2 unramified covering

$$\Lambda \left(= \Lambda_{\overline{W_{N,k'}}} \right) : \; W_{N,k'} \to \overline{W_{N,k'}} = W_{N,k'} / < \theta^{\frac{N}{2}} \sigma > \quad , \tag{16}$$

and define the line bundle over $\overline{W_{N,k'}}$:

$$\mathsf{E} = \left(W_{N,k'} \times \mathrm{C} \right) / \left(\mathrm{w} \times \mathrm{z} \sim \theta^{\frac{N}{2}} \sigma(\mathrm{w}) \times (-\mathrm{z}) \right). \tag{17}$$

Denote $\overline{\mathrm{p}} = \Lambda(\mathrm{p})$, $\overline{\mathrm{q}} = \Lambda(\mathrm{q})$, $\overline{\mathrm{b}_j} = \Lambda(\mathrm{b}_j)$, $\overline{\mathrm{b}_j'} = \Lambda\left(\mathrm{b}_j'\right)$ for $1 \leq j \leq \frac{N}{2}$. As the function t of $W_{N,k'}$ satisfies the relation $\left(\theta^{\frac{N}{2}} \sigma \right)^* t = -t$, it corresponds a meromorphic section of E with the divisor $\overline{\mathrm{q}} - \overline{\mathrm{p}}$. Hence we have

$$\mathcal{O}_{\overline{W_{N,k'}}}(\mathsf{E}) = \mathcal{O}_{\overline{W_{N,k'}}}(\overline{\mathrm{q}} - \overline{\mathrm{p}}) \; .$$

It is known that

$$\Lambda_* \Omega_{W_{N,k'}} \simeq \Omega_{\overline{W_{N,k'}}} \oplus \Omega_{\overline{W_{N,k'}}}(\mathsf{E}) \; ,$$

and we shall make the above identification for simplicity of notations.

Lemma 4. Under the isomorphism of Proposition 2, the expression

$$\left(\frac{\xi}{1 - \xi} \right)^{\frac{\ell - 1}{N}} \frac{d\xi}{\eta} \quad , \; 1 \leq \ell \leq N - 1,$$

for the curve (14) corresponds to a meromorphic form of $\overline{W_{N,k'}}$ for odd ℓ, and a meromorphic section of $\Omega_{\overline{W_{N,k'}}}(\mathsf{E})$ for even ℓ. The corresponding form for $\ell = 1, \ldots, N - 1$ has the divisor

$$(\ell - 1)\overline{\mathrm{q}} + (N - \ell - 1)\overline{\mathrm{p}} + 2 \sum_{j=1}^{\frac{N}{2}} \left(\overline{\mathrm{b}_j} - \overline{\mathrm{b}_j'} \right)$$

in $\overline{W_{N,k'}}$, and equals to the differential of second kind $N(-1)^{1+\frac{\ell-1}{N}}\left(\frac{1-k'}{1+k'}\right)^{2-\frac{\ell}{N}}\Upsilon_\ell$ of $W_{N,k'}$.

Proof. The proof is the same as Lemma 3, except we need the following identification:

$$\left(\frac{\xi}{1-\xi}\right)^{\frac{1}{N}} \leftrightsquigarrow \text{meromorphic section of E with divisor } \overline{\mathrm{q}}-\overline{\mathrm{p}}\,.$$

q.e.d.

With the same argument as Lemma 3, we obtain the similar conclusion for even N:

Lemma 5 . Let C be the path in the curve (14) from (ξ,η)=(0,0) to (1,0) with the real coordinates ξ,η and $0\leq\xi\leq 1$. Then for $1\leq\ell\leq N-1$,

$$F\left(-\frac{1}{2},\frac{\ell}{N};1;\frac{4k'}{(1+k')^2}\right)=\frac{1}{\Gamma(\frac{\ell}{N})\Gamma(1-\frac{\ell}{N})}\int_C\left(\frac{\xi}{1-\xi}\right)^{\frac{\ell-1}{N}}\frac{d\xi}{\eta}\ .$$

Combining Lemma 4 and 5, we have obtained Theorem 1 for even N, hence completed the proof of Theorem 1.

Section 4 . Prym varieties related to CP curves

In discussion of the previous section, the Prym variety for the 2–fold cover (16) has involved in the description of meromorphic forms in Lemma 4. In this section we shall study the other Prym varieties related to the CP curves $W_{N,k'}$ and we assume the parameter

$$k'\in\mathrm{C},\quad k'^2\neq 0,1\,.$$

for the rest of our discussion. For convenience of notations, we shall identify the Jacobian variety $J(C)$ of a Riemann surface C with its Picard variety $\mathrm{H}^1(C,\mathcal{O}^*)/\mathrm{H}^1(C,\mathrm{Z})$ through the isomorphism:

$$\begin{array}{rcl}\mathrm{H}^1(C,\mathcal{O}^*)/\mathrm{H}^1(C,\mathrm{Z}) & \simeq & J(C)=\mathrm{Hom}(\Gamma(C,\Omega),\mathrm{C})/\mathrm{H}_1(C,\mathrm{Z}) \\ \mathcal{O}_C(\mathrm{d}'-\mathrm{d}) & \leftrightsquigarrow & \mathrm{class}\left(\int_{\mathrm{d}}^{\mathrm{d}'}\text{holomorphic differential}\right)\end{array}$$

here d,d' are positive divisors of C with the same degree.

First we state a well-known fact in complex geometry and its proof can be found in standard text book in algebraic geometry.

Lemma 6 . Let X be a n−dimensional complex manifold, D a simple divisor of X (i.e., D=$\emptyset$, or submanifold of codimension 1). Suppose L is a complex line bundle over X such that L^d has a holomorphic section $\mathfrak{f}$ with D as the zero divisor. Let

$$\tilde{X} = \left\{ l \in \mathsf{L} \,|\, l^d \in \mathfrak{f} \subset \mathsf{L}^d \right\},$$

$$\pi : \tilde{X} \to X \text{ the restriction of the projection from } \mathsf{L} \text{ to } X,$$

$$r : \tilde{X} \to \tilde{X} \ , \ \tilde{x} \rightsquigarrow e^{\frac{2\pi i}{d}} \tilde{x} \ .$$

Then the relations of holomorphic functions and n-forms between X, $\tilde{X}$ are as follows.

$$\text{(i)} \quad \pi_* \mathcal{O}_{\tilde{X}} \simeq \mathcal{O}_X \oplus \mathcal{O}_X(\mathsf{L}^{-1}) \oplus \ldots \oplus \mathcal{O}_X\left(\mathsf{L}^{-d+1}\right),$$

$$\left\{ \rho \in \pi_* \mathcal{O}_{\tilde{X}} \,|\, \rho(r(\tilde{x})) = e^{j\frac{2\pi i}{d}} \rho(\tilde{x}) \right\} \simeq \mathcal{O}_X(\mathsf{L}^{-j})$$

for $0 \le j \le d-1$.

$$\text{(ii)} \quad \pi_* \Omega^n_{\tilde{X}} \simeq \Omega^n_X \oplus \Omega^n_X(\mathsf{L}) \oplus \ldots \oplus \Omega^n_X\left(\mathsf{L}^{d-1}\right),$$

$$\left\{ \varphi \in \pi_* \mathcal{O}_{\tilde{X}} \,|\, r^*(\varphi) = e^{j\frac{2\pi i}{d}} \varphi \right\} \simeq \Omega^n_X\left(\mathsf{L}^{d-j}\right)$$

for $1 \le j \le d$.

Remark.(i) In the situation for the free action of $< r >$ on $\tilde{X}$ (i.e., D=$\emptyset$), the above line bundle L is given by

$$\mathsf{L} = \left(\tilde{X} \times \mathbf{C}\right) / \left(\tilde{x} \times z \sim r(\tilde{x}) \times e^{\frac{-2\pi i}{d}} z\right) .$$

(ii) In the discussion of the paper, we consider only the case for $n = 1$, and write $\Omega_X = \Omega^1_X$. We shall denote the $\tilde{X}$ as

$$\widetilde{X} = (\mathfrak{f})^{\frac{1}{d}} \subset \mathsf{L} \ .$$

The following 2 propositions on Prym varieties for hyperelliptic curves are useful for later discussions.

Proposition 3 . Let C be a hyperelliptic curve of genus $g(C) \ge 2$, and σ_C be the hyperelliptic involution of C. Let φ be an order 2 automorphism of C which is not equal to σ_C. Denote the natural projections

$$\Lambda_B : C \to B := C / < \varphi > \quad \text{with branched locus } \mathbf{b}_B \subset B,$$

$$\Lambda_{B^*} : C \to B^* := C / < \varphi\sigma_C > \quad \text{with branched locus } \mathbf{b}_{B^*} \subset B^*.$$

Then

(i) The genus of C, B, B^* have the relation

$$g(B) + \frac{|\mathbf{b}_B|}{4} = g(B^*) + \frac{|\mathbf{b}_{B^*}|}{4} = \frac{g(C)+1}{2}$$

with even integers $|\mathbf{b}_B|$, $|\mathbf{b}_{B^*}|$ satisfying

$$|\mathbf{b}_B| + |\mathbf{b}_{B^*}| = 4.$$

(ii) The Prym variety $\mathrm{Prym}(\Lambda_B)$ for the covering Λ_B is a principal polarized abelian variety and under the induced morphism $(\Lambda_{B^*})^* : J(B^*) \to J(B)$,

$$\begin{array}{ll} J(B^*) \simeq \mathrm{Prym}(\Lambda_B) & \text{if } \mathbf{b}_{B^*} \neq \emptyset, \\ J(B^*)/\mathbf{Z}_2 \simeq \mathrm{Prym}(\Lambda_B) & \text{if } \mathbf{b}_{B^*} = \emptyset. \end{array}$$

(iii) The morphism

$$\begin{array}{ccc} J(B) \times J(B^*) & \to & J(C) \\ (\mathbf{d}_1, \mathbf{d}_2) & \rightsquigarrow & (\Lambda_B)^*\mathbf{d}_1 + (\Lambda_{B^*})^*\mathbf{d}_2 \end{array}$$

defines an isogeny of degree $2^{g(C)}$.

Proof. The hyperelliptic involution σ_C commutes with φ. Since $\varphi \neq \sigma_C$, both Λ_B, Λ_{B^*} are of degree 2. From Hurwitz Theorem, we have

$$g(B) + \frac{|\mathbf{b}_B|}{4} = g(B^*) + \frac{|\mathbf{b}_{B^*}|}{4} = \frac{g(C)+1}{2},$$

hence $|\mathbf{b}_B|$, $|\mathbf{b}_{B^*}|$ are even integers. As the degree of Λ_B equals to 2, it is known that $\mathrm{Prym}(\Lambda_B)$ is a principal polarized abelian variety and there is an isogeny

$$\begin{array}{ccc} J(B) \times \mathrm{Prym}(\Lambda_B) & \to & J(C) \\ (\mathbf{d}_1, \beta) & \rightsquigarrow & (\Lambda_B)^*\mathbf{d}_1 + \beta \end{array}$$

of degree

$$\left\{ \begin{array}{ll} 2^{2g(B)} & \text{if } \mathbf{b}_B \neq \emptyset, \\ 2^{2g(B)-1} & \text{if } \mathbf{b}_B = \emptyset, \end{array} \right.$$

(See [10]). Since

$$\begin{aligned} \mathrm{Prym}(\Lambda_B) &= \{\mathsf{F} \in J(C) \,|\, \varphi^*(\mathsf{F}) = \mathsf{F}^{-1}\} \\ &= \{\mathsf{F} \in J(C) \,|\, (\sigma\varphi)^*(\mathsf{F}) = \mathsf{F}\} \\ &= (\Lambda_{B^*})^* J(B^*), \end{aligned}$$

and

$$\mathrm{Kernel}\big((\Lambda_{B^*})^* : J(B^*) \to J(B)\big) = \left\{ \begin{array}{ll} 1 & \text{if } \mathbf{b}_{B^*} \neq \emptyset, \\ \mathbf{Z}/2\mathbf{Z} & \text{if } \mathbf{b}_{B^*} = \emptyset, \end{array} \right.$$

it follows (ii) and

$$\text{degree of isogeny in (ii)} = \begin{cases} 2^{2g(B)} & \text{for } \mathbf{b}_B \neq \emptyset \text{ and } \mathbf{b}_{B^*} \neq \emptyset, \\ 2^{2g(B)-1} & \text{for } \mathbf{b}_B = \emptyset \text{ and } \mathbf{b}_{B^*} \neq \emptyset, \\ 2^{2g(B)+1} & \text{for } \mathbf{b}_B \neq \emptyset \text{ and } \mathbf{b}_{B^*} = \emptyset. \end{cases}$$

From the equality

$$g(B) + g(B^*) = g(C),$$

we obtain (i), hence (iii). q.e.d.

We now show how the curve C in the above Proposition is obtained from B through the description of Lemma 6. The hyperelliptic involution σ_C of C induces ones for the hyperelliptic curves B, B^*, denoted by σ_B, σ_{B^*} respectively. Denote the hyperelliptic projections

$$\begin{aligned} \Pi_C : C \to \mathbf{P}^1 &= C / < \sigma_C >, \\ \Pi_B : B \to \mathbf{P}^1 &= B / < \sigma_B >, \\ \Pi_{B^*} : B^* \to \mathbf{P}^1 &= B^* / < \sigma_{B^*} >, \end{aligned}$$

We shall make the identification:

$$C / < \sigma_C, \varphi > = B / < \sigma_B > = B^* / < \sigma_{B^*} >$$

and let γ be the degree 2 morphism of projective lines:

$$\mathbf{P}^1 = C / < \sigma_C > \xrightarrow{\gamma} C / < \sigma_C, \varphi > = \mathbf{P}^1 \quad .$$

One easily see that in C

$$C^{\sigma_C} \cap C^{\varphi} = C^{\sigma_C} \cap C^{\varphi\sigma_C} = C^{\varphi} \cap C^{\varphi\sigma_C} = \emptyset\,, \quad (18)$$

hence σ_C acts freely on $C^{\varphi} \cup C^{\varphi\sigma_C}$, and the orbits give the critical points of γ. Since $|C^{\varphi}| = |\mathbf{b}_B|$, $|C^{\varphi\sigma_C}| = |\mathbf{b}_{B^*}|$, by Proposition 3 (i), we have $|C^{\varphi}| + |C^{\varphi\sigma_C}| = 4$, therefore

$$\mathbf{c} := \gamma\Pi_C(C^{\varphi} \cup C^{\varphi\sigma_C}) \qquad (19)$$

consists of 2 elements which are the branched locus of γ. By (18),

$$\begin{aligned} &\mathbf{c} = \text{disjoint union of } \mathbf{c}_B \text{ and } \mathbf{c}_{B^*}\,, \\ &\mathbf{c}_B := \gamma\Pi_C(C^{\varphi}),\ \mathbf{c}_{B^*} := \gamma\Pi_C(C^{\varphi\sigma_C}). \end{aligned} \qquad (20)$$

As C^{σ_C} corresponds to the branched locus of Π_C, it is mapped 2–1 onto

$$\mathbf{e} := \gamma\Pi_C(C^{\sigma_C}) \qquad (21)$$

under γ. The data $\mathbf{e}, \mathbf{c}_B, \mathbf{c}_{B^*}$ shall determine the structure of the hyperelliptic curves B, B^* as follows. Consider the commutative diagram:

$$\begin{array}{ccc} C & \stackrel{\Lambda_B}{\rightarrow} & B \\ \Pi_C \downarrow & & \downarrow \Pi_B \\ \mathsf{P}^1 = C/<\sigma_C> & \stackrel{\gamma}{\rightarrow} & B/<\sigma_B> = \mathsf{P}^1. \end{array}$$

By (18), φ acts freely on $C^{\sigma_C} \cup C^{\varphi\sigma_C}$. Then in B,

$$\begin{gathered} \mathbf{b}_B = \Pi_B^{-1}(\mathbf{c}_B), \\ B^{\sigma_B} = \Pi_B^{-1}(\mathbf{c}_{B^*}) \cup \Pi_B^{-1}(\mathbf{e}) \stackrel{\Pi_B}{\leftrightsquigarrow} \mathbf{c}_{B^*} \cup \mathbf{e}, \\ \Pi_B^{-1}(\mathbf{c}_{B^*}) = \Lambda_B(C^{\varphi\sigma_C}),\ \Pi_B^{-1}(\mathbf{e}) = \Lambda_B(C^{\sigma_C}). \end{gathered}$$

The only possibilities for the decomposition of the branched locus of γ is the following:

$$\begin{array}{lr} |\mathbf{c}_B| = 2,\ |\mathbf{c}_{B^*}| = 0,\ (\Leftrightarrow |\mathbf{b}_B| = 4, \quad |\Pi_B^{-1}(\mathbf{c}_{B^*})| = 0), & (22)_1 \\ |\mathbf{c}_B| = 1,\ |\mathbf{c}_{B^*}| = 1,\ (\Leftrightarrow |\mathbf{b}_B| = 2, \quad |\Pi_B^{-1}(\mathbf{c}_{B^*})| = 1), & (22)_2 \\ |\mathbf{c}_B| = 0,\ |\mathbf{c}_{B^*}| = 2,\ (\Leftrightarrow |\mathbf{b}_B| = 0, \quad |\Pi_B^{-1}(\mathbf{c}_{B^*})| = 2). & (22)_3 \end{array}$$

Let L be the line bundle over B defined as follows:

$$\begin{array}{ll} (22)_1: & \mathcal{O}_B(\mathsf{L}) = (\Pi_B)^*\mathcal{O}_{\mathsf{P}^1}(1)\,, \\ (22)_2: & \mathcal{O}_B(\mathsf{L}) = \mathcal{O}_B(\Pi_B^{-1}(\mathbf{c}_{B^*})), \\ (22)_3: & \mathcal{O}_B(\mathsf{L}) = \mathcal{O}_B(\alpha - \beta) \ \text{ with } \ \Pi_B^{-1}(\mathbf{c}_{B^*}) = \{\alpha, \beta\}. \end{array} \tag{23}$$

Then there is a section $\mathfrak{f} \in \Gamma(B, \mathsf{L}^2)$ with $\mathrm{div}(\mathfrak{f}) = \Pi_B^{-1}(\mathbf{c}_B)$ and C is obtained from $(\mathsf{L}, \mathfrak{f})$ by $C = (\mathfrak{f})^{\frac{1}{2}} \subset \mathsf{L}$ (in Lemma 6). Therefore we have shown the following result:

Proposition 4 . With C, B, $B^*, \mathbf{b}_B, \mathbf{b}_{B^*}$ the same as Proposition 3, let $\mathbf{c}, \mathbf{c}_B, \mathbf{c}_{B^*}, \mathbf{e}$ be the subsets of $C/<\varphi, \sigma_C>$ $(=\mathsf{P}^1)$ defined by (19), (20). Then B is the hyperelliptic curve with 2-fold cover over $C/<\varphi, \sigma_C>$ branched at $\mathbf{c}_{B^*} \cup \mathbf{e}$, and the curve C

$$C = (\mathfrak{f})^{\frac{1}{2}} \subset \mathsf{L}$$

with L defined by (23) and $\mathrm{div}(\mathfrak{f}) = \Pi_B^{-1}(\mathbf{c}_B)$.

Lemma 7 . (i) The fixed points of ι, $\iota\sigma$ in $W_{N,k'}$ are

$$(W_{N,k'})^{\iota} = \begin{cases} \left\{(t, \mathrm{w}) = \left(1, \frac{\pm 2k'i}{(1-k'^2)^{\frac{1}{2}}}\right)\right\} & \text{for odd } N, \\ \left\{(t, \mathrm{w}) = \left(1, \frac{\pm 2k'i}{(1-k'^2)^{\frac{1}{2}}}\right), \left(-1, \frac{\pm 2k'i}{(1-k'^2)^{\frac{1}{2}}}\right)\right\} & \text{for even } N. \end{cases}$$

$$(W_{N,k'})^{\iota\sigma} = \begin{cases} \left\{(t,\mathrm{w}) = \left(-1, \frac{\pm 2}{(1-k'^2)^{\frac{1}{2}}}\right)\right\} & \text{for odd } N, \\ \emptyset & \text{for even } N. \end{cases}$$

(ii) For even N, The fixed points of $\iota\theta^{\frac{N}{2}}$, $\iota\theta^{\frac{N}{2}}\sigma$ are

$$(W_{N,k'})^{\iota\theta^{\frac{N}{2}}} = \emptyset \quad \text{for odd } \frac{N}{2},$$
$$= \left\{(t,\mathrm{w}) = \left(i, \frac{\pm 2k'i}{(1-k'^2)^{\frac{1}{2}}}\right), \left(-i, \frac{\pm 2k'i}{(1-k'^2)^{\frac{1}{2}}}\right)\right\} \quad \text{for even } \frac{N}{2},$$

$$(W_{N,k'})^{\iota\theta^{\frac{N}{2}}\sigma} = \emptyset \quad \text{for even } \frac{N}{2};$$
$$= \left\{(t,\mathrm{w}) = \left(i, \frac{\pm 2k'i}{(1-k'^2)^{\frac{1}{2}}}\right), \left(-i, \frac{\pm 2k'i}{(1-k'^2)^{\frac{1}{2}}}\right)\right\} \quad \text{for odd } \frac{N}{2}.$$

Proof. By the definition of ι, $\iota\sigma$, we have

$$(t,\mathrm{w}) \in W^{\iota}_{N,k'} \Leftrightarrow t^2 = 1,\ t^N \mathrm{w} = \mathrm{w}\ ,$$
$$(t,\mathrm{w}) \in W^{\iota\sigma}_{N,k'} \Leftrightarrow t^2 = 1,\ t^N \mathrm{w} = -\mathrm{w}\ .$$

Then (i) following from the equation (4) of $W_{N,k'}$. With the same argument, we obtain (ii). q.e.d.

We are going to describe the various Prym varieties attached to the curve $W_{N,k'}$ by applying Proposition 3, 4 on hyperelliptic curves related to it.

For odd N, consider the double coverings

$$\begin{aligned} \Lambda_{V_{N,k'}} &: W_{N,k'} \to V_{N,k'} := W_{N,k'}/<\iota>\ , \\ \Lambda_{V^*_{N,k'}} &: W_{N,k'} \to V^*_{N,k'} := W_{N,k'}/<\iota\sigma> \end{aligned} \tag{24}$$

which have 2 branched points, and the genus

$$g(V_{N,k'}) = g(V^*_{N,k'}) = \frac{N-1}{2}$$

by Proposition 3 (i). With the coordinate t of $W_{N,k'}/<\sigma>$ in (3) , consider the coordinate s of $W_{N,k'}/<\iota,\sigma>$ given by

$$\begin{array}{ccc} \mathbf{P}^1 = W_{N,k'}/<\sigma> & \to & W_{N,k'}/<\iota,\sigma> = \mathbf{P}^1 \\ t & \rightsquigarrow & s = t + \frac{1}{t}\ . \end{array} \tag{25}$$

By Lemma 7 (i) the subsets $c_{V_{N,k'}}, c_{V^*_{N,k'}}, \mathbf{e}$ of $W_{N,k'}/ < \iota, \sigma >$ in Proposition 4 for this situation are

$$\mathbf{e} = \left\{ s = e^{j\frac{2\pi i}{N}} \left(\frac{1+k'}{1-k'} \right)^{\frac{1}{N}} + e^{-j\frac{2\pi i}{N}} \left(\frac{1-k'}{1+k'} \right)^{\frac{1}{N}} \right\}_{j=1}^{N},$$

$$c_{V^*_{N,k'}} = \{s = -2\},$$

$$c_{V_{N,k'}} = \{s = 2\}.$$

Therefore by Proposition 3 and 4, we have the following result:

Theorem 2. For odd N, let

$$V = V_{N,k'}, \quad V^* = V^*_{N,k'}$$

be the hyperelliptic curves of genus $\frac{N-1}{2}$ defined by (24), and s the coordinate of $W_{N,k'}/ < \iota, \sigma >$ defined by (25). Then

(i) The double covers

$$\Pi_V : V \to \mathbf{P}^1 (= W_{N,k'}/ < \iota, \sigma >),$$

$$\Pi_{V^*} : V^* \to \mathbf{P}^1 (= W_{N,k'}/ < \iota, \sigma >),$$

have the branched loci

$$\{s = -2\} \bigcup \left\{ s = e^{j\frac{2\pi i}{N}} \left(\frac{1+k'}{1-k'} \right)^{\frac{1}{N}} + e^{-j\frac{2\pi i}{N}} \left(\frac{1-k'}{1+k'} \right)^{\frac{1}{N}} \right\}_{j=1}^{N},$$

$$\{s = 2\} \bigcup \left\{ s = e^{j\frac{2\pi i}{N}} \left(\frac{1+k'}{1-k'} \right)^{\frac{1}{N}} + e^{-j\frac{2\pi i}{N}} \left(\frac{1-k'}{1+k'} \right)^{\frac{1}{N}} \right\}_{j=1}^{N}$$

respectively.

(ii) The projections (24) induce an isogeny

$$J(V) \times J(V^*) \to J(W_{N,k'})$$

of degree 2^{N-1}, and Prym(Λ_V)=$J(V^*)$, Prym(Λ_{V^*})=$J(V)$.

(iii) The 2–fold covering of $W_{N,k'}$ over V, V^* are described by

$$W_{N,k'} = (\mathfrak{f})^{\frac{1}{2}} \subset \mathbf{L}$$

with

$$\mathrm{div}(\mathfrak{f}) = \Pi_V^{-1}(2), \ \mathcal{O}_V(\mathbf{L}) = \mathcal{O}_V(\Pi_V^{-1}(-2)) \ \text{ for } V,$$

$$\mathrm{div}(\mathfrak{f}) = \Pi_{V^*}^{-1}(-2), \ \mathcal{O}_{V^*}(\mathbf{L}) = \mathcal{O}_{V^*}(\Pi_{V^*}^{-1}(2)) \ \text{ for } V^*.$$

We now consider the case of even N. Let $\overline{W_{N,k'}}$ be the same as in (15), and define

$$\Lambda_{\overline{W_{N,k'}}^*} : W_{N,k'} \to \overline{W_{N,k'}}^* := W_{N,k'}/ < \theta^{\frac{N}{2}} >, \qquad (26)$$
$$\Lambda_{\overline{W_{N,k'}}} : W_{N,k'} \to \overline{W_{N,k'}} := W_{N,k'}/ < \theta^{\frac{N}{2}}\sigma > .$$

$\Lambda_{\overline{W_{N,k'}}^*}$ has 4 branched points, hence by Proposition 3 (i),

$$g\left(\overline{W_{N,k'}}\right) = \frac{N}{2}, \; g\left(\overline{W_{N,k'}}^*\right) = \frac{N}{2} - 1. \qquad (27)$$

The coordinate t' of $W_{N,k'}/ < \theta^{\frac{N}{2}}, \sigma >$ $(= \mathbf{P}^1)$ is given by

$$\begin{array}{ccc} \mathbf{P}^1 = W_{N,k'}/ < \sigma > & \to & W_{N,k'}/ < \theta^{\frac{N}{2}}, \sigma > = \mathbf{P}^1 \\ t & \rightsquigarrow & t' = t^2 \end{array} . \qquad (28)$$

The subsets $\mathbf{c}_{\overline{V_{N,k'}}}, \mathbf{c}_{\overline{V_{N,k'}}^*}, \mathbf{e}$ of $W_{N,k'}/ < \theta^{\frac{N}{2}}, \sigma >$ for Proposition 4 are

$$\mathbf{e} = \left\{ t' = e^{2j\frac{2\pi i}{N}} \left(\frac{1+k'}{1-k'}\right)^{\frac{2}{N}}, \; e^{2j\frac{2\pi i}{N}} \left(\frac{1-k'}{1+k'}\right)^{\frac{2}{N}} \right\}_{j=1}^{\frac{N}{2}},$$
$$\mathbf{c}_{\overline{W_{N,k'}}^*} = \left\{ t' = 0, \infty \right\},$$
$$\mathbf{c}_{\overline{W_{N,k'}}} = \emptyset.$$

As ι commutes with $\theta^{\frac{N}{2}}\sigma$, it induces an order 2 automorphism $\bar{\iota}$ of $\overline{W_{N,k'}}$. Let $\bar{\sigma}$ be the hyperelliptic involution of $\overline{W_{N,k'}}$ induced by σ, and define

$$\Lambda_{\overline{V_{N,k'}}} : \overline{W_{N,k'}} \to \overline{V_{N,k'}} := \overline{W_{N,k'}}/ < \bar{\iota} >, \qquad (29)$$
$$\Lambda_{\overline{V_{N,k'}}^*} : \overline{W_{N,k'}} \to \overline{V_{N,k'}}^* := \overline{W_{N,k'}}/ < \bar{\iota}\bar{\sigma} > .$$

By Lemma 7 and Proposition 3 (i), the numbers of the branched loci for $\Lambda_{\overline{V_{N,k'}}}, \Lambda_{\overline{V_{N,k'}}^*}$ are

$$\left|\mathbf{b}_{\Lambda_{\overline{V_{N,k'}}}}\right| = \left|\mathbf{b}_{\Lambda_{\overline{V_{N,k'}}^*}}\right| = 2 \quad \text{for even } \frac{N}{2},$$
$$\left|\mathbf{b}_{\Lambda_{\overline{V_{N,k'}}}}\right| = 4, \; \left|\mathbf{b}_{\Lambda_{\overline{V_{N,k'}}^*}}\right| = 0 \quad \text{for odd } \frac{N}{2},$$

and

$$g\left(\overline{V_{N,k'}}\right) = g\left(\overline{V_{N,k'}}^*\right) = \frac{N}{4} \qquad \text{for even } \frac{N}{2},$$
$$g\left(\overline{V_{N,k'}}\right) = \frac{N-2}{4}, \; g\left(\overline{V_{N,k'}}^*\right) = \frac{N+2}{4} \quad \text{for odd } \frac{N}{2}. \qquad (30)$$

The coordinate s of $\overline{W_{N,k'}}/<\overline{\iota},\overline{\sigma}>$ $(=\mathbf{P}^1)$ is given by

$$\begin{array}{ccc} \mathbf{P}^1=\overline{W_{N,k'}}/<\overline{\sigma}> & \to \overline{W_{N,k'}}/<\overline{\iota},\overline{\sigma}> & =\mathbf{P}^1 \\ t' & \leadsto & s=t'+\dfrac{1}{t'} \end{array} \quad . \tag{31}$$

The subsets $\mathbf{c}_{\overline{V_{N,k'}}}, \mathbf{c}_{\overline{V_{N,k'}}^{*}}, \mathbf{e}$ of $\overline{W_{N,k'}}/<\overline{\iota},\overline{\sigma}>$ in Proposition 4 are

$$\mathbf{e}=\{s=\infty\}\bigcup\left\{s=e^{2j\frac{2\pi i}{N}}\left(\frac{1+k'}{1-k'}\right)^{\frac{2}{N}}+e^{-2j\frac{2\pi i}{N}}\left(\frac{1-k'}{1+k'}\right)^{\frac{2}{N}}\right\}_{j=1}^{\frac{N}{2}},$$

and

$$\mathbf{c}_{\overline{V_{N,k'}}^{*}}=\emptyset,\ \mathbf{c}_{\overline{V_{N,k'}}}=\{s=\pm 2\} \text{ for odd } \frac{N}{2},$$
$$\mathbf{c}_{\overline{V_{N,k'}}^{*}}=\{s=-2\},\ \mathbf{c}_{\overline{V_{N,k'}}}=\{s=2\} \text{ for even } \frac{N}{2}.$$

Therefore by Proposition 3 and 4, we have the following result:

Theorem 3 . For even N, let

$$\overline{W}=\overline{W_{N,k'}},\ \overline{W}^{*}=\overline{W_{N,k'}}^{*},\ \overline{V}=\overline{V_{N,k'}},\ \overline{V}^{*}=\overline{V_{N,k'}}^{*}$$

be the hyperelliptic curves defined by (26) and (29) with the genus given by (27) and (30). Let t', s the coordinate of $W_{N,k'}/<\theta^{\frac{N}{2}},\sigma>$, $\overline{W}/<\overline{\iota},\overline{\sigma}>$ defined by (28) and (31). Then

(i) The double covers

$$\Pi_{\overline{W}}:\overline{W}\ \to \mathbf{P}^1\left(=W_{N,k'}/<\theta^{\frac{N}{2}},\sigma>\right),$$
$$\Pi_{\overline{W}^{*}}:\overline{W}^{*}\ \to \mathbf{P}^1\left(=W_{N,k'}/<\theta^{\frac{N}{2}},\sigma>\right),$$

have the branched loci

$$\{t'=0,\infty\}\bigcup\left\{t'=\ e^{2j\frac{2\pi i}{N}}\left(\frac{1+k'}{1-k'}\right)^{\frac{2}{N}},\ e^{2j\frac{2\pi i}{N}}\left(\frac{1-k'}{1+k'}\right)^{\frac{2}{N}}\right\}_{j=1}^{\frac{N}{2}},$$
$$\left\{t'=\ e^{2j\frac{2\pi i}{N}}\left(\frac{1+k'}{1-k'}\right)^{\frac{2}{N}},\ e^{2j\frac{2\pi i}{N}}\left(\frac{1-k'}{1+k'}\right)^{\frac{2}{N}}\right\}_{j=1}^{\frac{N}{2}},$$

respectively. And the double covers

$$\Pi_{\overline{V}}:\overline{V}\ \to \mathbf{P}^1\,(=\overline{W}/<\overline{\iota},\overline{\sigma}>),$$
$$\Pi_{\overline{V}^{*}}:\overline{V}^{*}\ \to \mathbf{P}^1\,(=\overline{W}/<\overline{\iota},\overline{\sigma}>),$$

have the branched loci

$$\{s=\infty\}\bigcup\left\{s=\ e^{2j\frac{2\pi i}{N}}\left(\frac{1+k'}{1-k'}\right)^{\frac{2}{N}}+e^{-2j\frac{2\pi i}{N}}\left(\frac{1-k'}{1+k'}\right)^{\frac{2}{N}}\right\}_{j=1}^{\frac{N}{2}}$$

$$\{s=\pm 2,\infty\}\bigcup\left\{s=\ e^{2j\frac{2\pi i}{N}}\left(\frac{1+k'}{1-k'}\right)^{\frac{2}{N}}+e^{-2j\frac{2\pi i}{N}}\left(\frac{1-k'}{1+k'}\right)^{\frac{2}{N}}\right\}_{j=1}^{\frac{N}{2}}$$

$$\text{for odd } \frac{N}{2},$$

$$\{s=-2,\infty\}\bigcup\left\{s=\ e^{2j\frac{2\pi i}{N}}\left(\frac{1+k'}{1-k'}\right)^{\frac{2}{N}}+e^{-2j\frac{2\pi i}{N}}\left(\frac{1-k'}{1+k'}\right)^{\frac{2}{N}}\right\}_{j=1}^{\frac{N}{2}}$$

$$\{s=2,\infty\}\bigcup\left\{s=\ e^{2j\frac{2\pi i}{N}}\left(\frac{1+k'}{1-k'}\right)^{\frac{2}{N}}+e^{-2j\frac{2\pi i}{N}}\left(\frac{1-k'}{1+k'}\right)^{\frac{2}{N}}\right\}_{j=1}^{\frac{N}{2}}$$

$$\text{for even } \frac{N}{2}\ ,$$

respectively.

(ii) The projections of (26) and (29) induce the isogenies

$$J(\overline{W})\times J\left(\overline{W^*}\right)\to J(W_{N,k'}),$$
$$J(\overline{V})\times J\left(\overline{V^*}\right)\to J(\overline{W})$$

of degree 2^{N-1}, $2^{\frac{N}{2}}$ respectively. We have the following description of Prym varieties:

$$\mathrm{Prym}(\Lambda_{\overline{W}})=J\left(\overline{W^*}\right),$$
$$\mathrm{Prym}(\Lambda_{\overline{W^*}})=J(\overline{W})/\mathbf{Z}_2,$$
$$\mathrm{Prym}(\Lambda_{\overline{V}})=J\left(\overline{V^*}\right),$$
$$\mathrm{Prym}(\Lambda_{\overline{V^*}})=\begin{cases} J(\overline{V})/\mathbf{Z}_2 & \text{for odd } \frac{N}{2}, \\ J(\overline{V}) & \text{for even } \frac{N}{2}. \end{cases}$$

(iii) The 2–fold cover of $W_{N,k'}$ over $\overline{W}$, $\overline{W^*}$ are described by

$$W_{N,k'}=(\mathfrak{f})^{\frac{1}{2}}\subset \mathsf{L}$$

with

$$\mathrm{div}(\mathfrak{f})=\emptyset,\ \mathcal{O}_{\overline{W}}(\mathsf{L})=\mathcal{O}_{\overline{W}}\left(\Pi_{\overline{W}}^{-1}(0)-\Pi_{\overline{W}}^{-1}(\infty)\right)\ \text{for } \overline{W};$$
$$\mathrm{div}(\mathfrak{f})=\Pi_{\overline{W^*}}^{-1}(\{0,\infty\}),\ \mathcal{O}_{\overline{W^*}}(\mathsf{L})=(\Pi_{\overline{W^*}})^*\mathcal{O}_{\mathbf{P}^1}(1)\ \text{for } \overline{W^*}\ .$$

And the 2-fold covers of $\overline{W}$ over $\overline{V}$, $\overline{V}^*$ are described by

$$\overline{W} = (\mathfrak{g})^{\frac{1}{2}} \subset \mathsf{L}$$

with

$$\operatorname{div}(\mathfrak{g}) = \Pi_{\overline{V}}^{-1}(\pm 2),\ \mathcal{O}_{\overline{V}}(\mathsf{L}) = (\Pi_{\overline{V}})^* \mathcal{O}_{\mathbf{P}^1}(1) \quad \text{for } \overline{V},$$
$$\operatorname{div}(\mathfrak{g}^*) = \emptyset,\ \mathcal{O}_{\overline{V}^*}(\mathsf{L}) = \mathcal{O}_{\overline{V}^*}\left(\Pi_{\overline{V}^*}^{-1}(2) - \Pi_{\overline{V}^*}^{-1}(-2)\right) \quad \text{for } \overline{V}^*$$

when $\frac{N}{2}=$ odd integer,

$$\overline{W} = (\mathfrak{g})^{\frac{1}{2}} \subset \mathsf{L}$$

with

$$\operatorname{div}(\mathfrak{g}) = \Pi_{\overline{V}}^{-1}(2),\ \mathcal{O}_{\overline{V}}(\mathsf{L}) = \mathcal{O}_{\overline{V}}\left(\Pi_{\overline{V}}^{-1}(-2)\right) \quad \text{for } \overline{V},$$
$$\operatorname{div}(\mathfrak{g}^*) = \Pi_{\overline{V}^*}^{-1}(-2),\ \mathcal{O}_{\overline{V}^*}(\mathsf{L}) = \mathcal{O}_{\overline{V}^*}\left(\Pi_{\overline{V}^*}^{-1}(2)\right) \quad \text{for } \overline{V}^*$$

when $\frac{N}{2}=$ even integer.

Remark . The curve $\overline{W}^*$ $\left(= \overline{W_{N,k'}}^*\right)$ in the above theorem is a genus $\frac{N}{2}-1$ hyperelliptic curve. It is a CP $\frac{N}{2}$-curve. The order $\frac{N}{2}$ automorphism of $\overline{W_{N,k'}}^*$ with 4 fixed elements is the one induced from θ of $W_{N,k'}$. Hence we can split the Jacobian of $\overline{W_{N,k'}}^*$ using Theorem 2 and 3. Proceeding the procedure inductively, we obtain the splitting of the Jacobian of $W_{N,k'}$ into the Jacobians of quotients of it.

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