Twisting of quantum (super)algebras. Connection of Drinfeld's and Cartan-Weyl realizations for quantum affine algebras

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Abstract

We show that some factors of the universal R-matrix generate a family of twistings for a Hopf structure of any quantized contragredient Lie (super)algebra of finite growth. As an application we prove that any two isomorphic superalgebras with the different Cartan matrices have isomorphic q-deformations (as associative superalgebras) and their standard comultiplications are connected by such twisting. We present also an explicit relation between the generators from second Drinfeld's realization and Cartan-Weyl generators of quantized affine nontwisted Kac-Moody algebras. We show that Drinfeld's formula of comultiplication for the second realization is a twisting of standard comultiplication by a factor of the universal R-matrix. Properties of the Drinfeld's comultiplication are discussed.

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1 Introduction

A number of applications of quantum algebras is based on the fact that quantized enveloping algebras have nontrivial algebraic and coalgebraic structures as Hopf algebras. In addition this gives a possibility to use for their study not only the automorphisms of algebras but also the twistings of coalgebraic structure (which can not to change the structure of multiplication at all). More generally, the notion of twisting for quasi-Hopf algebras was introduced and successfully applied in classification theorems by V. Drinfeld [D1]. N. Reshetikhin remarked [R] that one can use, analogously to [D1], a two-tensor $F \in U_q(g) \otimes U_q(g)$ as a twisting operator and obtain as a result a new Hopf algebra (without 'quasi' prefix) if F satisfies some natural conditions. He showed also that multiparameter deformations of the quantum enveloping algebra $U_q(g)$ of simple Lie algebra g can be defined via twisting which depends on Cartan subalgebra of $U_q(g)$. Such type of twisting was used by Ya. Soibelman [S] for quantization of Lie-Poisson structure in compact Lie groups. The other type of twisting was considered by B. Enriques [E]. He showed that the usual (non-deformed) enveloping algebra $U(g)$ of a simple Lie algebra g can be done noncocommutative by a twisting while the algebraic structure is not changed. Such a deformation of coalgebraic sector can be also defined for some non-semisimple algebras. For instance it was shown in [LNRT] that universal enveloping algebra of the classical Poincare algebra admits a family of twistings of coalgebraic sector without changing of algebraic sector.

We consider here the twistings of quantized contragredient Lie (super)algebras of finite growth. (These (super)algebras are q-analogs of all finite-dimensional simple Lie algebras, classical superalgebras and of all infinite-dimensional affine Kac-Moody (super)algebras). All these quantum (super)algebras $U_q(g)$ are quasitriagular, i.e. they have the universal R-matrix. Explicit formula for the universal R-matrix looks like a product of factors over positive root system of a Lie (super)algebra. We show that the factors of the universal R-matrix define a family of twistings for $U_q(g)$ and demonstrate their connection with twistings by means of Lusztig automorphisms [DeCK]. This is known in mathematical folklore for Drinfeld-Jimbo deformations of simple finite-dimensional Lie algebras. In other cases we prove as consequences the following important results.

First we exhibit a connection between Drinfeld-Jimbo quantizations (see [KT1]) of two isomorphic contragredient Lie superalgebras g and g'. More precisely, we show that their exists an isomorphism $\omega : U_q(g') \mapsto U_q(g)$ of algebras (a superanalog of Lusztig automorphism [L], [DeCK]), and the standard comultiplications of $U_q(g)$ and $U_q(g')$ commute with ω modulo twisting by corresponding factors of the universal R-matrix for $U_q(g)$ or $U_q(g')$.

Next, we present a detailed study of the second Drinfeld's realization [D2] of quantum affine algebra $U_q(\hat{g})$ from viewpoints of Cartan-Weyl bases and of twistings. We write down an explicit relation between generators from second Drinfeld's realization and Cartan-Weyl generators for quantized affine nontwisted Kac-Moody algebras (see also [DF] for $\hat{gl}_n$ case). We show that Drinfeld's formula of comultiplication for the second realization is a twisting of the standard comultiplication by a factor of the universal R-matrix. This twisting is correctly defined for appropriate completion of $U_q(\hat{g}) \otimes U_q(\hat{g})$

and corresponds to a "virtual" longest element ω_0 of affine q-Weyl group. The origin one can see on quasiclassical level where ω_0 does not act on the elements of Lie algebra but interchange Manin triples which are responsible for two different quantizations of a current algebra.

We discuss also the properties of natural comultiplication in the second realization of quantum affine algebras [D3]. Unfortunately, this comultiplication is still out of common interest. We demonstrate the meaning of quasitriangularity conditions for this comultiplication, present the universal R-matrix and show that for concrete representations this universal R-matrix produces the solution of Yang-Baxter equation with entries being generalized functions of spectral parameter.

The paper is organized as follows. In Section 2 we remind the definition of any quantized finite-dimensional contragredient Lie (super)algebra g (or a quantum (super)algebra $U_q(g)$) in terms of Chevalley generators and q-(super)commutator and also in terms of the adjoint action.

In Section 3 we present a procedure of the construction of the quantum Cartan-Weyl basis and define some extensions of $U_q(g)$ and $U_q(g)\otimes U_q(g)$ which we need for the definition of twistings and of the universal R-matrix. The explicit formula for the universal R-matrix is presented in the Section 4.

In Section 5 we discuss at first some general properties of twistings for an arbitrary Hopf (super)algebra, then we consider twistigs by factors of the universal R-matrix. The Sections 6-8 are devoted to the second Drinfeld's realization of quantized affine algebras. In Appendices A,B one can find the details of the considerations for the case of $U_q(\hat{sl}_2)$.

2 Quantized Lie (super)algebras of finite growth

Let $g(A,\Upsilon)$ be any contragredient Lie (super)algebra of finite growth with symmetrizable Cartan matrix A ($A^{sym}=(a^{sym}_{ij})$ is a corresponding symmetrical matrix) and let $\Pi:=\{\alpha_1,\ldots,\alpha_r\}$ be a system of simple roots for $g(A,\Upsilon)$ [1]. The quantized (super)algebra $g:=g(A,\Upsilon)$ is an unital associative (super)algebra $U_q(g)$ with Chevalley generators $e_{\pm\alpha_i}$, $k^{\pm1}_{\alpha_i}=q^{\pm h_{\alpha_i}}$, ($i\in \mathrm{I}:=\{1,2,\ldots,r\}$), and the defining relations [T1,KT1,KT2]

$$[k^{\pm1}_{\alpha_i},k^{\pm1}_{\alpha_j}]=0\ ,\qquad k_{\alpha_i}e_{\pm\alpha_j}=q^{\pm(\alpha_i,\alpha_j)}e_{\pm\alpha_j}k_{\alpha_i}, \tag{2.1}$$

$$[e_{\alpha_i},e_{-\alpha_j}]=\delta_{ij}\frac{k_{\alpha_i}-k^{-1}_{\alpha_i}}{q-q^{-1}}\ , \tag{2.2}$$

$$(\tilde{\mathrm{ad}}_{q'}e_{\pm\alpha_i})^{n_{ij}+1}e_{\pm\alpha_j}=0\qquad \text{for } i\neq j,\ q'=q,q^{-1}\ , \tag{2.3}$$

$$\begin{aligned}\deg(k_{\alpha_i})=\deg(e_{\pm\alpha_j})=\bar{0}\qquad &\text{for } i\in \mathrm{I}\ ,\ j\notin\Upsilon\ ,\\ \deg(e_{\pm\alpha_i})=\bar{1}\qquad &\text{for } i\in\Upsilon\subset \mathrm{I}\ ,\end{aligned} \tag{2.4}$$

[1] These (super)algebras are all finite-dimensional simple Lie (super)algebras and all infinite-dimensional affine Kac-Moody (super)algebras [K1]

where

$$n_{ij} = \begin{cases} 0 \quad \text{if } a_{ii}^{sym} = a_{ij}^{sym} = 0 \ , \\ 1 \quad \text{if } a_{ii}^{sym} = 0, \ a_{ij}^{sym} \neq 0 \ , \\ -2(a_{ij}^{sym}/a_{ii}^{sym}) \quad \text{if } a_{ii}^{sym} \neq 0 \ . \end{cases} \tag{2.5}$$

Moreover, there are the following additional triple relations [KT1]

$$[[e_{\pm\alpha_i}, e_{\pm\alpha_j}]_{q'}, [e_{\pm\alpha_j}, e_{\pm\alpha_l}]_{q'}]_{q'} = 0 \ , \qquad \text{for } \ q' = q, \ q^{-1} \ , \tag{2.6}$$

if the three simple roots $\alpha_i, \alpha_j, \alpha_l \in \Pi$ satisfy the condition

$$(\alpha_j, \alpha_j) = (\alpha_i, \alpha_l) = (\alpha_j, \alpha_i + \alpha_l) = 0 \ . \tag{2.7}$$

Here the bracket $[\cdot,\cdot]$ is an usual supercommutator, $\tilde{\mathrm{ad}}_{q'}$ and $[\cdot,\cdot]_q$ denote a deformed supercommutator (q-supercommutator) in $U_q(g)$:

$$(\tilde{\mathrm{ad}}_{q'} e_\alpha) e_\beta \equiv [e_\alpha, e_\beta]_{q'} = e_\alpha e_\beta - (-1)^{\theta(e_\alpha)\theta(e_\beta)} q'^{(\alpha,\beta)} e_\beta e_\alpha \ , \tag{2.8}$$

where (α, β) is a scalar product of the roots α and β: $(\alpha_i, \alpha_j) = a_{ij}^{sym}$. In the formula (2.8) and below we use the short notation

$$\theta(\gamma) := \theta(e_\gamma) \equiv \deg(e_\gamma) \ . \tag{2.9}$$

Remarks. (i) The triple relations (2.6) may appear only in supercase for the following situation in the Dynkin diagram:

$$\begin{array}{ccccc} \alpha_i & & \alpha_j & & \alpha_l \\ \cdot & \text{———} & \otimes & \text{———} & \cdot \end{array} \tag{2.10}$$

where α_j is a grey root and the roots α_i and α_l are not connected and they can be of any color: white, grey or dark.

(ii) The outer q-supercommutator in (2.6) is actually a usual one since $(\alpha_i+\alpha_j, \alpha_j+\alpha_l) = 0$.

(iii) The triple relations have evident classical counterpart.

The quantum (super)algebra $U_q(g)$ is a Hopf (super)algebra with respect to a comultiplication $\Delta_{q'}$, an antipode $S_{q'}$ and a counit ε defined as

$$\Delta_{q'}(k_{\alpha_i}) = k_{\alpha_i} \otimes k_{\alpha_i} \ , \tag{2.11}$$

$$\Delta_{q'}(e_{\alpha_i}) = e_{\alpha_i} \otimes 1 + k'_{\alpha_i} \otimes e_{\alpha_i} \ , \tag{2.12}$$

$$\Delta_{q'}(e_{-\alpha_i}) = e_{-\alpha_i} \otimes k'^{-1}_{\alpha_i} + 1 \otimes e_{-\alpha_i} \ , \tag{2.13}$$

$$\begin{gathered} S_{q'}(k'_{\alpha_i}) = k'^{-1}_{\alpha_i} \ , \\ S_{q'}(e_{\alpha_i}) = -k'^{-1}_{\alpha_i} e_{\alpha_i} \ , \qquad S_{q'}(e_{-\alpha_i}) = -e_{-\alpha_i} k'_{\alpha_i} \ , \end{gathered} \tag{2.14}$$

$$\varepsilon(k_{\alpha_i}) = \varepsilon(e_{\alpha_i}) = \varepsilon(e_{-\alpha_i}) = 0 \ , \qquad \varepsilon(1) = 1 \ , \tag{2.15}$$

where $k'_\alpha = q'^{h_\alpha}$ and q' may be chosen as $q' = q$ or $q' = q^{-1}$.

We may rewrite the defining relation by means of an adjoint action of $U_q(g)$ on itself. For this aim we introduce new Chevalley generators $\hat{e}_{\pm\alpha_i}$ by the following formulas

$$\hat{e}_{\alpha_i} = e_{\alpha_i}, \qquad \hat{e}_{-\alpha_i} = q'^{-1} e_{-\alpha_i} k'_{\alpha_i} \ . \tag{2.16}$$

In this basis the relations (2.2), (2.3), (2.6) take the following form [KT3]

$$(\mathrm{ad}_{q'}\hat{e}_{\alpha_i})\hat{e}_{-\alpha_j} = [\hat{e}_{\alpha_i}, \hat{e}_{-\alpha_j}]_{q'} = \delta_{ij}\frac{1 - k'^2_{\alpha_i}}{1 - q'^2} \ , \tag{2.17}$$

$$(\mathrm{ad}_{q'}\hat{e}_{\pm\alpha_i})^{n_{ij}}\hat{e}_{\pm\alpha_j} = 0 \ , \qquad (i \neq j) \ , \tag{2.18}$$

$$[(\mathrm{ad}_{q'}\hat{e}_{\pm\alpha_i})\hat{e}_{\pm\alpha_j}, (\mathrm{ad}_{q'}\hat{e}_{\pm\alpha_j})\hat{e}_{\pm\alpha_l}] = 0 \ . \tag{2.19}$$

The last relation holds for the condition (2.7). Here $\mathrm{ad}_{q'}$ is an adjoint action (see details in [KT3]) defined by

$$(\mathrm{ad}_{q'}\, a)x := ((id \otimes S_{q'})\Delta_{q'}(a)) \circ x \tag{2.20}$$

for all homogeneous elements $a, x \in U_q(g)$, where the operation $\circ$ is defined by the rule

$$(a \otimes b) \circ x = (-1)^{\theta(b)\theta(x)} axb \ . \tag{2.21}$$

Below we denote by a symbol $(^*)$ an anti-involution in $U_q(g)$, defined as $(k_{\alpha_i})^* = k_{\alpha_i}^{-1}$, $(e_{\pm\alpha_i})^* = e_{\mp\alpha_i}$, $(q)^* = q^{-1}$. We also use the standard notations $U_q(\kappa)$ and $U_q(b_\pm)$ for the Cartan and Borel subalgebras, generated by $k_{\alpha_i}^{\pm 1}$ and $e_{\pm\alpha_i}$, k_{α_i}, $k_{\alpha_i}^{-1}$ correspondingly. We write also

$$exp_q(x) := 1 + x + \frac{x^2}{(2)_q!} + \ldots + \frac{x^n}{(n)_q!} + \ldots = \sum_{n\geq 0} \frac{x^n}{(n)_q!} \ , \tag{2.22}$$

$$(a)_q := \frac{q^a - 1}{q - 1} \ , \qquad [a]_q := \frac{q^a - q^{-a}}{q - q^{-1}} \ , \qquad q_\alpha := (-1)^{\theta(\alpha)} q^{(\alpha,\alpha)} \ . \tag{2.23}$$

Now we proceed to a description of the Cartan-Weyl basis for the quantum (super)algebras $U_q(g)$.

3 Cartan-Weyl basis for $U_q(g)$

Let Δ_+ be the system of all positive roots for $g(A, \Upsilon)$ with respect to Π. We denote by $\underline{\Delta}_+$ *the reduced root system* which is obtained from Δ_+ by removing such *odd* roots α for which $\alpha/2$ are roots.

Our procedure of a construction of the quantum Cartan-Weyl basis for $U_q(g)$ is in agreement with a choice of normal ordering in $\underline{\Delta}_+$. We remind the definition of normal ordering in $\underline{\Delta}_+$ [AST,T2,T3].

We say that the system $\underline{\Delta}_+$ is in normal ordering if each composite root $\gamma = \alpha + \beta \in \underline{\Delta}_+$, where $\alpha \neq \lambda\beta$, α, $\beta \in \underline{\Delta}_+$, is written between its components α and β.

It should be noted that for any finite-dimensional simple Lie algebra there is one-to-one correspondence between normal orderings and a reduced decompositions of the longest element of the Weyl group [Z] (see Section 5). We have no such correspondence for Lie superalgebras and affine Lie algebras because the superalgebras have no "good" Weyl group and the affine algebras have not any longest element of the Weyl group.

We shall say that $\alpha < \beta$ if α is located on the left side of β in the normal ordering system $\underline{\Delta}_+$.

The quantum Cartan-Weyl basis is being constructed by using the following inductive algorithm [T1,KT1,KT2,TK].

Algorithm 3.1 *We fix some normal ordering in $\underline{\Delta}_+$ and put by induction*

$$e_\gamma := [e_\alpha, e_\beta]_q \ , \quad e_{-\gamma} := [e_{-\beta}, e_{-\alpha}]_{q^{-1}} \tag{3.1}$$

if $\gamma = \alpha + \beta$, $\alpha < \gamma < \beta$, and $[\alpha;\beta]$ is a minimal segment including γ, i.e. the segment has not another such roots α' and β' for which $\alpha' + \beta' = \gamma$. Moreover we put

$$k_\gamma := \prod_{i=1}^{r} k_{\alpha_i}^{l_i}, \tag{3.2}$$

if $\gamma = \sum_{i=1}^{r} l_i\alpha_i$, $(\alpha_i \in \Pi)$.

By this procedure one can construct the total quantum Cartan-Weyl basis for all finite-dimensional contragredient simple (super)algebras. In a case of infinite-dimensional affine Lie (super)algebras we use an additional constraint. Namely, we construct at first all root vectors by our procedure and then we redefine the root vectors of imaginary roots so that new imaginary root vectors commute if they are not conjugated. That is, e.g., let $e'^{(i)}_{\pm n\delta}$ be root vectors of imaginary roots $\pm n\delta$ [2], constructed by the procedure. It turns out that

$$[e'^{(i)}_{n\delta}, e'^{(j)}_{-m\delta}] \neq \delta_{m,-n} a_{ij}(n) \frac{k_\delta^n - k_\delta^{-n}}{q - q^{-1}} \ . \tag{3.3}$$

We introduce new vectors $e^{(i)}_{\pm n\delta}$:

$$e^{(i)}_{\pm n\delta} = p(e'^{(i)}_{\pm\delta}, e'^{(i)}_{\pm 2\delta} \ , \ldots, e'^{(i)}_{\pm n\delta}) \tag{3.4}$$

which will satisfy the relation

$$[e^{(i)}_{n\delta}, e^{(j)}_{-m\delta}] = \delta_{m,-n} a_{ij}(n) \frac{k_\delta^n - k_\delta^{-n}}{q - q^{-1}} \ . \tag{3.5}$$

This relation agrees with its classical counterpart.

The quantum Cartan-Weyl generators constructed by the procedure are characterized by the following basic properties.

[2]In the case of a quantum affine algebra $U_q(g)$ the root vectors of imaginary roots $\gamma = \pm n\delta$ have to be labeled by an additional index i: $e'^{(i)}_{\pm n\delta}$, $i = 1, 2, \ldots, mult$, where $mult$ is a multiplicity of the imaginary root $\pm n\delta$.

Theorem 3.1 *The root vectors $e_{\pm\gamma} \in U_q(g)$ and the Cartan elements $k_\gamma \in U_q(g)$ for all $\gamma \in \underline{\Delta}_+$ satisfy the following relations:*

$$(e_{\pm\gamma})^* = e_{\mp\gamma} \ , \qquad k_\alpha^{\pm 1} e_\gamma = q^{\pm(\alpha,\gamma)} e_\gamma k_\alpha^{\pm 1} \ , \tag{3.6}$$

$$[e_\gamma, e_{-\gamma}] = a(\gamma) \frac{k_\gamma - k_\gamma^{-1}}{q - q^{-1}} \ , \tag{3.7}$$

$$[e_\alpha, e_\beta]_q = \sum_{\alpha<\gamma_1<\ldots<\gamma_m<\beta} C_{n_j,\gamma_j} e_{\gamma_1}^{n_1} e_{\gamma_2}^{n_2} \cdots e_{\gamma_m}^{n_m} \ , \tag{3.8}$$

for $\alpha, \beta \in \underline{\Delta}_+$, where $\sum_j n_j\gamma_j = \alpha + \beta$, and the coefficients $C_{\ldots}$ are rational functions of q and ones do not depend on the Cartan elements k_{α_i}, $i = 1, 2, \ldots r$. Moreover

$$[e_\beta, e_{-\alpha}] = \sum C'_{n_j,\gamma_j;n'_j,\gamma'_j} e_{-\gamma_1}^{n_1} e_{-\gamma_2}^{n_2} \cdots e_{-\gamma_m}^{n_m} e_{\gamma'_1}^{n'_1} e_{\gamma'_2}^{n'_2} \cdots e_{\gamma'_l}^{n'_l} \tag{3.9}$$

where the sum is taken on $\gamma_1, \ldots, \gamma_m, \gamma'_1, \ldots, \gamma'_l$ and $n_1, \ldots, n_m, n'_1, \ldots, n'_l$ such that

$$\gamma_1 < \ldots < \gamma_m < \alpha < \beta < \gamma'_1 < \ldots < \gamma'_l \ ,$$

$$\sum_j (n'_j\gamma'_j - n_j\gamma_j) = \beta - \alpha \ ,$$

and the coefficients $C'_{\ldots}$ are rational functions of q and k_α^{-1} or k_β^{-1}. The monomials $e_{\gamma_1}^{n_1} e_{\gamma_2}^{n_2} \cdots e_{\gamma_m}^{n_m}$ and $e_{-\gamma_1}^{n_1} e_{-\gamma_2}^{n_2} \cdots e_{-\gamma_m}^{n_m}$, $(\gamma_1 < \gamma_2 < \cdots < \gamma_m)$, generate (as a linear space over $U_q(\kappa)$) subalgebras $U_q(b_+)$ and $U_q(b_-)$ correspondingly. The monomials

$$e_{-\gamma_1}^{n_1} e_{-\gamma_2}^{n_2} \cdots e_{-\gamma_m}^{n_m} e_{\gamma'_1}^{n'_1} e_{\gamma'_2}^{n'_2} \cdots e_{\gamma'_l}^{n'_l} \ , \tag{3.10}$$

where $\gamma_1 < \gamma_2 < \cdots < \gamma_m$ and $\gamma'_1 < \gamma'_2 < \cdots < \gamma'_l$), generate $U_q(g)$ over $U_q(\kappa)$.

If there are imaginary root vectors in the relations (3.6)-(3.9) then we should use additional index for such vectors. For example, the relation (3.7) for $\gamma = \pm n\delta$ has the form (3.5).

We can transform the root vectors $e_{\pm\gamma}$ in new ones such that the coefficients $C_{\ldots}$ and $C'_{\ldots}$ in (3.8) and (3.9) will not depend on the Cartan elements k_γ. For this goal we extend a notation of normal ordering for $\underline{\Delta}_+$ to "circular" normal ordering for the reduced system of all roots, $\underline{\Delta} := \underline{\Delta}_+ \cup (-\underline{\Delta}_+)$.

Let $\gamma_1, \gamma_2, \ldots, \gamma_N$ be a normal ordering in $\underline{\Delta}_+$ then a circular normal ordering in $\underline{\Delta}$ means that the roots of $\underline{\Delta}$ are located on a circular by the following way (see [KT3])

$$\gamma_1, \gamma_2, \ldots, \gamma_N, -\gamma_1, -\gamma_2, \ldots, -\gamma_N, \gamma_1 \tag{3.11}$$

We shall say that $\alpha < \beta$, where $\alpha, \beta \in \underline{\Delta}$, if the circular normal ordering segment $[\alpha, \beta]$ of (3.1) does not contain the opposite roots $-\alpha$ and $-\beta$.

We introduce two type of the circular root vectors $\hat{e}_\gamma$ and $\check{e}_\gamma$ by the following formulas

$$\hat{e}_\gamma := e_\gamma \ , \qquad \hat{e}_{-\gamma} := -k_\gamma^{-1} e_{-\gamma} \ , \qquad \forall \gamma \in \underline{\Delta}_+ \ , \tag{3.12}$$

and

$$\check{e}_\gamma := -e_\gamma k_\gamma\ , \qquad \check{e}_{-\gamma} := e_{-\gamma}\ , \qquad \forall\, \gamma \in \underline{\Delta}_+\ . \tag{3.13}$$

In terms of these circular generators the relations (3.8) and (3.9) are rewritten in the united form

$$[\hat{e}_\alpha, \hat{e}_\beta]_q = \sum_{\alpha<\gamma_1<\ldots<\gamma_m<\beta} C_{n_j,\gamma_j} \hat{e}_{\gamma_1}^{n_1} \hat{e}_{\gamma_2}^{n_2} \cdots \hat{e}_{\gamma_m}^{n_m}\ , \tag{3.14}$$

if $\alpha \in \underline{\Delta}_+$ and $\alpha < \beta$ in a sense of circular normal ordering. We have also

$$[\check{e}_\alpha, \check{e}_\beta]_q = \sum_{\alpha<\gamma_1<\ldots<\gamma_n<\beta} C'_{m_j,\gamma_j} \check{e}_{\gamma_1}^{m_1} \check{e}_{\gamma_2}^{m_2} \cdots \check{e}_{\gamma_n}^{m_n}\ , \tag{3.15}$$

if $-\alpha \in \underline{\Delta}_+$ and $\alpha < \beta$ in a sense of circular normal ordering. The coefficients $C_{\ldots}$ and $C'_{\ldots}$ in (3.14) and (3.15) are rational functions of q and do not depend on the elements k_γ. It should be noted that we can construct the circular root vectors $\hat{e}_{\pm\gamma}$ (up to scalar factors) applying q-commutator algorithm to the Chevalley elements $\hat{e}_{\pm\alpha_i}$.

Now we want to consider some extensions of $U_q(g)$, $U_q(b_+) \otimes U_q(b_-)$, $U_q(g) \otimes U_q(g)$ since, for example, the universal R-matrix is element of two last extensions.

Let Fract$\,(U_q(\kappa))$ be a field of fractions over $U_q(\kappa)$, i.e. Fract$\,(U_q(\kappa))$ is an associative algebra of rational functions of the elements $k_{\alpha_i}^{\pm 1}$, $(i = 1, 2, \ldots, r)$. Let us construct a formal Taylor series on the following monomials

$$e_{-\beta}^{n_\beta} \cdots e_{-\gamma}^{n_\gamma} e_{-\alpha}^{n_\alpha}\, e_\alpha^{m_\alpha} e_\gamma^{m_\gamma} \cdots e_\beta^{m_\beta} \tag{3.16}$$

with coefficients from Fract$\,(U_q(\kappa))$, where $\alpha < \gamma < \cdots < \beta$ in a sense of the fixed normal ordering in $\underline{\Delta}_+$ and nonnegative integers $n_\beta, \ldots, n_\alpha, m_\alpha, \ldots, m_\beta$ are subjected to the constraints

$$\Big|\sum_{\alpha\in\underline{\Delta}_+} (n_\alpha - m_\alpha) c_i^{(\alpha)} \Big| \leq \mathrm{const}, \qquad i = 1, 2, \cdots, r, \tag{3.17}$$

where $c_i^{(\alpha)}$ are coefficients in a decomposition of the root α with respect to the system Π of simple roots. Let $T_q(g)$ be a linear space of all such formal series, then this space is an associative algebra with respect to a multiplication of formal series and it is called the Taylor extension of $U_q(g)$ (see KT2]).

Let Fract$\,(U_q(\kappa \otimes \kappa))$ be a field of fractions generated by the following elements $1 \otimes k_{\alpha_i}$, $k_{\alpha_i} \otimes 1$ and $q^{h_{\alpha_i} \otimes h_{\alpha_j}}$, $(i, j = 1, 2, \ldots, r)$. Let us consider a formal Taylor series of the following monomials

$$e_{-\beta}^{n_\beta} \cdots e_{-\gamma}^{n_\gamma} e_{-\alpha}^{n_\alpha} \otimes e_\alpha^{m_\alpha} e_\gamma^{m_\gamma} \cdots e_\beta^{m_\beta} \tag{3.18}$$

with coefficients from Fract$\,(U_q(\kappa \otimes \kappa))$, where $\alpha < \gamma < \cdots < \beta$ in a sense of the fixed normal ordering in $\underline{\Delta}_+$ and nonnegative integers $n_\beta, \ldots, n_\alpha, m_\alpha, \ldots, m_\beta$ are subjected to the constraint (3.17). Let $T_q(b_+ \otimes b_-)$ be a linear space of all such formal series. Then this space is an associative algebra with respect to a multiplication of formal series and it is called the Taylor extension of $U_q(b_+) \otimes U_q(b_-)$ (see [KT2]).

At last we take a formal Taylor series of the following monomials

$$e_{-\beta}^{n_\beta} \cdots e_{-\gamma}^{n_\gamma} e_{-\alpha}^{n_\alpha} e_\alpha^{m_\alpha} e_\gamma^{m_\gamma} \cdots e_\beta^{m_\beta} \otimes e_{-\beta}^{n'_\beta} \cdots e_{-\gamma}^{n'_\gamma} e_{-\alpha}^{n'_\alpha} e_\alpha^{m'_\alpha} e_\gamma^{m'_\gamma} \cdots e_\beta^{m'_\beta} \tag{3.19}$$

with coefficients from Fract $(U_q(\kappa \otimes \kappa))$, where $\alpha < \gamma < \cdots < \beta$ in a sense of the fixed normal ordering in $\underline{\Delta}_+$ and nonnegative integers $n_\beta, \ldots, n_\alpha, m_\alpha, \ldots, m_\beta$ and $n'_\beta, \ldots, n'_\alpha$, $m'_\alpha, \ldots, m'_\beta$ are subjected to the constraints

$$| \sum_{\alpha \in \Delta_+} (n_\alpha + n'_\alpha - m_\alpha - m'_\alpha) c_i^{(\alpha)} | \leq \text{const}, \qquad i = 1, 2, \cdots, r. \tag{3.20}$$

Let $T_q(g \otimes g)$ be a linear space of all such formal series, then this space is an associative algebra with respect to a multiplication of formal series and it is called the Taylor extension of $U_q(g) \otimes U_q(g)$ (see [KT2]).

The following embedding holds [KT2]

$$T_q(g \otimes g) \supset T_q(b_+ \otimes b_-),$$

$$T_q(g \otimes g) \supset T_q(g) \otimes T_q(g) \supset \Delta_{q'}(T_q(g)). \tag{3.21}$$

Now we introduce a natural topology on the space $T_q(g \otimes g)$, where basic open neighborhoods of zero, Ω_l, are defined as linear spans of such the series generated the monomials (3.19) with the additional constraint

$$\sum_{i=1}^{r} \sum_{\alpha \in \Delta_+} (n_\alpha + m_\alpha + n'_\alpha + m'_\alpha) c_i^{(\alpha)} \geq l\ . \tag{3.22}$$

Such topology will be called the formal series (FS) topology.

For the goals of Section 8 we introduce also two other topologies in $U_q(g) \otimes U_q(g)$. Namely, let Ω_l^+ and Ω_l^- be linear spans of monomials from $U_q(g) \otimes U_q(g)$ (3.19) with the additional constraints

$$\sum_{i=1}^{r} \sum_{\alpha \in \underline{\Delta}_+} (n_\alpha + m_\alpha - n'_\alpha - m'_\alpha) c_i^{(\alpha)} \geq l\ , \quad \text{and} \quad \sum_{i=1}^{r} \sum_{\alpha \in \underline{\Delta}_+} (-n_\alpha - m_\alpha + n'_\alpha + m'_\alpha) c_i^{(\alpha)} \geq l \tag{3.23}$$

correspondingly. Then Ω_l^+ and Ω_l^- generated two different topologies in $U_q(g) \otimes U_q(g)$. We denote their formal completions (or, equivalently, their closures in $T_q(g \otimes g)$) by $T_q^+(g \otimes g)$ and $T_q^-(g \otimes g)$.

4 Universal R-matrix

Any quantum (super)algebra $U_q(g)$ is a non-cocommutative Hopf (super)algebra which has an intertwining operator called the universal R-matrix.

By definition [D4], the universal R-matrix for the Hopf (super)algebra $U_q(g)$ is an invertible element R of the Taylor extension $T_q(b_+ \otimes b_-)$, satisfying the equations

$$\tilde{\Delta}_{q^{-1}}(a) = R \Delta_{q^{-1}}(a) R^{-1}, \qquad \forall\, a \in U_q(g)\ , \tag{4.1}$$

$$(\Delta_{q^{-1}} \otimes id) R = R^{13} R^{23}\ , \qquad (id \otimes \Delta_{q^{-1}}) R = R^{13} R^{12}, \tag{4.2}$$

where $\tilde{\Delta}_{q'}$ is an opposite comultiplication: $\tilde{\Delta}_{q'} = \sigma\Delta_{q'}$, $\sigma(a \otimes b) = (-1)^{\deg a \deg b} b \otimes a$ for all homogeneous elements $a, b \in U_q(g)$. In (4.2) we use standard notation $R^{12} = \sum a_i \otimes b_i \otimes id$, $R^{13} = \sum a_i \otimes id \otimes b_i$, $R^{23} = \sum id \otimes a_i \otimes b_i$ if R has a form $R = \sum a_i \otimes b_i$.

Fix some normal ordering in $\underline{\Delta}_+$, and let e_α be the corresponding Cartan-Weyl generators constructed by our procedure. The following statement holds for any quantized contragredient Lie (super)algebra of finite growth (see [KT2]).

Theorem 4.1 *The equation (4.1) has a unique (up to a multiplicative constant) invertible solution in the space $T_q(b_+ \otimes b_-)$ and this solution (for a certain value of the constant) has the form*

$$R = \Big(\overrightarrow{\prod_{\alpha \in \underline{\Delta}_+}} R_\alpha \Big) \cdot K \ , \tag{4.3}$$

where the order in the product coincides with the chosen normal ordering of $\underline{\Delta}_+$ and the elements R_α and K are defined by the formulae:

$$R_\alpha = \exp_{q_\alpha^{-1}} \Big((-1)^{\theta(\alpha)} (q - q^{-1}) (a(\alpha))^{-1} (e_\alpha \otimes e_{-\alpha}) \Big) \tag{4.4}$$

for any real root $\alpha \in \underline{\Delta}_+$ and

$$R_{n\delta} = \exp((-1)^{\theta(n\delta)} (q - q^{-1}) \sum_{i,j}^{mult} c_{ij}(n) (e_{n\delta}^{(i)} \otimes e_{-n\delta}^{(j)})) \tag{4.5}$$

for any imaginary root $n\delta \in \underline{\Delta}_+$ and

$$K = q^{\sum_{i,j} d_{ij} (h_{\alpha_i} \otimes h_{\alpha_j})} \ , \tag{4.6}$$

where $a(\alpha)$ is a factor from the relation (3.7) and $(c_{ij}(n))$ is an inverse to the matrix $(a_{ij}(n))$ with the elements determined from the relation (3.5), and d_{ij} is an inverse matrix for a symmetrical Cartan matrix (a_{ij}^{sym}) if (a_{ij}^{sym}) is not degenerated. (In a case of a degenerated (a_{ij}^{sym}) we extend it up to a non-degenerated matrix $(\tilde{a}_{ij}^{sym})$ and take an inverse to this extended matrix (see [KT1,TK]*)). Moreover the solution (4.3) is the universal R-matrix, i.e. it satisfies the equations (4.2) also.*

The proof of this theorem was given in [KT2] for all quantized contragredient Lie algebras of finite-dimensional growth. The explicit formula for the universal R-matrix was obtained in [Ro,KR,LS] for the case of quantized simple Lie algebras and in [KT1] for the supercase, and in [TK,KT2] for the affine case.

5 Twisting of the Hopf structure for $U_q(g)$

In this section we consider at first some general properties of twisting for an arbitrary Hopf (super)algebra and then return to the quantum (super)algebra $U_q(g)$ again.

(i) Twisting by Two-Tensor.

Let $\mathcal{H}_{\mathcal{A}} := (\mathcal{A}, \Delta, S, \varepsilon)$ be a (super)algebra Hopf with comultiplication Δ, antipode S and counit ϵ. Let F be an invertible even element of some extension $T(\mathcal{A}\otimes\mathcal{A})$ of $\mathcal{A}\otimes\mathcal{A}$, such that the formula

$$\Delta^{(F)}(a) := F\Delta(a)F^{-1}, \qquad \forall\, a \in \mathcal{A}, \tag{5.1}$$

determine a new comultiplication, i.e. $\Delta^{(F)}$ satisfies the coassociativity

$$(\Delta^{(F)} \otimes id)\Delta^{(F)} = (id \otimes \Delta^{(F)})\Delta^{(F)}. \tag{5.2}$$

Then the comultiplication $\Delta^{(F)}$ is called the twisted coproduct. One can prove the following simple proposition (see [R]).

Proposition 5.1 *If a invertible even element $F = \sum_i f_i \otimes f^i \in T(\mathcal{A}\otimes\mathcal{A})$ satisfies the condition*

$$(F \otimes id)(\Delta \otimes id)F = (id \otimes F)(id \otimes \Delta)F, \tag{5.3}$$

then the element $u := ((id \otimes S)F) \circ 1 = \sum_i f_i S(f^i)$ is invertible and the set

$$(\mathcal{A},\ \Delta^{(F)}, S^{(F)}, \varepsilon) \tag{5.4}$$

is a new Hopf algebra $\mathcal{H}_{\mathcal{A}}^{(F)}$, where

$$\Delta^{(F)}(a) := F\Delta(a)F^{-1}, \qquad S^{(F)}(a) := uS(a)u^{-1} \tag{5.5}$$

for any $a \in \mathcal{A}$.

(In (5.3) the comultiplication Δ acts on components of F).
The Hopf (super)algebra $\mathcal{A}^{(F)}$ is called the twisted one by the two-tensor F (or twisting of the type I). One should stress that for such twisting the algebraic sector $\mathcal{A}$ of $\mathcal{H}_{\mathcal{A}}$ is not changed but the coalgebraic sector of $\mathcal{H}_{\mathcal{A}}$ is changed.

(ii) Twisting by Automorphism.

We can obtain a twisting of coalgebraic sector a Hopf (super)algebra $\mathcal{H}_{\mathcal{A}} = (\mathcal{A},\ \Delta, S, \epsilon)$ by using any automorphism in algebraic sector $\mathcal{A}$. Namely, let $\omega : \mathcal{A} \mapsto \mathcal{A}$ be an even automorphism of a linear and multiplicative structure, i.e.

$$\omega(xa + yb) = x\omega a + y\omega b\ , \qquad \omega(ab) = (\omega a)(\omega b) \tag{5.6}$$

for any $a, b \in \mathcal{A}$ and any $x, y \in \mathbf{C}$. The following simple proposition holds.

Proposition 5.2 *Let $\Delta^{(\omega)} : \mathcal{A} \mapsto \mathcal{A}\otimes\mathcal{A}$ and $S^{(\omega)} : \mathcal{A} \mapsto \mathcal{A}$ be defined as follows*

$$\Delta^{(\omega)}(a) := (\omega\otimes\omega)\Delta(\omega^{-1}a)\ , \qquad S^{(\omega)}(a) := \omega S(\omega^{-1}a)\ , \tag{5.7}$$

for any $a \in \mathcal{A}$. Then $\mathcal{H}_{\mathcal{A}}^{(\omega)} := (\mathcal{A}\ , \Delta^{(\omega)}, S^{(\omega)}, \varepsilon)$ is a new Hopf (super)algebra isomorphic to $\mathcal{H}_{\mathcal{A}} = (\mathcal{A},\ \Delta, S, \epsilon)$. If $\mathcal{H}_{\mathcal{A}}$ is quasitriangular with an universal R-matrix R then $\mathcal{H}_{\mathcal{A}}^{(\omega)}$ is also quasitriangular with the universal R-matrix $R^{(\omega)}$:

$$R^{(\omega)} = (\omega\otimes\omega)R\ . \tag{5.8}$$

The Hopf (super)algebra $\mathcal{H}_{\mathcal{A}}^{(\omega)}$ is called the twisted Hopf (super)algebra by automorphism ω (or twisting of the type II). Note that the algebra structure for the twisting of the type II does not change also.

(iii) Twisting for $U_q(g)$ by some Factors of the Universal R-matrix.

Now we want to show how the factors of the universal R-matrix generate a family of twistings for $U_q(g)$.

For a fixed normal ordering in $\underline{\Delta}_+$ and for any $\gamma \in \underline{\Delta}_+$ we put

$$F_\gamma := \prod_{\alpha<\gamma} R_\alpha^{21} \tag{5.9}$$

and

$$F'_\gamma := (\prod_{\gamma<\alpha} R_\alpha^{21})K \tag{5.10}$$

where R_α are the factors of the universal R-matrix (4.3) and the product in (5.9) (and (5.10)) is taken over all roots α which are less (more) γ in a sense of the normal ordering. The following theorem is valid.

Theorem 5.1 *For any roots $\gamma \in \underline{\Delta}_+$ the two sets*

$$(U_q(g)\ ,\Delta_{q^{-1}}^{(F_\gamma)}, S_{q^{-1}}^{(F_\gamma)}, \varepsilon)\ , \quad and \quad (U_q(g)\ ,\Delta_{q^{-1}}^{(F'_\gamma)}\ , S_{q^{-1}}^{(F'_\gamma)}, \varepsilon) \tag{5.11}$$

are two Hopf algebras, where $\Delta_{q^{-1}}^{(F_\gamma)}$, $S_{q^{-1}}^{(F_\gamma)}$, and $\Delta_{q^{-1}}^{(F'_\gamma)}$, $S_{q^{-1}}^{(F'_\gamma)}$ are determined by the formulas

$$\Delta_{q^{-1}}^{(F_\gamma)}(a) := (F_\gamma)^{-1}\Delta_{q^{-1}}(a)F_\gamma\ , \qquad S_{q^{-1}}^{(F_\gamma)}(a) = u_\gamma S_{q^{-1}}(a)(u_\gamma)^{-1}, \tag{5.12}$$

and

$$\Delta_{q^{-1}}^{(F'_\gamma)}(a) := (F'_\gamma)\Delta_{q^{-1}}(F'_\gamma)^{-1}, \qquad S_{q^{-1}}^{(F'_\gamma)}(a) = u'_\gamma S_{q^{-1}}(a)(u'_\gamma)^{-1}. \tag{5.13}$$

for any $a \in U_q(g)$. (Here in (5.12) and (5.13) the elements u_γ and u'_γ are determined by the formulas similar to "u" in the Proposition 5.1).

Proof. By using the Proposition 8.3 of [KT1], which is valid indeed for any quantized contragredient Lie (super)algebra of finite growth, we prove at first that $\Delta_{q^{-1}}^{(F_\gamma)}$ and $\Delta_{q^{-1}}^{(F'_\gamma)}$ satisfy the coassociativity (5.3) after that we apply Proposition 5.1.

Now we would like to show that for the case of a quantized simple finite-dimensional Lie algebra g the twisting by the two-tensor F_γ coincides with the twisting by the Lusztig automorphism of $U_q(g)$. First we remind that there is one-to-one correspondence between the set of all normal orderings of Δ_+ and the set of reduced decompositions of the longest element w_o of the Weyl group $W(g)$. Namely, the following proposition is valid (see [Z]).

Proposition 5.3 *Let s_{α_i}, $(i = 1, 2, \ldots, r)$, be the elementary reflections of $W(g)$, corresponding to the simple roots α_i and let*

$$\gamma_1,\ \gamma_2,\ \gamma_3,\ \ldots,\ \gamma_n,\ \ldots,\ \gamma_N \tag{5.14}$$

be some fixed normal ordering in $\underline{\Delta}_+$ *, then all roots of the chain*

$$\alpha_{i_1}=\gamma_1,\ \ \alpha_{i_2}=s^{-1}_{\alpha_{i_1}}(\gamma_2),\ \ \alpha_{i_3}=s^{-1}_{\alpha_{i_2}}s^{-1}_{\alpha_{i_1}}(\gamma_3),\ \ \ldots\ \ ,$$

$$\alpha_{i_n}=s^{-1}_{\alpha_{i_{n-1}}}\cdots s^{-1}_{\alpha_{i_2}}s^{-1}_{\alpha_{i_1}}(\gamma_n),\ \ \ldots\ \ ,\alpha_{i_N}=s^{-1}_{\alpha_{i_{N-1}}}\cdots s^{-1}_{\alpha_{i_2}}s^{-1}_{\alpha_{i_1}}(\gamma_N) \tag{5.15}$$

are simple, and $w_o=s_{\alpha_{i_1}}s_{\alpha_{i_2}}\cdots s_{\alpha_{i_N}}$ *is a reduced decomposition of* w_o.
On the contrary, for any reduced decomposition of w_o, $w_o=s_{\alpha_{i_1}}s_{\alpha_{i_2}}\cdots s_{\alpha_{i_N}}$, *the sequence*

$$\alpha_{i_1},\ \ s_{\alpha_{i_1}}(\alpha_2),\ \ s_{\alpha_{i_2}}s_{\alpha_{i_1}}(\alpha_3),\ \ \ldots\ \ ,s_{\alpha_{i_{N-1}}}\cdots s_{\alpha_{i_2}}s_{\alpha_{i_1}}(\gamma_N) \tag{5.16}$$

is a normal ordering in Δ_+*,*

(It should be noted that there are identical simple roots in (5.14) because $r<N$).
For every root γ_n, $(n=1,2,\ldots,N)$, of the normal ordering (5.14) we put into correspondence the element w_{γ_n} of the Weyl group $W(g)$, which is an initial segment of the corresponding reduced decomposition of w_o, i. e.

$$w_{\gamma_n}:=s_{\alpha_{i_1}}s_{\alpha_{i_2}}\cdots s_{\alpha_{i_{n-1}}}\ , \qquad (w_{\gamma_n}(\alpha_{i_n})=\gamma_n)\ . \tag{5.17}$$

Following [L], [DeCK], we define an action of a braid group $\hat{W}(g)$ in $U_q(g)$ by means of the Lusztig automorphisms (see (2.3))

$$\begin{aligned}
\hat{s}_{\alpha_i}(k^{\pm1}_{\alpha_i})=k^{\mp1}_{\alpha_i}\ , \qquad & \hat{s}_{\alpha_i}(k^{\pm1}_{\alpha_j})=k^{\pm1}_{\alpha_j}k^{\mp n_{ij}}_{\alpha_i}\ , \quad (i\neq j)\ ,\\
\hat{s}_{\alpha_i}(e_{\alpha_i})=-e_{-\alpha_i}k_{\alpha_i}\ , \qquad & \hat{s}_{\alpha_i}(e_{-\alpha_i})=-k^{-1}_{\alpha_i}e_{\alpha_i}\ ,\\
\hat{s}_{\alpha_i}(e_{\pm\alpha_j})=N^{-\frac{1}{2}}_{ij}&(\tilde{\mathrm{ad}}_q e_{\pm\alpha_i})^{n_{ij}}e_{\pm\alpha_j}\ , \quad (i\neq j)\ ,
\end{aligned} \tag{5.18}$$

where the positive integer n_{ij} are determined by the formula (2.5) (the last line for this case); the normalizing factors N_{ij} can be determined from results of the work [KT1] and they have the form

$$N_{ij}=q^{(\alpha_i,\alpha_j)}([(\alpha_i,\alpha_j)]_q)^{n_{ij}}\prod_{s=1}^{n_{ij}-1}\frac{[(s\alpha_i+\alpha_j,s\alpha_i+\alpha_j)]_q}{[((s-1)\alpha_i+\alpha_j,s\alpha_i+\alpha_j)]_q}\ . \tag{5.19}$$

For the element w_{γ_n} (5.17) we put

$$\hat{w}_{\gamma_n}:=\hat{s}_{\alpha_{i_1}}\hat{s}_{\alpha_{i_2}}\cdots\hat{s}_{\alpha_{i_{n-1}}}\ . \tag{5.20}$$

The following statement (known in quantum group folklore) holds.

Proposition 5.4 *For any normal ordering (5.14) of any simple Lie algebra g the twisting of a Hopf algebra structure of the quantum algebra* $U_q(g)$ *by the two-tensor* F_{γ_n} *(see (5.9), (5.12)) coincides with the twisting by the Lusztig automorphism* $\hat{w}_{\gamma_n}$ *(5.19).*

Proof. We prove the statement for a root γ_n of (5.14) by induction on n. For $n=2$ we have that $\gamma_1=\alpha_1$ is a simple root, $F_{\gamma_2}=\exp_{q_{\alpha_1}^{-1}}((q-q^{-1})(e_{-\alpha_1}\otimes e_{-\alpha_1})$, and $\hat{w}_{\gamma_2}$ is the Lusztig automorphism, corresponding to the simple reflection s_{α_1}. It is clear from the definitions that

$$\Delta_{q^{-1}}^{(F_{\gamma_2})}(e_{\alpha_i})=\Delta_{q^{-1}}^{(\hat{w}_{\gamma_2})}(e_{\alpha_i})=e_{\alpha_i}\otimes 1+k_{\alpha_i}^{-1}\otimes e_{\alpha_i}\ , \tag{5.21}$$

$$\Delta_{q^{-1}}^{(F_{\gamma_2})}(e_{-\alpha_i})=\Delta_{q^{-1}}^{(\hat{w}_{\gamma_2})}(e_{-\alpha_i})=e_{-\alpha_i}\otimes k_{\alpha_i}+1\otimes e_{-\alpha_i}\ , \tag{5.22}$$

for any simple root α_i such that $(\alpha_1,\alpha_i)=0$. Now let α_i be a simple root such that $(\alpha_1,\alpha_i)\neq 0$ then $\beta_i:=\hat{w}_{\alpha_1}(\alpha_i)$ is a positive root and one can choose a normal ordering

$$\gamma_1',\ \gamma_2',\ \gamma_3',\ \ldots,\ \gamma_N' \tag{5.23}$$

such that $\gamma_1'=\alpha_1$, $\gamma_2'=\beta_i$. Using the Proposition 8.3 from [KT1] we have

$$\Delta_{q^{-1}}^{(F_{\gamma_2})}(e_{\beta_i})=\Delta_{q^{-1}}^{(\hat{w}_{\gamma_2})}(e_{\beta_i})=e_{\beta_i}\otimes 1+k_{\beta_i}^{-1}\otimes e_{\beta_i}\ , \tag{5.24}$$

$$\Delta_{q^{-1}}^{(F_{\gamma_2})}(e_{-\beta_i})=\Delta_{q^{-1}}^{(\hat{w}_{\gamma_2})}(e_{-\beta_i})=e_{-\beta_i}\otimes k_{\beta_i}+1\otimes e_{-\beta_i}\ , \tag{5.25}$$

The equations (5.21), (5.22) and (5.24), (5.25) are sufficient to conclude that

$$\Delta_{q^{-1}}^{(F_{\gamma_2})}=\Delta_{q^{-1}}^{(\hat{w}_{\gamma_2})}\ . \tag{5.26}$$

For $n>2$ the statement follows immediately from the multiplicative structure of $\hat{w}_{\gamma_n}$ taking in mind that

$$F_{\gamma_n}=F_{\gamma_{n-1}}R_{\gamma_{n-1}}^{21}=F_{\gamma_{n-1}}(\hat{w}_{\gamma_{n-1}}\otimes\hat{w}_{\gamma_{n-1}})\exp_{q_{\alpha_{n-1}}^{-1}}((q-q^{-1})(e_{-\alpha_{n-1}}\otimes e_{-\alpha_{n-1}})) \tag{5.27}$$

in the notations of (5.17).

An analog of Proposition 5.4 is also valid for quantized superalgebras. In this case the Lusztig automorphisms should be considered as isomorphisms between different quantized superalgebras. Let us consider this in detail.

Let $g(A,\Upsilon)$ and $g(A',\Upsilon')$ be two isomorphic finite-dimensional contragredient Lie superalgebra. We consider non-trivial case when the superalgebras have non-equivalent Cartan matrices A and A', i. e. $A'\neq BDAB^{-1}$ for any nonsingular matrix B and any diagonal matrix D. Such superalgebras have the same reduced system of all roots and different reduced system of positive roots.

Let $\Pi:=\{\alpha_1,\alpha_2,\ldots,\alpha_r\}$ and $\Pi':=\{\alpha_1',\alpha_2',\ldots,\alpha_r'\}$ be systems of simple roots for $g(A,\Upsilon)$ and $g(A',\Upsilon')$ correspondingly. Following Serganova (see [LSS]) one can define an "elementary reflection" s_{α_i} for any $\alpha_i\in\Pi$ as follows

$$s_{\alpha_i}(\alpha_i):=\alpha_i'=-\alpha_i\ ,\qquad s_{\alpha_i}(\alpha_j):=\alpha_j'=\alpha_j-n_{i,j}\alpha_i,\qquad (i\neq j)\ . \tag{5.28}$$

The following theorem is valid (see [LSS]).

Theorem 5.2 (V.Serganova). (i) *Let Π be a system of simple roots. Then the set $s_{\alpha_i}(\Pi)$ may be considered as a system of simple roots, moreover $s_{\alpha_i}(\Pi)$ is not equivalent to Π iff the root α_i is grey.*

(ii) *For any two isomorphic superalgebras $g(A,\Upsilon)$ and $g(A',\Upsilon')$ with non-equivalent Cartan matrices A and A' there exist a sequence of simple root systems $\Pi_1,\Pi_2,\ldots,\Pi_n$ and roots $\alpha^{(i)}\in\Pi_i$ such that $s_{\alpha^{(i)}}(\Pi_i)=\Pi_{i+1}$, and moreover $\Pi_1=\Pi$, $\Pi_n=\Pi'$.*

Remark. It should be noted that in a general case the system of all roots is not invariant with respect to the "elementary reflection" s_{α_i}, i. e. $s_{\alpha_i}(\Delta)$ is a root system which should not coincide with Δ (see, for example, the root system of the superalgebra $D(2,1;\alpha)$ [K2]). (This is why the words "elementary reflection" are putted in quotation marks). Therefore it is better to consider "elementary reflection" as change of variables or as a map from one linear space to another. An accurate formulation leads to a notation Weyl grouppoid instead of a Weyl group.

Let simple root systems Π and Π' are connected by one "elementary reflection" s_{α_i} only, i.e. $s_{\alpha_i}(\Pi)=\Pi'$, and let $\{e_{\pm\alpha_j},\ k_{\alpha_j}^{\pm1}\}$ and $\{e'_{\pm\alpha'_i},\ k'^{\pm1}_{\alpha'_i}\}$, $(i=1,2,\ldots,r)$ be the Chevalley generators of the quantum algebras $U_q(g(A,\Upsilon))$ and $U_q(g(A',\Upsilon'))$ correspondingly. We define Lusztig isomorphism $\hat{s}_{\alpha_i}$ as isomorphism $\hat{s}_{\alpha_i}:U_q(g(A',\Upsilon'))\mapsto U_q(g(A,\Upsilon))$ by the relations analogous to (5.16):

$$\begin{gathered}\hat{s}_{\alpha_i}(k'^{\pm1}_{\alpha'_i})=k^{\mp1}_{\alpha'_i}, \qquad \hat{s}_{\alpha_i}(k'^{\pm1}_{\alpha'_j})=k^{\pm1}_{\alpha_j}k^{\mp n_{ij}}_{\alpha_i}, \quad (i\neq j)\,, \\ \hat{s}_{\alpha_i}(e'_{\alpha'_i})=-e_{-\alpha_i}k_{\alpha_i}\,, \qquad \hat{s}_{\alpha_i}(e'_{-\alpha'_i})=-k^{-1}_{\alpha_i}e_{\alpha_i}\,, \\ \hat{s}_{\alpha_i}(e'_{\pm\alpha'_j})=((-1)^{\theta(n_{ij}\alpha_i)\theta(\alpha_j)}N_{ij})^{-\frac{1}{2}}(\tilde{\mathrm{ad}}_q e_{\pm\alpha_i})^{n_{ij}}e_{\pm\alpha_j}\,, \quad (i\neq j)\,,\end{gathered} \tag{5.29}$$

where the normalizing factors N_{ij} are given by the formula (5.19).

If $\Delta_{q^{-1}}$ and $\Delta'_{q^{-1}}$ are the standard comultiplications of $U_q(g(A,\Upsilon))$ and $U_q(g(A',\Upsilon'))$ (see (2.11)-(2.13)) then just as in a even case using the Proposition 8.3 from [KT1] we have

$$(\hat{s}_{\alpha_i}\otimes\hat{s}_{\alpha_i})\Delta'_{q^{-1}}(a')=(\hat{s}_{\alpha_i}\otimes\hat{s}_{\alpha_i})\Delta_{q^{-1}}(\hat{s}^{-1}_{\alpha_i}a)=F^{-1}_{\alpha_i}\Delta_{q^{-1}}(a)F_{\alpha_i} \tag{5.30}$$

for any $a\in U_q(g(A,\Upsilon))$, where

$$F_{\alpha_i}=R^{21}_{\alpha_i}:=\exp_{q^{-1}_{\alpha_i}}((q-q^{-1})(e_{-\alpha_i}\otimes e_{-\alpha_{n-1}}))\ . \tag{5.31}$$

Now let Π and Π' be an arbitrary non-equivalent systems of isomorphic superalgebras $U_q(g(A,\Upsilon))$ and $U_q(g(A',\Upsilon'))$ correspondingly. Then according to the Serganova's theorem there is a chain $w:=s_{\alpha^{(1)}}s_{\alpha^{(2)}}\cdots s_{\alpha^{(n)}}$ of the elementary reflections (5.28), such that $\alpha'_i=w(\alpha_i)$, $(i=1,2,\ldots,r)$. (Here we do not distinguish systems of simple roots which differ an enumeration of roots). We define Lusztig isomorphism $\hat{w}:U_q(g(A',\Upsilon'))\mapsto U_q(g(A,\Upsilon))$ as

$$\hat{w}:=\hat{s}_{\alpha^{(1)}}\hat{s}_{\alpha^{(2)}}\cdots\hat{s}_{\alpha^{(n)}}\ . \tag{5.32}$$

The relation (5.30) turns to

$$(\hat{w}\otimes\hat{w})\Delta'_{q^{-1}}(a')=(\hat{w}\otimes\hat{w})\Delta_{q^{-1}}(\hat{w}^{-1}a)=F^{-1}_{w}\Delta_{q^{-1}}(a)F_{w} \tag{5.33}$$

for any $a \in U_q(g(A,\Upsilon))$, where the twisting two-tensor F_w is defined by the formula

$$F_w = F_{\alpha(1)}((\hat{s}_{\alpha(1)} \otimes \hat{s}_{\alpha(1)})F_{\alpha(2)}) \cdots ((\hat{s}_{\alpha(1)} \cdots \hat{s}_{\alpha(n-1)} \otimes \hat{s}_{\alpha(1)} \cdots \hat{s}_{\alpha(n-1)})F_{\alpha(n)}) \ . \tag{5.34}$$

It should be noted that the factors $((\hat{s}_{\alpha(1)} \cdots \hat{s}_{\alpha(l-1)} \otimes \hat{s}_{\alpha(1)} \cdots \hat{s}_{\alpha(l-1)})F_{\alpha(l)})$, $(l = 1, 2, \dots, n)$, belong to $T_q(g(A,\Upsilon) \otimes g(A,\Upsilon)$ and are composed from factors of the universal R-matrix of $U_q(g(A,\Upsilon))$.

We can summarize our considerations as the theorem.

Theorem 5.3 *Let $g(A,\Upsilon)$ and $g(A',\Upsilon')$ be two isomorphic finite-dimensional contragredient Lie superalgebras. Then there exists an isomorphism of algebras, $\hat{w} : U_q(g(A',\Upsilon')) \mapsto U_q(g(A,\Upsilon))$, such that a comultiplication $\Delta^{(\omega)}_{q^{-1}}$ of $U_q(g(A,\Upsilon))$ induced from $\Delta'_{q^{-1}}$ of $U_q(g(A',\Upsilon'))$ by the isomorphism ω differs from the initial comultiplication $\Delta_{q^{-1}}$ with a twisting by some factors of the universal R-matrix for $U_q(g(A,\Upsilon))$.*

6 Drinfeld's realization of quantum affine algebras

In this and last sections we consider quantized nontwisted affine algebras only.

Let $\hat{g}$ be nontwisted affine Lie algebra and $\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_r\}$ be a system of simple roots for $\hat{g}$. We assume that the roots $\Pi_0 = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ generate the system $\Delta_+(g)$ of positive roots of the corresponding finite-dimensional Lie algebra g.

In the paper "A new realization of Yangians and quantized affine algebras" [D2] V.G. Drinfeld suggested another realization of the nontwisted affine algebra $U_q(\hat{g})$. In this description the algebra $U_q(\hat{g})$ is generated by the elements:

$$k_c, \ \chi_{i,l}, \ \xi^{\pm}_{i,l} \ , \qquad (\text{for } i = 1, 2, \dots, r; \quad l \in \mathbf{Z}) \ , \tag{6.1}$$

with the defining relations (we modify them a little for technical convenience):

$$[k_c, everything] = 0 \ , \qquad \chi_{i,0}\xi^{\pm}_{j,m} = q^{\pm(\alpha_i,\alpha_j)}\xi^{\pm}_{j,m}\chi_{i,0} \ , \tag{6.2}$$

$$[\chi_{i,l}, \chi_{j,m}] = \delta_{l,-m} a_{ij}(l) \frac{k_c^l - k_c^{-l}}{q - q^{-1}} \ , \tag{6.3}$$

$$[\chi_{i,l}, \xi^{\pm}_{j,m}] = \pm a_{ij}(l) k_c^{(-l \pm |l|)/2} \xi^{\pm}_{j,l+m} \ , \tag{6.4}$$

$$\xi^{\pm}_{i,l+1}\xi^{\pm}_{j,m} - q^{\pm(\alpha_i,\alpha_j)}\xi^{\pm}_{j,m}\xi^{\pm}_{i,l+1} = q^{\pm(\alpha_i,\alpha_j)}\xi^{\pm}_{i,l}\xi^{\pm}_{j,m+1} - \xi^{\pm}_{j,m+1}\xi^{\pm}_{i,l} \ , \tag{6.5}$$

$$[\xi^{+}_{i,l}, \xi^{-}_{j,m}] = \delta_{i,j} \frac{\psi_{i,l+m}k_c^m - \phi_{i,l+m}k_c^l}{q - q^{-1}} \ , \tag{6.6}$$

$$\mathrm{Sym}\left(\sum_{s=o}^{n'_{ij}} (-1)^s C^s_{n'_{ij}}(q^{(\alpha_i,\alpha_j)}) \xi^{\pm}_{i,l_1} \cdots \xi^{\pm}_{i,l_s} \xi^{\pm}_{j,m} \xi^{\pm}_{i,l_{s+1}} \cdots \xi^{\pm}_{i,l_{n'_{ij}}}\right) = 0 \quad \text{for } i \neq j \ , \tag{6.7}$$

where

$$a_{ij}(l) = \frac{q^{l(\alpha_i,\alpha_j)} - q^{-l(\alpha_i,\alpha_j)}}{l(q - q^{-1})} \ , \tag{6.8}$$

the elements $\phi_{i,p}$, $\psi_{i,p}$ are defined from the relations:

$$\sum_p \psi_{i,p} u^{-p} = \chi_{i,0}^{-1} \exp((q-q^{-1})\sum_{p>0} \chi_{i,p} u^{-p}) \ , \tag{6.9}$$

$$\sum_p \phi_{i,p} u^{-p} = \chi_{i,0} \exp((q-q^{-1})\sum_{p<0} \chi_{i,p} u^{-p}) \ , \tag{6.10}$$

the q-binomial coefficients $C_n^s(q)$ are determined by the formula

$$C_n^s(q) = \frac{[n]_q!}{[s]_q![n-s]_q!} \ , \tag{6.11}$$

the symbol "Sym" in (6.7) denotes a symmetrization on $l_1, l_2, \ldots, l_{n_{ij}}$, and $n'_{ij} := 1 - (\alpha_i, \alpha_j)$.

Drinfeld has shown that the elements $\chi_{i,0}^{\pm}$ and $\xi_{i,0}^{\pm}$ are connected with the Chevalley generators e_{α_i}, h_{α_i} in the following way

$$h_{\alpha_i} = \chi_{i,0}, \qquad e_{\pm\alpha_i} = \xi_{i,0}^{\pm} \tag{6.12}$$

(for $i = 1, 2, \ldots, r$), and there are more complicated formulas for e_{α_0} and h_{α_i} which we do not write down here (see [D2]). Drinfeld suggested also another formulas of comultiplication for $U_q(g)$, which have the following form [D3]

$$\Delta^{(D)}(k_c) = k_c \otimes k_c \ , \tag{6.13}$$

$$\Delta^{(D)}(\chi_{i,l}) = \chi_{i,l} \otimes 1 + k_c^{-l} \otimes \chi_{i,l} \ , \tag{6.14}$$

$$\Delta^{(D)}(\chi_{i,-l}) = \chi_{i,-l} \otimes k_\delta^{-l} + 1 \otimes \chi_{i,-l} \tag{6.15}$$

for $l > 0$, and

$$\Delta^{(D)}(\xi_{i,l}^{+}) = \xi_{i,l}^{+} \otimes 1 + \sum_{m\geq 0} k_c^m \phi_{i,m} \otimes \xi_{i,l+m}^{+} \ , \tag{6.16}$$

$$\Delta^{(D)}(\xi_{i,l}^{-}) = 1 \otimes \xi_{i,l}^{-} + \sum_{m\geq 0} \xi_{i,l-m}^{-} \otimes \psi_{i,m} k_c^m \ , \tag{6.17}$$

for any $l \in \mathbf{Z}$.

Now we want to show how the generators k_c, $\chi_{i,l}$, $\xi_{i,l}^{\pm}$ can be expressed via the Cartan-Weyl generators constructed by our procedure, and in the Section 8 we show how to obtain the formulas (6.14)-(6.17) by a twisting of the standard comultiplication (2.11)-(2.13) using some factor of the universal R-matrix.

7 Connection of the Drinfeld's realization with the Cartan-Weyl basis for $U_q(\hat{g})$

We fix some special normal ordering in $\Delta_+(\hat{g}) := \Delta_+$, which satisfies the following additional constraint:

$$l\delta + \alpha_i < (m+1)\delta < (n+1)\delta - \alpha_j) \tag{7.1}$$

for any simple roots $\alpha_i, \alpha_j \in \Delta_+(g)$, and $l, m, n \geq 0$. Here δ is a minimal positive imaginary root. For given normal ordering we apply our procedure for construction of the Cartan-Weyl generators for $U_q(\hat{g})$. Furthermore we put

$$e_\delta^{(i)} = [e_{\alpha_i}, e_{\delta-\alpha_i}]_q \ , \tag{7.2}$$

$$e_{n\delta+\alpha_i} = (-1)^n([(\alpha_i,\alpha_i)]_q)^{-n}(\tilde{\mathrm{ad}}\ e_\delta^{(i)})^n e_{\alpha_i} \ , \tag{7.3}$$

$$e_{(n+1)\delta-\alpha_i} = ([(\alpha_i,\alpha_i)]_q)^{-n}(\tilde{\mathrm{ad}}\ e_\delta^{(i)})^n e_{\delta-\alpha_i} \ , \tag{7.4}$$

$$e'^{(i)}_{(n+1)\delta} = [e_{n\delta+\alpha_i}, e_{\delta-\alpha_i}]_q \ , \tag{7.5}$$

(for $n > 0$), where $(\tilde{\mathrm{ad}}\, x)y = [x, y]$ is a usual commutator. The imaginary root vectors $e'^{(i)}_{\pm n\delta}$ do not satisfy the relation (3.5). We introduce new vectors $e^{(i)}_{\pm n\delta}$ by the following (Schur) relations:

$$e'^{(i)}_{n\delta} = \sum_{p_1+2p_2+\ldots+np_n=n} \frac{(q-q^{-1})^{\sum p_i - 1}}{p_1!\cdots p_n!}(e_\delta^{(i)})^{p_1}\cdots(e_{n\delta}^{(i)})^{p_n}. \tag{7.6}$$

In terms of the generating functions

$$E'_i(z) = (q-q^{-1})\sum_{m\geq 1} e'^{(i)}_{m\delta} z^m \tag{7.7}$$

and

$$E_i(z) = (q-q^{-1})\sum_{m\geq 1} e^{(i)}_{m\delta} z^m \tag{7.8}$$

the relation (7.6) may be rewritten in the form

$$E'_i(z) = -1 + \exp E_i(z) \tag{7.9}$$

or

$$E_i(z) = \ln(1 + E'_i(z)) \ . \tag{7.10}$$

¿From this we have the inverse formula to (7.6)

$$e^{(i)}_{n\delta} = \sum_{p_1+2p_2+\ldots+np_n=n} \frac{(q^{-1}-q)^{\sum p_i - 1}(\sum_{i=1}^n p_i - 1)!}{p_1!\cdots p_n!}(e'^{(i)}_\delta)^{p_1}\cdots(e'^{(i)}_{n\delta})^{p_n}. \tag{7.11}$$

The rest of the real root vectors we construct in accordance with the Algorithm 3.1 using the root vectors $e_{n\delta+\alpha_i}$, $e_{(n+1)\delta-\alpha_i}$, $e^{(i)}_{(n+1)\delta}$, $(i = 1, 2, \ldots, r;\ n \in \mathbf{Z}_+)$. The root vectors of negative roots are obtained by the Cartan involution $(^*)$:

$$e_{-\gamma} = (e_\gamma)^* \tag{7.12}$$

for $\gamma \in \underline{\Delta}_+(\hat{g})$. Using the explicit relations (7.2)-(7.5), (7.11) and (7.12) we can prove the following proposition.

Proposition 7.1 *The root vectors $e^{(i)}_{n\delta}$, $e_{n\delta-\alpha_i}$ and $e_{n\delta+\alpha_i}$ satisfy the following relations*

$$[e^{(i)}_{n\delta}, e^{(j)}_{m\delta}] = (-1)^{n\delta_{ij}}\delta_{n,-m}a_{ij}(n)\frac{k_\delta^n - k_\delta^{-n}}{q-q^{-1}} \tag{7.13}$$

for n, $m \neq 0$,

$$[e^{(i)}_{n\delta}, e_{(m+1)\delta-\alpha_j}] = (-1)^{n\delta_{ij}}a_{ij}(n)e_{(n+m+1)\delta-\alpha_i} \ , \tag{7.14}$$

$$[e^{(i)}_{n\delta}, e_{m\delta+\alpha_j}] = -(-1)^{n\delta_{ij}}a_{ij}(n)e_{(n+m)\delta+\alpha_i} \tag{7.15}$$

for n, $m > 0$, where the matrix elements $a_{ij}(n)$ are defined by the formula (6.8) and k_δ is a central element of $U_q(\hat{g})$.

It should be noted that the matrix $(a_{ij}(n))$ with the elements (6.8) may be considered as a q-analog of a "level n" for the matrix Cartan (a^{sym}_{ij}).

The following theorem states the connection between the Cartan-Weyl and Drinfeld's generators for the quantum nontwisted affine algebra $U_q(\hat{g})$.

Theorem 7.1 *Let some function π: $\{\alpha_1, \alpha_2 \ldots, \alpha_r\} \mapsto \{0, 1\}$ be chosen such that $\pi(\alpha_i) \neq \pi(\alpha_j)$ if $(\alpha_i, \alpha_j) \neq 0$ and let the root vectors $\hat{e}_{\pm\gamma}$ and $\check{e}_{\pm\gamma}$ of the real roots $\gamma \in \Delta_+(\hat{g})$ be the circular Cartan-Weyl generators (3.12), (3.13) and $e^{(i)}_{n\delta}$ be imaginary root vectors of $U_q(\hat{g})$. Then the elements*

$$k_c := k_\delta \ , \qquad \chi_{i,0} := k_{\alpha_i} \ , \qquad \xi^{\pm}_{i,0} := \hat{e}_{\pm\alpha_i} \ , \tag{7.16}$$

$$\chi_{i,\pm n} := e^{(i)}_{\pm n\delta} \ , \qquad \xi^{\pm}_{i,n} = \tau_{i,n}\hat{e}_{n\delta\pm\alpha_i} \ , \qquad \xi^{\pm}_{i,-n} = \tau_{i,n}\check{e}_{n\delta\pm\alpha_i} \ , \tag{7.17}$$

where $\tau_{i,n} = (-1)^{n\pi(\alpha_i)}$ and $n > 0$, satisfy the relations (6.2)-(6.7), i.e. the elements (7.16) and (7.17) are the generators of the Drinfeld's realization of $U_q(\hat{g})$.

Remark. In terms of the Cartan-Weyl generators the relations (6.2)-(6.7) can be interpreted as follows:

(i) The "Serre" relations (6.7) are equivalent to the following corollary of Proposition 3.1:

$$[e_\alpha, e_\beta]_q = 0 \tag{7.18}$$

if the roots α and β are neighboring and $\alpha < \beta$ in a sense of fixed normal ordering of the root system Δ_+.

(ii) The defining relations (7.2)-(7.5) may be easily generalized to the identities

$$[e_{n\delta+\alpha_i}, e_{(m+1)\delta-\alpha_i}] = c\delta_{ij}e'^{(i)}_{(n+m+1)\delta} \tag{7.19}$$

for $n, m \in \mathbf{Z}$ where c is a constant. The relations (7.19) rewritten by means of (7.6) in terms of generators $e^{(i)}_{l\delta\pm\alpha_i}$ give us (6.6).

(iii) The formulas (6.5) define quadratic relations between vectors $e_{n\delta+\alpha_i}$ or between $e_{n\delta-\alpha_i}$, $(i = 1, 2, \ldots, r)$. We write down them explicitly for $U_q(\hat{sl}_2)$ in Appendix A (see (A.10)-(A.21)).

8 The second Drinfeld's realization as twisting of Hopf structure in $U_q(\hat{g})$

The second Drinfeld's realization of the quantum affine algebra $U_q(\hat{g})$ was originally obtained as a quantization of Lie bialgebra structure in affine Lie algebra $\hat{g}$, which is equivalent to presentation of $\hat{g} \oplus \kappa$ (where κ is the Cartan subalgebra of $g = n_+ \oplus \kappa \oplus n_-$) as a classical double of the current algebra $\hat{n}_+ := n_+[t, t^{-1}] \oplus \kappa[t] \oplus \mathbb{C}d$. It turned out that multiplicative structure of this quantization is isomorphic to $U_q(\hat{g})$ but no any evident connection of comultiplication structures was found. Here we make clear this connection.

The algebra $\hat{n}_+$ may be considered as an image of Borel subalgebra $\hat{b}_+$ under a action of a limiting longest element w_0 of the Weyl group $W(\hat{g})$ of $\hat{g}$. This limiting element w_0 does not act on vectors in the Cartan subalgebra or on the root vectors in $\hat{g}$, nevertheless the twisting of a Hopf structure in $U_q(\hat{g})$ by w_0 is well defined as a limit of twistings by finite elements of Weyl group $W(\hat{g})$ in the FS topology of $T_q(\hat{g} \otimes \hat{g})$ (just as in $T_q^+(\hat{g} \otimes \hat{g})$, see the Section 3). We prove that Drinfeld's comultiplication in his second realization can be obtained as a twisting by w_0 of a standard comultiplication in $U_q(\hat{g})$. Analogously to the Theorem 5.1 we state also that this twisting can be presented as conjugation by an infinite product of factors of the universal R-matrix.

Theorem 8.1 (i) *For any fixed element $x \in U_q(\hat{g})$ an expression*

$$\Delta^{(D)}(a) = (\prod_{\gamma<\delta} R_\gamma^{21})^{-1} \Delta_{q^{-1}}(a) (\prod_{\gamma<\delta} R_\gamma^{21}) , \qquad \forall\, a \in U_q(\hat{g}) , \tag{8.1}$$

is well defined element of $T_q(\hat{g} \otimes \hat{g})$ (just as in $T_q^+(\hat{g} \otimes \hat{g})$).

(ii) *A map $\Delta^{(D)}$: $U_q(\hat{g}) \to T_q(\hat{g} \otimes \hat{g})$ (just as $\Delta^{(D)}$: $U_q(\hat{g}) \to T_q^+(\hat{g} \otimes \hat{g})$) is homomorphism and it satisfies all the axioms of the Hopf algebra.*

(iii) *Explicit formulas for Δ^D look as follows:*

$$\Delta^{(D)}(e_{\alpha_i}) = e_{\alpha_i} \otimes 1 + k_{\alpha_i} \otimes e_{\alpha_i} - (q - q^{-1}) \sum_{m \geq 1} e'^{(i)}_{-m\delta} k_{\alpha_i} \otimes e_{m\delta+\alpha_i} , \tag{8.2}$$

$$\Delta^{(D)}(e_{-\alpha_i}) = e_{-\alpha_i} \otimes k_{\alpha_i}^{-1} + 1 \otimes e_{\alpha_i} - (q - q^{-1}) \sum_{m \geq 1} e_{-m\delta-\alpha_i} \otimes e'^{(i)}_{m\delta} k_{\alpha_i}^{-1} , \tag{8.3}$$

$$\Delta^{(D)}(e^{(i)}_{n\delta}) = e^{(i)}_{n\delta} \otimes 1 + k_\delta^{-n} \otimes e^{(i)}_{n\delta} , \qquad \Delta^{(D)}(e^{(i)}_{-n\delta}) = e^{(i)}_{-n\delta} \otimes k_\delta^{n} + 1 \otimes e^{(i)}_{-n\delta}. \tag{8.4}$$

In terms of Drinfeld generators the comultiplication $\Delta^{(D)}$ has the form (6.14)-(6.17).

We shall give a complete proof of this theorem for the case of $U_q(\hat{sl}_2)$ in Appendix B. The general case has no essential changes. The crucial idea is to look to twistings by powers of the Lusztig automorphisms that correspond to translations in the affine Weyl group. It is possible to control the main terms of twisted coproduct and see what is left in topological limit.

Remarks. (i) We can also obtain the explicit expression of the right part of (8.1) by some direct application of q-analogs of H'Adamard identities (see [KT1]). Note that only linear terms (or one-step q-commutators) in H'Adamard identities give nonzero contribution in the formula (8.1) and corresponding terms appear with a constant $(q-q^{-1})$ proportional to a "Planck constant" h. In this sense the comultiplication $\Delta^{(D)}$ has quasiclassical nature.

(ii) The technique developed here allows to obtain the connection between two comultiplications only in one direction. We cannot invert the procedure and obtain $\Delta_{q^{-1}}$ in $U_q(\hat{g})$ from $\Delta^{(D)}$.

Let $U_q^{(D)}(\hat{g})$ denote the second Drinfeld realization of the quantum algebra $U_q(\hat{g})$ with the comultiplication $\Delta^{(D)}$: $U_q(\hat{g}) \mapsto T_q(\hat{g} \otimes \hat{g})$. A natural question arises whether this Hopf algebra is quasitriangular and what is the formula for the universal R-matrix for $U_q^{(D)}(\hat{g})$. Let us at first remind what is going on with standard Hopf algebra structure in $U_q(\hat{g})$. This Hopf algebra is quasitriangular and for given normal ordering satisfying (7.1) the universal R-matrix can be presented in a form (see (4.3)-(4.6)):

$$R = R_+ R_0 R_- K \tag{8.5}$$

where

$$R_+ = \prod_{\alpha<\delta} R_\alpha \ , \qquad R_- = \prod_{\delta>\alpha} R_\alpha \ , \tag{8.6}$$

$$R_0 = \exp((q-q^{-1}) \sum_{n>0} \sum_{i,j}^{mult} c_{ij}(n)(e_{n\delta}^{(i)} \otimes e_{-n\delta}^{(j)})) \tag{8.7}$$

The products in (8.6) are taken over all real roots located only on the left side and only on the right one of imaginary roots in the normal ordering of Δ_+.

By using (8.1) and (4.1) we have that

$$\tilde{\Delta}^{(D)}(a) = (R^{(D)})^{-1} \Delta^{(D)}(a) R^{(D)}, \qquad \forall \ a \in U_q(\hat{g}) \ , \tag{8.8}$$

where

$$R^{(D)} = R_0 R_- K R_+^{21} \tag{8.9}$$

and $R^{(D)}$ may be considered as universal R-matrix for $U_q^{(D)}(\hat{g})$.

Unfortunately, an interpretation of the equality (8.8) in concrete representations is not so simple. Indeed, let V be a finite-dimensional representation of $U_q(g)$, V_{z_1} and V_{z_2} be corresponding two representations of $U_q(\hat{g})$ shifted by z_1 and z_2 (see [FR] for definitions). Then the expressions $\Delta^{(D)}(a)(v_{z_1} \otimes v_{z_2})$, $R_0 R_- K(v_{z_1} \otimes v_{z_2})$ (where $v_{z_1} \in V_{z_1}$, $v_{z_2} \in V_{z_2}$) are regular for $|z_1| < |z_2|$ and singular for $|z_1| > |z_2|$, and vice versa the expressions $\tilde{\Delta}^{(D)}(a)(v_{z_1} \otimes v_{z_2})$, $R_+^{21}(v_{z_1} \otimes v_{z_2})$ are regular for $|z_1| > |z_2|$ and singular for $|z_1| < |z_2|$.

We can rewrite the equality (8.8) as the follows

$$(R_+^{21}(\tilde{\Delta}^{(D)})(a)(R_+^{21})^{-1}(v_{z_1} \otimes v_{z_2}) = (R_0 R_- K)^{-1} \Delta^{(D)}(a) R_0 R_- K(v_{z_1} \otimes v_{z_2}) \tag{8.10}$$

with the left side being originally defined for $|z_1| < |z_2|$ and the right side for $|z_1| > |z_2|$. (The point is that now both sides of (8.10) have no singularities only on diagonal $z_1 = z_2$ and the equality (8.10) has rigorous sense).

Thus we see that there is no definite sense for the representations of $T_q(\hat{q} \otimes \hat{g})$ in $V_{z_1} \otimes V_{z_2}$. On the other hand, the algebra $T_q^+(\hat{q} \otimes \hat{g})$ acts on $V_{z_1} \otimes V_{z_2}$ for $|z_1| < |z_2|$ and the algebra $T_q^-(\hat{q}\otimes\hat{g})$ acts on $V_{z_1}\otimes V_{z_2}$ for $|\ z_1\ |>|\ z_2\ |$. In this context one can consider the comultiplication $\Delta^{(D)}$ as a map $U_q(\hat{g})\to T_q^+(\hat{g}\otimes\hat{g})$ and the opposite comultiplication $\tilde{\Delta}^{(D)}$ as a map $U_q(\hat{g})\to T_q^-(\hat{g}\otimes\hat{g})$ and the universal R-matrix $R^{(D)}$ as the operator: $T_q^-(\hat{q}\otimes\hat{g})\to T_q^+(\hat{g}\otimes\hat{g})$. In terms of $V_{z_1}\otimes V_{z_2}$ the operator $R^{(D)}$ has entries being generalized functions of $z=\frac{z_1}{z_2}$.

For illustration we consider concrete example $g=sl_2$. Let ρ be a two-dimensional representation, then modulo scalar function (see [KT2]) we have the following formulas

$$(\rho_{z_1}\otimes\rho_{z_2})R_+=1\otimes 1+(\sum_{n\geq 0}z^n)(e_{12}\otimes e_{21})\ , \tag{8.11}$$

$$\begin{aligned}(\rho_{z_1}\otimes\rho_{z_2})R_0 &= e_{11}\otimes e_{11}+e_{22}\otimes e_{22}+\\ &+(\exp\sum_{n>0}\frac{q^{2n}-1}{n}z^n)(e_{11}\otimes e_{22})+(\exp\sum_{n>0}\frac{1-q^{-2n}}{n}z^n)(e_{22}\otimes e_{11})\ ,\end{aligned} \tag{8.12}$$

$$(\rho_{z_1}\otimes\rho_{z_2})R_-=1\otimes 1+(\sum_{n\geq 0}z^n)(e_{21}\otimes e_{12})\ , \tag{8.13}$$

$$(\rho_{z_1}\otimes\rho_{z_2})K=q^{\frac{1}{2}}(e_{11}\otimes e_{11}+e_{22}\otimes e_{22})+q^{-\frac{1}{2}}(e_{11}\otimes e_{22}+e_{22}\otimes e_{11})\ , \tag{8.14}$$

$$(\rho_{z_1}\otimes\rho_{z_2})R_+^{21}=1\otimes 1+(\sum_{n\geq 0}z^{-n})(e_{21}\otimes e_{12})\ , \tag{8.15}$$

and also

$$\begin{aligned}(\rho_{z_1}\otimes\rho_{z_2})R^{(D)} &= (q^{\frac{1}{2}}(e_{11}\otimes e_{11}+e_{22}\otimes e_{22})+\frac{1-z}{q^{\frac{1}{2}}(1-q^2z)}(e_{11}\otimes e_{22})+\\ &+\frac{1-q^{-2}z}{q^{\frac{1}{2}}(1-z)}(e_{22}\otimes e_{11})\cdot(1\otimes 1+\delta(z)(e_{21}\otimes e_{12}))\end{aligned} \tag{8.16}$$

where $z:=\frac{z_1}{z_2}$, $\delta(z):=\sum_{-\infty}^{\infty}z^n$.

Appendices.

In this section we exhibit the construction of Cartan-Weyl basis and give the complete list of commutation relations between Cartan-Weyl generators for $U_q(\hat{sl}_2)$. We also demonstrate here the proof of the Theorem 8.1 for this case.

A.The Cartan-Weyl basis of $U_q(\hat{sl}_2)$.

Let α and $\beta:=\delta-\alpha$ are simple roots for the affine algebra $\hat{sl}_2$ then $\delta=\alpha+\beta$ is a minimal imaginary root. We fix the following normal ordering in Δ_+:

$$\alpha,\ \delta+\alpha,\ldots,\infty\delta+\alpha,\ \delta,\ 2\delta,\ldots,\infty\delta,\ \infty\delta-\alpha,\ldots,2\delta-\alpha,\ \delta-\alpha\ . \tag{A.1}$$

The another normal ordering is an inverse to (A.1):

$$\delta-\alpha,\ 2\delta-\alpha,\ldots,\infty\delta-\alpha,\ \delta,\ 2\delta,\ldots,\infty\delta,\ \infty\delta+\alpha,\ldots,\delta+\alpha,\ \alpha\ . \tag{A.2}$$

In accordance with our procedure for construction of the Cartan-Weyl basis we put ($n = 1, 2, \ldots$)

$$e_{\delta} = [e_{\alpha}, e_{\delta-\alpha}]_q \ , \tag{A.3}$$

$$e_{n\delta+\alpha} = (-1)^n \left([(\alpha,\alpha)]_q\right)^{-n} (\mathrm{ad}\ e_{\delta})^n e_{\alpha} \ , \tag{A.4}$$

$$e_{(n+1)\delta-\alpha} = \left([(\alpha,\alpha)]_q\right)^{-n} (\mathrm{ad}\ e_{\delta})^n e_{\delta-\alpha} \ , \tag{A.5}$$

$$e'_{(n+1)\delta} = [e_{n\delta+\alpha}, e_{\delta-\alpha}]_q \tag{A.6}$$

and then we redefine the imaginary roots $e'_{n\delta}$ by means of the Schur polynomials:

$$e'_{n\delta} = \sum_{p_1+2p_2+\ldots+np_n=n} \frac{(q-q^{-1})^{\sum p_i - 1}}{p_1!\ldots p_n!} e_{\delta}^{p_1} e_{2\delta}^{p_2} \ldots e_{n\delta}^{p_n} \ . \tag{A.7}$$

We take also $e_{-\gamma} = e^{*}_{\gamma}$, $(\gamma \in \Delta_+)$.

The following formulas are a total list of the relations for the Cartan-Weyl generators $e_{\pm\gamma}$, $(\gamma \in \Delta_+)$:

$$k_{\gamma} e_{\pm\gamma'} = q^{\pm(\gamma,\gamma')} e_{\pm\gamma'} k_{\gamma} \ , \qquad \gamma,\ \gamma' \in \Delta_+ \ , \tag{A.8}$$

$$[e_{\gamma}, e_{-\gamma}] = \frac{k_{\gamma} - k_{\gamma}^{-1}}{q - q^{-1}} \ , \qquad \gamma \neq n\delta \ , \tag{A.9}$$

$$[e_{n\delta}, e_{m\delta}] = \delta_{n,-m} a(n) \frac{k_{\delta}^{n} - k_{\delta}^{-n}}{q - q^{-1}} \ , \qquad n,\ m \neq 0 \ , \tag{A.10}$$

$$[e_{n\delta+\alpha}, e_{m\delta-\alpha}]_q = e'_{(n+m)\delta} \ , \qquad n \geq 0,\ m > 0 \ , \tag{A.11}$$

$$[e_{n\delta+\alpha}, e_{-m\delta-\alpha}] = -e'_{(n-m)\delta} k_{\delta}^{-m} k_{\alpha}^{-1} \ , \qquad n > m \geq 0 \ , \tag{A.12}$$

$$[e_{n\delta-\alpha}, e_{-m\delta+\alpha}] = k_{\delta}^{m} k_{\alpha}^{-1} e'_{(n-m)\delta} \ , \qquad n > m\ > 0 \ , \tag{A.13}$$

$$[e_{n\delta+\alpha}, e_{m\delta}] = a(m) e_{(n+m)\delta+\alpha} \ , \qquad n \geq 0,\ m > 0, \tag{A.14}$$

$$[e_{n\delta}, e_{m\delta-\alpha}] = a(n) e_{(n+m)\delta-\alpha} \ , \qquad n,\ m > 0 \ , \tag{A.15}$$

$$[e_{n\delta+\alpha}, e_{-m\delta}] = a(n) e_{(n-m)\delta+\alpha} k_{\delta}^{n} \ , \qquad n \geq m > 0 \ , \tag{A.16}$$

$$[e_{-n\delta}, e_{m\delta-\alpha}] = a(n) e_{(m-n)\delta-\alpha} k_{\delta}^{-n} \ , \qquad m \geq n > 0 \ , \tag{A.17}$$

$$[e_{n\delta+\alpha}, e_{(n+2m-1)\delta+\alpha}]_q = (q_{\alpha}^{2} - 1) \sum_{l=1}^{m-1} q_{\alpha}^{-l} e_{(n+l)\delta+\alpha} e_{(n+2m-1-l)\delta+\alpha} \ , \tag{A.18}$$

$$\begin{aligned}[e_{n\delta+\alpha}, e_{(n+2m)\delta+\alpha}]_q &= (q_{\alpha} - 1) q_{\alpha}^{(m-1)} e^{2}_{(n+m)\delta-\alpha} + \\ &+ (q_{\alpha}^{2} - 1) \sum_{l=1}^{m-1} q_{\alpha}^{-l} e_{(n+l)\delta+\alpha} e_{(n+2m-l)\delta+\alpha} \ ,\end{aligned} \tag{A.19}$$

for any $n \geq 0,\ m > 0$

$$[e_{(n+2m-1)\delta-\alpha}, e_{(n\delta-\alpha}]_q = (q_{\alpha}^{2} - 1) \sum_{l=1}^{m-1} q_{\alpha}^{-l} e_{(n+2m-1-l)\delta-\alpha} e_{(n+l)\delta-\alpha} \ , \tag{A.20}$$

$$[e_{(n+2m\delta-\alpha}, e_{n\delta+\alpha}]_q = (q_{\alpha} - 1) q_{\alpha}^{(m-1)} e^{2}_{(n+m)\delta-\alpha} +$$

$$+(q_\alpha^2-1)\sum_{l=1}^{m-1} q_\alpha^{-l} e_{(n+l)\delta+\alpha} e_{(n+2m-l)\delta+\alpha} \ , \tag{A.21}$$

for any $n,\ m>0$. Here in (A.10), (A.14)-(A.17) the coefficient $a(n)$ is determined by the formula (6.8) with $\alpha_i=\alpha_j=\alpha$.

In order to obtain the rest of the relations between root vectors we have to extend the relations (A.11)-(A.21) to arbitrary values of n. This can be done if we use the circular generators $\hat{e}_{\pm\gamma}$ and $\check{e}_{\pm\gamma}$ (see (3.12), (3.13)). More precisely, let

$$\hat{e}_{n\delta+\alpha}=e_{n\delta+\alpha} \ , \qquad \hat{e}_{-(n+1)\delta+\alpha}=-k_\alpha k_\delta^{-n} e_{-n\delta+\alpha}, \ \ n\geq 0 \ , \tag{A.22}$$

$$\check{e}_{(n+1)\delta-\alpha}=-e_{(n+1)\delta-\alpha}k_\alpha^{-1}k_\delta^n \ , \qquad \check{e}_{-n\delta-\alpha}=e_{-n\delta-\alpha}, \ \ n\geq 0 \ . \tag{A.23}$$

Then the relations (A.11)-(A.21) transform to the same formulas where $e_{n\delta+\alpha}$ replaced everywhere by $\hat{e}_{n\delta+\alpha}$, $e_{n\delta-\alpha}$ replaced by $\check{e}_{(n+1)\delta-\alpha}$ with the only restriction $n\geq 0$. Now we have after conjugation by Cartan involution (*) the complete list of the relations for Cartan-Weyl generators.

Remark. We can observe that the relations (A.14)-(A.17) may be rewritten in quadratic form if we rewrite the relations in terms of $e'_{n\delta}$, for instance,

$$[\hat{e}_{n\delta+\alpha}, e'_{m\delta}]=q_\alpha^{-(m-1)}\hat{e}_{(n+m)\delta+\alpha}+(q_\alpha^2-1)\sum_{l}^{n-1} q_\alpha^{-l}\hat{e}_{(n+l)\delta+\alpha}e'_{(m-l)\delta} \tag{A.24}$$

The Drinfeld's generators in the case of $U_q(\hat{sl}_2)$ have the form:

$$\xi_n^+=\hat{e}_{n\delta+\alpha} \ , \qquad \xi_n^-=\hat{e}_{n\delta-\alpha} \ , \qquad \xi_0=k_\alpha^{-1} \ , \tag{A.25}$$

for any $n\in\mathbf{Z}$ and

$$\psi_n=(q-q^{-1})k_\alpha e'_{n\delta}, \qquad \phi_{-n}=(q-q^{-1})k_\alpha^{-1}e'_{-n\delta} \tag{A.26}$$

for $n>0$.

B. The connection between two comultiplications for $U_q(\hat{sl}_2)$. The proof of the Theorem 8.1.

Let s_α and $s_{\delta-\alpha}$ are the elementary reflections of the Weyl group of $\hat{sl}_2$. The explicit formulas for the Lusztig automorphisms $\hat{s}_\alpha$ and $\hat{s}_{\delta-\alpha}$ in $U_q(\hat{sl}_2)$ look as follows:

$$\hat{s}_\alpha(k_\alpha)=k_\alpha^{-1} \ , \qquad \hat{s}_{\delta-\alpha}(k_{\delta-\alpha})=k_{\delta-\alpha}^{-1} \ , \tag{B.1}$$

$$\hat{s}_\alpha(k_{\delta-\alpha})=k_\alpha k_\delta \ , \qquad \hat{s}_{\delta-\alpha}(k_\alpha)=k_\alpha k_\delta \ , \tag{B.2}$$

$$\hat{s}_\alpha(e_\alpha)=-e_{-\alpha}k_\alpha \ , \qquad \hat{s}_{\delta-\alpha}(e_{\delta-\alpha})=-e_{-\delta+\alpha}k_{\delta-\alpha} \ , \tag{B.3}$$

$$\hat{s}_\alpha(e_{-\alpha})=-k_\alpha^{-1}e_\alpha \ , \qquad \hat{s}_{\delta-\alpha}(e_{-\delta+\alpha})=-k_{\delta-\alpha}^{-1}e_{\delta-\alpha} \ , \tag{B.4}$$

$$\hat{s}_\alpha(\tilde{e}_{n\delta\pm\alpha})=e_{n\delta\mp\alpha} \ , \qquad \hat{s}_{\delta-\alpha}(e_{(n\mp1)\delta\pm\alpha})=\tilde{e}_{(n\pm1)\delta+\alpha} \ , \tag{B.5}$$

$$\hat{s}_\alpha(\tilde{e}_{n\delta})=e_{n\delta} \ , \qquad \hat{s}_{\delta-\alpha}(e_{n\delta})=\tilde{e}_{n\delta} \ , \tag{B.6}$$

for any integers $n \neq 0$. Here in (B.5) and (B.6) the root vectors e_γ are constructed in accordance with the normal ordering (A.1) and the root vectors $\tilde{e}_\gamma$ in accordance with the inverse normal ordering (A.2).

If we put $\hat{t}_{2\delta} := \hat{s}_\alpha \hat{s}_\beta$ ($t_{2\delta}$ is a translation in the Weyl group) then we have from (B.1)-(B.4) the relations

$$\hat{t}_{2\delta}(k_\alpha) = k_\alpha k_\delta^2 , \qquad \hat{t}_{2\delta}(k_\beta) = k_\beta k_\delta^{-2} , \tag{B.7}$$

$$\hat{t}_{2\delta}(k_\delta) = k_\delta , \qquad \hat{t}_{2\delta}(e_{n\delta}) = e_{n\delta} , \tag{B.8}$$

$$\hat{t}_{2\delta}(e_{\delta-\alpha}) = -e_{-\delta-\alpha} k_\alpha k_\delta , \qquad \hat{t}_{2\delta}(e_{-\delta+\alpha}) = k_\alpha^{-1} k_\delta^{-1} e_{-\delta-\alpha} , \tag{B.9}$$

$$\hat{t}_{2\delta}(e_{2\delta-\alpha}) = -e_{-\alpha} k_\alpha , \qquad \hat{t}_{2\delta}(e_{-2\delta+\alpha}) = k_\alpha^{-1} e_{-\alpha} , \tag{B.10}$$

$$\hat{t}_{2\delta}(e_{(n\mp 1)\delta\pm\alpha}) = e_{(n\pm 1)\delta\pm\alpha} , \tag{B.11}$$

(for any integers $n \neq 0$).

Using general arguments of the Section 8 we have

$$(\hat{t}_{2\delta}^n \otimes \hat{t}_{2\delta}^n)\Delta_{q^{-1}} \hat{t}_{2\delta}^{-n}(a) = \left(R^{21}_{(2n-1)\delta+\alpha}\right)^{-1} \cdots \left(R^{21}_\alpha\right)^{-1} \Delta_{q^{-1}}(a) R^{21}_\alpha \cdots R^{21}_{(2n-1)\delta+\alpha} \tag{B.12}$$

for any $a \in U_q(\hat{sl}_2)$. Now we want to investigate the limits of both sides of (B.12) when $n \to \infty$. For case $a = e_\delta$ we have

$$\Delta_{q^{-1}}(e_\delta) = e_\delta \otimes 1 + k_\delta^{-1} \otimes e_\delta + (q_\alpha - q_\alpha^{-1}) e_\beta k_\alpha^{-1} \otimes e_\alpha \tag{B.13}$$

and

$$(\hat{t}_{2\delta}^n \otimes \hat{t}_{2\delta}^n \Delta_{q^{-1}}(\hat{t}_{2\delta}^{-n}(e_\delta)) =$$
$$= e_\delta \otimes 1 + k_\delta^{-1} \otimes e_\delta + (q_\alpha - q_\alpha^{-1}) e_{(-2n+1)\delta-\alpha} k_\alpha^{-1} \otimes e_{2n\delta+\alpha} . \tag{B.14}$$

The last summand tends to zero in the FS topology so we have

$$\lim_{n\to\infty} (\hat{t}_{2\delta}^n \otimes \hat{t}_{2\delta}^n)\Delta_{q^{-1}}(\hat{t}_{2\delta}^{-n}(e_\delta)) = \Delta^{(D)}(e_\delta) = e_\delta \otimes 1 + k_\delta^{-1} \otimes e_\delta \tag{B.15}$$

and analogously for other imaginary root vectors.

Now let us consider the real root vectors, for example, $a = e_\alpha$. We have $\hat{t}_{2\delta}^{-n}(e_\alpha) = -k_\alpha k_\delta^{-2n} e_{-2n\delta+\alpha}$ and have to investigate behavior of the element $\Delta_{q^{-1}}(e_{-2n\delta+\alpha})$ for large n. By induction we see that $\Delta_{q^{-1}}(e_{-2n\delta+\alpha})$ consists of the following monomials $e_{-2n\delta+\alpha} \otimes k_\alpha k_\delta^{-2n}$, and

$$a \otimes b = e_{-\alpha}^{l_0} e_{-\delta-\alpha}^{l_1} \cdots e_{-2m\delta-\alpha}^{l_{2m}} e_{-\delta}^{p_1} \cdots e_{-2m\delta}^{p_{2m}} \otimes e_{-\delta+\alpha}^{l'_1} \cdots e_{-2m\delta+\alpha}^{l'_{2m}} ,$$

with coefficients from $\mathrm{Frac}(U_q(\kappa \otimes \kappa))$ (see Section 3), where $m \geq n$. Further we have

$$(\hat{t}_{2\delta}^n \otimes \hat{t}_{2\delta}^n)(a \otimes b) = e_{-2n\delta-\alpha}^{l_0} \cdots e_{-2(m+n)\delta-\alpha}^{l_{2m}} e_{-\delta}^{p_1} \cdots e_{-2m\delta}^{p_{2m}} \otimes e_{(2n-1)\delta+\alpha}^{l'_1} \cdots e_{2(n-m)\delta+\alpha}^{l'_{2m}}$$

¿From weight analysis it is clear that the only nonvanishing terms in the FS topology (or for topology in $T^+(\hat{g} \otimes \hat{g})$) for $\Delta^{(D)}(e_\alpha)$ are

$$e_\alpha \otimes 1 \qquad \text{and} \qquad e_{-m\delta} \otimes e_{m\delta+\alpha}, \qquad (m \geq 0) ,$$

with coefficients from $\mathrm{Frac}(U_q(\kappa \otimes \kappa))$. After inductive calculation of these coefficients we have (8.2) and the statement of the theorem.

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